Ark. Mat., **37** (1999), 171–182

Norm convergence of normalized iterates and the growth of Kœnigs maps

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Abstract. Let ϕ be an analytic function defined on the unit disk $\mathbf{D}$, with $\phi(\mathbf{D})\subset\mathbf{D}$, $\phi(0)=0$, and $\phi'(0)=\lambda\neq 0$. Then by a classical result of G. Kœnigs, the sequence of normalized iterates ϕ_n/λ^n converges uniformly on compact subsets of $\mathbf{D}$ to a function σ analytic in $\mathbf{D}$ which satisfies $\sigma\circ\phi=\lambda\sigma$. It is of interest in the study of composition operators to know if, whenever σ belongs to a Hardy space H^p, the sequence ϕ_n/λ^n converges to σ in the norm of H^p. We show that this is indeed the case, generalizing a result of P. Bourdon obtained under the assumption that ϕ is univalent.

When ϕ is inner, P. Bourdon and J. Shapiro have shown that σ does not belong to the Nevanlinna class, in particular it does not belong to any H^p. It is natural to ask, how bad can the growth of σ be in this case? As a partial answer we show that σ always belongs to some Bergman space L^p_a.

1. Introduction

Let ϕ be an analytic function defined on $\mathbf{D}$, with $\phi(\mathbf{D})\subset\mathbf{D}$, $\phi(0)=0$, and $\phi'(0)=\lambda\neq 0$. Kœnigs's Theorem provides an analytic map σ on $\mathbf{D}$ which intertwines ϕ with multiplication by λ,

$$\sigma\circ\phi=\lambda\sigma. \tag{1.1}$$

The map σ is obtained as the limit, uniform on compact subsets of $\mathbf{D}$, of the sequence of normalized iterates ϕ_n/λ^n, where $\phi_n=\phi\circ\ldots\circ\phi$, n times. Originally introduced to study the behavior of ϕ near the origin, the function σ has recently found applications in the study of the composition operator $C_\phi(f)=f\circ\phi$, induced by ϕ on the analytic functions f of $\mathbf{D}$, mainly because, by (1.1), σ is a formal

(1) The author is partially supported by NSF Grant DMS 97-06408 and wishes to thank Professor P. Bourdon for sharing results and conjectures.

eigenfunction of C_ϕ. Recall that for $0<p<\infty$, the Hardy space H^p is the family of analytic functions f defined on $\mathbf{D}$ which satisfy

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta/2\pi < \infty. \tag{1.2}$$

The operator C_ϕ is known to be bounded on the Hardy spaces H^p, i.e. $C_\phi(H^p) \subset H^p$ for $0<p<\infty$ (this is known as Littlewood's subordination principle). On the other hand, the Kœnigs map σ does not always belong to H^p. When ϕ is univalent, then σ is also univalent, so $\sigma \in H^p$ at least for $0<p<\frac{1}{2}$. However, when ϕ is an inner function, P. Bourdon and J. Shapiro in [BS] show that $\sigma \notin \bigcup_{p>0} H^p$ (actually they prove that σ is not even in the Nevanlinna class). Conversely, in [P1] we showed that when ϕ is not inner, σ is always in some H^p, for some $p>0$. Hence, the property of ϕ being inner provides a dichotomy for the growth of σ. A further description of this phenomenon will be provided below (see Remark 3.2).

In this context, P. Bourdon in [B] recently asked the following question. When is the sequence of normalized iterates ϕ_n/λ^n convergent to σ in the norm of H^p?

Clearly a necessary condition is that σ be in H^p. Bourdon shows that this condition is also sufficient if one assumes that ϕ is univalent. We show that the univalence requirement can be dropped.

Theorem 1.1. *For every $p>0$ such that $\sigma \in H^p$ we have*

$$\left\| \frac{\phi_n}{\lambda^n} - \sigma \right\|_{H^p} \to 0, \quad \text{as } n \to \infty.$$

As mentioned above, when ϕ is inner σ is in no H^p space. So it is natural to ask the following question. How bad can the growth of σ be in this case? For instance, how does σ behave with respect to the Bergman spaces? Recall that for $0<p<\infty$ the Bergman space L_a^p is the family of analytic functions f defined on $\mathbf{D}$ such that

$$\|f\|_{L_a^p}^p = \int_{\mathbf{D}} |f(z)|^p \, dA(z) < \infty,$$

where dA is area measure normalized so that $A(\mathbf{D})=1$.

In the second part of the paper we show that for arbitrary self-maps ϕ (inner or non-inner), the Kœnigs map σ is always in L_a^p for some $p>0$.

In order to study the behavior of Kœnigs maps with respect to the Bergman spaces, we consider the "growth spaces" G^p, for $0<p<\infty$, consisting of all analytic functions f defined on $\mathbf{D}$ such that

$$\|f\|_{G^p} = \sup_{z \in \mathbf{D}} |f(z)|(1-|z|)^{1/p} < \infty. \tag{1.3}$$

These spaces are related to the Bergman spaces L_a^p by the following inclusions. For $0<p<\infty$ and $0<\varepsilon<p$,

$$L_a^{p-\varepsilon} \supset G^p \supset L_a^{2p}. \tag{1.4}$$

The first inclusion is clear; for the second one, see for instance [R, Theorem 7.2.5, p. 128]. The space

$$A^{-\infty} = \bigcup_{p>0} G^p = \bigcup_{p>0} L_a^p$$

is much bigger than the Nevanlinna class. But some parts of the classical theory can be extended to $A^{-\infty}$, see the work of Korenblum [K]. In particular, the zero sets of functions in $A^{-\infty}$ have been studied extensively. We prove the following result.

Theorem 1.2. *If ϕ is an analytic function of $\mathbf{D}$, with $\phi(\mathbf{D})\subset\mathbf{D}$, $\phi(0)=0$, and $\phi'(0)=\lambda\neq 0$, and σ is the associated Kœnigs map, then*

$$\sigma \in \bigcup_{p>0} L_a^p.$$

Corollary 1.3. *Let $Z=\{a_n\}\subset\mathbf{D}\setminus\{0\}$ be a Blaschke sequence, i.e. a sequence such that $\sum_{n=1}^{\infty}(1-|a_n|)<\infty$, and let ϕ be a function as in Theorem 1.2 which vanishes on Z. Then the set $\bigcup_{n=1}^{\infty}\phi_n^{-1}(Z)$ obtained by repeated backward iterations is an $A^{-\infty}$-zero set.*

Proof of Corollary 1.3. Suppose $w\in\mathbf{D}$ and $\phi_n(w)\in Z$ for some $n\geq 1$. Then

$$\sigma(w) = \frac{\lambda^{n+1}\sigma(w)}{\lambda^{n+1}} = \frac{\sigma(\phi_{n+1}(w))}{\lambda^{n+1}} = \frac{\sigma(0)}{\lambda^{n+1}} = 0.$$

Hence, by Theorem 1.2, $\bigcup_{n=1}^{\infty}\phi_n^{-1}(Z)$ is a subset of an $A^{-\infty}$-zero set. So Corollary 1.3 follows from Corollary 2, p. 129, of [K]. □

Question 1.4. Is Theorem 1.2 sharp? i.e. for every $p>0$, are there Kœnigs maps σ that are not in L_a^p?

Added in proof: P. Bourdon has recently answered this question in the affirmative (personal communication).

Question 1.5. Does Corollary 1.3 still hold if we drop the requirement that $\phi'(0)\neq 0$?

Finally, in view of Theorem 1.1, one may ask if the sequence of normalized iterates ϕ_n/λ^n converges to σ in the norm of L_a^p whenever $\sigma\in L_a^p$. This turns out to be a quick generalization of Theorem 1.1.

In Section 2 we describe the key example that led us to the proof of Theorem 1.1. In Section 3 we introduce the radial maximal function of σ and prove Theorem 1.1. Then in Sections 4 and 5 we deal with the Bergman spaces situation.

2. A Kœnigs map obtained from a covering Riemann surface

Before proving Theorem 1.1, we are going to describe the example that provided the intuition for the proof. In the univalent case, ϕ_n/λ^n and σ have radial limits almost everywhere on $\partial\mathbf{D}$, so one can restrict the attention to the boundary functions. Bourdon then shows that ϕ_n/λ^n converges almost everywhere on $\partial\mathbf{D}$ to σ, and uses the fact that σ is one-to-one together with Koebe's $\frac{1}{4}$-theorem to establish

$$\left|\frac{\phi_n}{\lambda^n}-\sigma\right|\leq C|\sigma|, \quad \text{a.e. on } \partial\mathbf{D}, \tag{2.1}$$

for some constant $C>0$. So by the dominated convergence theorem, he obtains that whenever $\sigma\in H^p$, the sequence ϕ_n/λ^n converges to σ in H^p.

It is therefore natural to ask if (2.1) holds in general, for non-univalent functions. First observe that when ϕ is not univalent, the set $\phi^{-1}(0)$ could be infinite and there could be a point $0\neq z_0\in\phi^{-1}(0)$ for which $\phi_n^{-1}(z_0)$ is non-empty for all $n\geq 1$. Pick $z_n\in\phi_n^{-1}(z_0)$. Then $|\phi_n(z_n)/\lambda^n|=|z_0|\,|\lambda|^{-n}\to\infty$. However, $\sigma(z_n)$ must be zero, by (1.1), since

$$\lambda^{n+1}\sigma(z_n)=\sigma(\phi_{n+1}(z_n))=\sigma(0)=0.$$

This suggests that (2.1) may not hold for non-univalent maps. Hence, we set out to construct an example of ϕ for which (2.1) fails.

As mentioned above, the sequence of normalized iterates ϕ_n/λ^n converges to a map σ which solves the functional equation (1.1). Kœnigs's theorem, however, also has a uniqueness part. Namely, whenever σ is a function on $\mathbf{D}$ satisfying $\sigma\circ\phi=\lambda\sigma$, then σ must be a constant multiple of $\lim_{n\to\infty}\phi_n/\lambda^n$ (see [S, p. 90]). So, in order to construct a counter-example, we first produce a map σ by geometric means, such that $\sigma(0)=0$ and $\sigma'(0)\neq 0$. Then we check that there is a map ϕ which satisfies $\sigma\circ\phi=\lambda\sigma$. Thus, by uniqueness, $\sigma=\sigma'(0)\lim_{n\to\infty}\phi_n/\lambda^n$.

Consider the region Ω obtained from the right half-plane $\Pi=\{z\in\mathbf{C}:\operatorname{Re} z>0\}$ by punching out the sequence of points $\{2^n\}_{n=0}^{\infty}$. Then the universal cover of Ω is a simply connected open Riemann surface $\mathcal{W}$ standing over the right half-plane with branch points at 2^n for $n=0, 1, 2, 3, \ldots$. We let π be the canonical projection of $\mathcal{W}$ onto the complex plane. By cutting $\mathcal{W}$ along all points that project onto $[1,+\infty)$ we obtain countably many simply connected sheets which look like $\Pi\setminus[1,+\infty)$. Pick one such sheet S_0 and extend it by the Schwarz reflection principle to cover $(\Pi\setminus[1,+\infty))\cup\mathbf{D}$. We obtain a new simply connected open Riemann surface $\mathcal{S}\supset\mathcal{W}$ covering $\Omega\cup\mathbf{D}$ and branched at $\{2^n\}_{n=0}^{\infty}$. Moreover, points in Ω have infinitely many preimages in $\mathcal{S}$, while every point in $\mathbf{D}\cap\{z:\operatorname{Re} z\leq 0\}$ has a unique preimage.

A good way to picture this surface is to imagine a dictionary lying over the right half-plane with infinitely many pages ligated at the points $\{2^n\}_{n=0}^{\infty}$, and such that the first page has a tag in the form of $\mathbf{D}$ sticking out over the left half-plane.

Let $\theta\in\mathcal{S}$ be the point that projects onto 0, and let G be an analytic and one-to-one map of $\mathbf{D}$ onto $\mathcal{S}$ such that $G(0)=\theta$. Set $\sigma=\pi\circ G$. Then $\sigma(0)=0$ and $\sigma'(0)\neq 0$. The branch $\sigma^{-1}|_{\mathbf{D}}$ is uniquely defined, analytic and one-to-one onto a neighborhood of the origin $V_0=\sigma^{-1}|_{\mathbf{D}}(\mathbf{D})\subset\mathbf{D}$, and sends 0 to 0. Define $\phi_0(z)=\sigma^{-1}|_{\mathbf{D}}\big(\frac{1}{2}\sigma(z)\big)$. Then ϕ_0 is analytic and one-to-one on V_0, fixes the origin, and has derivative equal to $\frac{1}{2}$ at 0. For all $z\in\mathbf{D}\setminus V_0$, $\sigma(z)\in\Omega$, hence $\frac{1}{2}\sigma(z)\in\Omega$. So, in a disk Δ of radius $\frac{1}{2}d(\sigma(z),\partial\Omega)$ centered at $\frac{1}{2}\sigma(z)$ one can define a branch of $\sigma^{-1}|_{\Delta}$ to be one-to-one and analytic. By letting $\phi_z=\sigma^{-1}|_{\Delta}\big(\frac{1}{2}\sigma(z)\big)$ for some appropriate choice of the branch $\sigma^{-1}|_{\Delta}$, ϕ_0 can be analytically continued along any path in $\mathbf{D}$ starting at 0. Since $\mathbf{D}$ is simply connected, ϕ_0 extends to a map ϕ analytic on all of $\mathbf{D}$ by the monodromy theorem. Note that ϕ is locally one-to-one at every point of $\mathbf{D}$, hence $\phi'(z)\neq 0$ for all $z\in\mathbf{D}$. Also, at every $z\in\mathbf{D}$, $\phi(z)=\sigma^{-1}\big(\frac{1}{2}\sigma(z)\big)$ for some choice of σ^{-1}, so $\phi(\mathbf{D})\subset\mathbf{D}$ and $\sigma\circ\phi(z)=\frac{1}{2}\sigma(z)$. Therefore $\sigma'(0)2^n\phi_n$ converges uniformly on compact subsets of $\mathbf{D}$ to σ. Moreover, the range of σ is contained in $\{z:\operatorname{Re} z>-1\}$. So, by subordination $\sigma\in H^p$ for $0<p<1$. Finally, ϕ is not inner because, for instance, it has modulus strictly less than one on the arc of $\partial\mathbf{D}$ which is sent by σ onto the left half-circle $\partial\mathbf{D}\cap\{z:\operatorname{Re} z<0\}$ (or apply Theorem 1.5 of [P1]).

We now need to identify a sequence of sheets besides S_0. Let $\gamma_n:[0,1]\to\mathcal{S}$ be the path starting at θ whose projection on $\mathbf{C}$ describes a circle in the positive direction with diameter $[0,3\cdot 2^{n-1}]$. Let S_n be the sheet containing $\gamma_n(t)$ for t near 1. Note that for $n\geq 1$, S_n projects onto $\Pi\setminus[1,+\infty)$. Let I_n be the segment of ∂S_n projecting onto $[-i,i]$. Then G^{-1} extends continuously to ∂S_n, by the Schwarz reflection principle, and $E_n=G^{-1}(I_n)\subset\partial\mathbf{D}$. Observe that multiplication by $\frac{1}{2}$ sends the path $\pi(\gamma_n)$ onto the path $\pi(\gamma_{n-1})$. By construction the map $\frac{1}{2}z$ lifts to the map $\tilde{\phi}=G\circ\phi\circ G^{-1}$ of $\mathcal{S}$ into itself. So $\tilde{\phi}(S_0)\subset S_0$ and for all $n\geq 1$, $\tilde{\phi}(S_n)\subset S_{n-1}$. In particular, for all $n\geq 1$ we have $\tilde{\phi}_n(I_{n+1})=I_1$. Thus $\phi_n(E_{n+1})=E_1\subset\partial\mathbf{D}$ and $|2^n\phi_n|=2^n$ on E_{n+1}. On the other hand, $|\sigma|\leq 1$ on E_{n+1}. So for all $R>0$ we can always find a set $E\subset\partial\mathbf{D}$ of positive measure where $|\sigma'(0)2^n\phi_n-\sigma|\geq R|\sigma|$, i.e. (2.1) cannot hold.

Nevertheless, consider the cross-cut $J\subset\mathcal{S}$ which projects onto $(2^n,2^{n+1})$ and separates S_0 from S_n. Fix $\zeta\in E_n$, then the path $G(r\zeta)$, for $0<r<1$, starts in S_0 and ends in S_n, so it must cross J. Therefore, $\sigma(r\zeta)=\pi\circ G(r\zeta)$ must intersect the segment $(2^n,2^{n+1})$. This means that if we consider the radial maximal function of σ, $\sigma^\star(\zeta)=\sup_{0<r<1}|\sigma(r\zeta)|$, we have $\sigma^\star(\zeta)\geq 2^n$, on E_n. Thus the function $\sigma^\star$ grows in size like the sequence of normalized iterates. This is the observation that suggested to us the proof of Theorem 1.1.

3. The radial maximal function of the Kœnigs map

Motivated by the example of the last section, we now prove the following result.

Theorem 3.1. *Let ϕ be an analytic map on $\mathbf{D}$ such that $\phi(\mathbf{D})\subset\mathbf{D}$, $\phi(0)=0$ and $\phi'(0)=\lambda\neq0$. Let σ be its Kœnigs map, i.e. $\sigma=\lim_{n\to\infty}\phi_n/\lambda^n$. For $\zeta\in\partial\mathbf{D}$, let $\sigma^\star(\zeta)=\sup_{0<r<1}|\sigma(r\zeta)|$ be the radial maximal function of σ. Then, there exists a constant $C>0$ independent of n and ζ such that*

$$\left|\frac{\phi_n(\zeta)}{\lambda^n}\right|\leq C\sigma^\star(\zeta)\quad \text{for a.e. } \zeta\in\partial\mathbf{D}. \tag{3.1}$$

Moreover, if $\sigma\in H^p$ for some $p>0$, then σ can be defined on $\partial\mathbf{D}$ and ϕ_n/λ^n converges to σ almost everywhere on $\partial\mathbf{D}$.

From this we deduce immediately Theorem 1.1.

Proof of Theorem 1.1. Suppose $\sigma\in H^p$ for some $p>0$. By a well-known theorem of Hardy and Littlewood, see [G, p. 57 (and top of p. 59)], it follows that $\sigma^\star\in L^p(\partial\mathbf{D})$. By Theorem 3.1, there is a constant $C>0$ such that

$$\left|\frac{\phi_n}{\lambda^n}-\sigma\right|\leq(C+1)\sigma^\star,\quad \text{a.e. on } \partial\mathbf{D}.$$

Also, by Theorem 3.1, ϕ_n/λ^n converges to σ almost everywhere on $\partial\mathbf{D}$. Thus Theorem 1.1 follows from Lebesgue's dominated convergence theorem. □

Remark 3.2. When ϕ is inner, [BS] show that σ does not belong to the Nevanlinna class. In view of Theorem 3.1, we see that actually $\sigma^\star=\infty$ almost everywhere on $\partial\mathbf{D}$, since in this case ϕ_n/λ^n clearly converges to infinity almost everywhere on $\partial\mathbf{D}$, as n tends to infinity.

Proof of Theorem 3.1. We first show (3.1). For $\alpha>0$, let Ω_α be the component of $\{z\in\mathbf{D}:|\sigma(z)|<\alpha\}$ containing the origin. Since $\sigma'(0)=1$ and $\sigma(0)=0$, σ is one-to-one on some disk $\Delta\subset\bar{\Delta}\subset\mathbf{D}$ containing the origin, and $\sigma(\Delta)$ is an open neighborhood of 0. So there is $\delta>0$ such that the open disk of radius δ at 0, $B(0,\delta)$, satisfies $B(0,\delta)\subset\overline{B(0,\delta)}\subset\sigma(\Delta)$, that is to say $\Omega_\delta\subset\Delta$. Let us write σ^{-1} for $(\sigma|_\Delta)^{-1}$. Since the derivative of σ^{-1} is 1 at 0, $\sigma^{-1}(z)-z=z^2g(z)$, for some g analytic in a neighborhood of $\overline{B(0,\delta)}$. Let $M=M(\delta)$ be the maximum of $|g|$ on $\overline{B(0,\delta)}$. Then for $z\in B(0,\delta)$, $|\sigma^{-1}(z)|\leq|z|(1+M|z|)\leq(1+M\delta)|z|$. Write $C=1+M\delta$. Then for all $w\in\Delta$ such that $|\sigma(w)|<\varrho<\delta$, we have $|w|=|\sigma^{-1}(\sigma(w))|\leq C\varrho$. In other words, for every $0<\varrho<\delta$,

$$\Omega_\varrho\subset B(0,C\varrho). \tag{3.2}$$

Since ϕ is bounded there exists a set $W \subset \partial\mathbf{D}$ of full measure such that every iterate ϕ_n has a radial limit at each point of W. Fix a point $\zeta \in W$. If $\sigma^\star(\zeta)=\infty$, then (3.1) holds trivially. So suppose that $\sigma^\star(\zeta)<\infty$. Then there is an integer $N=N(\zeta)$ such that

$$\delta|\lambda|^{-N+1} \le \sigma^\star(\zeta) < \delta|\lambda|^{-N}. \tag{3.3}$$

For $n=1, \dots, N-1$,

$$|\lambda|^{-n}|\phi_n(\zeta)| \le |\lambda|^{-n} \le |\lambda|^{-N+1} \le \frac{1}{\delta}\sigma^\star(\zeta). \tag{3.4}$$

On the other hand, by the second inequality of (3.3), for $n \ge N$,

$$\sup_{0<r<1} |\sigma(\phi_n(r\zeta))| = \sup_{0<r<1} |\lambda^n \sigma(r\zeta)| = |\lambda|^n \sigma^\star(\zeta) < \delta|\lambda|^{n-N}. \tag{3.5}$$

Thus $L(r)=\phi_n(r\zeta)$ for $0<r<1$, is a path starting at 0 which stays in $\{z\in\mathbf{D}:|\sigma(z)|<\delta|\lambda|^{n-N}\}$, and hence stays in $\Omega_{\delta|\lambda|^{n-N}}$. So, by (3.2), $\phi_n^\star(\zeta) \le C\delta|\lambda|^{n-N}$, and, for $n \ge N$,

$$|\lambda|^{-n}|\phi_n(\zeta)| \le |\lambda|^{-n}\phi_n^\star(\zeta) \le C\delta|\lambda|^{-N} \le C|\lambda|^{-1}\sigma^\star(\zeta), \tag{3.6}$$

where we used the first inequality of (3.3). Let $C_1=\max\{1/\delta, C|\lambda|^{-1}\}$, and notice that C_1 depends only on λ and δ. Then by (3.4) and (3.6), for every $\zeta \in W$ and for all $n \ge 1$,

$$|\lambda^{-n}\phi_n(\zeta)| \le C_1 \sigma^\star(\zeta).$$

This establishes (3.1).

Now assume that $\sigma \in H^p$ for some $p>0$. Hence, $\sigma^\star<\infty$ almost everywhere on $\partial\mathbf{D}$, and without loss of generality $\sigma^\star(\zeta)<\infty$ for all $\zeta\in W$. For each $\zeta \in W$ choose $N=N(\zeta)$ as in (3.3). Then, for $n \ge N$, $\phi_n(\zeta)\in\Omega_\delta$, by (3.5). Since $\overline{\Omega}_\delta$ is a compact subset of $\mathbf{D}$, ϕ_n/λ^n converges to σ uniformly in $\overline{\Omega}_\delta$. So,

$$\frac{\phi_n(\zeta)}{\lambda^n} = \frac{\phi_{n-N}(\phi_N(\zeta))}{\lambda^{n-N}\lambda^N} \to \frac{\sigma(\phi_N(\zeta))}{\lambda^N} = \sigma(\zeta),$$

as $n\to\infty$. Therefore, ϕ_n/λ^n tends to σ pointwise everywhere on W. $\square$

4. Kœnigs maps are always in some Bergman space

In this section we prove Theorem 1.2, which says that Kœnigs maps are always in some Bergman space. Our method of proof is analogous to the one used to prove

Theorem 2.1 of [P1], which says that the Kœnigs map of a non-inner self-map ϕ is always in some Hardy space. If ψ is an analytic function defined on $\mathbf{D}$, we introduce the following level sets: for $\alpha>0$, let $\Omega_\alpha(\psi)$ be the component of $\{z\in\mathbf{D}:|\psi(z)|<\alpha\}$ containing the origin, and let $F_\alpha(\psi)=\partial\Omega_\alpha\cap\mathbf{D}$. In [P1] we studied the Hardy class of ψ by looking at the rate of decay as α tends to infinity of the harmonic measure of $F_\alpha(\psi)$ at 0.

Theorem 4.1. ([P1, Theorem 2.1]) *Let ϕ be an analytic map such that $\phi(\mathbf{D})\subset\mathbf{D}$, $\phi(0)=0$ and $0<|\phi'(0)|<1$, and let σ be its Kœnigs map. Let ω_α be the harmonic measure of the level set $F_\alpha(\sigma)$ at 0 in $\mathbf{D}$. Then the following limit exists,*

$$\lim_{\alpha\to\infty}\frac{\log(1/\omega_\alpha)}{\log\alpha}=\mu(\sigma).$$

Moreover, $\sigma\in H^p$ if and only if $0<p<\mu(\sigma)$, and $\mu(\sigma)=0$ if and only if ϕ is inner.

Now, instead of harmonic measure we will use hyperbolic distance. Recall that the hyperbolic distance between two points $a,b\in\mathbf{D}$ is defined by

$$\varrho_{\mathbf{D}}(a,b)=\log\frac{1+|(b-a)/(1-\bar{b}a)|}{1-|(b-a)/(1-\bar{b}a)|}.$$

For $\alpha>0$, we let

$$\varrho_\alpha=\varrho_{\mathbf{D}}(0,F_\alpha(\psi))=\inf_{\zeta\in F_\alpha(\psi)}\varrho_{\mathbf{D}}(0,\zeta).$$

Our convention is that ϱ_α is infinite when $F_\alpha(\psi)$ is empty.

We reformulate the definition of the spaces G^p, defined in the introduction, in terms of ϱ_α.

Lemma 4.2. *Let ψ and ϱ_α be defined as above. Then, for $0<p<\infty$, $\psi\in G^p$ if and only if there is a constant $C>0$ such that*

$$\frac{\varrho_\alpha}{\log\alpha}\ge p-\frac{C}{\log\alpha}. \tag{4.1}$$

Proof. Define $M(\psi,r)=\max_{|z|=r}|\psi(z)|$ for $0<r<1$. Then, $\psi\in G^p$ if and only if there is a constant $C>0$ such that

$$M(\psi,r)(1-r)^{1/p}\le C \tag{4.2}$$

for all $0<r<1$. Choose z_0 with $|z_0|=r$ such that $|\psi(z_0)|=M(\psi,r)$. Then by the maximum principle, $|\psi(z)|<|\psi(z_0)|$ for all z such that $|z|<r$. Hence, setting $\alpha=$

$M(\psi, r)$, we have $r\mathbf{D} \subset \Omega_\alpha(\psi)$ and $z_0 \in F_\alpha(\psi)$. In particular, $\varrho_\alpha = \varrho_{\mathbf{D}}(0, z_0)$. Since $\varrho_{\mathbf{D}}(0, z_0) \asymp -\log(1-r)$, (4.2) can be rewritten to yield (4.1). $\square$

Let ϕ be a self-map of $\mathbf{D}$ as in the introduction and let σ be its Kœnigs map. Consider the level sets $F_\alpha = F_\alpha(\sigma)$ and $\Omega_\alpha = \Omega_\alpha(\sigma)$. Then $\{\Omega_\alpha\}$ is an increasing family of non-empty (because $\sigma(0)=0$) simply connected regions. The sets F_α are disjoint for different α's, and if $\alpha_1 < \alpha_2$, F_{α_1} separates F_{α_2} from 0 in Ω_{α_2}. Thus

$$\varrho_{\alpha_2} \geq \varrho_{\alpha_1} + \varrho_{\mathbf{D}}(F_{\alpha_1}, F_{\alpha_2}). \tag{4.3}$$

Observe that, since σ satisfies equation (1.1), the following properties hold for all $\alpha > 0$,

$$\phi(\Omega_\alpha) \subset \Omega_{|\lambda|\alpha} \quad \text{and} \quad \phi(F_\alpha) \subset F_{|\lambda|\alpha}. \tag{4.4}$$

Suppose E is a closed set in $\mathbf{D} \setminus \Omega_\alpha$ for some $\alpha > 0$, so that F_α separates E from 0 in $\mathbf{D}$. Then by (4.4), $F_{\alpha/|\lambda|}$ separates $\phi^{-1}(E)$ from 0 in $\mathbf{D}$, and by the invariant form of Schwarz's lemma (Theorem I.4.1 of [CG]),

$$\varrho_{\mathbf{D}}(F_\alpha, E) \leq \varrho_{\mathbf{D}}(F_{\alpha/|\lambda|}, \phi^{-1}(E)). \tag{4.5}$$

Using (4.3) and (4.5) we prove the following theorem.

Theorem 4.3. *Let ϕ be an analytic map such that $\phi(\mathbf{D}) \subset \mathbf{D}$, $\phi(0)=0$ and $\lambda = \phi'(0) \neq 0$, and let σ be its Kœnigs map. Then the following limit exists strictly positive,*

$$\lim_{\alpha \to \infty} \frac{\varrho_\alpha}{\log \alpha} = \eta(\sigma) > 0.$$

Moreover, $\sigma \in G^p$ for $0 < p < \eta(\sigma)$ and $\sigma \notin G^p$ for $\eta(\sigma) < p < \infty$.

From Theorem 4.3 and the inclusions (1.4) we obtain that $\sigma \in L^p_a$ for $0 < p < \eta(\sigma)$ and $\sigma \notin L^p_a$ for $2\eta(\sigma) < p < \infty$. In particular, since $\eta(\sigma) > 0$, Theorem 1.2 follows.

Question 4.4. Is σ in L^p_a for $\eta(\sigma) \leq p \leq 2\eta(\sigma)$?

Remark 4.5. Theorem 4.3 leaves open the question whether $\sigma \in G^p$ for $p = \eta(\sigma)$. Equation (4.10) below implies that

$$\frac{\varrho_\alpha}{\log \alpha} \leq \eta(\sigma) + \frac{C}{\log \alpha}$$

which is the other direction of (4.1), and hence does not help. We suspect that one can find examples in both cases by looking at univalent maps.

Proof of Theorem 4.3. Without loss of generality, $\bar{\Omega}_1 \subset s\mathbf{D}$ for some $0<s<1$ (multiply σ by a large enough constant). Fix $\beta>1$ and find an integer $N\geq 1$ such that

$$|\lambda|^{-N+1} \leq \beta < |\lambda|^{-N}. \tag{4.6}$$

For all $\alpha>\beta$, there is an integer $H\geq 1$ such that

$$\beta|\lambda|^{-(H-1)N} \leq \alpha < \beta|\lambda|^{-HN}. \tag{4.7}$$

Having set the scales in which we are measuring the sizes of α and β we let $T_h = F_{\beta|\lambda|^{-hN}}$ and $W_h = \Omega_{\beta|\lambda|^{-hN}}$ for $h=-1, 0, \dots, H-1$. Iterating (4.3), we obtain

$$\varrho_\alpha \geq \sum_{h=0}^{H-1} \varrho_{\mathbf{D}}(T_{h-1}, T_h). \tag{4.8}$$

Notice that by (4.4), $\phi_{hN}^{-1}(F_\beta) \cap W_h = \emptyset$ and $\phi_{hN}^{-1}(F_\beta) \supset T_h$. So,

$$\varrho_{\mathbf{D}}(T_{h-1}, T_h) = \varrho_{\mathbf{D}}(T_{h-1}, \phi_{hN}^{-1}(F_\beta)).$$

Then, by (4.5), for $h=0, \dots, H-1$,

$$\varrho_{\mathbf{D}}(T_{h-1}, \phi_{hN}^{-1}(F_\beta)) \geq \varrho_{\mathbf{D}}(T_{-1}, F_\beta).$$

Finally, using the fact that, by (4.6), $\beta|\lambda|^N<1$ and thus $T_{-1} \subset \bar{\Omega}_1 \subset s\mathbf{D}$, (4.8) becomes

$$\varrho_\alpha \geq H \varrho_{\mathbf{D}}(\bar{\Omega}_1, F_\beta).$$

Write R_β for $\varrho_{\mathbf{D}}(\bar{\Omega}_1, F_\beta)$. Using (4.7), we obtain

$$\frac{\varrho_\alpha}{\log \alpha} \geq \frac{H R_\beta}{HN \log(1/|\lambda|) + \log \beta}.$$

Letting α tend to infinity, H also tends to infinity. Hence,

$$\liminf_{\alpha\to\infty} \frac{\varrho_\alpha}{\log \alpha} \geq \frac{R_\beta}{N \log(1/|\lambda|)}.$$

By (4.6), $N \log(1/|\lambda|) \leq \log \beta + \log(1/|\lambda|)$, so

$$\liminf_{\alpha\to\infty} \frac{\varrho_\alpha}{\log \alpha} \geq \frac{R_\beta}{\log \beta + \log(1/|\lambda|)}. \tag{4.9}$$

This estimate will be used below.

Since σ is bounded on $s\bar{\mathbf{D}}$, there exists $\beta_0>1$ such that $s\bar{\mathbf{D}}\subset\Omega_\beta$ for all $\beta>\beta_0$. By the triangle inequality, the constant $C_0=\log((1+s)/(1-s))$ is such that

$$R_\beta \geq \varrho_\beta - C_0$$

for $\beta>\beta_0$. Thus

$$\liminf_{\alpha\to\infty} \frac{\varrho_\alpha}{\log\alpha} \geq \frac{\varrho_\beta - C_0}{\log\beta + \log(1/|\lambda|)}. \tag{4.10}$$

Letting β tend to infinity in (4.10), we obtain

$$\liminf_{\alpha\to\infty} \frac{\varrho_\alpha}{\log\alpha} \geq \limsup_{\beta\to\infty} \frac{\varrho_\beta}{\log\beta}.$$

So, the limit $\eta(\sigma)$ exists.

Moreover, for $\beta>\beta_0$, $R_\beta>0$. Therefore, (4.9) implies that we always have $\eta(\sigma)>0$. $\square$

5. More on norm convergence

In this section we show that Theorem 1.1 can be extended to the Bergman space case. We proceed as in Section 3 and define the maximal function

$$\sigma^\star(z) = \sup_{0<r<1} |\sigma(rz)|.$$

But now we let z be any point of $\mathbf{D}$. Then, Theorem 3.1 still holds (the proof goes through verbatim), i.e. there exists a constant $C>0$ independent of n and z such that

$$\left|\frac{\phi_n(z)}{\lambda^n}\right| \leq C\sigma^\star(z) \quad \text{for all } z\in\mathbf{D}.$$

Notice also that $\sigma\in L^p_a$ implies $\sigma^\star\in L^p(\mathbf{D}, dA)$. In fact, for every function f defined on $\mathbf{D}$ and $0<r<1$ let $f_r(z)=f(rz)$, then

$$(\sigma^\star)_r = (\sigma_r)^\star.$$

Hence,

$$\int_{\mathbf{D}} |\sigma^\star(z)|^p\, dA(z) = \int_0^1 \|(\sigma^\star)_r\|_{L^p(\partial\mathbf{D})}\, r\, dr \leq C \int_0^1 \|\sigma_r\|_{L^p(\partial\mathbf{D})}\, r\, dr = C\|\sigma\|_{L^p_a},$$

where $C>0$ is the constant provided by the theorem of Hardy and Littlewood. Therefore, Lebesgue's dominated convergence theorem yields the following statement.

Theorem 5.1. *Let ϕ be an analytic map such that $\phi(\mathbf{D})\subset\mathbf{D}$, $\phi(0)=0$ and $\lambda=\phi'(0)\neq0$, and let σ be its Kœnigs map. For every $p>0$ such that $\sigma\in L^p_a$, we have*

$$\left\|\frac{\phi_n}{\lambda^n}-\sigma\right\|_{L^p_a}\to0, \quad \textit{as } n\to\infty.$$

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Received August 4, 1997

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