Ark. Mat., **37** (1999), 345–356

A Dirichlet principle for the complex Monge–Ampère operator

Leif Persson

Abstract. A solution to a Dirichlet problem for the complex Monge–Ampère operator is characterized as a minimizer of an energy functional. A mutual energy estimate and a generalization of Hölder's inequality is proved. A comparison is made with corresponding results in classical potential theory.

1. Introduction

In classical potential theory, there is a correspondence between positive measures and Green potentials on a domain in $\mathbf{R}^n$, and the Green potential is the unique solution to a Dirichlet problem for the Laplace operator Δ on the domain. The Dirichlet principle states that the Green potential of a measure μ can be characterized in an alternative way, as the unique minimizer of the energy functional

$$J_\mu(u)=\int(-u)\left(\tfrac{1}{2}\Delta u-\mu\right). \tag{1}$$

The main result of this paper is an analogous Dirichlet principle in pluripotential theory, where we consider positive measures and plurisubharmonic functions on a pseudoconvex domain in $\mathbf{C}^n$. The Laplace operator, which is linear, is replaced by the complex Monge–Ampère operator, which is nonlinear. More precisely, the complex Monge–Ampère operator acting on smooth plurisubharmonic functions u is a positive measure, given by

$$(dd^c u)^n=4^n n!\det\left(\frac{\partial u}{\partial z_j\partial \bar z_k}\right)\lambda,$$

where λ is the Lebesgue measure on $\mathbf{R}^{2n}$. Furthermore, the energy functional (1) is replaced by

$$J_\mu(u)=\int(-u)\left(\frac{1}{n+1}(dd^c u)^n-\mu\right). \tag{2}$$

The measure $(dd^c u)^n$ can be defined for bounded plurisubharmonic functions u, see [1]. It was shown in [3] that for $p \geq 1$, $(dd^c \cdot)^n$ is well defined on the sets $\mathcal{E}_p$ of plurisubharmonic functions u with least harmonic majorant 0 on Ω, and such that the *pluricomplex p-energy*

$$I_p(u) = \int (-u)^p (dd^c u)^n$$

is finite. The main theorem in [3] says that the corresponding Dirichlet problem

$$(dd^c u)^n = \mu, \quad u \in \mathcal{E}_p,$$

has a solution if and only if μ has *finite pluricomplex p-energy* in the sense that there is a constant $C > 0$ such that

$$\int (-u)^p \mu \leq C I_p(u)^{p/(p+n)}$$

for all bounded functions u in $\mathcal{E}_p$ with $\int (dd^c u)^n < \infty$. We denote the set of such measures by $\mathcal{M}_p$. For functions in $\mathcal{E}_1$, the energy functional (2) is defined and finite, provided $\mu \in \mathcal{M}_1$. Our main theorem is the Dirichlet principle (Theorem 3.13)

$$(dd^c u)^n = \mu \quad \text{if and only if} \quad J_\mu(u) = \min_{w \in \mathcal{E}_1} J_\mu(w).$$

Furthermore, we prove an estimate for the *mutual pluricomplex p-energy*

$$(u_0, \dots, u_n)_p = \int (-u_0)^p dd^c u_1 \wedge \dots \wedge dd^c u_n,$$

which is a generalization of the energy estimate in [4], namely (Theorem 3.4)

$$(u_0, \dots, u_n)_p \leq D_{n,p} I_p(u_0)^{p/(p+n)} I_p(u_1)^{1/(p+n)} \dots I_p(u_n)^{1/(p+n)},$$

where $D_{n,p}$ is a positive constant, independent of $u_0, \dots, u_n$.

This energy estimate is obtained from a generalization of the Hölder inequality (Theorem 4.1), which we state as a separate theorem, since we think it is of independent interest. This inequality is the third and last result of this paper.

Remark on notation. In pluripotential theory, d denotes the exterior derivative, so to avoid confusion, we have omitted d in the usual notation for integrals. Thus,

$$\int f \mu$$

means integration of the function f with respect to the measure μ.

2. Classical potential theory

Let us recall some facts from classical potential theory. We refer to [7] for proofs and original references. Let Ω be a domain in $\mathbf{R}^n$, let μ and ν be measures on Ω, let $u=G_\mu$ and $v=G_\nu$ denote the corresponding (subharmonic) Green potentials with respect to Ω, and let $\mathcal{M}$ denote the set of positive measures μ such that G_μ is subharmonic. The following theorem is an immediate consequence of the Riesz decomposition theorem ([7, Theorem 2.6]).

Theorem 2.1. (Dirichlet problem) *If $\mu\in\mathcal{M}$, then $u=G_\mu$ is the unique solution to the Dirichlet problem*

$$u \text{ subharmonic on } \Omega, \tag{3}$$

$$\Delta u = \mu, \tag{4}$$

$$\limsup_{z\to\partial\Omega} u(z) = 0. \tag{5}$$

Definition 2.2. (Energy of a measure) The *mutual energy* of two measures $\mu,\nu\in\mathcal{M}$ is the integral

$$(\mu,\nu) = \int -G_\mu \nu = \int (-u)\Delta v. \tag{6}$$

The *energy* of μ is $I_\mu=(\mu,\mu)$.

For the mutual energy we have the following result.

Theorem 2.3. (Energy estimate, [7, Theorem 4.2]) *If $\mu,\nu\in\mathcal{M}$, then*

$$(\mu,\nu) \le (\mu,\mu)^{1/2}(\nu,\nu)^{1/2}.$$

For signed measures μ, ν, the mutual energy is defined by Jordan decomposition and bilinearity, and the linear space $\mathcal{E}$ of Radon measures of finite energy thus obtained is a pre-Hilbert space with the mutual energy as inner product. In fact, we have the following theorem.

Theorem 2.4. (Energy principle for Green potentials, [7, Theorem 4.3]) *If μ has finite energy then $(\mu,\mu)\ge 0$, with equality if and only if $\mu=0$.*

However, $\mathcal{E}$ is not complete ([7, Theorem 5.15]), but the topological subspace $\mathcal{E}^+$ of positive measures in $\mathcal{E}$ *is* a complete metric space (see [7, Section 6.4], for details). Thus, by a slight extension of the Riesz representation theorem, we obtain the following result.

Theorem 2.5. (Measures of finite energy) *A positive measure μ has finite energy if and only if there is a constant $C>0$ such that*

$$(\mu,\nu)\leq C(\nu,\nu)^{1/2} \tag{7}$$

for all ν in some dense subset of $\mathcal{E}^+$.

Remark. For example, the set of all compactly supported ν such that G_ν is continuous form a dense subset of $\mathcal{E}^+$, cf. [7, Theorem 4.9].

Next, we make the following definition.

Definition 2.6. The energy functional is

$$J_\mu(u)=\int(-u)\left(\tfrac{1}{2}\Delta u-\mu\right). \tag{8}$$

Remark. We see by integration by parts that

$$J_\mu(u)=\int\tfrac{1}{2}\|\nabla u\|^2+u\mu$$

so if u is the potential of an electric field and μ is a charge distribution, J_μ is the sum of the energy in the electric field and the potential energy of the charge distribution in the electric field.

Moreover we have the Dirichlet principle.

Theorem 2.7. (Dirichlet principle) *If $\mu\in\mathcal{E}$, then G_μ is the unique minimum of J_μ over the set of Green potentials of measures of finite energy.*

Proof. The energy functional can be expressed in terms of the inner product

$$J_\mu(G_\nu)=-\tfrac{1}{2}(\mu,\mu)+(\nu-\mu,\nu-\mu) \tag{9}$$

and the theorem follows from this formula combined with the energy principle (Theorem 2.4). $\square$

3. Pluricomplex energy

The theory of the complex Monge–Ampère operator was originally developed by Bedford and Taylor in [1] and [2], and the subject is comprehensively developed in [6], which is a general reference for this section. The theory of pluricomplex energy was developed in [3] and we will recall some definitions and results from that paper. We assume for simplicity that Ω is a strictly pseudoconvex open set in $\mathbf{C}^n$; this condition can sometimes be weakened. For notational convenience, we define a class of *test functions*.

Definition 3.1. (Test functions) Let $\mathcal{E}_0$ denote the set of bounded negative plurisubharmonic functions u on Ω such that

$$\lim_{z\to\partial\Omega} u(z)=0 \quad \text{and} \quad \int (dd^c u)^n<\infty.$$

The analog of Green potentials is given by the following definition.

Definition 3.2. (Pluricomplex Green potentials) For $p\geq 1$, let $\mathcal{E}_p$ denote the class of plurisubharmonic functions u on Ω such that there is a decreasing sequence of test functions $\mathcal{E}_0\ni u_j\searrow u$, and such that

$$\sup_j \int (-u_j)^p (dd^c u_j)^n<\infty.$$

We say that functions in $\mathcal{E}_1$ are *pluricomplex Green potentials.*

We define analogies of energy and mutual energy in Definition 2.2 as follows.

Definition 3.3. (Pluricomplex energy) For $p\geq 1$, we define the *mutual pluricomplex p-energy* of $u_0, \dots, u_n\in\mathcal{E}_0$ to be

$$(u_0,\dots,u_n)_p=\int (-u_0)^p dd^c u_1\wedge\dots\wedge dd^c u_n \tag{10}$$

and the *pluricomplex p-energy* of $u\in\mathcal{E}_0$ to be

$$I_p(u)=\int (-u)^p (dd^c u)^n=(u,\dots,u)_p. \tag{11}$$

We say that $(u_0,\dots,u_n)_1$ and $I_1(u)$ are the mutual pluricomplex energy and the pluricomplex energy, respectively.

For pluricomplex Green potentials we have the following mutual energy estimate, which is an analog of Theorem 2.3. This is our first result, and generalizes the energy estimate in [4].

Theorem 3.4. (Energy estimate) *If* $u_0,\dots,u_n\in\mathcal{E}_0$, *then*

$$(u_0,\dots,u_n)_p\leq D_{n,p} I_p(u_0)^{p/(p+n)} I_p(u_1)^{1/(p+n)} \dots I_p(u_n)^{1/(p+n)}, \tag{12}$$

where $D_{n,1}=1$ *and* $D_{n,p}=p^{p\alpha(n,p)/(p-1)}$ *for* $p>1$ *and*

$$\alpha(n,p)\equiv(p+2)\left(\frac{p+1}{p}\right)^{n-2}-(p+1). \tag{13}$$

Analogous to the classical estimate (7), we make the following definition.

Definition 3.5. (Measures of finite pluricomplex energy) Assume that $p \geq 1$. We say that a positive measure μ has *finite pluricomplex p-energy* if there exists a constant $C>0$ such that

$$\int (-u)^p \mu \leq C I_p(u)^{p/(p+n)} \tag{14}$$

for all $u \in \mathcal{E}_0$. We denote the set of such measures by $\mathcal{M}_p$.

To state the Dirichlet problem, we need first to have $(dd^c \cdot)^n$ defined on all of $\mathcal{E}_p$; so far, it is only defined on locally bounded plurisubharmonic functions, see [1, Proposition 2.9]. To this end we have the following theorem.

Theorem 3.6. (Cegrell, [3]) *If $\mathcal{E}_0 \ni u_j \searrow u \in \mathcal{E}_p$ for some $p \geq 1$, then $(dd^c u_j)^n$ is a weakly convergent sequence, and the limit depends only on u, not on the choice of the approximating sequence u_j converging to u.*

Definition 3.7. The complex Monge–Ampère measure of $u \in \mathcal{E}_p$, denoted by $(dd^c u)^n$, is defined to be the unique measure obtained in Theorem 3.6.

With the same proof as for Theorem 3.6 we also obtain the following corollary.

Corollary 3.8. *If $\mathcal{E}_p \ni u_j \searrow u \in \mathcal{E}_p$, then $(dd^c u_j)^n \to (dd^c u)^n$ weakly as measures on Ω.*

Also, we need the following theorem.

Theorem 3.9. (Cegrell [3]) *If $u \in \mathcal{E}_1$, then*

$$\lim_{j \to \infty} I_1(u_j) = I_1(u)$$

for every sequence u_j in $\mathcal{E}_0$ which decreases to u.

The main theorem in [3] is the pluricomplex counterpart to Theorem 2.1.

Theorem 3.10. (Dirichlet problem) *The Dirichlet problem*

$$(dd^c u)^n = \mu, \quad u \in \mathcal{E}_p, \tag{15}$$

has a solution if and only if $\mu \in \mathcal{M}_p$, and if a solution exists, it is unique.

We consider the following analog of Definition 2.6.

Definition 3.11. (Energy functional)

$$J_\mu(u) = \int (-u)\left(\frac{1}{n+1}(dd^c u)^n - \mu\right). \tag{16}$$

Note that our energy integrals can be extended to $\mathcal{E}_1$.

Proposition 3.12. *If $\mu\in\mathcal{M}_1$, $u\in\mathcal{E}_1$ and $\mathcal{E}_0\ni u_j\searrow u$, then $\lim_{j\to\infty} J_\mu(u_j)$ is finite and depends only on u, not on the particular sequence approximating u.*

Proof. This follows immediately from Theorem 3.9, monotone convergence and the fact that $\mu\in\mathcal{M}_1$. □

Thus J_μ is extended to $\mathcal{E}_1$. In particular, the energy integral $I_1=(n+1)J_0$ is extended to $\mathcal{E}_1$, and the estimate (14) holds for all $u\in\mathcal{E}_1$.

Remark. The functional J_μ is convex, and, formally, $(dd^c u)^n=\mu$ is the Euler equation for J_μ. This functional was considered by Kalina [5] under the assumption that the minimizer is twice continuously differentiable and that μ is absolutely continuous with respect to Lebesgue measure, with a continuous density.

The main result of this paper is the following Dirichlet principle for the Dirichlet problem (15), which is an analog of Theorem 2.7.

Theorem 3.13. (Dirichlet principle) *If $\mu\in\mathcal{M}_1$ and $u\in\mathcal{E}_1$, then*

$$(dd^c u)^n=\mu \quad \textit{if and only if} \quad J_\mu(u)=\min_{w\in\mathcal{E}_1} J_\mu(w).$$

Proof. First note that if $u,v\in\mathcal{E}_1$ and $(dd^c u)^n=\mu$, $(dd^c v)^n=\nu$, then

$$\begin{aligned} J_\mu(v)-J_\mu(u) &= \frac{1}{n+1}\int(-v)\nu+\frac{n}{n+1}\int(-u)\mu-\int(-v)\mu \\ &\geq \frac{1}{n+1}\int(-v)\nu+\frac{n}{n+1}\int(-u)\mu \\ &\quad -\biggl(\int(-v)\nu\biggr)^{1/(n+1)}\biggl(\int(-u)\mu\biggr)^{n/(n+1)} \\ &\geq 0 \end{aligned} \tag{17}$$

by Theorem 3.4 and Young's inequality. This also proves the "only if" part of the theorem.

To prove the "if" part, suppose that v is a minimizer of J_μ over $\mathcal{E}_1$. Let $\nu=(dd^c v)^n$, then $\nu\in\mathcal{M}_1$ by Theorem 3.4. Also, by Theorem 3.10, there exist $u\in\mathcal{E}_1$ such that $(dd^c u)^n=\mu$. Since v is a minimizer,

$$J_\mu(v)\leq J_\mu(u) \tag{18}$$

so $J_\mu(u)=J_\mu(v)$ by (17), and we have equalities in (17). It follows from the last equality in (17) and the equality part of Young's inequality that

$$\int(-v)\nu=\int(-u)\mu, \tag{19}$$

and hence from the second equality in (17) that

$$\int(-v)\nu=\int(-u)\mu=\int(-v)\mu. \tag{20}$$

Furthermore, using the multilinearity of $dd^c\cdot\wedge\ldots\wedge dd^c\cdot$ and integration by parts, we get

$$0\leq\liminf_{t\to 0+}\frac{(J_\mu(v+tw)-J_\mu(v))}{t}=\int(-w)(\nu-\mu)$$

for all $w\in\mathcal{E}_1$. In particular, we have $\int(-u)(\nu-\mu)=0$, since if $\int(-u)(\nu-\mu)>0$ we would get a contradiction to Theorem 3.4. Thus, $J_\nu(u)=J_\nu(v)$, so reversing the roles of μ and ν, we get

$$\int(-w)(\nu-\mu)=0$$

for all $w\in\mathcal{E}_1$, so $\nu=\mu$. This proves the "if" part. □

It remains to prove the energy estimate (Theorem 3.4). This theorem is derived from a generalization of Hölder's inequality (Theorem 4.1), which is stated and proved in the next section.

4. A generalized Hölder inequality

The classical Hölder inequality states that

$$\int u_1u_2\,d\mu\leq\left(\int|u_1|^{p_1}\,d\mu\right)^{1/p_1}\left(\int|u_2|^{p_2}\,d\mu\right)^{1/p_2}, \tag{21}$$

where $1/p_1+1/p_2=1$, $1<p_j<\infty$. This can be generalized to more than two functions, of which a special case is

$$\begin{aligned}\int u_1^pu_2\ldots u_n\,d\mu\leq&\left(\int u_1^{p+n-1}\,d\mu\right)^{p/(p+n-1)}\times\left(\int u_2^{p+n-1}\,d\mu\right)^{1/(p+n-1)}\\&\times\ldots\times\left(\int u_n^{p+n-1}\,d\mu\right)^{1/(p+n-1)}.\end{aligned} \tag{22}$$

In this section we are going to prove a generalization of this result to functionals $F(u_1,\ldots,u_n)$ of a more general form. The only properties of $F(u_1,\ldots,u_n)$ required is that it commutes in the last $n-1$ arguments, and that a Hölder type inequality holds in the first two arguments, with the remaining arguments fixed.

Theorem 4.1. *Assume that X is a set, n a natural number ≥ 2 and that F is a nonnegative real-valued function on X^n such that F is commutative in the last $n-1$ variables. Also assume that $p \geq 1$ and that there exists a real constant $C \geq 1$ such that*

$$F(u_1, u_2, u_3, \dots, u_n) \leq C F(u_1, u_1, u_3, \dots, u_n)^{p/(p+1)} F(u_2, u_2, u_3, \dots, u_n)^{1/(p+1)}. \tag{23}$$

Then

$$\begin{aligned} F(u_1, \dots, u_n) \leq\, & C^{\alpha(n,p)} F(u_1, \dots, u_1)^{p/(p+n-1)} \\ & \times F(u_2, \dots, u_2)^{1/(p+n-1)} \dots F(u_n, \dots, u_n)^{1/(p+n-1)}, \end{aligned} \tag{24}$$

where

$$\alpha(n,p) \equiv (p+2)\left(\frac{p+1}{p}\right)^{n-2} - (p+1). \tag{25}$$

Proof. If $n=2$ there is nothing to prove. Fix $n>2$ and assume that (24) is true with n replaced by $n-1$. The proof is complete by induction if we prove (24) under these assumptions.

Then $(u_1, \dots, u_{n-1}) \mapsto F(u_1, \dots, u_{n-1}, u_n)$ fulfills the induction assumption for a fixed u_n, so we have

$$\begin{aligned} F(u_1, \dots, u_n) \leq\, & C^{\alpha(n-1,p)} F(u_1, \dots, u_1, u_n)^{p/(p+n-2)} \\ & \times F(u_2, \dots, u_2, u_n)^{1/(p+n-2)} \dots F(u_{n-1}, \dots, u_{n-1}, u_n)^{1/(p+n-2)}. \end{aligned} \tag{26}$$

Then by (23) and commutativity, we get

$$\begin{aligned} F(u, \dots, u, v) &= F(u, v, u, \dots, u) \\ &\leq C F(u, u, u, \dots, u)^{p/(p+1)} \cdot F(v, v, u, \dots, u)^{1/(p+1)} \end{aligned} \tag{27}$$

and

$$\begin{aligned} F(v, v, u, \dots, u) &= F(v, u, \dots, u, v) \\ &\leq C^{\alpha(n-1,p)} F(v, \dots, v, v)^{p/(p+n-2)} F(u, \dots, u, v)^{(n-2)/(p+n-2)}, \end{aligned} \tag{28}$$

by commutativity and the induction assumption (24) with n replaced by $n-1$. Now we substitute (28) in (27) and get

$$\begin{aligned} F(u, \dots, u, v) \leq\, & C^{(1+\alpha(n-1,p)/(p+1))(p+1)(p+n-2)/p(p+n-1)} \\ & \times F(u, \dots, u)^{(p+n-2)/(p+n-1)} F(v, \dots, v)^{1/(p+n-1)}. \end{aligned} \tag{29}$$

Note that $\alpha(n,p)$ in (25) is the solution to the recurrence equation

$$\alpha(n,p)=\alpha(n-1,p)+\left(1+\frac{\alpha(n-1,p)}{p+1}\right)\frac{p+1}{p} \tag{30}$$

with initial value $\alpha(2,p)=1$. Thus, $\alpha(n,p)$ also satisfies the recurrence inequality

$$\alpha(n,p)\geq\alpha(n-1,p)+\left(1+\frac{\alpha(n-1,p)}{p+1}\right)\frac{(p+1)(p+n-2)}{p(p+n-1)}. \tag{31}$$

We substitute (29) in (26) and get (24) with $\alpha(n,p)$ replaced by the right-hand side of (31). Then we use (31) to get (24). This completes the proof. □

5. An energy estimate for the complex Monge–Ampére operator

In this section, we prove Theorem 3.4, using Theorem 4.1 and integration by parts formulas, similar to those used in [4]. We assume that Ω is a strictly pseudoconvex domain in $\mathbf{C}^n$.

Lemma 5.1. *If u_j, $j=0,\dots,n$, are negative, locally bounded plurisubharmonic functions on Ω, $\lim_{z\to\partial\Omega}u_j(z)=0$, $j=0,\dots,n$, and $p>1$ then*

$$\begin{aligned}\int(-u_0)^p dd^cu_1\wedge T\leq p^{p/(p-1)}&\left(\int(-u_0)^p dd^cu_0\wedge T\right)^{p/(p+1)}\\ &\times\left(\int(-u_1)^p dd^cu_1\wedge T\right)^{1/(p+1)},\end{aligned} \tag{32}$$

where $T=dd^cu_2\wedge\dots\wedge dd^cu_n$.

Proof. By an approximation argument proved in [4], integration by parts is justified. Thus,

$$\begin{aligned}\int(-u_0)^p dd^cu_1\wedge T&=-\int du_1\wedge d^c((-u_0)^p)\wedge T\\ &=p\int(-u_0)^{p-1}du_1\wedge d^cu_0\wedge T\\ &=p\int(-u_1)\,d((-u_0)^{p-1})\wedge d^cu_0\wedge T\\ &\quad+p\int(-u_1)(-u_0)^{p-1}dd^cu_0\wedge T\\ &=-p\int(p-1)(-u_0)^{p-2}(-u_1)\,du_0\wedge d^cu_0\wedge T\\ &\quad+p\int(-u_1)(-u_0)^{p-1}dd^cu_0\wedge T\end{aligned} \tag{33}$$

and hence, by Hölder's inequality,

$$\begin{aligned}\int(-u_0)^p dd^c u_1\wedge T &\le p\int(-u_1)(-u_0)^{p-1} dd^c u_0\wedge T\\ &\le p\Big(\int(-u_0)^p dd^c u_0\wedge T\Big)^{(p-1)/p}\Big(\int(-u_1)^p dd^c u_0\wedge T\Big)^{1/p}.\end{aligned}\tag{34}$$

If we reverse the roles of u_0 and u_1 in (34), we get an estimate of $\int(-u_1)^p(dd^c u_0)\wedge T$. If we use this estimate in (34) to eliminate $\int(-u_1)^p(dd^c u_0)\wedge T$ and do some simplification, we get (32). The proof is complete. □

In the case $p=1$, we have the following result.

Lemma 5.2. *If u_j, $j=0,\dots,n$, are negative, locally bounded plurisubharmonic functions on Ω, $\lim_{z\to\partial\Omega} u_j(z)=0$, $j=0,\dots,n$, then*

$$\int(-u_0)dd^c u_1\wedge T\le\Big(\int(-u_0)dd^c u_0\wedge T\Big)^{1/2}\Big(\int(-u_1)dd^c u_1\wedge T\Big)^{1/2},\tag{35}$$

where $T=dd^c u_2\wedge\dots\wedge dd^c u_n$.

Proof. By an approximation argument proved in [4], integration by parts is justified. Integration by parts and Schwarz' inequality yield

$$\begin{aligned}\int(-u_0)dd^c u_1\wedge T &= \int du_0\wedge d^c u_1\wedge T\\ &\le\Big(\int du_0\wedge d^c u_0\wedge T\Big)^{1/2}\Big(\int du_1\wedge d^c u_1\wedge T\Big)^{1/2}.\end{aligned}\tag{36}$$

Now apply integration by parts again to complete the proof. □

Proof of Theorem 3.4. Let $F(u_0,\dots,u_n)=\int(-u_0)^p dd^c u_1\wedge\dots\wedge dd^c u_n$. Then, by Lemma 5.1 or Lemma 5.2, the assumptions of Theorem 4.1 are fulfilled with

$$C=p^{p/(p-1)}\quad\text{or}\quad C=1$$

and the energy estimate (12) follows. The proof is complete.

References

1. Bedford, E. and Taylor, B. A., The Dirichlet problem for a complex Monge–Ampère equation, *Invent. Math.* **37** (1976), 1–44.

2. BEDFORD, E. and TAYLOR, B. A., A new capacity for plurisubharmonic functions, *Acta Math.* **149** (1982), 1–40.
3. CEGRELL, U., Pluricomplex energy, *Acta Math.* **180** (1998), 187–217.
4. CEGRELL, U. and PERSSON, L., An energy estimate for the complex Monge–Ampère operator, *Ann. Polon. Math.* **67** (1997), 95–102.
5. KALINA, J., Some remarks on variational properties of inhomogeneous complex Monge–Ampère equation, *Bull. Polish Acad. Sci. Math.* **31** (1983), 9–13.
6. KLIMEK, M., *Pluripotential Theory,* Oxford Univ. Press, Oxford, 1992.
7. PORT, S. C. and STONE, C. J., *Brownian Motion and Classical Potential Theory,* Academic Press, New York, 1978.

Received August 11, 1997
in revised form June 15, 1998

Leif Persson
Department of Mathematics
Umeå University
SE-901 87 Umeå
Sweden
email: Leif.Persson@math.umu.se