Ark. Mat., **37** (1999), 357–371

Rigidity of holomorphic Collet–Eckmann repellers

Feliks Przytycki and Steffen Rohde(1)

Abstract. We prove rigidity results for a class of non-uniformly hyperbolic holomorphic maps. If a holomorphic Collet–Eckmann map f is topologically conjugate to a holomorphic map g, then the conjugacy can be improved to be quasiconformal. If there is only one critical point in the repeller, then g is Collet–Eckmann, too.

1. Introduction

Collet–Eckmann maps of the interval were introduced by P. Collet and J.-P. Eckmann as a large class of non-uniformly expanding maps for which a probability absolutely continuous invariant measure exists. A theory of rational Collet–Eckmann maps was originated in [P2] and continued in [P3], [GS1] and [PR]; see [PR] for a more detailed historical account. This paper is a continuation of [PR]. We consider repellers for holomorphic maps, without assuming the maps extend to rational maps.

Consider a compact set X in the Riemann sphere $\widehat{\mathbf{C}}$, together with a holomorphic map $f\colon U\to\widehat{\mathbf{C}}$ with $f(X)=X$, where U is a neighbourhood of X.

We call the pair (X,f) a *holomorphic repeller* if there exists a neighbourhood $V\subset U$ of X such that for every $x\in U$ the assumption $f^n(x)\in V$ for every $n=0,1,\dots$ implies $x\in X$. We do not assume *a priori* that X has empty interior. For instance, both $(\overline{\mathbf{D}},z^2)$ and (S^1,z^2) are repellers according to our definition.

A point c is called *f-critical* if $f'(c)=0$. The set of all f-critical points is denoted by Crit or Crit(f).

---

(1) The first author acknowledges support by Polish KBN Grant 2 P03A 025 12 "Iterations of Holomorphic Functions" and support of the Hebrew University of Jerusalem, where a part of the paper was written. The second author is grateful for the hospitality and support of the Caltech, where a part of the paper was written.

We call a holomorphic repeller (X,f) *Collet-Eckmann*, abbreviated CE, if there are constants $C>0$ and $\lambda>1$ such that for every f-critical point $c\in X$ such that its forward trajectory does not meet other critical points,

$$\text{(CE)} \qquad |(f^n)'(f(c))| \geq C\lambda^n$$

for all $n\geq 0$. See [CE], [N], [P2], [P3]. Here and in what follows derivatives and distances are always with respect to the spherical metric of $\widehat{\mathbf{C}}$.

If there is no critical point at all in X, we understand (X,f) as a CE-repeller, too. In particular, Julia sets of expanding rational maps provide examples of CE-repellers.

We are also concerned with the following notion, see Section 2 for a formal definition. We call a holomorphic repeller (X,f) *topological Collet–Eckmann*, abbreviated TCE, if the following holds: There is a constant $d\geq 1$ and for each $x\in X$ a set $G(x)$ of positive integers, of lower density $\geq \frac{1}{2}$, such that for every $n\in G(x)$ there is a connected neighborhood of x that is mapped properly by f^n to a large disc, with mapping degree bounded by d.

If (X,f) and (Y,g) are holomorphic repellers we say that they are *topologically conjugate* if they are topologically conjugate on neighbourhoods U_X, U_Y of X, Y respectively, i.e. there exists a homeomorphism $h\colon U_X\to U_Y$ such that $g\circ h=h\circ f$.

In Section 3 we prove the following.

Theorem A. *If (X,f) and (Y,g) are holomorphic repellers which are conjugate by an orientation preserving homeomorphism h_0 and if f is* TCE *then there exists a quasiconformal conjugacy h of f and g on neighbourhoods of X and Y satisfying $h|_X=h_0|_X$.*

Notice that TCE is a topological property, so the assumption that (X,f) is TCE immediately implies that (Y,g) is TCE, too. Theorem A for rational maps f and g that are expanding on their Julia sets is due to McMullen and Sullivan, [MS].

It is not hard to modify h_0 to become quasiconformal off X. The main idea of our proof of Theorem A is to show that for every $x\in X$ there is a sequence of discs centered at x, of radii converging to 0, that are mapped under h to boundedly distorted topological discs. Indeed, TCE implies the existence of a sequence of boundedly distorted topological discs around x mapped by f^n for $n\in G(x)$ to large round discs, mapped next by the topological conjugacy h onto boundedly distorted large discs and finally back as components of the preimages under g^n to small boundedly distorted discs centered at $h(x)$. We then apply the following result of Heinonen and Koskela [HK] to the effect that such homeomorphisms are quasiconformal.

Theorem HK. *Let $H<\infty$ and h be a homeomorphism of a domain D in $\widehat{\mathbf{C}}$ with*

$$\liminf_{r\to 0} \frac{\max\{|h(x)-h(y)| : |x-y|=r\}}{\min\{|h(x)-h(y)| : |x-y|=r\}} \leq H \tag{1.1}$$

for all $x\in D$. Then h is quasiconformal in D.

We proved in [PR] that for a rational map $f:\widehat{\mathbf{C}}\to\widehat{\mathbf{C}}$ and its Julia set X, CE implies TCE. With almost no changes the proof holds for every holomorphic repeller (X,f), see [P4, Proposition 4.1]. Therefore, from Theorem A we obtain the following corollaries.

Corollary B. *If (X,f) and (Y,g) are topologically conjugate holomorphic repellers, and if (X,f) is* CE, *then the conjugacy can be replaced by a quasiconformal conjugacy* (*without changing it on X*).

Corollary C. *If f and g are polynomials which are topologically conjugate by an orientation-preserving homeomorphism, if their filled-in Julia sets are connected and equal to their Julia sets, and if $(J(f),f)$ is* CE (*or* TCE), *then f and g are conjugate by a conformal affine map.*

Corollary C has been obtained independently by Jones and Smirnov. Indeed, it is shown by Graczyk and Smirnov in [GS1] (and later in [PR]) that for CE-polynomials f, the Fatou component at infinity is a Hölder domain. A conjugacy between f and another polynomial g easily leads to a conjugacy h conformal off the Julia set $J(f)$. An improvement of the removability result [J] of Jones, due to Jones and Smirnov [JS], now gives Corollary C. In fact, using this removability result Graczyk and Smirnov [GS2] were able to obtain Corollary C under an even weaker (summability) condition. However, an advantage of our approach (using Theorem HK as a removability statement) is that it does not need any assumptions on the geometry of the Julia set. In particular, it works in the rational case as well as for polynomials.

As the referee pointed out, the special case of quadratic polynomials f and g in Corollary C follows from Yoccoz' rigidity theorem. Indeed, from Proposition 2.5 below it follows that TCE repellers are not infinitely renormalizable.

In Section 4 we prove the following theorem.

Theorem D. *If (X,f) and (Y,g) are topologically conjugate holomorphic repellers, if X contains at most one critical point and if (X,f) is* CE, *then (Y,g) is* CE.

The proof uses the method of "shrinking neighbourhoods" from [P2] to control distortion, as well as Graczyk and Smirnov's [GS1] reversed telescope construction.

By Theorem A, we can assume the conjugacy to be quasiconformal. So this theorem corresponds to the theorem by Nowicki and Sands [NS], that f is CE implies that g is CE for S-unimodal maps of the interval if there is a quasisymmetric conjugacy.

In [P4] a stronger theorem([2]) is proved. For (X, f) such that X contains only one critical point, TCE implies CE. This holds also in the interval case [NP], hence one does not need to assume that the conjugacy above is quasisymmetric.

If there is more than one critical point in X, this is no longer true. In Section 5, we provide an example of a semihyperbolic polynomial (i.e. no critical point in the Julia set is recurrent; this is stronger than TCE, see Section 2) which is not Collet–Eckmann. Hence TCE does not imply CE in general. In our example, the forward trajectory of a critical point approaches a second critical point arbitrarily closely. This is similar to an example in [CJY] of a semihyperbolic map where a critical point is mapped into another critical point.

Acknowledgement. We would like to thank the referee for his careful reading and various comments.

2. Definitions

Fix numbers $\delta>0$ and $1\leq d<\infty$ and consider the disc $B=B(f^n(x),\delta)$ together with the component W of $f^{-n}(B)$ containing x. For $x\in X$ denote the set of integers n, for which the mapping degree of f^n on W is at most d (i.e. each point of B has at most d preimages in W, counted with multiplicities), by $G(x)$ or $G(x,\delta,d)$ (sometimes even $G(x,\delta,d,f)$, if the map is not clear from the context).

Definition 2.1. A repeller (X, f) is called *topological Collet–Eckmann* (TCE), if there are $\delta>0$ and $d<\infty$ such that

$$\frac{\#(G(x,\delta,d)\cap[1,n])}{n}\geq\frac{1}{2}$$

for all $x\in X$ and all $n\geq 1$.

Definition 2.2. A repeller (X, f) is called *expanding*, if there is $\delta>0$ such that

$$G(x,\delta,1)=\mathbf{N}$$

for all $x\in X$.

The standard definition of *expanding* requires the existence of constants $C>0$ and $\lambda>1$ such that $|(f^n)'(x)|\geq C\lambda^n$ for all $x\in X$ and all $n\in\mathbf{N}$. It is easy to see that

([2]) The results of [P4] were obtained after the results of the present paper.

this coincides with our definition, if X is nowhere dense. The nowhere density is discussed in the appendix.

In between the properties TCE and *expanding* is the notion of *semihyperbolicity.*

Definition 2.3. A repeller (X,f) is called *semihyperbolic*, if there are $\delta>0$ and $d<\infty$ such that

$$G(x,\delta,d)=\mathbf{N}$$

for all $x\in X$.

It can be shown that a repeller is expanding if and only if there is no critical point in X. In [M], [CJY], [P4] it has been shown that a repeller is semihyperbolic if and only if there are no *recurrent* critical points in X.

If f is a polynomial without parabolic periodic points and $X=J(f)$ is its Julia set, then (X,f) is semihyperbolic if and only if the basin of attraction to ∞ is a John domain by [CJY]. Next, (X,f) is TCE if and only if the basin is a Hölder domain, see [GS1], [PR, Section 3] and [P4, Section 4]. The basin of our example in Section 4 is a Hölder domain (it is even a John domain), so that the Hölder property does not imply CE in general. However, it follows from the aforementioned result of [P4] that the Hölder property of the basin at ∞ does imply CE if there is only one critical point in the Julia set.

Lemma 2.4. *Let (X,f) be a holomorphic repeller with X nowhere dense, and consider for every disc $B(x,2\delta)$ with δ small enough and $x\in X$ a component W of $f^{-n}(B(x,2\delta))$ that intersects X, together with a component $W'\subset W$ of $f^{-n}(B(x,\delta))$. If the degree of f on W is at most d, we have* $\operatorname{diam} W'\to 0$, *as $n\to\infty$, uniformly, i.e. not depending on x or the choice of W'.*

This has been shown by Mañé [M] for rational f. See [P4] for the adjustments to holomorphic repellers.

In the appendix, we will show that TCE repellers different from the Riemann sphere are nowhere dense. Hence Lemma 2.4 applies to this case. From this it is possible to conclude that diameters of preimages shrink to zero for TCE repeller. However, the following stronger statement holds.

Proposition 2.5. *If (X,f) is a* TCE *repeller, then there exist $\delta>0$ and $0<\xi<1$ such that for every $x\in X$ and $n\geq 0$,*

$$\operatorname{diam}\operatorname{Comp}_x f^{-n}(B(f^n(x),\delta))\leq\xi^n.$$

Here and in what follows we use the notation $\operatorname{Comp}_x M$ to denote the component of M that contains x. See [PR] for the rational case and [P4, Proposition 4.1] for the repeller case.

From Proposition 2.5 (or just from the fact that for all δ there exists δ' such that all $\operatorname{diam} \operatorname{Comp}_x f^{-n}(B(f^n(x),\delta'))\le\delta$ for $x\in X$—this property is called *backward Lyapunov stability*, see [L]) it follows that the density $\frac{1}{2}$ in our Definition 2.1 of TCE could be replaced by any other number between 0 and 1, at the cost of a respectively larger d and smaller δ. Indeed, set $n=n_{j+1}-n_j$ for any two consecutive integers in $G(x,\delta,d)$. Then the set $n_j+(G(f^{n_j}(x),\delta',d)\cap[1,n])$ is contained in $G(x,\delta',d^2)$. Now continue filling gaps of G as long as necessary.

Another consequence is that preimages of (small) discs will always be simply connected. If a component of $f^{-1}(B)$ were not simply connected, B would contain at least two distinct critical values for f.

Hence the degree of f^n on $\operatorname{Comp}_x f^{-n}(B(f^n(x),\delta))$ is controlled by the number of critical points, so that we get the following alternative definition of TCE.

There exist $M>0$, $P\ge1$ and $\delta>0$ such that for every $x\in X$ there exists an increasing sequence of positive integers n_j, $j=1,2,\dots$, such that $n_j\le Pj$ and

$$\#\{i:0\le i<n_j,\ \operatorname{Comp}_{f^i(x)} f^{-(n_j-i)}(B(f^{n_j}(x),\delta))\cap\operatorname{Crit}\neq\emptyset\}\le M \tag{2.1}$$

for each j.

Notice that this is equivalent to the following: There exists $\delta>0$, such that *for all* $P>1$ there is $M>0$ such that (2.1) holds for sufficiently large j and all x. The proof is easy. Of course, δ can be replaced by any smaller positive number.

3. Improving the conjugacy

Lemma 3.1. *Let E be a compact metric space, X a compact subset and U an open neighbourhood of X. Let $f:U\to E$ be a continuous and open map such that $f(X)=X$. Suppose that (X,f) is a repeller (i.e. there exists a neighbourhood $V=V_X\subset U$ of X such that for every $x\in V$ with $f^n(x)\in V$ for every $n=0,1,\dots$, we have $x\in X$).*

Then for every V as above there exists an open neighbourhood $W\subset V$ of X, such that $W'=(f|_{\overline{W}})^{-1}(\overline{W})\subset W$, and such that all components of W' intersect X.

Proof. This is similar to 1° in the proof of Proposition 1.1 in [P1], using [S] (in [P1] an attractor was considered, whereas here we have a repeller. So instead of f we apply f^{-1}). $\square$

The next lemma provides a quasiconformal conjugacy outside X, thus proving a part of Theorem A.

Lemma 3.2. *If (X, f) and (Y, g) are holomorphic repellers which are topologically conjugate by an orientation preserving homeomorphism h_0, then there exist neighbourhoods W_X and W_Y of X and Y respectively, and a homeomorphism $h: W_X \to W_Y$ conjugating f and g, such that h is quasiconformal on $W_X \setminus X$ and h is equal to h_0 on X.*

Proof. (Cf. [MS] in the rational expanding case.) We can assume that V is small enough so that $(V \setminus X) \cap \operatorname{Crit}(f) = \emptyset$, and that V is contained in the domain of h_0. Take $W_X = W$ from Lemma 3.1. By a small change we can assume that W_X has smooth boundary ∂_0. Let $H: [0,1] \times \partial_0 \to \widehat{\mathbf{C}}$ be a homotopy from $H|_{\{0\} \times \partial_0} = h_0|_{\partial_0}$ to a smooth embedding $h_1 = H|_{\{1\} \times \partial_0}$. We assume that each $h_t = H|_{\{t\} \times \partial_0}$ is C^0-close to h_0. Then we can lift h_t from ∂_0 to $\partial_1 := (f|_{\overline{W}})^{-1}(\partial_0)$, by requiring that on ∂_1 we have $h_t f = g h_t$ (this means that the same branch of g^{-1} has to be used; we have denoted the extension of h_t to ∂_1 again by h_t) and h_t is C^0-close to h_0. Moreover, the mutual positions of the curves ∂_0, ∂_1 and their identifications by f are the same as for their h_t images and the identifications by g.

Therefore, h_t can be extended to a homotopy $H: [0,1] \times \overline{W_X \setminus f^{-1}(W_X)} \to \widehat{\mathbf{C}}$, again C^0-close to h_0, and such that the restriction h_1 of H to $\{1\} \times \overline{W_X \setminus f^{-1}(W_X)}$ is a diffeomorphism, hence quasiconformal.

Next, by consecutive lifts of H via f^{-1} and g^{-1} we obtain a quasiconformal conjugacy h of f and g on neighbourhoods of X and Y (with the sets X and Y removed). The map h extends continuously to h_0 on X because $\operatorname{diam} H([0,1] \times \{x\}) \to 0$, as $n \to \infty$, for $x \in (f|_{\overline{W}})^{-n}(\overline{W_X \setminus f^{-1}(W_X)})$. The latter follows from the fact that for a neighbourhood N of $H([0,1] \times \{x\})$, for $x \in \overline{W_X \setminus f^{-1}(W_X)}$ and for every choice of branches F_n of f^{-n} on N, we have $F_n(N) \to X$. Hence the only limit functions of F_n are constant functions by Hurwitz' theorem. □

Lemma 3.3. (Distortion Lemma in finite criticality, see [P2] and Lemma 2.1 in [PR].) *For every $\varepsilon > 0$ every $1 \le d < \infty$ there are constants C_1 and C_2 such that the following holds for every $\frac{1}{2} \le t < 1$ and every holomorphic proper map $F: W \to \mathbf{D}$ of degree $\le d$ of a simply connected domain W in $\widehat{\mathbf{C}}$ to the unit disc $\mathbf{D}$, for which* $\operatorname{diam}(\widehat{\mathbf{C}} \setminus W) \ge \varepsilon$:

Assume that W' is a simply connected component of $F^{-1}(\{|z| < t\})$. Then for every $x \in W'$

$$|F'(x)| \operatorname{diam} W' < C_1 (1-t)^{-C_2}. \tag{3.1}$$

Furthermore, there exist $C_3 = C_3(\tau)$ and $C_4 = C_4(\tau)$ such that $C_3, C_4 \searrow 0$, as $\tau \to 0$, satisfying the following:

Let $t = \frac{1}{2}$ and $0 < \tau < \frac{1}{2}$, let W'' be a component of $F^{-1}(\{|z| < \tau\})$ in W'. Then

$$\operatorname{diam} W'' < C_3 \operatorname{diam} W'. \tag{3.2}$$

Moreover, for every $x \in W''$,

$$|F'(x)| \operatorname{diam} W'' < C_3. \tag{3.3}$$

On the other hand,

$$W'' \supset B(y, C_4 \operatorname{diam} W') \tag{3.4}$$

for every $y \in F^{-1}(0) \cap W''$.

Proof of Theorem A. Take a conjugacy h from Lemma 3.2. It satisfies the Heinonen–Koskela condition (1.1) for every $x \notin X$, with a suitable number $H < \infty$. Indeed, by quasiconformality it even satisfies the stronger quasisymmetry condition

$$\limsup_{r \to 0} \frac{\max\{|h(x)-h(y)| : |x-y| = r\}}{\min\{|h(x)-h(y)| : |x-y| = r\}} \le H$$

for all $x \notin X$.

Now take an arbitrary $x \in X$, fix $\delta > 0$ and $d < \infty$ satisfying Definition 2.1 of TCE and consider an increasing sequence of integers $n_j \in G(x, \delta, d, f)$ (cf. Section 2). By the uniform continuity of h on X there exists δ' such that $n_j \in G(h(x), \delta', d, g)$.

For an arbitrary j, set $W := \operatorname{Comp}_x f^{-n_j} B(f^{n_j}(x), \delta)$ and apply Lemma 3.3 to $F = f^{n_j} : W \to B(f^{n_j}(x), \delta)$ (with $\mathbf{D}$ replaced by $B(f^{n_j}(x), \delta)$). The existence of ε required in Lemma 3.3 follows from the fact that the diameters of the components of $f^{-n}(B(f^n(x), \delta)$ tend to zero, Lemma 2.4 or Proposition 2.5.

For τ small enough so that $C_3(\tau) < C_4\left(\frac{1}{2}\right)$, we obtain with

$$W' = \operatorname{Comp}_x f^{-n_j} B\left(f^{n_j}(x), \tfrac{1}{2}\delta\right) \quad \text{and} \quad W'' = \operatorname{Comp}_x f^{-n_j} B(f^{n_j}(x), \tau\delta),$$

that

$$W'' \subset B(x, r) \subset W'$$

for $r = C_4\left(\frac{1}{2}\right) \operatorname{diam} W'$.

By a compactness argument and the homeomorphy of h there exist constants $a_1 > a_2 > 0$ such that for every $y \in X$

$$\operatorname{diam} h\left(B\left(y, \tfrac{1}{2}\delta\right)\right) \le a_1 \quad \text{and} \quad h(B(y, \tau\delta)) \supset B(h(y), a_2).$$

Choosing δ small enough (without changing δ') we may assume $a_1 \le \delta'$ and obtain

$$\operatorname{diam} h(B(x, r)) \le \operatorname{diam} \operatorname{Comp}_{h(x)} g^{-n_j}(B(hf^{n_j}(x), a_1)) =: r'.$$

From (3.4) we conclude that $h(B(x,r))$ contains $B(h(x), C_4(a_2/2a_1)r')$.

Notice finally that $r\to 0$, as $n_j\to\infty$, by Lemma 2.4 or Proposition 2.5. Therefore (1.1) is satisfied and h is quasiconformal by Theorem HK. $\square$

Proof of Corollary C. We start the construction of h in Lemma 1.2 with a holomorphic conjugacy in a neighbourhood of ∞. This gives h in Theorem A quasiconformal in $\widehat{\mathbf{C}}$ and conformal outside the Julia set $J(f)$. By [P3] (for f CE), [GS1] or [PR], the area of $J(f)$ is 0. Hence h is 1-quasiconformal and therefore conformal. $\square$

4. Rigidity

Let us start with a quasiconformal version of the Koebe distortion theorem needed in this section. It is an immediate consequence from the Hölder continuity of quasiconformal selfmaps of a disc.

Lemma 4.1. *For every $\varepsilon>0$ and $K\geq 1$ there exist $C_5, C_6>0$ such that for every K-quasiconformal $\Phi:\mathbf{D}\to U$ and every conformal map $F:U\to\widehat{\mathbf{C}}$ with* $\operatorname{diam}(\widehat{\mathbf{C}}\setminus U)>\varepsilon$ *and* $\operatorname{diam}(\widehat{\mathbf{C}}\setminus F(U))>\varepsilon$*, for every $0<t<1$ and x,y with $|x|,|y|\leq t$, we have*

$$\frac{|F'(\Phi(x))|}{|F'(\Phi(y))|}\leq C_5(1-t)^{-C_6}.$$

Proof of Theorem D. We need to prove that (Y,g) is CE, namely to prove the condition (CE) in the introduction. Due to Corollary B we can assume that the conjugacy h is quasiconformal.

Step 1. *Derivatives for periodic orbits.* We have $|(g^m)'(y)|\geq\xi^{-m}$ for every periodic $y\in Y$ with $g^m(y)=y$. This follows immediately from the Hölder continuity of h and by the above property for f. The following is a more direct explanation of the same fact.

Consider the components B_n of $g^{-n}(B(y,\delta))$ that intersect the periodic orbit $O(y)$ of y. As the diameters of B_n tend to zero, we can choose δ small enough (depending on y) so that the B_n are disjoint from Crit. Now f being TCE implies that g is TCE. By Proposition 2.5 and the Koebe distortion theorem, applied to the branches of g^{-n} along $O(y)$, we have

$$|(g^n)'(z)|\geq C_7\operatorname{diam}(B_n)^{-1}\geq C_7\xi^{-n}$$

for $z\in g^{-n}(y)\cap O(y)$, with a constant C_7 depending on y. Applied to all multiples n of m, this gives $|(g^m)'(y)|\geq\xi^{-m}$.

Step 2. *Derivatives far from Crit.* Fix an arbitrary positive integer n. Denote by c^f (and c^g) the only f-critical (respectively g-critical) point in X (resp. Y). Let $0 \le i \le n$ be the largest integer such that $\operatorname{dist}(g^i(c^g), c^g) \le \delta_1$, for a positive constant δ_1 to be determined later. Arguing as above we obtain

$$|(g^{n-i-1})'(g^{i+1}(c^g))| \ge C_8 \xi^{-(n-i-1)}, \tag{4.1}$$

with a constant C_8 depending on δ_1.

Step 3. *Capture of a periodic orbit.* We shall write $c^f_j = f^j(c^f)$ and $c^g_j = g^j(c^g)$. Suppose that $i > 0$. Let $B = B(c^f_{i+1}, a)$ for $a := 4 \operatorname{dist}(c^f_{i+1}, c^f_1)$. By the continuity of h^{-1}, we may assume that a is as small as we need by choosing δ_1 sufficiently small in Step 2.

Now we consider preimages according to the "shrinking neighbourhoods" procedure.

Fix a subexponentially decreasing sequence $b_j > 0$ with $P := \prod_{j=1}^{\infty}(1-b_j) > \frac{1}{2}$. Consider the sequence of discs

$$B_s = B\biggl(c^f_{i+1}, a\prod_{j=1}^{s}(1-b_j)\biggr)$$

and preimages

$$W_s := \operatorname{Comp}_{c^f_{i+1-s}} f^{-s}(B_s).$$

Fix now s to be the largest positive integer such that $c^f \notin W_j$ for $2 \le j \le s$. Of course such s exists and $s \le i$ because $c^f \in W_{i+1}$ as $c^f = c^f_0$.

Thus $W_{s+1} \ni c^f$, hence $f(W_{s+1}) \ni c^f_1$ and f^{s-1} is univalent on W_s, hence f^s has only one critical point in W_s.

The point $f^s(c^f_1)$ is in B_{s+1} so the annulus $B_s \setminus B_{s+1}$ allows to control distortion. If we assume δ_1 (hence a) small enough, then B_s and W_s have complements of diameters at least ε (say half of the diameter of the sphere) and we can apply Lemma 3.3 (to $F = f^s$, $W = W_s$, $t = 1 - b_{s+1}$ and $W' = f(W_{s+1})$). We obtain

$$\frac{\operatorname{diam} f(W_{s+1})}{\operatorname{diam}(B_{s+1})} \le |(f^s)'(c^f_1)|^{-1} C_1 b_{s+1}^{-C_2}. \tag{4.2}$$

Due to $P \ge \frac{1}{2}$ we have also

$$c^f_1 \in \tfrac{1}{2} B_{s+1}, \tag{4.3}$$

where $\frac{1}{2}B(z,r)$ for any ball $B(z,r)$ denotes the ball $B\bigl(z, \frac{1}{2}r\bigr)$.

By (4.2) and (CE) applied to f,

$$\frac{\operatorname{diam} f(W_{s+1})}{\operatorname{diam}(B_{s+1})} \le C\lambda^{-s} b_{s+1}^{-C_2} \le C(\theta)\lambda^{-s}\theta^s. \tag{4.4}$$

Here $\theta>1$ is arbitrarily close to 1 and $C(\theta)$ depends on θ. This is possible because b_j shrink subexponentially.

By (4.3), if s is large enough, we obtain

$$f(W_{s+1}) \subset B_{s+1}.$$

By choosing δ_1 small enough we can assume that s is large. Indeed, the time s between two consecutive approaches of an orbit to c^f is long, otherwise there would be a periodic sink in X which is not possible by CE. More precisely: Set $d=2\max\{\operatorname{dist}(c^f, c^f_{i-s}), \operatorname{dist}(c^f, c^f_i)\}$; it is small for a small by (4.4). Then the ball $D=D(c^f, d)$ has the f^s-image of diameter at most $C(f)L^s(\operatorname{diam} D)^2$ (where L is the Lipschitz constant of f), hence D will be mapped into itself, $f^s(D)\subset D$, unless s is sufficiently large.

Thus $f^s\colon f(W_{s+1})\to B_{s+1}$ is polynomial-like, and we call $f^s\colon f(W_{s+1})\to B_{s+1}$ a *tube* of the reversed telescope construction.

Hence there exists a periodic point of period s in $f(W_{s+1})$. We consider next its h-image $p\in h(f(W_{s+1}))$.

Step 4. *Derivatives along a tube of the reversed telescope.* We already know that $|(g^s)'(p)|\ge\xi^{-s}$, see Step 1, and we want to estimate $|(g^s)'(c^g_{i-s+1})|$.

To this end, notice that $g^{s-1}\colon h(W_s)\to h(f^{s-1}(W_s))$ is univalent, hence its inverse G exists. By Lemma 4.1 (with $\Phi=h\circ f^{-1}\circ T\circ z^d$, where T is a Möbius transformation from the unit disc onto B_s mapping 0 to c^f_1, and d is the degree of f at c^f) we obtain

$$\frac{|G'(c^g_i)|}{|G'(g^{s-1}(p))|} \le C_9 b_{s+1}^{-C_6}.$$

Finally, recall that by the construction

$$\operatorname{dist}(c^f_1, c^f_{i+1}) \ge \tfrac14 \operatorname{dist}(c^f_1, h^{-1}(p)),$$

so that we obtain

$$\operatorname{dist}(c^f, c^f_i) \ge c\operatorname{dist}(c^f, f^{s-1}(h^{-1}(p)))$$

for some universal constant c.

Under the change of coordinates given by the quasiconformal map h, relative distances change by at most a constant, hence there exists $C>0$ such that $|g'(c_i^g)|\geq C|g'(g^{s-1}(p))|$. We conclude that

$$|(g^s)'(c^g_{i-s+1})|\geq C(\theta)^{-1}\theta^{-s}\xi^{-s}, \tag{4.5}$$

where again $\theta>1$ is arbitrarily close to 1 and $C(\theta)$ depends on θ.

Step 5. *Conclusion.* If $i-s>0$ we build a *tube* starting with $i-s$ rather than i. Notice that $\operatorname{dist}(c^f_{i-s+1},c^f_1)<a$ by (4.4), since s is large. Hence, we can repeat the argument of Step 3 with $a:=4\operatorname{dist}(c^f_{i-s+1},c^f_1)$. We find a new s for which we write s_2. Write s_1 for the first s. We continue until we end up with $\sum_{j\geq1}s_j=i$. This divides f^i into tubes $f^{s_1},f^{s_2},\dots,$ (each one from the critical value to the critical value) such that (4.5) holds for the h-image of each tube. Together with (4.1) this gives (CE) for g. $\square$

By choosing δ_1 sufficiently small, our proof shows that g is CE with λ arbitrarily close to ξ^{-1}.

5. An example

Let $f_1(z)=z^2-2$ and $f_2(z)=(z-25)^2+a$, with a an arbitrary real number in $[-2,2]$. Define $D:=\{|z|<100\}$, $D_1=f_1^{-1}(D)$ and $D_2=f_2^{-1}(D)$. Notice that

$$\{|z|<9\}\subset D_1\subset\{|z|<11\}\quad\text{and}\quad\{|z-25|<9\}\subset D_2\subset\{|z-25|<11\}.$$

Therefore, $D_1\cap D_2=\emptyset$ and $D_1,D_2\subset D$.

Now we choose a so that the lower Lyapunov characteristic exponent for f_1 at a is $-\infty$,

$$\underline{\chi}(f_1,a):=\liminf_{n\to\infty}\frac{1}{n}\log|(f_1^n)'(a)|=-\infty.$$

Such a exists by a Baire category argument. Let

$$A_n=\{x\in[-2,2]:|f_1^n(x)|<\exp(-\exp n)\}.$$

Then $A=\bigcap_{N=1}^\infty\bigcup_{n=N}^\infty A_n$ is nonempty as it is the intersection of a sequence of open, dense sets. Here we used the fact that for each N the set $\bigcup_{n=N}^\infty f^{-n}(0)$ is dense in $[-2,2]$. It follows from straightforward estimates that $\underline{\chi}(a)=-\infty$ for every $a\in A$.

Let F be defined as f_1 on D_1 and f_2 on D_2. This is a so called *generalized polynomial-like map.* It is well known that one can change coordinates on $\widehat{\mathbf{C}}$ by a quasiconformal homeomorphism h so that $G=h\circ F\circ h^{-1}$ extends to a polynomial, cf. [DH, Straightening Theorem], [LM, Lemma 7.1] and [LS, Lemma 2.1].

The critical points in the Julia set $J(G)$ are $c_1=h(0)$ and $c_2=h(25)$. We have $G^2(c_1)=h(2)$ which is a repelling fixed point, and the forward G-trajectory of c_2 is confined by $h([-2,2])$, so it stays far away from c_2. Therefore G is semihyperbolic.

Notice finally that h is Hölder continuous, so for a constant $\alpha>0$ we have $|G^n(h(a))-h(0)|=|G^n(G(c_2))-c_1|<\exp(-\alpha\exp n)$ for a sequence of n's. Hence $|(G^n)'(G(c_2))|<3\exp(-\alpha\exp n)$, and thus $\underline{\chi}(G,h(a))=-\infty$. Therefore the mapping G is not Collet–Eckmann.

Appendix

Proposition. *If (X,f) is a holomorphic repeller with the TCE property, then X is nowhere dense provided that $X\neq\widehat{\mathbf{C}}$.*

Proof. As f is an open map and X is forward invariant, every component C of $\operatorname{int}X$ is mapped into a component of $\operatorname{int}X$. By the repelling property, C is mapped properly onto a component of $\operatorname{int}X$.

Case 1. Suppose there exists a sequence $n_j\to\infty$ such that $\lim(f|_C)^{n_j}$ exists and is non-constant. Then there exists j such that the component N of $\operatorname{int}X$ containing $f^{n_j}(C)$ is periodic. As $f^n\neq\mathrm{id}$ on N (since X is a repeller), N is a Siegel disc or Herman ring by the usual proof of the classification theorem.

Consider a disc D of a definite small radius intersecting ∂N. By the repelling property of X, for every sequence of components D_n of $f^{-n}(D)$ which intersect N we have

$$\#(\mathrm{Crit}(f^n|_{D_n}))\to\infty$$

(counted with multiplicities) uniformly, i.e. independent of D_n, as $n\to\infty$. Otherwise there would exist branches g_{n_t} of f^{-n_t} on a subdisc of D for a sequence n_t, convergent to a constant in ∂X on the complement of X, and to the identity on N.

It follows that no point $x\in\partial N$ satisfies (2.1) because its forward trajectory stays in a finite number of discs D discussed above. This contradicts TCE.

Case 2. Suppose that all limit functions of the sequence $f^n|_C$ are constant. Then there exists a disc C' with closure in C such that $|(f^n)'|\to0$ on C'. Suppose C is wandering. Then we can assume that there are no critical points in $f^n(C)$ for $n\geq0$. Hence by Koebe's $\frac{1}{4}$ theorem

$$\mathrm{dist}(f^n(x_0),\partial f^n(C))\to0 \tag{A1}$$

for an arbitrary $x_0 \in C'$. If C is eventually periodic and $f^n(C') \to \partial X$, then (A1) holds automatically. The possibility that C is eventually periodic and $f^n(C') \not\to \partial X$ leads to an attracting periodic point, at which TCE is not satisfied.

Hence there exists a finite family of discs $B(j) = B(x_j, r)$ for an arbitrary $r > 0$ small enough, each intersecting $\widehat{\mathbf{C}} \setminus X$, such that for each large n there exists $j = j(n)$ and a component D_n of $f^{-n}(B(j(n)))$ that contains C'.

As in the first case we deduce that the criticality of f^n on D_n tends to ∞. Otherwise some branches g_{n_t} of f^{-n_t} tend to constants on a little disc outside X by the repeller property, but the sets $g_{n_t}(f^{n_t}(C'))$ are equal to C', i.e. they have a definite size.

The criticality tending to ∞ contradicts TCE.

References

[CJY] Carleson, L., Jones, P. and Yoccoz, J.-C., Julia and John, *Bol. Soc. Brasil. Mat.* **25** (1994), 1–30.

[CE] Collet, P. and Eckmann, J.-P., Positive Lyapunov exponents and absolute continuity for maps of the interval, *Ergodic Theory Dynamical Systems* **3** (1983), 13–46.

[DH] Douady, A. and Hubbard, J., On the dynamics of polynomial-like mappings, *Ann. Sci. École Norm. Sup.* **18** (1985), 287–343.

[GS1] Graczyk, J. and Smirnov, S., Collet, Eckmann and Hölder, *Invent. Math.* **133** (1998), 69–96.

[GS2] Graczyk, J. and Smirnov, S., Weak expansion and geometry of Julia sets, Preprint, 1997.

[HK] Heinonen, J. and Koskela, P., Definitions of quasiconformality, *Invent. Math.* **120** (1995), 61–79.

[J] Jones, P. W., On removable sets for Sobolev spaces in the plane, in *Essays on Fourier Analysis in Honor of Elias M. Stein* (Fefferman, C., Fefferman, R. and Wainger, S., eds.), Princeton Math. Ser. **42**, pp. 250–267, Princeton Univ. Press, Princeton, N. J., 1995.

[JS] Jones, P. W. and Smirnov, S., Removability theorems for Sobolev functions and quasiconformal maps, to appear in *Ark. Mat.*

[L] Levin, G., On backward stability of holomorphic dynamical systems, *Fund. Math.* **158** (1998), 97–107.

[LS] Levin, G. and van Strien, S., Local connectivity of the Julia set of real polynomials, *Ann. of Math.* **147** (1998), 471–541.

[LM] Lyubich, M. and Milnor, J., The unimodal Fibonacci map, *J. Amer. Math. Soc.* **6** (1993), 425–457.

[M] Mañé, R., On a theorem of Fatou, *Bol. Soc. Brasil. Mat.* **24** (1993), 1–12.

[MS] McMullen, C. and Sullivan, D., Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a holomorphic dynamical systems, *Adv. Math.* **135** (1998), 351–395.

[NP] NOWICKI, T. and PRZYTYCKI, F., Topological invariance of the Collet–Eckmann property for S-unimodal maps, *Fund. Math.* **155** (1998), 33–43.

[P1] PRZYTYCKI, F., On Ω-stability and structural stability of endomorphisms satisfying Axiom A, *Studia Math.* **60** (1977), 61–77.

[P2] PRZYTYCKI, F., Iterations of holomorphic Collet–Eckmann maps: conformal and invariant measures, *Trans. Amer. Math. Soc.* **350** (1998), 717–742.

[P3] PRZYTYCKI, F., On measure and Hausdorff dimension of Julia sets for holomorphic Collet–Eckmann maps, in *International Conference on Dynamical Systems* (Ledrappier, F., Lewowicz, J. and Newhouse, S., eds.), Pitman Research Notes in Math. Ser. **362**, pp. 167–181, Longman, Harlow, 1996.

[P4] PRZYTYCKI, F., Hölder implies CE, to appear in *Astérisque.*

[PR] PRZYTYCKI, F. and ROHDE, S., Porosity of Collet–Eckmann Julia sets, *Fund. Math.* **155** (1998), 189–199.

[SN] SANDS, D. and NOWICKI, T., Quasisymmetric conjugacies of Collet–Eckmann maps, to appear in *Ergodic Theory Dynamical Systems.*

[S] SMALE, S., The Ω-stability theorem, in *Global Analysis* (Chern, S. and Smale, S., eds.), Proc. Symp. Pure Math. **14**, pp. 289–297, Amer. Math. Soc., Providence, R. I., 1970.

Received August 8, 1997
in revised form June 22, 1998

Feliks Przytycki
Institute of Mathematics
Polish Academy of Sciences
Sniadeckich 8
PL-00-950 Warszawa
Poland
email: feliksp@impan.gov.pl

Steffen Rohde
Department of Mathematics
Technische Universität Berlin
Straße des 17. Juni 136
DE-10623 Berlin
Germany
email: rohde@math.tu-berlin.de