Ark. Mat., **37** (1999), 291–304

Removable sets for Sobolev spaces

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Abstract. We study removable sets for the Sobolev space $W^{1,p}$. We show that removability for sets lying in a hyperplane is essentially determined by their thickness measured in terms of a concept of p-porosity.

1. Introduction

Let Ω be an open set in $\mathbf{R}^n$, $n\geq 2$. Recall that $u\in W^{1,p}(\Omega)$ provided $u\in L^p(\Omega)$, $1\leq p<\infty$, and there are functions $\partial_j u\in L^p(\Omega)$, $j=1,\dots,n$, so that

$$\int_\Omega u\partial_j\psi\,dx=-\int_\Omega \psi\partial_j u\,dx \tag{1.1}$$

for each test function $\psi\in C_0^1(\Omega)$ and all $1\leq j\leq n$. If $E\subset\mathbf{R}^n$ is a closed set of zero Lebesgue n-measure, then we say that E is removable for $W^{1,p}$ if $W^{1,p}(\mathbf{R}^n\setminus E)=W^{1,p}(\mathbf{R}^n)$ as sets. As Sobolev functions are defined a priori only almost everywhere this does not seem to be much of a requirement. A look at (1.1) should soon convince the reader that the question is more subtle than one first expects. Indeed, the test function class for (1.1) changes when E is removed from Ω.

Let us continue with some simple observations. First of all, it is immediate that removability is a local question. That is, E is removable for $W^{1,p}$ if and only if for each $x\in E$ there is $r>0$ so that $W^{1,p}(B(x,r)\setminus E)=W^{1,p}(B(x,r))$ as sets. Moreover, if $E\subset\Omega$ for some open set Ω, then E is removable for $W^{1,p}$ if and only if $W^{1,p}(\Omega\setminus E)=W^{1,p}(\Omega)$ as sets. Secondly, as smooth functions are dense in $W^{1,p}(\Omega\setminus E)$ and $W^{1,p}(\Omega)$ is a Banach space, it suffices to verify (1.1) whenever $u\in C^1(\Omega\setminus E)\cap W^{1,p}(\Omega\setminus E)$ and $\psi\in C_0^1(\Omega)$. Thirdly, integrating by parts and using the Fubini theorem we notice that (1.1) remains true for all $\psi\in C_0^1(\Omega)$ provided the projections of E along the coordinate axes have vanishing $n-1$-dimensional

[1] The author was partially supported by the NSF and the Academy of Finland, SA-34082.

measure. Thus sets of vanishing $n-1$-dimensional measure are removable and there exist removable sets of Hausdorff dimension n (simply take for E the n-fold product of an appropriate set F of Hausdorff dimension 1 but of vanishing one dimensional measure). On the other hand, there are nonremovable sets of dimension $n-1$ as such a set may even separate Ω.

The structure of removable sets has been studied by several authors. Ahlfors and Beurling [AB] introduced the so called NED sets as the sets whose removal does not affect extremal length and proved that NED sets are the removable singularities for Dirichlet finite analytic functions and univalent functions. NED sets coincide with the sets removable for $W^{1,2}$ in the plane. In general, the removable sets for $W^{1,n}$ are removable singularities for quasiconformal mappings in the euclidean n-space (cf. [R, p. 188], [V2]). For recent work see [B1], [B2], [KW], [W]. The removable sets for $W^{1,p}$ can be characterized as the null sets for the extremal length or modulus of order p or as null sets for condenser capacities [AH], [AS], [GV1], [H], [Kl], [S], [V1], [Y]. These characterizations are rather difficult to apply in practice and one of the motivations for this paper is to produce more concrete criteria for removability. It is known that the complement of a set removable for $W^{1,p}$, $p>n$, is quasiconvex (the internal distance defined as infimum of lengths of curves is comparable to the euclidean distance). This follows easily from the results in [KR]. From (1.1) and the Hölder inequality we readily observe that sets removable for $W^{1,p}$ are removable for $W^{1,q}$ when $q>p$. Thus, the complement of a set removable for $W^{1,p}$ is always quasiconvex. Our main result is the following theorem.

Theorem A. *Let $E\subset\mathbf{R}^{n-1}$. If E is p-porous, $1<p\le n$, then E is removable for $W^{1,p}$ in $\mathbf{R}^n$. Moreover, for each $1<p\le n$ there is a p-porous $E\subset\mathbf{R}^{n-1}$ that is not removable for $W^{1,q}$ for any $q<p$.*

The restriction $p\le n$ above comes from the fact that any compact $E\subset\mathbf{R}^{n-1}$ without interior is removable in $\mathbf{R}^n$ for all $p>n$, see Section 2. The definition of p-porosity is given in Section 3. Notice that by Theorem A, the removability of a set E can really depend on the exponent p. This conclusion can also rather easily be deduced from a result of Hedberg on sets of uniqueness for Bessel potential spaces [AH, Theorem 11.3.2], [H, p. 200].

Theorem A has a consequence for the extendability of Sobolev functions. We say that Ω is a p-extension domain if there is a bounded linear operator $L: W^{1,p}(\Omega)\to W^{1,p}(\mathbf{R}^n)$ with $Lu|_\Omega=u$ for each $u\in W^{1,p}(\Omega)$ (cf. [GV2], [HrK1], [J], [M], [Z]).

Corollary B. *There is an n-extension domain $\Omega\subset\mathbf{R}^n$ that is not a p-extension domain for any $p<n$.*

By a result of Gol'dstein and Vodop'yanov [GV2] and Jones [J], a simply connected planar 2-extension domain is a p-extension domain for all p. In [HrK1] Herron and the author showed that each n-extension domain in $\mathbf{R}^n$ that is quasiconformally equivalent to a so called uniform domain is in fact a p-extension domain, for all p. Corollary B shows that one really needs some additional assumption on the domain besides of being an n-extension domain. On the other hand, an n-extension domain is a p-extension domain for each $p>n$, see [K]. Notice that Maz'ya [M] has constructed a simply connected planar domain whose exterior is an extension domain for all $p>2$ but for no $p\leq 2$ and whose interior is an extension domain for all $p<2$ but for no $p\geq 2$.

Our next result requires some terminology. We say that a metric space X equipped with a Borel measure μ is n-regular if there is a constant C so that

$$C^{-1}r^n \leq \mu(B(x,r)) \leq Cr^n$$

for each $x\in X$ and all $0<r<\operatorname{diam}(X)$. Let u be continuous in X. We say that a measurable function $g\geq 0$ is an upper gradient of u provided

$$|u(x)-u(y)| \leq \int_\gamma g\,dH^1$$

for all $x,y\in X$ and each rectifiable curve γ that joins x and y. Here H^1 denotes the 1-dimensional Hausdorff measure, normalized so that $H^1([0,1])=1$. Notice that if X is a domain in $\mathbf{R}^n$ and $u\in C^1(X)$, then $g=|\nabla u|$ is an upper gradient of u. Following [HK2], [HK3] we say that X supports a p-Poincaré inequality if there exist constants $C,\lambda\geq 1$ so that

$$\fint_{B(x,r)} |u-u_B|\,d\mu \leq Cr\left(\fint_{B(x,\lambda r)} g^p\,d\mu\right)^{1/p} \tag{1.2}$$

for each $x\in X$, all $0<r<\operatorname{diam}(X)$, and each bounded continuous u and every upper gradient g of u. Here u_B is the average of u in $B(x,r)$ and the barred integrals are averaged integrals, that is $\fint_A v\,d\mu=\mu(A)^{-1}\int_A v\,d\mu$.

Theorem C. *Let $E\subset\mathbf{R}^n$ be a closed set of measure zero. Equip $X=\mathbf{R}^n\setminus E$ with the restrictions of the euclidean distance and the Lebesgue measure. Then E is removable for $W^{1,p}$, $1<p<\infty$, if and only if X supports a p-Poincaré inequality.*

Corollary D. *Let $1<p\leq n$. There is a locally compact n-regular metric space that supports a p-Poincaré inequality but does not support a q-Poincaré inequality for any $1<q<p$.*

The existence of such a space was stated without proof in a recent paper of Heinonen and the author [HK3]. The spaces that support a Poincaré inequality are important in the theory of quasiconformal mappings [HK1], [HK2], [HK3].

If $X=\mathbf{R}^n$, then any continuous function that has an upper gradient g in L^p can be approximated by a sequence (ψ_j) of Lipschitz continuous functions with $(\nabla\psi_j)$ bounded in L^p by the L^p-norm of g. Thus the Poincaré inequality for u, g can be deduced from a Poincaré inequality satisfied by Lipschitz functions and their gradients. Recently Heinonen and the author [HK4] extended this by showing that a proper, quasiconvex n-regular metric space that supports a p-Poincaré inequality for all Lipschitz functions supports a p-Poincaré inequality for continuous functions. Here the properness of X means that each closed ball is compact. Based on Theorems A and C we see that the properness assumption is essential.

Corollary E. *Let $1<p\leq n$. There is a locally compact n-regular metric space that supports a p-Poincaré inequality for all Lipschitz functions but does not support a p-Poincaré inequality for continuous functions.*

The paper is organized as follows. As Theorem A admits a more elementary proof in the planar case, we begin by proving Theorem A in Section 2 in the plane. In Section 3 we describe the modifications necessary for handling the higher dimensional situation. Section 4 contains the proofs of Theorem C and the corollaries.

Acknowledgement. We wish to thank Jang-Mei Wu for the construction of the Sierpiński type set in Section 3 and Lars Inge Hedberg and Yuri Netrusov for their comments on the original version of this paper.

2. The planar case

Let $E\subset\mathbf{R}$ be a compact set. For simplicity, we assume that $E\subset]0,1[$ and we say that E is p-removable if E is removable for $W^{1,p}(\mathbf{R}^2)$. Let us begin with a simple reduction. Suppose that $u\in W^{1,p}(B(0,2)\setminus E)\cap C^1(B(0,2)\setminus E)$. Then the Fubini theorem and the fundamental theorem of calculus show that u has a finite limit $u^+(x)=\lim_{0<t\to 0}u(x_1,t)$ and a corresponding limit $u^-(x)$ for H^1-a.e. $x=(x_1,0)\in E$. Thus integration by parts and the Fubini theorem show that $u\in W^{1,p}(B(0,2))$ provided $u^+(x)=u^-(x)$ for H^1-a.e. $x\in E$. In fact, one can easily show that this condition is equivalent to p-removability.

Proposition 2.1. *If E has empty interior, then E is p-removable for each $p>2$.*

Proof. By the Sobolev embedding theorem u is uniformly Hölder continuous both in the upper half of $B(0,2)$ and in the lower half of $B(0,2)$. As E has empty interior, it follows that, in fact, $u^+(x)=u^-(x)$ for each $x\in E$, and the claim follows.

Thus the case $p>2$ is not very interesting as the size of the complementary intervals of E in $]0,1[$ plays no role. For $p\leq 2$ the situation is different. Let us begin the discussion with a nonremovability result.

Theorem 2.2. *Let $]0,1[\setminus E=\bigcup_{j=1}^{\infty} I_j$, where I_j are pairwise disjoint open intervals. Suppose that $H^1(E)>0$, $1\leq p<2$ and that $\sum_{j=1}^{\infty} H^1(I_j)^{2-p}<\infty$. Then E is not p-removable.*

Proof. For $x\in B(0,2)\setminus E$ with $x_2\geq 0$, set

$$u(x)=\min\left\{\frac{x_2}{d(x,E)},\frac{\sqrt{2}}{2}\right\},$$

where x_2 is the second coordinate of x. This defines u in the upper half of $B(0,2)\setminus E$, and we extend u as zero to the lower half. Then u is locally Lipschitz, and $|\nabla u|\leq M<\infty$ almost everywhere in $B(0,2)\setminus\bigcup_{j=1}^{\infty}\Delta_j$, where Δ_j is an isosceles right triangle in the upper half plane with hypotenuse I_j. As

$$\int_{\Delta_j}|\nabla u|^p\,dx\leq CH^1(I_j)^{2-p},$$

we conclude that $u\in W^{1,p}(B(0,2)\setminus E)$. One can easily check that u cannot be extended to a Sobolev function in $B(0,2)$ (notice that $u^+(x)=\sqrt{2}/2$ when $x\in E$).

Theorem 2.2 shows that E cannot be removable if the complementary intervals are small and $p<2$. A similar result holds for $p=2$; see Theorem 3.1 below. We next define a sufficient condition for removability in terms of the complementary intervals.

We say that E is *p-porous*, $1<p<2$, if for H^1-a.e. $x=(x_1,0)\in E$ there is a sequence of numbers (r_i) and a constant C_x such that $r_i\to 0$, as $i\to\infty$, and each interval $]x_1-r_i,x_1+r_i[$ contains an interval $I_i\subset[0,1]\setminus E$ with $H^1(I_i)\geq C_x r_i^{1/(2-p)}$. We call E 2-porous if above $H^1(I_i)\geq C_x r_i\exp(-1/C_x r_i)$.

Theorem 2.3. *If E is p-porous, $1<p\leq 2$, then E is p-removable.*

Proof. By the usual covering theorems

$$\lim_{r\to 0}\frac{1}{r}\int_{B(x,r)}|\nabla u|^p\,dx=0, \tag{2.1}$$

for H^1-a.e. $x\in B(0,2)$ (cf. [Z, p. 118]).

Suppose first that $1<p<2$. Fix $x\in E$ so that the upper and lower limits $u^+(x)$ and $u^-(x)$ exist and (2.1) holds and so that the p-porosity condition holds for x.

It suffices to show that $u^+(x)=u^-(x)$. Suppose that these one-sided limits do not coincide. Without loss of generality we may assume that $u=0$ in $A^+=\{(x_1,t):0<t<r_i\}$ and $u=1$ in $A^-=\{(x_1,t):-r_i<t<0\}$. Fix r_i as in the definition and write $I^+=\{y\in I_i:u(y)\geq\frac{1}{2}\}$ and $I^-=I_i\setminus I^+$. By symmetry, we may assume that $H^1(I^+)\geq\frac{1}{2}C_x r_i^{1/(2-p)}$. Let $z=(x_1+d(x,I_i)+2H^1(I_i),-H^1(I_i))$. Then $|y-z|$ is comparable to $H^1(I_i)$, for each $y\in I_i$. Let J_y be the line segment on the line through z, y that joins y to A^+. Then the fundamental theorem of calculus and the Hölder inequality give

$$\frac{1}{2}\leq\int_{J_y}|\nabla u|\,dH^1\leq\left(\int_{J_y}|w-z|^{-1/(p-1)}\,dH^1\right)^{(p-1)/p}\left(\int_{J_y}|\nabla u|^p(w)|w-z|\,dH^1\right)^{1/p}.$$

Integrating in polar coordinates we obtain

$$\int_{B(x,2r_i)}|\nabla u|^p\,dy\geq CH^1(I_i)^{2-p}\geq C_x r_i,$$

and a contradiction follows by letting i tend to infinity.

Finally, let $p=2$. We use the above notation. Fix x and r_i as above. Since we wish to obtain a contradiction by estimating the integral of $|\nabla u|^2$ from below, we may replace u by a function with minimal energy; that is by a non-negative function v, harmonic in $B(0,2)\setminus(E\cup A^+\cup A^-)$, and with boundary values 0 in A^+ and 1 in A^-. Let x_i be the midpoint of I_i. By the Harnack inequality for positive harmonic functions, $\max_B v\leq 10\min_B v$, where $B=B\bigl(x_i,\frac{1}{4}H^1(I_i)\bigr)$. If $v(x)\leq\frac{1}{100}$ for some $x\in B$, then $v\leq\frac{1}{10}$ in B. Otherwise, $v\geq\frac{1}{100}$ in B. Thus either $v\geq\frac{1}{100}$ or $1-v\geq\frac{1}{100}$ in B, where $\operatorname{diam}(B)\geq\frac{1}{2}H^1(I_i)$. By symmetry we may assume that $v\geq\frac{1}{100}$. Then there is a constant C so that

$$4\int_{B(x,2r_i)}|\nabla v|^2\,dy\geq\frac{\pi}{\log(Cr_i/H^1(I_i))}\geq C_x r_i,$$

as seen from the standard capacity or extremal length estimates, and a contradiction follows by letting i tend to infinity.

The proof above shows that one could somewhat weaken the definition of p-porosity and still conclude p-removability. On the other hand, p-porosity is an essentially sharp condition for p-removability for sufficiently regular sets as seen in the proof of Theorem A.

Proof of Theorem A in the planar case. It suffices to construct a p-porous Cantor set $E\subset[0,1]$ such that E has positive length and $\sum_{j=1}^{\infty}H^1(I_j)^{2-q}<\infty$ for each $1<q<p$. Here the intervals I_j are the complementary intervals of E on $[0,1]$.

Let first $1<p<2$. The set E is obtained by the following Cantor construction. Let $0<s<\frac{1}{3}$ be a small constant to be determined momentarily. We begin by deleting an open interval of length $s2^{-1/(2-p)}$ from the middle of $J=[0,1]$. We are then left with two closed intervals of equal length. Assume that we have constructed 2^i closed intervals of equal length. We remove from the middle of each of these intervals an open interval of length $s2^{(-i-1)/(2-p)}$. By induction we obtain a nested sequence of closed intervals. We define E as the intersection of all these closed intervals. The total length of the removed intervals I_j (the complementary intervals of E) is

$$s2^{-1/(2-p)}\sum_{i=0}^{\infty}2^i2^{-i/(2-p)}=\frac{s2^{-1/(2-p)}}{1-2^{-(p-1)/(2-p)}}<1,$$

when s is sufficiently small. Thus E has positive length when s is sufficiently small. Moreover, given $x\in E$ and $j\geq 1$ the construction provides us with a complementary interval J_j of length $s2^{-j/(2-p)}$ and with $d(x,J_j)\leq 2^{-j}$. The p-porosity of E follows. Finally, the convergence of $\sum_{j=1}^{\infty}H^1(I_j)^{2-q}$, for $1<q<p$, is obvious as the collection of the intervals I_j consists of groups of 2^i intervals each of length $s2^{-i/(2-p)}$ with $i\geq 1$.

When $p=2$, we remove intervals of length $s2^{-i}\exp(-2^i)$. We again obtain a set E of positive length and it is easy to check that $\sum_{j=1}^{\infty}H^1(I_j)^{2-q}$ converges for each $1<q<2$. We leave the details to the reader.

3. The higher dimensional case

The nonremovable sets from Section 2 and the corresponding functions can be used to construct similar examples in higher dimensions. Indeed, if E_p is the set we constructed for $1<p<2$, then $E_p\times E_p$ is nonremovable in 3-space as seen by considering the function $v(x_1,x_2,x_3)=u_p(x_1,x_3)u_p(x_2,x_3)$, where u_p is the function from Theorem 2.2. However, one can check using the Fubini theorem, Proposition 2.1, and induction that each totally disconnected closed set $E\subset\mathbf{R}^{n-1}$ is p-removable for $p>2$ (here and in what follows, p-removability means removability for $W^{1,p}$ in $\mathbf{R}^n$). Indeed, the restriction of u to $T\setminus E$ belongs to $W^{1,p}(T\setminus E)$ for almost all hyperplanes T parallel to the coordinate axes and the removability follows by integrating by parts with the help of Proposition 2.1 provided $n=3$. Use induction to cover the case of dimensions larger than 3.

Thus such nonremovable sets cannot be Cantor sets for $p>2$. Moreover, a construction similar to that used in the proof of Theorem 2.2 cannot give a nonextendable Sobolev function as the boundary values of a Sobolev function of the upper half space cannot be the characteristic function of a bounded set of positive $(n-1)$-dimensional measure when $p \geq 2$ (cf. [HrK2]).

We again obtain p-removability from p-porosity. We say that $E \subset \mathbf{R}^{n-1}$ is *p-porous*, $1<p<n-1$, if for H^{n-1}-a.e. $x \in E$ there is a sequence (r_i) and a constant C_x such that $r_i \to 0$, as $i \to \infty$, and each $(n-1)$-dimensional ball $B(x, r_i)$ contains a ball $B_i \subset B(x, r_i) \setminus E$ of radius no less than $C_x r_i^{(n-1)/(n-p)}$. We define p-porous sets for $n-1 \leq p < n$ by replacing the balls B_i by continua F_i of diameters no less than $C_x r_i^{(n-1)/(n-p)}$. We call E n-porous if the diameter of F_i is no less than $C_x r_i \exp(-1/C_x r_i)$.

Notice that our definition is consistent with the definition given in Section 2.

We begin with a version of Theorem 2.2. For a cube Q and a positive real number a we write aQ for the concentric cube whose side length is a times that of Q.

Theorem 3.1. *Suppose that $I^{n-1} \setminus E = \bigcup_{i=1}^{\infty} Q_i$, where $I =]0,1[$, and each Q_i is an open cube in I^{n-1}. If $1<p<n$,*

$$\sum_{i=1}^{\infty} \operatorname{diam}(Q_i)^{n-p} < \infty, \quad \textit{and} \quad H^{n-1}\Big(I^{n-1} \setminus \bigcup_{i=1}^{\infty} 2Q_i\Big) > 0,$$

then E is not p-removable. If

$$\sum_{i=1}^{\infty} (\log(1/\operatorname{diam}(Q_i)))^{1-n} < \infty, \quad \textit{and} \quad H^{n-1}\Big(I^{n-1} \setminus \bigcup_{i=1}^{\infty} \operatorname{diam}(Q_i)^{-1/2} Q_i\Big) > 0,$$

then E is not n-removable.

Proof. Set $\Omega = I^{n-1} \times]-1,1[$. Suppose first that $1<p<n$. For each cube Q_i from our collection, let the set $W_i = 2Q_i \times]-\operatorname{diam}(Q_i), \operatorname{diam}(Q_i)[$. Define $f_i(x) = \operatorname{diam}(Q_i)^{-1} \chi_{W_i}(x)$ and set $g(x) = \sum_{i=1}^{\infty} f_i(x)$. Set $y = \left(\frac{1}{2}, \dots, \frac{1}{2}, -1\right)$, and define

$$u(x) = \inf_{\gamma_x} \int_{\gamma_x} g(x)\, dH^1$$

for $x \in \Omega \setminus E$, where the infimum is taken over all rectifiable curves that join x to y in $(I^{n-1} \times [-1,1]) \setminus E$. Then u is locally Lipschitz, and $|\nabla u| \leq g$ almost everywhere. Moreover, $\int_\Omega |\nabla u|^p\, dx \leq C \sum_{i=1}^{\infty} \operatorname{diam}(Q_i)^{n-p} < \infty$, and, consequently, $u \in W^{1,p}(\Omega \setminus E)$. As $u \geq 1$ in the upper half of Ω and $\lim_{0>t\to 0} u(x', t) = 0$ for all $x' \in I^{n-1} \setminus \bigcup_{i=1}^{\infty} 2Q_i$, E is not removable for u.

The case $p=n$ is similar. In this case the intersection of W_i with $\mathbf{R}^{n-1}$ will be $\operatorname{diam}(Q_i)^{-1/2} Q_i$ and $f_i(x) = \log(1/\operatorname{diam}(Q_i)) |x - x_i|^{-1} \chi_{W_i \setminus \operatorname{diam}(Q_i)^{1/2} W_i}(x)$, where x_i is the center point of Q_i. We leave the necessary computations to the reader.

Theorem 3.2. *If E is p-porous, then E is p-removable.*

Proof. Let $E\subset I^{n-1}$. Given $u\in W^{1,p}(\Omega\setminus E)$, we let v be the p-harmonic function with the Dirichlet data given by u. Then $u-v\in W_0^{1,p}(\Omega\setminus E)$, and it suffices to show that E is removable for v. As v is p-harmonic, and $\int_{\Omega\setminus E}|\nabla v|^p\,dx<\infty$, v has one-sided upper and lower non-tangential limits almost everywhere in E; see [KMV]. As in the proof of Theorem 2.3 it suffices to show that

$$\int_{B^n(x,2r_i)}|\nabla v|^p\,dx\geq Cr_i^{n-1}$$

whenever $x\in E$ is such that the one-sided (non-tangential) limits do not coincide at x and the porosity condition holds at x. Here $B^n(x,2r_i)$ is the n-dimensional ball corresponding to the $(n-1)$-dimensional ball $B(x_i,r_i)$ from the porosity condition. Assume again that the one-sided limits are 0 and 1. Let V_i be an n-dimensional ball of radius $R=\operatorname{diam}(G_i)$ that contains the set G_i for x from the definition of porosity (so $G_i=B_i$ or $G_i=F_i$).

Suppose first that $p<n$. By symmetry and p-porosity, we may assume that the upper limit is 0 and that $v\geq\frac{1}{2}$ in a set $A\subset B_i$ with $H_\infty^{n-1}(A)\geq\frac{1}{2}H_\infty^{n-1}(B_i)$ or in a set $A\subset F_i$ with $H_\infty^1(A)\geq\frac{1}{2}H_\infty^1(F_i)$. Here $H_\infty^\lambda(A)$ is the λ-dimensional Hausdorff content of A; notice that $H_\infty^{n-1}(A)=H^{n-1}(A)$ for a set $A\in\mathbf{R}^{n-1}$. Extend the restriction of v to the upper half space to all of $\mathbf{R}^n$ by reflection. Write w for this new function. It clearly suffices to establish the above inequality for w.

Assume first that the average $w_{V_i}\leq\frac{1}{4}$, and let $\varepsilon>0$. As $u\geq\frac{1}{2}$ on A, standard estimates (cf. [HK3, 5.9]) show that

$$R^{p\varepsilon}C\int_{2V_i}|\nabla w|^p\geq H_\infty^{n-p+p\varepsilon}(A),$$

where C depends only on p, n, and ε. When $p\leq n-1$, we let $\varepsilon=(p-1)/p$. The above inequality and the estimate on the size of A then give

$$C_1\int_{2V_i}|\nabla w|^p\geq R^{n-p}\geq C_2r_i^{n-1},$$

where C_1 and C_2 depend only on C, p, n, and C_x. When $n-1<p<n$, we let $\varepsilon=(p+1-n)/p$ and the above inequality again follows.

Suppose then that $w_{V_i}\geq\frac{1}{4}$. As u has the non-tangential upper limit zero at x, we find for small r_i a ball $U_i\subset B^n(x,r_i)$ of radius comparable to r_i so that $w_{U_i}\leq\frac{1}{8}$. By the Sobolev–Poincaré inequality

$$\left(\int_B|w-w_B|^{pn/(n-p)}\right)^{(n-p)/n}\leq C\int_B|\nabla w|^p,$$

where C depends only on p, n, and $B=B^n(x,2r_i)$. Because the diameters of U_i and V_i are comparable to r_i, it then easily follows that

$$C\int_{B^n(x,2r_i)} |\nabla w|^p \geq R^{n-p},$$

where C depends only on p and n.

Combining the above two cases and noting the definition of w we conclude that

$$C\int_{B^n(x,r_i)} |\nabla v|^p \geq r_i^{n-1},$$

where C is independent of i. The claim follows.

The case $p=n$ is similar: let $\varepsilon=1/n$ and use the Trudinger inequality instead of the Sobolev–Poincaré inequality.

Proof of Theorem A in higher dimensions. Let $1<p\leq n$. By Theorem 3.2 it suffices to construct a p-porous compact set $E\subset[0,1]^{n-1}=I^{n-1}$ such that E is not removable when $q<p$ in $\mathbf{R}^n$. Again, the situation is slightly different depending on whether $p<n$ or $p=n$.

Let first $p<n$ and set $A=(n-1)/(n-p)$. In what follows $\frac{1}{4}\leq s\leq\frac{1}{2}$ can change its value from line to line. We begin by deleting a cube Q_1 of side length $s2^{-A}$ from the center of I^{n-1}. We subdivide $I^{n-1}\setminus Q_1$ into cubes of two different sizes: 2^{n-1} of them of size $l_1=\frac{1}{2}(1-s2^{-A})$ and the rest of size $s2^{-A}$. The cubes of size $s2^{-A}$ correspond to translating the central cube along the coordinate directions. This determines the value of s at this stage as we need $2^A/s$ to be an odd integer. Write $\mathcal{W}^1$ for the collection of all the cubes in the subdivision. We continue by deleting a cube of side length $s2^{-2A}$ for an appropriate s from the center of each cube in $\mathcal{W}^1$ whose size is at least $\frac{1}{2}l_1$. Our new collection $\mathcal{W}^2$ consists of the cubes in $\mathcal{W}^1$ whose side lengths are less than $\frac{1}{2}l_1$ and from the cubes obtained from the subdivisions of the cubes subject to central deletion. The subdivision of such a cube results in cubes of two sizes: 2^{n-1} cubes of size $\frac{1}{2}(l(Q)-s2^{-2A})$, the rest of size $s2^{-2A}$. Let l_2 be the largest side length of a cube in $\mathcal{W}^2$. We repeat the construction inductively: at stage i we delete a cube of size $s2^{-iA}$ from the center of each cube in $\mathcal{W}^i$ of size at least $\frac{1}{2}l_i$, we subdivide (again the value of s is determined by the requirement that $l(Q)2^{iA}/s$ be an odd integer) and let l_{i+1} be the size of largest cube obtained. Notice that l_i is comparable to 2^{-i}. Let $E=\bigcap_{i=1}^{\infty}\mathcal{W}^i$. Then E is clearly p-porous, and a simple counting argument shows that $\sum_{i=1}^{\infty}\operatorname{diam}(Q_i)^{n-q}<\infty$ for each $q<p$.

When $p=n$ we begin by deleting a cube Q_1 of side length $s\exp(-1)$. We then let $l_1=\frac{1}{2}(1-s\exp(-1))$. In the ith step we delete a cube of size $s2^{-i}\exp(-2^i)$ from the center of each cube whose size is at least $\frac{1}{2}l_i$. The claim follows as above.

Remark 3.3. One can modify the construction used in the proof of Theorem A above so as to obtain a compact set $E\subset\mathbf{R}^{n-1}$ that is not n-removable but that is p-removable for all $p>n$.

4. Proofs of Theorem C and the corollaries

Corollary B immediately follows from Theorem A and as Corollary D follows directly from Theorems A and C, we only present the proofs of Theorem C and Corollary E.

Proof of Theorem C. We define $X=\mathbf{R}^n\setminus E$, and equip X with the euclidean distance and with the restriction of the euclidean volume. Let u be bounded and continuous in X and g be an upper gradient of u. We first establish (1.2) for the pair u, g assuming that E is p-removable.

Fix a ball $B(x,r)$. If g is not in $L^p(B(x,r))$, there is nothing to be shown. Otherwise, one can check that $u\in W^{1,p}(B(x,r)\setminus E)$ (cf. [HjK], [KM]) and that $|\nabla u|\leq g$ almost everywhere in $B(x,r)$. Thus the desired p-Poincaré inequality follows from the usual p-Poincaré inequality in $\mathbf{R}^n$.

Suppose then that X supports a p-Poincaré inequality. Let u be smooth and bounded in $X=\mathbf{R}^n\setminus E$ with $|\nabla u|\in L^p(B\setminus E)$ for each ball B; notice that $L^p(B\setminus E)=L^p(B)$ as $|E|=0$. Notice that $|\nabla u|$ is an upper gradient of u.

Fix j. We cover $\mathbf{R}^n$ with balls $5B_i$, each B_i of radius 2^{-j} and centered in X and so that the balls B_i are pairwise disjoint. We pick a partition of unity ψ_i, $i\geq 1$, so that $0\leq\psi_i\leq 1$, $\psi_i=1$ in the euclidean ball B_i, $\psi_i=0$ in $\mathbf{R}^n\setminus 10B_i$, and $|\nabla\psi_i|\leq C2^j$. Here C is independent of i. Define $u_j=\sum_{i=1}^{\infty}a_i\psi_i$, where $a_i=\fint_{10B_i}u\,dx$. Clearly u_j is smooth. Fix k. Then

$$|\nabla u_j(x)|=\left|\nabla\sum_{i=1}^{\infty}(a_i-a_k)\psi_i(x)\right|.$$

If $x\in 10B_k$, we conclude that

$$|\nabla u_j(x)|\leq C2^j\sum_{i=1}^{\infty}|a_i-a_k|,$$

where the sum is taken over those i with $10B_i\cap 10B_k\neq\emptyset$. For each such i,

$$|a_i-a_k|\leq C2^{-j}\left(\fint_{30\lambda B_k}|\nabla u|^p\,dx\right)^{1/p},$$

and so

$$\int_{10B_k} |\nabla u_j|^p \, dx \le C \int_{30\lambda B_k} |\nabla u|^p \, dx;$$

notice that there is only a uniformly bounded number of indices i with $30B_i \cap 30B_k \neq \emptyset$ and that $|\nabla u|$ is an upper gradient of u. As the balls B_k are pairwise disjoint, the balls $30\lambda B_k$ also have uniformly bounded overlap. Fix an arbitrary ball B. It follows that $\int_B |\nabla u_j|^p \, dx \le M < \infty$, where M is independent of j. Moreover, as $u(x) = \sum_{i=1}^{\infty} u(x)\psi_i(x)$ and $u_j(x) = \sum_{i=1}^{\infty} a_i \psi_i(x)$, where a_i is the average of u on $10B_i$, one easily checks using the p-Poincaré inequality for u that $u_j \to u$ in $L^1(B)$ (in fact $\|u - u_j\|_{L^1(B)} \le C 2^{-j} \operatorname{diam}(B)^{n(p-1)/p} \|\nabla u\|_{L^p(2B)}$). Hence the sequence (u_j) is bounded in $L^1(B)$ and thus the usual p-Poincaré inequality in B shows that (u_j) is bounded in $L^p(B)$ as well. Thus (u_j) is bounded in $W^{1,p}(B)$. Consequently, a subsequence converges weakly to some $v \in W^{1,p}(B)$ and as the functions u_j tend to u in $L^1(B)$, we obtain $u \in W^{1,p}(B)$. The claim follows.

Proof of Corollary E. Let $X = \mathbf{R}^n \setminus E$, where $E \subset \mathbf{R}^{n-1}$ is removable for all $q > n$, but not removable for n; see Remark 3.3.

Let u be continuous and let g be an upper gradient of u. If u is Lipschitz, then $u \in W^{1,q}(\Omega \setminus E)$ for all $q > n$ and each bounded domain Ω. Then $u \in W^{1,q}(\Omega)$ and consequently $u \in W^{1,p}(\Omega)$ for each bounded Ω. Thus the usual p-Poincaré inequality holds for u and $|\nabla u|$. On the other hand, it is easy to check that $|\nabla u| \le g$ almost everywhere, and we obtain the desired p-Poincaré inequality for the pair u, g. The rest of the claim follows from Theorem C.

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Received May 21, 1997
in revised form April 20, 1998

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