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A generalization of Bochner's extension theorem and its application

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1. Introduction

Let M be a smooth paracompact real manifold. Let Z be a closed smooth submanifold of M. Let P be a differential operator (or a matrix of differential operators) on M with smooth coefficients. In the following m denotes the order of P. (This notation will be used through this paper.) For a distribution f on M, we consider solutions of the differential equation

$$Pu=f \tag{1.1}$$

defined on $M\setminus Z$.

In [B], Bochner proved the following extension theorem.

Theorem 1.1. ([B]) *Let u be a locally integrable function on M which satisfies*

$$Pu=0 \tag{1.2}$$

outside Z. Let $x\in Z$. Let $p>1$ and assume that

$$m\le d(1-1/p)$$

with $d=\operatorname{codim} Z$. If

$$\int_U |u|^p\,dx<\infty \tag{1.3}$$

on a neighborhood U of x, u satisfies the equation $Pu=0$ on U.

In [B], the apparent singular locus Z is not assumed to be a smooth submanifold (see also [HP]) and (1.3) is replaced by a weaker condition. See [EP], [HP], [P] and

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the references cited there for the works which generalize the result of Bochner [B] in connection with potential theory.

In this paper, we restrict ourselves to the case when Z is a smooth submanifold and consider equation (1.1). If the right-hand side f is locally integrable on M, we can easily prove the same result as above for $Pu=f$ by the same proof. We treat the case when f is not necessarily locally integrable. At the same time, as we state in Theorem 2.2, we weaken the boundary condition (1.3) on u (which controls the growth of u near the boundary) from a microlocal point of view. The proof presented in this paper is accordingly based on microlocal analysis, and it gives also a new proof of Theorem 1.1.

In Section 6, as an application of Theorem 2.2, we extend Theorem 1.1 to semilinear differential equations.

Notation. By $\mathcal{D}'$ we denote the sheaf of distributions on M. For $p\geq 1$, $L_{p,\mathrm{loc}}$ denotes the sheaf of locally p-integrable functions on M.

2. Main result

Let T^*M denote the cotangent bundle of M and $\pi: T^*M \to M$ the projection.

Definition 2.1. Let f be a distribution on M. For $p\geq 1$ and $z\in T^*M$, f is microlocally p-integrable (or microlocally integrable, if $p=1$) at z if there exists an $L_{p,\mathrm{loc}}$ germ g at $\pi(z)$ such that $f-g$ is microlocally smooth at z (in the usual sense). We set

$$\mathcal{L}_p(z) = \{f \in \mathcal{D}'_{\pi(z)} \mid f \text{ is microlocally } p\text{-integrable at } z\}.$$

Let Z be a closed submanifold of M of class C^∞ and let $T^*_Z M$ denote the conormal bundle of Z.

Let f be a distribution on M.

Theorem 2.2. *Let u be a distribution on M which satisfies*

$$Pu=f \tag{2.1}$$

outside Z. Let $x\in Z$. Let $z\in (T^*_Z M)_x$ and assume that f is microlocally integrable at z. Let $p>1$ and assume that

$$m\leq d(1-1/p) \tag{2.2}$$

with $d=\operatorname{codim} Z$. If

$$u\in \mathcal{L}_p(z), \tag{2.3}$$

then u satisfies the equation $Pu=f$ on a neighborhood of x.

This theorem implies that, if f is microlocally integrable and u satisfies the microlocal p-integrability condition (2.3) instead of (1.3), the solution u of (2.1) outside Z is locally extendable to Z. In particular, we have the following corollary.

Corollary 2.3. *Let $x \in Z$. Let f be a locally integrable function on M. Let u be a distribution on $M \setminus Z$ which satisfies $Pu=f$. Let $p>1$ and assume that $m \leq d(1-1/p)$, with $d=\operatorname{codim} Z$. If*

$$\int_{U \setminus Z} |u|^p \, dx < \infty \tag{2.4}$$

on a coordinate neighborhood U of x, then u is extendable to a neighborhood of x as a distribution solution of $Pu=f$.

3. Proof of Theorem 2.2 (Reduction to the delta function)

The proof of Theorem 2.2 will be given in this and the next sections. In this section, we reduce Theorem 2.2 to a microlocal estimate of the delta function.

Let $n=\dim M$.

Notation. Let $p>1$ and $s \in \mathbf{R}$. For an open subset U of M, $\mathcal{L}^s_{p,\mathrm{loc}}(U)$ denotes the space of distributions on U locally in L^s_p. (Here L^s_p denotes the L_p Sobolev space of order s on $\mathbf{R}^n$.)

Definition 3.1. Let f be a distribution on M. For $p>1$, $s \in \mathbf{R}$ and $z \in T^*M$, if there exists a distribution germ g at $\pi(z)$ such that $f-g$ is microlocally smooth at z and g is locally in L^s_p, we say that f is microlocally in L^s_p at z. We set

$$\mathcal{L}^s_p(z) = \{f \in \mathcal{D}'_{\pi(z)} \mid f \text{ is microlocally in } L^s_p \text{ at } z\}.$$

Let us take a local coordinate $(x_1, \dots, x_n)$ of M so that

$$Z = \{(x_1, \dots, x_n) \mid x_{n-d+1} = \dots = x_n = 0\}$$

and $x=0$. We denote $x'=(x_1, \dots, x_{n-d})$, $x''=(x_{n-d+1}, \dots, x_n)$ and use the notation

$$D' = (D_1, \dots, D_{n-d}), \quad D'' = (D_{n-d+1}, \dots, D_n),$$

where $D_k = \partial/\partial x_k$, $k=1, \dots, n$.

By the structure theorem of distributions with support in Z, we have

$$Pu-f=Q(x',D'')\delta(x'') \tag{3.1}$$

for a differential operator $Q(x',D'')$ with distribution coefficients in x' and independent of D', where $\delta(x'')$ denotes the delta function in x''.

Let $z\in(T_Z^*M)_0$ such that f is microlocally integrable at z and u microlocally p-integrable at z. Then Pu is microlocally in L_p^{-m} at z ($m=\operatorname{ord}P$), and locally

$$f=g+h, \quad z\notin \mathrm{WF}(h)$$

with $g\in L_{1,\mathrm{loc}}(M)$. (Here $\mathrm{WF}(h)$ denotes the C^∞ wavefront set of h as usual.) Hence, from (3.1), we have

$$Q(x',D'')\delta(x'')+g(x)\in\mathcal{L}_p^{-m}(z) \tag{3.2}$$

with some $g\in L_{1,\mathrm{loc}}(M)$ compactly supported in a coordinate neighborhood.

We shall prove that (3.2) implies $Q=0$ if $m\le d(1-1/p)$. From (3.1), this completes the proof of Theorem 2.2.

We may assume that $z=(0,dx_n)$ (by a linear coordinate transform).

Let us take a coordinate neighborhood U as $U'\times U''$ with U' being a bounded open neighborhood of $x'=0$ in $\mathbf{R}^{n-d}$ and U'' of $x''=0$ in $\mathbf{R}^d$. Let us take a test function $\varphi(x')\in\mathcal{D}(U')$. It is sufficient to prove that the distribution

$$\int\varphi(x')Q(x',D'')\delta(x'')\,dx' \tag{3.3}$$

is zero in a neighborhood of $x''=0$ (for any φ).

We now fix $\varphi\in\mathcal{D}(U')$ and consider the integration

$$\int_\varphi:\mathcal{D}'(U'\times U'')\longrightarrow\mathcal{D}'(U''),\quad g\longmapsto\int\varphi(x')g(x',x'')\,dx'.$$

For $p>1$ and $s\in\mathbf{R}$, let us set

$$\begin{aligned}\mathcal{L}_{p,\mathrm{loc}}^s(U\times\{dx_n\})=\{u\in\mathcal{D}'(U)\mid u=u_0+u_1,\ &\text{with } u_0\in\mathcal{L}_{p,\mathrm{loc}}^s(U) \text{ and}\\ &u_1\in\mathcal{D}'(U) \text{ such that } \mathrm{WF}(u_1)\cap(U\times\{dx_n\})=\emptyset\}\end{aligned}$$

and $\mathcal{L}_{p,\mathrm{loc}}^s(U''\times\{dx_n\})$ in the same manner. Then we have the following lemma.

Lemma 3.2. *We have*

$$\int_\varphi \mathcal{L}^s_{p,\mathrm{loc}}(U\times\{dx_n\})\subset\mathcal{L}^s_{p,\mathrm{loc}}(U''\times\{dx_n\}).$$

Proof of Lemma 3.2. This follows from the microlocal theory of integration and the fact that

$$\int_\varphi \mathcal{L}^s_{p,\mathrm{loc}}(U)\subset\mathcal{L}^s_{p,\mathrm{loc}}(U''),\quad \text{if } p>1.$$

The proof of this fact will be given in Section 5.

Since the distribution (3.3) is of the form $Q(D'')\delta(x'')$, by Lemma 3.2 above, we are reduced to proving that $Q=0$ if

$$Q(D)\delta(x)\in\mathcal{L}^{-m}_p(0,dx_n)+L_{1,\mathrm{loc}}(U),\quad \text{with } m\le n(1-1/p),$$

on $\mathbf{R}^n$.

4. End of the proof of Theorem 2.2

In this section, we prove the following proposition, which completes the proof of Theorem 2.2. Let $\delta(x)$ denote the delta function on $\mathbf{R}^n$ with support at 0.

Proposition 4.1. *Let $Q(D)$ be a differential operator with constant coefficients in n variables. If*

$$Q(D)\delta(x)\in\mathcal{L}^{-m}_p(0,dx_n)+L_{1,\mathrm{loc}}(U),$$

U being a neighborhood of 0, then $\operatorname{ord}Q<m-n(1-1/p)$. *In particular, if $m\le n(1-1/p)$, we have $Q=0$.*

For the definition of $\mathcal{L}^{-m}_p(0,dx_n)$, see Definition 3.1.

In what follows, we denote by L_p the Banach space $L_p(\mathbf{R}^n,dx)$, dx being the standard volume element on $\mathbf{R}^n$, and by $\|\cdot\|_{L_p}$ the norm of L_p. For $s\in\mathbf{R}$, L^s_p denotes the L_p Sobolev space of order s with respect to dx. In this section, the symbol D denotes $-i\partial/\partial x$.

Let us first recall the notion of Besov spaces. Let $\Phi=\Phi(\xi)$ be a function for the Littlewood–Paley decomposition, that is, Φ a smooth function satisfying the conditions of [BL, Lemma 6.1.7], and set $\Psi(\xi)=1-\sum_{\nu\in\mathbf{N}}\Phi(\xi/2^\nu)$. Let $p\ge1$, $q\ge1$. The Besov space $B^0_{p,q}$ on $\mathbf{R}^n$ is defined as

$$B^0_{p,q}=\left\{f\in\mathcal{S}'(\mathbf{R}^n)\;\middle|\;\Psi(D)f\in L_p,\ \Phi\left(\frac{D}{2^\nu}\right)f\in L_p,\ \nu\in\mathbf{N},\ \sum_{\nu\in\mathbf{N}}\left\|\Phi\left(\frac{D}{2^\nu}\right)f\right\|^q_{L_p}<\infty\right\},$$

where $\mathcal{S}'(\mathbf{R}^n)$ denotes the space of tempered distributions. We then have the following lemma.

Lemma 4.2. ([BL, Theorem 6.4.4]) *Let* $p>1$. *For* $q=\max\{p, 2\}$, *we have* $L_p \subset B^0_{p,q}$ *and this embedding is bounded.*

The following lemma is a microlocalization of the fact that one point of an n-dimensional manifold has q-capacity zero for $1<q\leq n$ if $n\geq 2$ (see [EP, Section 2]).

Let $\chi(\xi)$ be a smooth function, homogeneous of degree 0 on $|\xi|>1$.

Lemma 4.3. *Let* $p>1$. *If* χ *is not identically zero on a neighborhood of* ∞, *then* $\chi(D)\delta(x)$ *is in* L^s_p *if and only if* $s+n(1-1/p)<0$.

Proof. Assume $\chi(D)\delta(x)\in L^s_p$. We may assume from the beginning that $\chi=0$ identically on a neighborhood of $\xi=0$. We then have, by definition, $\chi(D)|D|^s\delta(x)\in L_p$. By Lemma 4.2, letting $q=\max\{p, 2\}$, we have

$$\sum_{\nu\geq 0} \|\Phi(D/2^\nu)\chi(D)|D|^s\delta(x)\|^q_{L_p} < \infty. \tag{4.1}$$

By putting $\xi=2^\nu\eta$ in the Fourier integral, we have

$$\|\Phi(D/2^\nu)\chi(D)|D|^s\delta(x)\|_{L_p} = 2^{[s+n(1-1/p)]\nu}C, \quad \text{if } \nu\gg 1, \tag{4.2}$$

with $C>0$, independent of ν. Hence (4.1) yields $s+n(1-1/p)<0$.

The if part also follows from (4.2) (since $B^0_{p,1}\subset L_p$). □

In the following proof, $\mathcal{L}(L^s_p)$ denotes the Banach space of bounded operators on L^s_p with the operator norm.

Proof of Proposition 4.1. From the hypothesis, by multiplying by a cut-off function, we may assume that

$$Q(D)\delta(x)+g(x)\in \mathcal{L}^{-m}_{p,\mathrm{loc}}(U\times\{dx_n\})$$

with some $g\in L_{1,\mathrm{loc}}(U)$. Moreover we may assume that $\operatorname{supp} g$ is compact and $\|g\|_{L_1}\leq 1/A$ for some sufficiently large A.

Let $\varphi(\xi)$ denote the Fourier transform of g; then $|\varphi(\xi)|\leq 1/A$. For $f\in\mathcal{S}'$, let

$$\varphi(D)f = F^{-1}[\varphi(\xi)Ff(\xi)],$$

where F denotes the Fourier transform. The operator $\varphi(D)$ is a bounded linear operator on L^s_p, for any $s\in\mathbf{R}$, and

$$\|\varphi(D)\|_{\mathcal{L}(L^s_p)} \leq 1/A.$$

We then have

$$[Q(D)+\varphi(D)]\delta(x)\in \mathcal{L}^{-m}_{p,\mathrm{loc}}(U\times\{dx_n\}).$$

Hence, the left-hand side being compactly supported, there is an open convex cone Δ of $\mathbf{R}^n$ and a smooth function $\beta(\xi)$ such that $dx_n\in\Delta$, $\beta\equiv 1$ on $\Delta(B)$, where $B>0$ and $\Delta(B)=\{\xi\in\Delta\,|\,|\xi|\geq B\}$, and

$$\beta(D)[Q(D)+\varphi(D)]\delta(x)\in L^{-m}_p. \tag{4.3}$$

Let $q=\operatorname{ord} Q$; we can then find an open convex cone Γ of $\mathbf{R}^n$, $R\geq B$ and $\delta>0$ so that $\Gamma\subset\Delta$ and

$$|Q(\xi)|\geq\delta|\xi|^q \quad \text{on } \Gamma(R),$$

where $\Gamma(R)=\{\xi\in\Gamma\,|\,|\xi|\geq R\}$. Hence $|\varphi(\xi)|\leq\frac{1}{2}|Q(\xi)|$ on $\Gamma(R)$, if $R\geq 1$ and $A\geq 2/\delta$; in particular, $Q+\varphi\neq 0$ on $\Gamma(R)$.

Now let $R''>R'>R$. Let $\chi(\xi)$ be a smooth function, homogeneous of degree 0 on $|\xi|>R''$ and supported in $\Gamma(R')$. Then $\chi/(Q+\varphi)$ defines, as Fourier multiplier, a continuous linear mapping $(\chi/(Q+\varphi))(D)\colon L^s_p\to L^{s+q}_p$ for any s,

$$\frac{\chi}{Q+\varphi}(D)f=F^{-1}\left[\frac{\chi}{Q+\varphi}(\xi)Ff(\xi)\right]. \tag{4.4}$$

To see this, let $\chi_1(\xi)$ be a smooth function having the same properties as χ (for some R' and R'') and assume that $|\chi_1(\xi)|\leq 1$ and $\chi_1=1$ identically on a neighborhood of $\operatorname{supp}\chi$. Then we have

$$\frac{\chi}{Q+\varphi}=\frac{\chi}{Q}\sum_{\nu\geq 0}\left(-\frac{\chi_1\varphi}{Q}\right)^{\nu}.$$

Since $|\varphi/Q|\leq\frac{1}{2}$ on $\Gamma(R)$, the right-hand side converges uniformly. Moreover, since $|D^\alpha(\varphi/Q)|$ is bounded on $\Gamma(R)$ for any α, any derivative converges uniformly in the right-hand side. Hence

$$\frac{\chi}{Q+\varphi}(D)=\frac{\chi}{Q}(D)\sum_{\nu\geq 0}\left(-\frac{\chi_1\varphi}{Q}\right)^{\nu}(D). \tag{4.5}$$

It follows from the L_p-boundedness theorem of Fourier multipliers (cf. [T, Chapter XI, Section 1]) that the first factor χ/Q defines a continuous linear mapping $L^s_p\to L^{s+q}_p$. On the other hand, since $\|\varphi(D)\|_{\mathcal{L}(L^s_p)}\leq 1/A$, we have

$$\left\|\frac{\chi_1\varphi}{Q}(D)\right\|_{\mathcal{L}(L^s_p)}\leq\left\|\frac{\chi_1}{Q}(D)\right\|_{\mathcal{L}(L^s_p)}\|\varphi(D)\|_{\mathcal{L}(L^s_p)}\leq\frac{1}{2},$$

if $A \gg 1$. Hence the infinite sum

$$\sum_{\nu\geq 0}\left[-\frac{\chi_1\varphi}{Q}(D)\right]^{\nu}$$

converges in $\mathcal{L}(L_p^s)$ (for any s), and the second factor of the right-hand side of (4.5) defines a bounded operator on L_p^s. Thus (4.4) defines a continuous linear mapping $(\chi/(Q+\varphi))(D)\colon L_p^s \to L_p^{s+q}$.

Hence, from (4.3), we have

$$\chi(D)\delta(x) \in L_p^{-m+q}.$$

By Lemma 4.3, we have $q<m-n(1-1/p)$. This completes the proof. □

5. Integration of distributions of Sobolev class

In this section, we give a proof to the following proposition, by which the proof of Lemma 3.2 is completed. Let $p>1$ and $s\in\mathbf{R}$. Recall that we denote by $\mathcal{L}_{p,\mathrm{loc}}^s$ the sheaf of distributions locally in L_p^s on a manifold.

Proposition 5.1. *Let U' and U'' be real manifolds. Let dx' be a volume element on U'. Let $g\in\mathcal{L}_{p,\mathrm{loc}}^s(U'\times U'')$ with compact support. Then the distribution*

$$\int_{U'} g(x',x'')\,dx'$$

belongs to $\mathcal{L}_{p,\mathrm{loc}}^s(U'')$.

Proof. We may assume from the beginning that U' and U'' are open subsets of $\mathbf{R}^n$ and $\mathbf{R}^k$, respectively, and that dx' is the standard volume element of $\mathbf{R}^n$. Let x' and x'' denote the coordinate of $\mathbf{R}^n$ and $\mathbf{R}^k$, respectively. The symbol D' (resp. D'') denotes $-i\partial/\partial x'$ (resp. $-i\partial/\partial x''$).

For $u\in\mathbf{R}$, we set $\langle D'\rangle^u=(1+(D')^2)^{u/2}$, $\langle D''\rangle^u=(1+(D'')^2)^{u/2}$.

Let us take $\varphi(x')\in\mathcal{D}(U')$ such that $\varphi\equiv 1$ on a neighborhood of $\operatorname{supp} g$. Then we have, by integration by parts,

$$\langle D''\rangle^s\int_{U'} g(x',x'')\,dx' = \int_{\mathbf{R}^n}((\langle D'\rangle^{-u}\varphi)(x')\langle D'\rangle^u\langle D''\rangle^s g(x',x'')\,dx'$$

for any $u\in\mathbf{R}$. Since $\langle D'\rangle^{\min\{s,0\}}\langle D''\rangle^s\langle D\rangle^{-s}$ is a bounded linear operator from $L_p(\mathbf{R}^{n+k})$ to itself (see [T, Chapter XI, Theorem 1.5]), if we put $u=\min\{s,\ 0\}$, the right-hand side is in $L_p(\mathbf{R}^k)$ by Hölder's inequality. This completes the proof. □

6. Application

In this section, we extend Theorem 1.1 to semilinear differential equations.

As in Section 1, let P be a differential operator (or a matrix of differential operators) of order m with smooth coefficients on a manifold M. Let $X_1, \dots, X_r$ be smooth complex vector fields on M. For presentation simplicity, we assume that they are commutative to one another. Letting u be the unknown function, we consider the semilinear differential equation

$$P(x,D)u = F(x, X^\alpha u; |\alpha| \le h), \tag{6.1}$$

where $h<m$, and the index α ranges through $\{\alpha \in \mathbf{N}^r \,|\, |\alpha| \le h\}$ in the nonlinear term F. (For the index α, $X^\alpha u$ denotes $X_1^{\alpha_1} \dots X_r^{\alpha_r} u$.) The nonlinear term

$$F = F(x, u^\alpha)$$

is a continuous function in the local coordinate x of M and in u^α, $|\alpha| \le h$, and we assume that there exist $q \ge 1$, $A \in L_{\infty,\mathrm{loc}}(M)$ and $B \in L_{1,\mathrm{loc}}(M)$ such that

$$|F(x,u^\alpha)| \le A(x) \sum_{|\alpha| \le h} |u^\alpha|^q + B(x)$$

on M. If a locally integrable function u satisfies

$$\sum_{|\alpha| \le h} \int |X^\alpha u|^q \, dx < \infty \tag{6.2}$$

locally on an open subset U, with dx being a volume element on M, then $F(x, X^\alpha u)$ defines a locally integrable function on U.

For $x \in M$, let $\mathcal{F}(x)$ denote the $\mathbf{C}$-vector subspace of $\mathbf{C} \otimes T_x M$ generated by $X_1(x), \dots, X_r(x)$.

We then have the following extension theorem of weak solutions of the semilinear differential equation (6.1).

Theorem 6.1. *Let Z be a closed submanifold of M of class C^∞ of codimension $d \ge 2$. Let u be a locally integrable function on M.* (i) *Let U be an open subset of M. Let $p \ge d/(d-1)$. Assume that*

$$\sum_{|\alpha| \le h} \int_{U \setminus Z} |X^\alpha u|^p \, dx < \infty. \tag{6.3}$$

Then $X^\alpha u$ is in $L_{p,\mathrm{loc}}(U)$ for $|\alpha| \le h$.

(ii) *Assume that, for all* $x \in Z$, $\mathcal{F}(x) \not\subset \mathbf{C} \otimes T_x Z$. *Suppose that* u *satisfies* (6.2) *locally on* $M \setminus Z$ *and equation* (6.1) *in the distribution sense. Let* $p > 1$ *and assume that* $p \geq q$ *and*

$$m - h \leq d(1 - 1/p). \tag{6.4}$$

If u *satisfies* (6.3) *for a neighborhood* U *of every* $x \in Z$, *then* u *satisfies equation* (6.1) *in* $\mathcal{D}'(M)$.

Proof. (i) Let $u^\alpha = X^\alpha u$ and let f^β be the locally p-integrable function on U defined by $X^\beta u$ on $U \setminus Z$ and by 0 on Z. Let $\beta = \alpha + e_i$, with e_i being the i-th unit vector of $\mathbf{N}^r$, $i = 1, \dots, r$, and apply Corollary 2.3 to

$$X_i u^\alpha = f^\beta.$$

Then we have $X_i f^\alpha = f^\beta$ for any α, $|\alpha| < h$. This implies that $f^\alpha = X^\alpha u$.

(ii) Let $x \in Z$. We may assume that X_1 is not tangent to Z at x; we can then find $z \in (T_Z^* M)_x$ such that X_1 is elliptic at z: $\sigma(X_1)(z) \neq 0$. By (i), we have

$$u \in \mathcal{L}_p^h(z) \tag{6.5}$$

and $F(x, X^\alpha u)$ is a locally integrable function on M. Then the theorem is a consequence of Theorem 2.2 with (2.2) and (2.3) replaced by (6.4) and (6.5), respectively. This modification of Theorem 2.2 is verified if we remark that P is allowed to be a pseudo-differential operator and apply Theorem 2.2 with $P(1+|D|)^{-h}$ as P and $(1+|D|)^h u$ as u. □

This theorem recovers the result of Eells and Polking [EP] on the weak extension of harmonic maps in the case where the singular locus is a submanifold.

References

[BL] Bergh, J. and Löfström, J., *Interpolation Spaces*, Springer-Verlag, Berlin–New York, 1976.

[B] Bochner, S., Weak solutions of linear partial differential equations, *J. Math. Pures Appl.* **35** (1956), 193–202.

[EP] Eells, J. and Polking, J. C., Removable singularities of harmonic maps, *Indiana Univ. Math. J.* **33** (1984), 859–871.

[HP] Harvey, R. and Polking, J. C., Removable singularities of solutions of linear partial differential equations, *Acta Math.* **125** (1970), 39–56.

[P] Polking, J. C., A survey of removable singularities, in *Seminar on Nonlinear Partial Differential Equations* (Chern, S. S., ed.), Math. Sci. Res. Inst. Publ. **2**, pp. 261–292, Springer-Verlag, Berlin–Heidelberg–New York, 1984.

[T] TAYLOR, M. E., *Pseudodifferential Operators*, Princeton Univ. Press, Princeton, N. J., 1981.

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