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A sharp weighted L^2-estimate for the solution to the time-dependent Schrödinger equation

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Abstract. For $\xi \in \mathbf{R}^n$, $t \in \mathbf{R}$ and $f \in \mathcal{S}(\mathbf{R}^n)$ define $(S^2 f)(t)\hat{\ }(\xi) = \exp(it|\xi|^2)\hat{f}(\xi)$. We determine the optimal regularity s_0 such that

$$\int_{\mathbf{R}^n} \|(S^2 f)[x]\|^2_{L^2(\mathbf{R})} \frac{dx}{(1+|x|)^b} \le C\|f\|^2_{H^s(\mathbf{R}^n)}, \quad s > s_0,$$

holds where C is independent of $f \in \mathcal{S}(\mathbf{R}^n)$ or we show that such optimal regularity does not exist. This problem has been treated earlier, e.g. by Ben-Artzi and Klainerman [2], Kato and Yajima [4], Simon [6], Vega [9] and Wang [11].

Our theorems can be generalized to the case where the $\exp(it|\xi|^2)$ is replaced by $\exp(it|\xi|^a)$, $a \neq 2$.

The proof uses Parseval's formula on $\mathbf{R}$, orthogonality arguments arising from decomposing $L^2(\mathbf{R}^n)$ using spherical harmonics and a uniform estimate for Bessel functions. Homogeneity arguments are used to show that results are sharp with respect to regularity.

0. Introduction

0.1. Let u denote the solution to the free time-dependent Schrödinger equation $\Delta_x u = i\partial_t u$ with initial data f. At least for $f \in \mathcal{S}(\mathbf{R}^n)$ it is represented by an oscillatory integral with quadratic phase. In this note we are interested in estimating the double integral

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u(x,t)|^2 \frac{dt\,dx}{(1+|x|)^b}$$

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from above by inhomogeneous Sobolev space norms. Here b is a real number larger than 1. A related estimate has been used by Sjölin [7] to derive an L^2_{loc}-estimate for the maximal function $u^*(x)=\sup_{0\le t\le 1}|u(x,t)|$. See [10, pp. 487–488]. The method in those papers uses interpolation for mixed norm spaces and the continuity of the embedding $H^{1/2+\varepsilon}(\mathbf{R})\subseteq L^\infty(\mathbf{R})$, $\varepsilon>0$. The objective here is different, namely to characterize the decay of u expressed by estimates for the above double integral. This is a problem of interest independent of estimates for the maximal function u^*.

0.2. Some earlier results. Let us consider the inequality

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}}|((|\partial_t|^\alpha u)(x,t)|^2\,\frac{dt\,dx}{(1+|x|)^b}\le C\|f\|^2_{H^{2(\alpha-1/2)+b/2}(\mathbf{R}^n)}, \tag{0.1}$$

where C is assumed to be independent of $f\in\mathcal{S}(\mathbf{R}^n)$ and $\alpha\ge 0$. (The real power of the modulus of the time-derivation in the left-hand side can be defined in a proper way using the Fourier transform, and the Sobolev space norm on the right-hand side is defined in (1.1b) below.) For $\alpha=0$ and $1<b<2$, Wang [11] has showed that (0.1) does not hold.

It has been argued (see [9, p. 874 line 3 and 4 from below]) that (0.1) can be used in conjunction with the continuity of the embedding $H^{1/2+\varepsilon}(\mathbf{R})\subseteq L^\infty(\mathbf{R})$, $\varepsilon>0$ to prove that there is a number C independent of $f\in\mathcal{S}(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n}|u^*(x)|^2\,\frac{dx}{(1+|x|)^b}\le C\|f\|^2_{H^{1/2+\varepsilon}(\mathbf{R}^n)},\quad \varepsilon>0, \tag{0.2}$$

holds. That argument uses the inequality

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}}|(([1+|\partial_t|^\alpha]u)(x,t)|^2\,\frac{dt\,dx}{(1+|x|)^b}\le C\|f\|^2_{H^{2(\alpha-1/2)+b/2}(\mathbf{R}^n)}, \tag{0.3}$$

where C is independent of $f\in\mathcal{S}(\mathbf{R}^n)$, α is slightly larger than $\frac{1}{2}$ and b is slightly larger than 1.

It is one of the purposes of this paper to show that (0.3) does not hold for any $b<2$. We refer the reader to Theorems 2.1 and 2.2 below. These theorems are characterizations. We use spherical harmonics and uniform estimates for integrals of squares of Bessel functions $J_{n/2+k-1}$ to prove the sufficiency parts. One crucial fact for the necessity part is that there is no number C independent of $f_0\in L^1(0,1)$ such that

$$\int_0^1\left(\int_\varrho^\infty J_{n/2-1}(r)^2r^{1-b}\,dr\right)\varrho^{b-2}f_0(\varrho)\,d\varrho\le C\|f_0\|_{L^1(0,1)},\quad f_0\ge 0, \tag{0.4}$$

holds, where b is any number strictly less than 2. (When $n=2$ we cannot allow $b=2$.)

Our results are contradictory to [9, Theorem 3, p. 874]. In fact (0.4) easily provides us with the following counterexample to (0.3) (cf. [11]): Take $f_0(\varrho)=\varrho^{-b/2}$ for $0<\varrho<1$ and $f_0(\varrho)=0$ for $\varrho\geq 1$. Set $\hat{f}(\xi)=|\xi|^{-n/2+1/2}f_0(|\xi|)^{1/2}\varphi_N(\xi)$, where φ_N is a sequence of non-negative functions in $C_0^\infty(\mathbf{R}^n)$, $\operatorname{supp}\varphi_N\subseteq\dot{B}^n$ tending pointwise to the characteristic function of $\dot{B}^n$, the punctured open unit ball of $\mathbf{R}^n$, as N tends to infinity. Then the right-hand side of (0.3) will be bounded by a number independent of N whereas the left-hand side will tend to infinity with N. This counterexample does not require that one analyses the rôle of the regularity $2\left(\alpha-\frac{1}{2}\right)+\frac{1}{2}b$ in the right-hand side of (0.3). It is rather the low frequencies of f which completely determine the weight $(1+|x|)^{-b}$. See Theorems 2.1(b) and 2.2(b).

One may object that although (0.3) does not hold, its modification

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}}|([1+|\partial_t|^\alpha]u)(x,t)|^2\,\frac{dt\,dx}{(1+|x|)^b}\leq C\|f_\psi\|^2_{H^{2(\alpha-1/2)+b/2}(\mathbf{R}^n)},\tag{0.5}$$

obtained by disregarding the low frequencies holds. Here u is the solution to the free time-dependent Schrödinger equation with initial data f_ψ, $\hat{f}_\psi(\xi)=\psi(|\xi|)\hat{f}(\xi)$ and ψ is defined in Section 1.3. However, it is on the whole misleading to consider such an inequality. The reason is that

$$\|f_\psi\|_{\dot{H}^{-1+b/2}(\mathbf{R}^n)}$$

which is equivalent to $\|f_\psi\|_{H^{-1+b/2}(\mathbf{R}^n)}$, *increases* with b whereas the norm

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}}|u(x,t)|^2\,\frac{dt\,dx}{(1+|x|)^b}\tag{0.6}$$

decreases with b. (Choose $\alpha=0$ in (0.5).)

What might be relevant here is to replace (0.6) by

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}}|u(x,t)|^2\,\frac{dt\,dx}{|x|^b}$$

and to consider initial data in $\dot{H}^{-1+b/2}(\mathbf{R}^n)$, so that the resulting estimate scales. See e.g. Ben-Artzi and Klainerman [2, Theorem 1(a), p. 26]. This and related matters will, however, be further discussed elsewhere.

The decomposition $f=f_\psi+(f-f_\psi)$ is used in Ben-Artzi and Devinatz [1] to handle more general symbols in the representation formula for u.

Despite the fact that (0.3) does not hold for any $b<2$, (0.2) holds for all $b>1$ and a correct proof of this can be found using Soljanik's theorem [10, Theorem 2.2, p. 487].

0.3. The plan of this communication. In Section 1 we introduce some notation. Our theorems are stated in Section 2 and proofs are prepared in Section 3 where we cite some well-known results and prove a corollary to one of them. Finally we prove our theorems in Section 4.

1. Notation

1.1. Representation formula for u, the Fourier transform. For x and ξ in $\mathbf{R}^n$ we let $x\xi=x_1\xi_1+\ldots+x_n\xi_n$. If $m\in L^\infty(\mathbf{R}^n\times\mathbf{R}_+)$, if a is a real positive number and if f is in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ we define

$$(S^a_m f)[x](t)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} m(x,|\xi|)e^{i(x\xi+t|\xi|^a)}\hat{f}(\xi)\,d\xi,\quad t\in\mathbf{R}.$$

If $m=1$ we will write $S^a_m=S^a$. Here $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(\xi)=\int_{\mathbf{R}^n} e^{-ix\xi}f(x)\,dx.$$

Note that $u(x,t)=(S^2f)[x](t)$. Throughout the present communication we will restrict ourselves to the case $a=2$. This restriction is not serious as will be clear below when we perform a certain change of variables. See (4.3).

1.2. Sobolev spaces. We introduce fractional Sobolev spaces

$$\text{(1.1a)}\quad \dot{H}^s(\mathbf{R}^n)=\left\{f\in\mathcal{S}'(\mathbf{R}^n):\|f\|^2_{\dot{H}^s(\mathbf{R}^n)}=\int_{\mathbf{R}^n}|\xi|^{2s}|\hat{f}(\xi)|^2\,d\xi<\infty\right\},$$

$$\text{(1.1b)}\quad H^s(\mathbf{R}^n)=\left\{f\in\mathcal{S}'(\mathbf{R}^n):\|f\|^2_{H^s(\mathbf{R}^n)}=\int_{\mathbf{R}^n}(1+|\xi|^2)^s|\hat{f}(\xi)|^2\,d\xi<\infty\right\}.$$

The embedding property alluded to in Section 0.1 is merely Cauchy–Schwartz' inequality applied to Fourier's inversion formula

$$f(x)=\frac{1}{2\pi}\int_{\mathbf{R}} e^{ix\xi}(1+\xi^2)^{-(1/4+\varepsilon)}(1+\xi^2)^{1/4+\varepsilon}\hat{f}(\xi)\,d\xi.$$

1.3. Auxiliary notation. By B^n and Σ^{n-1} we denote the open unit ball and the unit sphere in $\mathbf{R}^n$ respectively. (The open unit interval B^1 will be denoted by B.) The symbol $d\sigma$ is used when we integrate with respect to the surface measure induced by the Lebesgue measure on $\mathbf{R}^n$.

In Section 4 we will use auxiliary functions χ and ψ such that $\chi\in C_0^\infty(\mathbf{R})$ is even, non-negative and equal to 1 on B and 0 outside $2B$ and $\psi=1-\chi$.

Numbers denoted by C may be different at each occurrence.

Unless otherwise explicitly stated all functions f are supposed to belong to $\mathcal{S}(\mathbf{R}^n)$.

2. The theorems

2.1. Theorem.
(a) *If $b>2$, then there is a number C independent of $f\in\mathcal{S}(\mathbf{R}^2)$ such that*

$$\int_{\mathbf{R}^2} \|(S_m^2 f)[x]\|_{L^2(\mathbf{R})}^2 \frac{dx}{(1+|x|)^b} \leq C\|f\|_{H^{-1/2}(\mathbf{R}^2)}^2.$$

(b) *Assume that there is a number C independent of $f\in\mathcal{S}(\mathbf{R}^2)$ such that*

$$\int_{\mathbf{R}^2} \|(S^2 f)[x]\|_{L^2(\mathbf{R})}^2 \frac{dx}{(1+|x|)^b} \leq C\|f\|_{L^2(\mathbf{R}^n)}^2, \quad \operatorname{supp}\hat{f}\subseteq B^2. \tag{2.1}$$

Then $b>2$.

2.2. Theorem. *Let $n\geq 3$.*
(a) *There is a number C independent of f such that*

$$\int_{\mathbf{R}^n} \|(S_m^2 f)[x]\|_{L^2(\mathbf{R})}^2 \frac{dx}{(1+|x|)^2} \leq C\|f\|_{H^{-1/2}(\mathbf{R}^n)}^2.$$

(b) *Assume that there is a number C independent of f such that*

$$\int_{\mathbf{R}^n} \|(S^2 f)[x]\|_{L^2(\mathbf{R})}^2 \frac{dx}{(1+|x|)^b} \leq C\|f\|_{L^2(\mathbf{R}^n)}^2, \quad \operatorname{supp}\hat{f}\subseteq B^n. \tag{2.2}$$

Then $b\geq 2$.

2.3. Remark. In the case when $m=1$ Theorems 2.1(a) and 2.2(a) have been proved earlier by Ben-Artzi and Klainerman [2, Corollary 2, p. 28], Kato and Yajima [4, (1.5), p. 482] and Simon [6].

2.4. Discussion. It is to be observed that the estimates in Theorems 2.1 and 2.2 are global in t and global in x with an appropriate weight (whose admissible decay is characterized when $m=1$) and that the weighted norm of $\|(S_m^2 f)[x]\|_{L^2(\mathbf{R})}$ can be controlled not only by $\|f\|_{\dot{H}^{-1/2}(\mathbf{R}^n)}$, but also by $\|f\|_{H^{-1/2}(\mathbf{R}^n)}$ once a careful analysis of the *low frequencies* of the initial data is carried out. Corresponding results hold true for the phase $\xi\mapsto|\xi|^a$ (at least when $a>1$); the number of derivatives in the *inhomogeneous* Sobolev space norm is $\frac{1}{2}(1-a)$ in this more general case.

For the discussion in this paragraph cf. Kenig, Ponce and Vega [5, Section 4, Remark (b), p. 56].

3. Some preparation

3.1. In this section we introduce some notation and collect some well-known results which will be used in the proof of our theorem. Standard references are given.

3.2. Notation. For integers l we define the Bessel function of order l by

$$J_l(r)=\frac{1}{2\pi}\int_0^{2\pi} e^{i(r\sin\omega-l\omega)}\,d\omega \tag{3.1}$$

and for real numbers $\lambda>-\frac{1}{2}$ the Bessel function of order λ by

$$J_\lambda(r)=\frac{r^\lambda}{2^\lambda\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{irv}(1-v^2)^{\lambda-1/2}\,dv. \tag{3.2}$$

Here Γ is the well-known Gamma function. The Poisson representation (3.2) is consistent with (3.1). (See e.g. Stein and Weiss [8, Lemma 3.1, p. 153].)

3.3. Notation. Throughout this communication P will denote a *solid spherical harmonic* (cf. [8, pp. 140–141]) of non-negative degree k such that $\|P\|_{L^2(\Sigma^{n-1})}=1$.

Let $\mathfrak{H}_k(\mathbf{R}^n)$ be the linear space of all finite linear combinations of functions of the form

$$\xi\longmapsto P(\xi)f_0(|\xi|)|\xi|^{-n/2-k+1/2},$$

where $f_0\in L^2(\mathbf{R}_+)$. (Cf. [8, p. 138].) The space $\mathfrak{H}_k(\mathbf{R}^n)$ is a Hilbert subspace of $L^2(\mathbf{R}^n)$ (with the inherited inner product).

3.4. Theorem. ([8, Theorem 3.10, p. 158].) *Let $f\in\mathfrak{H}_k(\mathbf{R}^n)$ be given by $f(\xi)=P(\xi)f_0(|\xi|)|\xi|^{-n/2-k+1/2}$. Then*

$$\hat f(x)=\frac{(2\pi)^{n/2}P(x)}{i^k|x|^{\nu(k)}}\int_0^\infty f_0(\varrho)J_{\nu(k)}(\varrho|x|)\varrho^{1/2}\,d\varrho,\quad \nu(k)=\frac{n}{2}+k-1.$$

3.5. Corollary. *The tempered distribution μ_P defined by*

$$\mu_P(\varphi)=\int_{\Sigma^{n-1}}\varphi(\xi')P(\xi')\,d\sigma(\xi'),\quad \varphi\in\mathcal{S}(\mathbf{R}^n),$$

has Fourier transform $\hat\mu_P$ given by

$$\hat\mu_P(x)=\frac{(2\pi)^{n/2}P(x)}{i^k|x|^{\nu(k)}}J_{\nu(k)}(|x|).$$

3.6. Theorem. ([8, Lemma 2.18, p. 151].) *The complete orthogonal decomposition*

$$L^2(\mathbf{R}^n) = \bigoplus_{k=0}^{\infty} \mathfrak{H}_k(\mathbf{R}^n)$$

holds in the sense that

(a) *each subspace $\mathfrak{H}_k(\mathbf{R}^n)$ is closed;*

(b) *$\mathfrak{H}_{k_1}(\mathbf{R}^n)$ is orthogonal to $\mathfrak{H}_{k_2}(\mathbf{R}^n)$ if $k_1 \neq k_2$;*

(c) *every f can be written as a sum*

$$f = \sum_{k=0}^{\infty} f_k, \quad f_k \in \mathfrak{H}_k(\mathbf{R}^n),$$

with convergence in $L^2(\mathbf{R}^n)$.

3.7. Theorem. (Asymptotics of Bessel functions, [8, Lemma 3.11, p. 158].) *If $\lambda > -\frac{1}{2}$, then there exists a number C depending on λ but independent of r such that*

$$\left| J_\lambda(r) - \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right)\right| \le C r^{-3/2}, \quad r \ge 1.$$

3.8. Theorem. (Cf. Hörmander [3, Theorem 7.1.26, p. 173].) *If $f \in L^2(\Sigma^{n-1})$ and if μ_f is the tempered distribution defined by*

$$\mu_f(\varphi) = \int_{\Sigma^{n-1}} \varphi(\xi') f(\xi')\, d\sigma(\xi'), \quad \varphi \in \mathcal{S}(\mathbf{R}^n),$$

then there is a number C independent of f and ϱ such that

$$\int_{|x|\le\varrho} |\hat{\mu}_f(x)|^2\, dx \le C\varrho \|f\|^2_{L^2(\Sigma^{n-1})}.$$

3.9. Corollary. *The mean value*

$$\frac{1}{\varrho}\int_0^\varrho J_\lambda(r)^2 r\, dr$$

is bounded from above by a number independent of $\varrho > 0$ and $\lambda \in \frac{1}{2}\mathbf{N}$.

Proof. Choose $f = P$ in Theorem 3.8. We get

$$\begin{aligned}\|P\|^2_{L^2(\Sigma^{n-1})}\int_0^\varrho J_{\nu(k)}(r)^2 r\, dr &= \int_{|x|\le\varrho} \frac{|P(x)|^2}{|x|^{2\nu(k)}} J_{\nu(k)}(|x|)^2\, dx \\ &= \frac{1}{(2\pi)^n}\int_{|x|\le\varrho} |\hat{\mu}_P(x)|^2\, dx \le C\varrho\|P\|^2_{L^2(\Sigma^{n-1})},\end{aligned}$$

where the last equality follows from Corollary 3.5 and where C is independent of ϱ and k by Theorem 3.8.

3.10. Lemma. *Let $b>1$. There exists a number C independent of $\varrho\geq 1$ such that*

$$\int_\varrho^\infty J_{\nu(k)}(r)^2 r^{1-b}\,dr \leq C\varrho^{1-b}.$$

Proof. Let $\varrho\geq 1$. We have

$$\begin{aligned}\int_\varrho^\infty J_{\nu(k)}(r)^2 \frac{dr}{r^{b-1}} &= \sum_{j=0}^\infty (2^{j+1}\varrho)\frac{1}{2^{j+1}\varrho}\int_{2^j\varrho}^{2^{j+1}\varrho} J_{\nu(k)}(r)^2 r\,\frac{dr}{r^b}\\ &\leq \sum_{j=0}^\infty \frac{2^{j+1}\varrho}{(2^j\varrho)^b}\frac{1}{2^{j+1}\varrho}\int_0^{2^{j+1}\varrho} J_{\nu(k)}(r)^2 r\,dr\\ &\leq \sum_{j=0}^\infty \frac{2^{j+1}\varrho}{(2^j\varrho)^b}C \leq \frac{2C}{\varrho^{b-1}}\left(\sum_{j=0}^\infty 2^{j(1-b)}\right),\end{aligned}$$

where the number C may be chosen to be independent of k by Corollary 3.9.

4. Proofs

4.1. In this section we will prove Theorems 2.1 and 2.2. We will start with Theorem 2.2. The proof of Theorem 2.1 will then be obtained by a slight modification.

4.2. Proof of 2.2(a). Let $b=2$. Define

(4.1)
$$(\widetilde{S}_m^2 f)[x](t) = \frac{1}{(1+|x|)^{b/2}}\int_{\mathbf{R}^n} m(x,|\xi|)e^{i(x\xi+t|\xi|^2)}(1+|\xi|^2)^{-s/2} f(\xi)\,d\xi, \quad t\in\mathbf{R}.$$

Our theorem follows if we can show that there is a number C independent of f such that

$$\|\widetilde{S}_m^2 f\|_{L^2(\mathbf{R}^{n+1})} \leq C\|f\|_{L^2(\mathbf{R}^n)}. \tag{4.2}$$

4.2.1. For $\varrho>0$ we define

$$\tilde{f}[x](\varrho) = \frac{(1+\varrho)^{-s/2}\varrho^{(n-2)/2}}{2(1+|x|)^{b/2}}\int_{\Sigma^{n-1}} e^{i\varrho^{1/2}x\xi'} f(\varrho^{1/2}\xi')\,d\sigma(\xi'), \quad \varrho>0,$$

(and by $\tilde{f}[x](\varrho)=0$ for $\varrho\leq 0$). The formula

$$(\widetilde{S}_m^2 f)[x](t) = \int_0^\infty e^{it\varrho} m(x,\varrho^{1/2})\tilde{f}[x](\varrho)\,d\varrho \tag{4.3}$$

follows by polar coordinates and change of variables in (4.1) and the estimate

$$\|\widetilde{S}_m^2 f\|_{L^2(\mathbf{R}^{n+1})} \leq (2\pi)^{1/2}\|m\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}_+)}\|\tilde{f}\|_{L^2(\mathbf{R}^{n+1})}$$

follows from Parseval's formula on $\mathbf{R}$ applied to (4.3). Hence, to prove (4.2) (with C independent of f) it is sufficient to prove that

$$\|\tilde{f}\|_{L^2(\mathbf{R}^{n+1})} \le C\|f\|_{L^2(\mathbf{R}^n)}, \tag{4.4}$$

where C is independent of f.

4.2.2. If $f_j \in (\mathfrak{H}_{k_j} \cap \mathcal{S})(\mathbf{R}^n)$, $k_1 \neq k_2$, we have

$$\int_{\Sigma^{n-1}} \tilde{f}_1[rx'](\varrho)\overline{\tilde{f}_2[rx'](\varrho)}\, d\sigma(x') = 0$$

for all $r>0$ and all $\varrho>0$. In fact

$$\begin{aligned}\int_{\Sigma^{n-1}} & \tilde{f}_1[rx'](\varrho)\overline{\tilde{f}_2[rx'](\varrho)}\, d\sigma(x') \\ &= \frac{(1+\varrho)^{-s}\varrho^{n-2}}{4(1+r)^b}\int_{\Sigma^{n-1}}\biggl(\int_{\Sigma^{n-1}} e^{i\varrho^{1/2}rx'\xi'} f_1(\varrho^{1/2}\xi')\, d\sigma(\xi') \\ &\qquad \times \int_{\Sigma^{n-1}} e^{-i\varrho^{1/2}rx'\xi'}\overline{f_2(\varrho^{1/2}\xi')}\, d\sigma(\xi')\biggr)\, d\sigma(x').\end{aligned}$$

Each of the inner integrals in the right-hand side will by Corollary 3.5 produce a linear combination of products of spherical harmonics of degree k_j times certain functions of (r,ϱ). Because of the assumption $k_1 \neq k_2$ the spherical harmonics (or rather the *surface* spherical harmonics) will be orthogonal in $L^2(\Sigma^{n-1})$, which proves our assertion.

4.2.3. We have the orthogonality relation

$$\langle \tilde{f}_1, \tilde{f}_2\rangle_{L^2(\mathbf{R}^{n+1})} = 0, \quad f_j \in \mathfrak{H}_{k_j}(\mathbf{R}^n),\ k_1 \neq k_2.$$

In fact

$$\int_{\Sigma^{n-1}} \langle \tilde{f}_1[rx'], \tilde{f}_2[rx']\rangle_{L^2(\mathbf{R})}\, d\sigma(x') = \int_0^\infty \int_{\Sigma^{n-1}} \tilde{f}_1[rx'](\varrho)\overline{\tilde{f}_2[rx'](\varrho)}\, d\sigma(x')\, d\varrho = 0,$$

where we have applied the orthogonality assertion in Section 4.2.2 in the last equality. Integrating from 0 to ∞ with respect to the measure $r^{n-1}dr$ now proves our assertion.

4.2.4. For any element[2]

$$f : \xi \longmapsto \sum_{(f_0,P)} P(\xi) f_0(|\xi|)|\xi|^{-n/2-k+1/2}, \quad f_0 \in C_0(\mathbf{R}_+), \tag{4.5}$$

[2] Recall the notation in Section 3.3. The summations in (4.5) and (4.6) are performed over finite sets.

in $\mathfrak{H}_k(\mathbf{R}^n)$ we can assume that the *surface* spherical harmonics $P|_{\Sigma^{n-1}}$ are orthogonal in $L^2(\Sigma^{n-1})$ and hence, by polar coordinates, that the terms in (4.5) are orthogonal in $\mathfrak{H}_k(\mathbf{R}^n)$. Then the terms of $\tilde{f}$ will be orthogonal in $L^2(\mathbf{R}^{n+1})$, since the terms of $x\mapsto\tilde{f}[rx](\varrho)$ restricted to Σ^{n-1} will be orthogonal in $L^2(\Sigma^{n-1})$ for fixed (r,ϱ). (Cf. Sections 4.2.2 and 4.2.3.) Also†

$$\|f\|^2_{L^2(\mathbf{R}^n)}=\sum_{f_0}\|f_0\|^2_{L^2(\mathbf{R}_+)}. \tag{4.6}$$

4.2.5. It is a consequence of orthogonality (see Section 4.2.3 and Theorem 3.6) that it is sufficient to prove the estimate (4.4) for $f\in\mathfrak{H}_k(\mathbf{R}^n)\cap\mathcal{S}(\mathbf{R}^n)$, where the number C has to be independent of f *and* k. In turn, it is a consequence of orthogonality (see Section 4.2.4) that it is sufficient to prove the estimate (4.4) for

$$f:\xi\longmapsto P(\xi)f_0(|\xi|)|\xi|^{-n/2-k+1/2},\quad f_0\in C_0(\mathbf{R}_+),$$

where the number C has to be independent of f *and* k. Straightforward computations using change of variables, Corollary 3.5 and polar coordinates show that for such f

$$\|\tilde{f}\|^2_{L^2(\mathbf{R}^{n+1})}=\frac{(2\pi)^n}{2}\int_0^\infty\int_0^\infty J_{\nu(k)}(\varrho r)^2\frac{|f_0(\varrho)|^2r}{(1+\varrho^2)^s}\frac{dr\,d\varrho}{(1+r)^b}. \tag{4.7}$$

Hence it is sufficient to prove the estimate

$$\begin{aligned}\int_0^\infty\int_0^\infty J_{\nu(k)}(\varrho r)^2\frac{f_0(\varrho)r}{(1+\varrho^2)^s}\frac{dr\,d\varrho}{(1+r)^b}&=\int_0^\infty\int_0^\infty J_{\nu(k)}(r)^2\frac{f_0(\varrho)\varrho^{b-2}r}{(1+\varrho^2)^s}\frac{dr\,d\varrho}{(\varrho+r)^b}\\&\le C\|f_0\|_{L^1(\mathbf{R}_+)},\end{aligned}$$

where $f_0\ge0$ and C is independent of $f_0\in C_0(\mathbf{R}_+)$ and k. This estimate will be proved by replacing $f_0(\varrho)$ by $\chi(2\varrho)f_0(\varrho)$ (in Section 4.2.6) and $\psi(2\varrho)f_0(\varrho)$ (in Section 4.2.7).

4.2.6. Estimate for $\chi(2\varrho)f_0(\varrho)$. We shall derive the estimate

$$\int_0^\infty\int_0^\infty J_{\nu(k)}(r)^2\chi(2\varrho)f_0(\varrho)\varrho^{b-2}r\frac{dr\,d\varrho}{(\varrho+r)^b}\le C\int_0^\infty\chi(2\varrho)f_0(\varrho)\,d\varrho, \tag{4.8}$$

where C may be chosen to be independent of $f_0\in C_0^\infty(\mathbf{R}_+)$ and k. Recall that $b=2$. We have

$$\begin{aligned}&\int_0^\infty\int_0^\infty J_{\nu(k)}(r)^2\chi(2\varrho)f_0(\varrho)\varrho^{b-2}r\frac{dr\,d\varrho}{(\varrho+r)^b}\\&\qquad\le\int_0^\infty\chi(2\varrho)f_0(\varrho)\biggl(\int_0^\infty J_{\nu(k)}(r)^2r^{1-b}\,dr\biggr)\,d\varrho.\end{aligned}$$

Here we split the integration with respect to r into two pieces and use the Poisson representation (3.2) for $0\le r\le 1$ and Lemma 3.10 (with $\varrho=1$) for $r\ge 1$ to conclude that the inner integral in the right hand side is bounded from above by a number C independent of k. We have proved (4.8).

4.2.7. Estimate for $\psi(2\varrho)f_0(\varrho)$. We shall derive the estimate

$$\int_0^\infty\int_0^\infty J_{\nu(k)}(r)^2\psi(2\varrho)f_0(\varrho)\varrho^{b-2-2s}r\,\frac{dr\,d\varrho}{(\varrho+r)^b}\le C\int_0^\infty \psi(2\varrho)f_0(\varrho)\,d\varrho, \tag{4.9}$$

where C may be chosen to be independent of $f_0\in C_0^\infty(\mathbf{R}_+)$ and k. Recall that $b=2$ and that $s=-\frac{1}{2}$. We have

$$\begin{aligned}\int_0^\infty\int_0^\infty & J_{\nu(k)}(r)^2\psi(2\varrho)f_0(\varrho)\varrho^{b-2-2s}r\,\frac{dr\,d\varrho}{(\varrho+r)^b}\\ &\le\int_0^\infty \psi(2\varrho)f_0(\varrho)\varrho^{b-1}\left(\int_0^\infty J_{\nu(k)}(r)^2r\,\frac{dr}{(\varrho+r)^b}\right)d\varrho.\end{aligned}$$

Here we use ϱ to split the integration with respect to r into two pieces and use Corollary 3.9 for $0\le r\le\varrho$ and Lemma 3.10 for $r\ge\varrho$ to conclude that the inner integral in the right-hand side is bounded from above by a number C independent of k. We have proved (4.9).

4.3. Remark. The technique used in Section 4.2.1 (one-dimensional change of variable, Parseval's formula on $\mathbf{R}$) has been used before. See e.g. Kenig, Ponce and Vega [5, p. 57]. In this context it is also appropriate to draw the reader's attention to the discussion in Section 2.4.

4.4. Proof of 2.2(b). We have the estimate

$$\begin{aligned}\int_0^\infty\int_0^\infty & J_{\nu(0)}(\varrho r)^2\chi(\varrho)f_0(\varrho)r\,\frac{dr\,d\varrho}{(1+r)^b}\\ &\ge 2^{-b}\int_0^\infty\left(\int_{r\ge 1} J_{\nu(0)}(\varrho r)^2(\varrho r)^{1-b}d(\varrho r)\right)\varrho^{b-2}\chi(\varrho)f_0(\varrho)\,d\varrho\\ &=2^{-b}\int_0^\infty\left(\int_\varrho^\infty J_{\nu(0)}(r)^2r^{1-b}\,dr\right)\varrho^{b-2}\chi(\varrho)f_0(\varrho)\,d\varrho.\end{aligned}$$

Now assume that the estimate (2.2) holds with a number C independent of f. It follows from the discussion in Section 4.2.5 that we then have the inequality

$$\int_0^\infty\left(\int_\varrho^\infty J_{\nu(0)}(r)^2r^{1-b}\,dr\right)\varrho^{b-2}\chi(\varrho)f_0(\varrho)\,d\varrho\le C\|\chi f_0\|_{L^1(\mathbf{R}_+)}, \tag{4.10}$$

where C is independent of $f_0\in C_0(\mathbf{R}_+)$. From this statement it is clear that we cannot allow $b<2$.

4.5. Proof of 2.1(a). We repeat the proof of Theorem 2.2(a); the estimate for $\chi(2\varrho)f_0(\varrho)$ has to be carried out in a slightly different way when $n=2$ and $b>2$.

4.5.1. Estimate for $\chi(2\varrho)f_0(\varrho)$. We shall derive the estimate

$$\int_0^\infty\int_0^\infty J_{\nu(k)}(\varrho r)^2\chi(2\varrho)f_0(\varrho)r\,\frac{dr\,d\varrho}{(1+r)^b}\le C\int_0^\infty \chi(2\varrho)f_0(\varrho)\,d\varrho,$$

where C may be chosen to be independent of f_0 and k. Cf. (4.8). Here we split the integration with respect to r into two pieces and use the Poisson representation (3.2) for $0\le r\le 1$. For the remaining part we estimate as follows,

$$\begin{aligned}
\int_0^\infty\int_0^\infty & J_{\nu(k)}(\varrho r)^2\chi(2\varrho)f_0(\varrho)\psi(2r)r\,\frac{dr\,d\varrho}{(1+r)^b}\\
&\le\int_0^\infty\biggl(\int_{r\ge 1/2} J_k(\varrho r)^2(\varrho r)^{1-b}d(\varrho r)\biggr)\varrho^{b-2}\chi(2\varrho)f_0(\varrho)\,d\varrho\\
&=\int_0^\infty\biggl(\int_{\varrho/2}^\infty J_k(r)^2r^{1-b}\,dr\biggr)\varrho^{b-2}\chi(2\varrho)f_0(\varrho)\,d\varrho\\
&\le\int_0^\infty\biggl(C\int_{\varrho/2}^1 r^{1-b}\,dr+\int_1^\infty J_k(r)^2r^{1-b}\,dr\biggr)\varrho^{b-2}\chi(2\varrho)f_0(\varrho)\,d\varrho.
\end{aligned}$$

Here we have used the Poisson representation (3.2) in the last inequality. The number C may be chosen to be independent of k. Recall that $b>2$. By Lemma 3.10 with $\varrho=1$ the expression in the parentheses can be estimated from above by $C\varrho^{2-b}$ for some number C independent of $\varrho\le 1$ and k.

4.5.2. Estimate for $\psi(2\varrho)f_0(\varrho)$. We replace $b=2$ in Section 4.2.7 by $b>2$ and check the estimates in that paragraph.

4.6. Proof of 2.1(b). We repeat the proof of 2.2(b) and observe that there is no number C independent of $f_0\in\mathcal{C}_0(\mathbf{R}_+)$ such that the inequality

$$\int_0^\infty\biggl(\int_\varrho^\infty J_0(r)^2r^{-1}\,dr\biggr)\chi(\varrho)f_0(\varrho)\,d\varrho\le C\|\chi f_0\|_{L^1(\mathbf{R}_+)}$$

holds. Cf. (4.10). This means that we cannot allow $b=2$. We cannot allow any smaller b either.

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