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The negative discrete spectrum of the operator $(-\Delta)^l - \alpha V$ in $L_2(\mathbf{R}^d)$ for d even and $2l \geq d$

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Abstract. We study the asymptotic behaviour of $\mathcal{N}(\alpha)$—the number of negative eigenvalues of the operator $(-\Delta)^l - \alpha V$ in $L_2(\mathbf{R}^d)$ for an even d and $2l \geq d$. This is the only case where the previously known results were far from being complete. In order to describe our results we introduce an auxiliary ordinary differential operator (system) on the semiaxis. Depending on the spectral properties of this operator we can distinguish between three cases where $\mathcal{N}(\alpha)$ is of the Weyl-type, $\mathcal{N}(\alpha)$ is of the Weyl-order but not the Weyl-type coefficient and finally where $\mathcal{N}(\alpha) = O(\alpha^q)$ with $q > d/2l$.

0. Introduction

0.1. We consider in $L_2(\mathbf{R}^d)$ the selfadjoint operator

$$A(\alpha) = (-\Delta)^l - \alpha V, \quad l \in \mathbf{N}, \ \alpha > 0. \tag{0.1}$$

Here V is the operator of multiplication by a real function (potential) $V(x)$; the parameter α is a coupling constant. The operator (0.1) is accurately defined by its quadratic form (see Subsection 3.1). It is assumed that V has a qualified convergence to zero at infinity, so that the negative spectrum of the operator $A(\alpha)$ is discrete and, moreover, finite for all $\alpha > 0$. Let us denote by $\mathcal{N}(\alpha)$ the number of negative eigenvalues of the operator (0.1). Estimates of the function $\mathcal{N}(\alpha)$ and especially its asymptotics as $\alpha \to \infty$ is our main interest. More specifically, we consider the case

$$2l \geq d, \quad d \text{ even}, \tag{0.2}$$

which, in a sense, happens to be the most complicated. If V is a bounded and rapidly decreasing function, then for all l and d the asymptotics is of the Weyl-type

$$\lim_{\alpha \to \infty} \alpha^{-\varkappa} \mathcal{N}(\alpha) = c_d \int V_+^{\varkappa} \, dx, \quad \varkappa = d/2l, \tag{0.3}$$

where $V_+ = \max\{V, 0\}$ and

$$c_d = (2\pi)^{-d} \operatorname{vol}\{x \in \mathbf{R}^d : |x| < 1\}. \tag{0.4}$$

On the other hand the order of growth of $\mathcal{N}(\alpha)$ can be arbitrarily greater than $\alpha^{\varkappa}$ (for reference see [BS5]) when V decays slowly; it is also possible that $\mathcal{N}(\alpha)=\infty$ for all $\alpha>0$. The "Weyl-type case" can be separated from the others very simply if $2l<d$. Here the sufficient (and for $V=V_+$ even necessary) condition for (0.3) to be true, is the inclusion

$$V \in L_{\varkappa}(\mathbf{R}^d). \tag{0.5}$$

(For $l=1$ this is based on the well-known Rosenblum–Lieb–Cwikel estimate [R1], [R2], [L], [C]; see [RS]. For $l>1$ the corresponding estimates were obtained in [R1], [R2] and are actually contained in [C]. The case $l \geq 1$ was discussed in [BS3].) On the contrary, it is well known that if $2l \geq d$, then the inclusion (0.5) does not imply (0.3). Moreover, one cannot even guarantee the estimate

$$\mathcal{N}(\alpha) = O(\alpha^{\varkappa}).$$

For an odd d and $2l>d$, a sufficient condition for (0.3) to be true was pointed out in [BS5] ($V \in G(d,l)$, see below Subsection 3.3). Here the local conditions on V are minimal, i.e. $V \in L_{1,\mathrm{loc}}$, but it is unlikely that the global condition $V \in G(d,l)$ is necessary. However counterexamples are not constructed so far.

Added in proof. The problem mentioned above was investigated in the recent paper by K. Naimark and M. Solomyak [NS]. The case $d=l=1$ was considered, then $\varkappa=\frac{1}{2}$. In particular, a criterion for $\mathcal{N}(\alpha)=O(\alpha^{1/2})$ was given there. The corresponding class for V is wider than $G(1,1)$. This confirms the conjecture made above and provides one with the desired counterexamples.

0.2. In the case (0.2) the condition $V \in G(d,l)$ does not provide (0.3). The present paper is concerned with the explanation of this fact and discussions of the various possibilities. It was preceded by the papers [S2], [S3] and [BL] where some observations have already been collected. The special role of the behaviour of the operator (0.1) on functions depending only on $|x|$ became clear in the case $2l=d$. An auxiliary ordinary differential operator on the semiaxis $\mathbf{R}_+$ with effective potential

$$Q(r) = \frac{1}{\operatorname{meas} \mathbf{S}^{d-1}} \int_{\mathbf{S}^{d-1}} V(r,\omega)\, d\omega, \quad r=|x|,\ \omega = x/r$$

was considered. It was clarified that the spectrum of this operator is responsible for possible failure of (0.3). In [S3] this was done on the level of two-sided spectral estimates and in [BL] on the level of spectral asymptotics. In particular, it was shown in [BL] that for $2l=d=2$ (then $\varkappa=1$) there exist potentials V such that $\mathcal{N}(\alpha)\sim c\alpha^q$ for any $q>1$. Moreover, there are examples where $\mathcal{N}(\alpha)\sim c\alpha$ is of the Weyl-type order but the coefficient c is greater than $c_2\int V_+\,dx$.

Using the scheme of the paper [BL] we consider here similar problems in the general case (0.2). The following new facts become clear. (1) The possible violation of (0.3) is caused by the behaviour of an auxiliary differential operator of order $2l$ on $\mathbf{R}_+$ acting on vector-valued functions. Its description is closely connected with singling out a "special" subspace of functions where the standard Hardy inequality fails (see [S1]). (2) The case $\frac{1}{2}d\leq l<d$ is very similar to the case $2l=d=2$ which was studied in [BL]. Here an active competition between the Weyl-type asymptotics (0.3) and the contribution given by the auxiliary operator is possible. The new fact here is a possibility to reduce the auxiliary operator in the case $l>1$ to its lowest order part which is an operator of the second order. Only this lowest order part of the operator on the semiaxis is responsible for a possible contribution to the asymptotic formula. (3) If $l\geq d$ the situation is somewhat different. The Weyl-type order $\mathcal{N}(\alpha)=O(\alpha^{\varkappa})$ can also be violated, but only for potentials V irregular at infinity (for example, lacunary). A slight additional regularity condition on V in the case $l\geq d$ gives the asymptotics (0.3).

0.3. We use the variational approach in coordinate representation. By using the standard method the study of $\mathcal{N}(\alpha)$ is reduced to a study of the eigenvalue distribution function of a compact operator (spectrum of a variational quotient). We present our results in these terms. The return to $\mathcal{N}(\alpha)$ is automatic. Therefore we rarely duplicate the statements, with the exception of Theorem 4.7 which is a translation of our main Theorem 4.2 into the language of the function $\mathcal{N}(\alpha)$. The latter theorem has a "semieffective" character: a conclusion about the asymptotics of the spectrum for the studied variational quotient depends on the asymptotics of the spectrum of an auxiliary operator acting on the semiaxis. The operator on the semiaxis needs a special study since it is not included in the well-known standard cases. We were forced to extract these discussions in a different paper [BLS]. On the basis of the results obtained in [BLS] we succeeded in giving Theorem 4.2 a concrete analytical form (Theorems 6.3 and 6.6).

Let us notice that the problem considered in this paper is of some interest, for example, in the study of a periodic Schrödinger operator perturbed by a decreasing potential. There (0.1) can appear as a model operator.

The structure of this paper is as follows: In Section 1 we collect preparatory material, mainly of an operator-theoretical character. In Section 2 we discuss some necessary facts from real analysis, in particular, the Hardy inequality. In Section 3 the operator (0.1) is accurately described, and the problem of the behaviour of $\mathcal{N}(\alpha)$ is reduced to the one concerning the spectrum of the operator generated by an appropriate variational quotient. The main Theorem 4.2 is proved in Section 4. In Section 5 we give information about the spectral properties of a vector problem on the semiaxis, borrowed from [BLS]. Finally in Section 6, the results from Sections 4 and 5 are combined and this completes the paper.

0.4. The necessary notation is introduced throughout our paper. Here we only mention some of the most frequently used notation. In what follows $\mathbf{N}$ is the set of positive integers, $\mathbf{Z}_+=\{0\}\cup\mathbf{N}$; $\mathbf{R}_+=(0,\infty)$, $\mathring{\mathbf{R}}^d=\mathbf{R}^d\setminus\{0\}$, $B_R=\{x\in\mathbf{R}^d:|x|<R\}$ and $|\nabla^l u|^2:=\sum_{|\beta|=l}(l!/\beta!)|D^\beta u|^2$. Let $H^l(\Omega)$ be the Sobolev space in a domain $\Omega\subseteq\mathbf{R}^d$, $H^l_0(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^l(\Omega)$. By C and c we denote different constants whose values are unimportant. Sometimes C is supplied by a subindex which coincides with the number of the formula where this constant appeared for the first time. The symbol $\asymp$ denotes a two sided estimate.

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1. Preliminary information

1.1. We begin with notation concerning numerical sequences. The distribution function of an arbitrary sequence $\mathbf{h}=\{h_j\}$, $j\in\mathbf{Z}_+$ of complex numbers is defined as

$$n(\lambda,\mathbf{h})=\#\{j:|h_j|>\lambda\},\quad \lambda>0.$$

Given $q>0$, we define $\|\mathbf{h}\|_q=\|\mathbf{h}\|_{l_q}$ and

$$\|\mathbf{h}\|_{q,\infty}=\sup_{\lambda>0}\lambda(n(\lambda,\mathbf{h}))^{1/q}. \tag{1.1}$$

The space $l_{q,\infty}$ (by another terminology the "weak l_q-space", $l_{q,w}$) is defined as

$$l_{q,\infty}=\{\mathbf{h}:\|\mathbf{h}\|_{q,\infty}<\infty\}.$$

This is a complete linear *quasinormed* space with respect to the quasinorm (1.1). The space $l_{q,\infty}$ is nonseparable.

The following (nonlinear) functionals are finite and continuous on $l_{q,\infty}$

$$\Delta_q(\mathbf{h}) = \limsup_{\lambda \to 0} \lambda^q n(\lambda, \mathbf{h}), \quad \delta_q(\mathbf{h}) = \liminf_{\lambda \to 0} \lambda^q n(\lambda, \mathbf{h}).$$

Obviously

$$0 \leq \delta_q(\mathbf{h}) \leq \Delta_q(\mathbf{h}) \leq \|\mathbf{h}\|_{q,\infty}^q.$$

The set

$$l_{q,\infty}^0 := \{\mathbf{h} \in l_{q,\infty} : \Delta_q(\mathbf{h}) = 0\}$$

is a linear closed separable subspace in $l_{q,\infty}$. The set of sequences with a finite number of nonzero elements is dense in this subspace.

1.2. Here we present some important facts from the elementary theory of Hilbert space. By $\mathcal{B} = \mathcal{B}(\mathfrak{H})$ and $\mathcal{C}(\mathfrak{H})$ we denote the spaces of all bounded and all compact linear operators, acting in a Hilbert space $\mathfrak{H}$. Furthermore $\mathcal{C}_{\mathrm{sa}} = \{T \in \mathcal{C}: T = T^*\}$ and $\mathcal{C}^+ = \{T \in \mathcal{C}_{\mathrm{sa}} : T \geq 0\}$. The sequence of singular numbers of an operator $T \in \mathcal{C}$ is denoted by $\mathbf{s}(T) = \{s_k(T)\}$ and the sequence of positive eigenvalues of an operator $T \in \mathcal{C}_{\mathrm{sa}}$ is denoted by $\boldsymbol{\lambda}^+(T) = \{\lambda_k^+(T)\}$. Multiplicities are taken into account in both cases. Moreover $\boldsymbol{\lambda}^-(T) := \boldsymbol{\lambda}^+(-T)$. We write $\boldsymbol{\lambda}(T)$ instead of $\boldsymbol{\lambda}^+(T)$ if $T \in \mathcal{C}^+$. Obviously $\boldsymbol{\lambda}(T) = \mathbf{s}(T)$ for $T \in \mathcal{C}^+$.

The respective distribution functions are denoted by

$$n(\lambda, T) := n(\lambda, \mathbf{s}(T)), \qquad T \in \mathcal{C}, \tag{1.2}$$

$$n_\pm(\lambda, T) := n(\lambda, \boldsymbol{\lambda}^\pm(T)), \quad T \in \mathcal{C}_{\mathrm{sa}}. \tag{1.3}$$

For $T \in \mathcal{C}_{\mathrm{sa}}$ we obviously have $n(\lambda, T) = n_+(\lambda, T) + n_-(\lambda, T)$. The classes $\mathcal{C}_q$ and the "weak classes" $\mathcal{C}_{q,\infty}$ are introduced for an arbitrary $q > 0$ as

$$\mathcal{C}_q = \{T \in \mathcal{C} : \mathbf{s}(T) \in l_q\}, \quad \mathcal{C}_{q,\infty} = \{T \in \mathcal{C} : \mathbf{s}(T) \in l_{q,\infty}\}. \tag{1.4}$$

The quasinorms in $\mathcal{C}_q$ and $\mathcal{C}_{q,\infty}$ are induced by the definition (1.4)

$$\|T\|_q := \|\mathbf{s}(T)\|_q, \quad \|T\|_{q,\infty} := \|\mathbf{s}(T)\|_{q,\infty}.$$

The set

$$\mathcal{C}_{q,\infty}^0 = \{T \in \mathcal{C} : \mathbf{s}(T) \in l_{q,\infty}^0\}$$

is a closed separable subspace of the nonseparable space $\mathcal{C}_{q,\infty}$. Clearly $\mathcal{C}_q \subset \mathcal{C}_{q,\infty}^0$. The set of all finite rank operators is dense in $\mathcal{C}_{q,\infty}^0$.

Put

$$(1.5)\qquad \Delta_q(T):=\Delta_q(\mathbf{s}(T)),\quad \delta_q(T):=\delta_q(\mathbf{s}(T))\quad \text{for } T\in\mathcal{C}_{q,\infty},$$

and

$$(1.6)\qquad \Delta_q^{\pm}(T):=\Delta_q(\boldsymbol{\lambda}^{\pm}(T)),\quad \delta_q^{\pm}(T):=\delta_q(\boldsymbol{\lambda}^{\pm}(T))\quad \text{for } T\in\mathcal{C}_{\mathrm{sa}}\cap\mathcal{C}_{q,\infty}.$$

The next useful statement is systematically used throughout the paper; see its proof, e.g. in [BS4, Theorems 11.6.6 and 11.6.11].

Proposition 1.1. *The functionals defined in* (1.5) *and* (1.6) *are continuous in the topology of the space* $\mathcal{C}_{q,\infty}$. *Moreover, they are continuous in the topology of the quotient space* $\mathcal{C}_{q,\infty}/\mathcal{C}^0_{q,\infty}$.

Let us explain the meaning of the last statement. The functionals (1.5) and (1.6) do not change when we add an operator from the class $\mathcal{C}^0_{q,\infty}$ to T (by lemmas of K. Fan and H. Weyl). Thus they are well defined on $\mathcal{C}_{q,\infty}/\mathcal{C}^0_{q,\infty}$. The continuity means that $\Delta_q(T_n-T)\to 0$ implies $D_q(T_n)\to D_q(T)$, where D_q is any of the functionals (1.5) and (1.6).

1.3. It is convenient to define and study selfadjoint operators in $\mathfrak{H}$ with the help of quadratic forms. By $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the scalar product and the norm in $\mathfrak{H}$. Let $a[u]$ be a quadratic form defined on a linear set $\mathfrak{d}=\mathfrak{d}[a]$ dense in $\mathfrak{H}$. Let us assume that $a[u]$ is semibounded from below and closed in $\mathfrak{H}$. (Closedness means the completeness of the Hilbert space $\mathfrak{d}$ with respect to the metric form $a[u]+\gamma\|u\|^2$. Here γ is chosen in such a way that $\gamma+m(a)>0$, where $m(a)$ is the lower bound of a.) By Friedrichs' theorem a unique selfadjoint semibounded operator A in $\mathfrak{H}$ can be associated with a. Let E_A be its spectral measure and

$$N(A):=\operatorname{rank} E_A(\mathbf{R}_-).$$

The function $\mathcal{N}(\alpha)$ introduced in Section 0, coincides with $N(A(\alpha))$, where $A(\alpha)$ is the operator given by (0.1).

Let now $b[\cdot]$ be a real quadratic form defined on $\mathfrak{d}[a]$. Let us assume that *the form b is compact in* $\mathfrak{d}[a]$. In other words, the operator generated by the form b in the Hilbert space $\mathfrak{d}$ is compact.

Proposition 1.2. *Let the form $a[u]$ be nonnegative and closed in $\mathfrak{H}$. Let a real form $b[u]$ be compact with respect to the metric defined by the form $a[u]+\|u\|^2$. Then for any $\alpha>0$ the form*

$$(1.7)\qquad a(\alpha):=a-\alpha b,\quad \mathfrak{d}[a(\alpha)]=\mathfrak{d}[a],\ \alpha>0$$

is semibounded from below and closed in $\mathfrak{H}$. The negative spectrum of the corresponding operator $A(\alpha)$ is discrete.

1.4. Let us now assume that $a>0$, i.e. $a[u]>0$ for $u\in\mathfrak{d}\setminus\{0\}$. We denote by $\mathfrak{H}(a)$ the Hilbert space obtained by completion of $\mathfrak{d}$ with respect to the metric defined by the form a. In order to avoid complications in the following statements we shall assume that *the topologies in* $\mathfrak{H}$ *and in* $\mathfrak{H}(a)$ *are compatible* (in our applications this will always be the case). Let us assume that the form $b[\cdot]$ can be extended by continuity to a compact form in $\mathfrak{H}(a)$. The notation for the extended form remains the same. Denote by $T(a,b,\mathfrak{H}(a))$ the operator generated by b in $\mathfrak{H}(a)$. According to the variational principle the distribution functions (1.2) and (1.3) can be expressed in terms of the "variational quotient"

$$b[u]/a[u], \quad u\in\mathfrak{H}(a). \tag{1.8}$$

So

$$n_\pm(\lambda, T(a,b,\mathfrak{H}(a))) = \max_{\mathcal{F}} \dim\{\mathcal{F}\subset\mathfrak{H}(a) : \pm b[u]/a[u] > \lambda \text{ for } u\in\mathcal{F}\setminus\{0\}\}. \tag{1.9}$$

The eigenvalues $\lambda_n^\pm(T(a,b,\mathfrak{H}(a)))$ coincide with the sequential maxima of the quotient $\pm b/a$. Notice that here, as well as in (1.8) and (1.9), we can replace $\mathfrak{H}(a)$ by any of its a-dense subsets (for example, by $\mathfrak{d}$).

We often use (in intermediate calculations) the simplified notation of the type

$$n_\pm(\lambda,(1.8)) = n_\pm(\lambda, T(a,b,\mathfrak{H}(a))),$$
$$\Delta_q^\pm((1.8)) = \Delta_q^\pm(T(a,b,\mathfrak{H}(a))),$$
$$\|(1.8)\|_{q,\infty} = \|T(a,b,\mathfrak{H}(a))\|_{q,\infty}.$$

This notation is sufficiently expressive and convenient.

The following statement, as well as Proposition 1.2, was obtained by Birman [B]; see also an exposition in [BS6].

Proposition 1.3. *Let $a>0$ be a closed form in $\mathfrak{H}$ and b be a real compact form in $\mathfrak{H}(a)$. Then the negative spectrum of the operator $A(\alpha)$, generated by the form* (1.7), *is finite. The following identity is fulfilled*

$$N(A(\alpha)) = n_+(\alpha^{-1}, T(a,b,\mathfrak{H}(a))). \tag{1.10}$$

1.5. One has sometimes to consider variational quotients of the type (1.8) independently of the original space $\mathfrak{H}$. It is often convenient to speak about a variational triple $(a,b,\mathcal{H})$, where $\mathcal{H}$ is a complete Hilbert space, $a>0$ is a form defining a metric in $\mathcal{H}$ which is equivalent to the original one and b is a compact form in $\mathcal{H}$. The selfadjoint operator corresponding to the quotient $b[u]/a[u]$, $u\in\mathcal{H}$ is denoted by $T = T(a,b,\mathcal{H})$. Sometimes we shall write $\Delta_q^{\pm}(a,b,\mathcal{H})$ instead of $\Delta_q^{\pm}(T(a,b,\mathcal{H}))$ etc. In particular, the notion of variational triple is useful when one perturbs the forms a or b, or replaces $\mathcal{H}$ by a wider or a narrower Hilbert space. Here we give two statements which allow one to compare the spectral characteristics of the corresponding operators. The first of these propositions is a reformulation of Lemma 1.2 from [BS2] (see also [BS3, Lemma 1.15]), the second one is a reformulation of Lemma 1.16 from [BS3].

Proposition 1.4. *Let us consider a pair of variational triples* $(a,b,\mathcal{H})$, $(a_1,b_1,\mathcal{H}_1)$ *and let their respective operators* T, T_1 *be compact. Assume that there exists an operator* $\Gamma \in \mathcal{B}(\mathcal{H},\mathcal{H}_1)$, *such that*

$$b[u] = b_1[\Gamma u], \quad u \in \mathcal{H}, \tag{1.11}$$

and for some $t > 0$

$$a[u] \geq t a_1[\Gamma u]. \tag{1.12}$$

Then for any $\lambda > 0$

$$n_{\pm}(\lambda, T) \leq n_{\pm}(\lambda t, T_1). \tag{1.13}$$

Proposition 1.5. *Let* $(a,b,\mathcal{H})$ *be a variational triple. Let* a_1 *be a form on* $\mathcal{H}$, *such that* $a_1[u] > 0$ *for* $u \in \mathcal{H} \setminus \{0\}$. *Assume that the form* $a - a_1$ *is compact in* $\mathcal{H}$. *Then the forms* a *and* a_1 *define equivalent metrics on* $\mathcal{H}$. *Besides*

$$\Delta_q^{\pm}(a,b,\mathcal{H}) = \Delta_q^{\pm}(a_1,b,\mathcal{H}) \quad \text{and} \quad \delta_q^{\pm}(a,b,\mathcal{H}) = \delta_q^{\pm}(a_1,b,\mathcal{H}).$$

1.6. We need the Orlicz space $L \log L$ when estimating the function $\mathcal{N}(\alpha)$ for the operator (0.1) with $2l = d$. We say that a measurable function f defined on a set $E \subset \mathbf{R}^d$ of finite measure, belongs to the class $L \log L(E)$ if

$$\int_E |f|(1 + \log_+ |f|)\, dx < \infty.$$

The class $L \log L(E)$ is a linear normed space. The most convenient way for us to introduce a norm on $L \log L$ is as follows. Let us consider the Orlicz function

$$\mathcal{B}(s) = (1+|s|)\log(1+|s|) - |s|$$

and also the function dual to $\mathcal{B}(s)$,

$$\mathcal{A}(s)=e^{|s|}-1-|s|.$$

We define the "average Orlicz norm"

$$\|f\|^{(\mathrm{av})}_{L\log L(E)}=\sup_g\left\{\left|\int_E fg\,dx\right|:\int_E \mathcal{A}(g(x))\,dx\le \operatorname{meas} E\right\}. \tag{1.14}$$

The norm (1.14) is more convenient for our purposes than the standard Orlicz norm (see [KR]) since it has a useful homogeneity property with respect to the dilations of $\mathbf{R}^d$. Namely, if ξ is a dilation, then for an arbitrary set $E\subset\mathbf{R}^d$ of a finite measure, we have

$$(\operatorname{meas} E)^{-1}\|f\circ\xi\|^{(\mathrm{av})}_{L\log L(E)}=(\operatorname{meas}\xi(E))^{-1}\|f\|^{(\mathrm{av})}_{L\log L(\xi(E))}. \tag{1.15}$$

The norm $\|\cdot\|^{(\mathrm{av})}$ was introduced in [S2].

2. Function classes. The Hardy inequality

Here we have collected relatively simple facts about the function classes used in the main part of the paper. *It is always assumed in what follows that*

$$2l\ge d.$$

2.1. Let us temporarily remove the assumption of d being even. This will allow us to successively compare results for even and odd d. We introduce the quadratic forms

$$\mathcal{J}_l[u]=\int|\nabla^l u|^2\,dx \tag{2.1}$$

and

$$\mathcal{J}_{l,\gamma}[u]=\int(|\nabla^l u|^2+\gamma|x|^{-2l}|u|^2)\,dx,\quad \gamma\ge 0, \tag{2.2}$$

defining them for all those $u\in H^l_{\mathrm{loc}}(\mathbf{R}^d)$ whose corresponding integrals are finite. The form (2.2) coincides with (2.1) for $\gamma=0$. For $\gamma>0$ the finiteness of the integral in (2.2) implies

$$D^\alpha u(0)=0\quad\text{for } |\alpha|<l-\tfrac12 d. \tag{2.3}$$

The number of independent conditions in (2.3) is

$$\#\{\alpha\in\mathbf{Z}_+^d:|\alpha|<l-\tfrac{1}{2}d\}=\binom{l+\left[\frac{1}{2}(d-1)\right]}{d}. \tag{2.4}$$

We denote by $\mathcal{H}_1^l=\mathcal{H}_1^l(\mathring{\mathbf{R}}^d)$ the complete Hilbert space

$$\mathcal{H}_1^l(\mathring{\mathbf{R}}^d)=\{u\in H^l_{\mathrm{loc}}(\mathbf{R}^d):\mathcal{J}_{l,1}[u]<\infty\} \tag{2.5}$$

supplied with the metric form $\mathcal{J}_{l,1}$. The forms $\mathcal{J}_{l,\gamma}$ with different $\gamma>0$ define equivalent metrics on $\mathcal{H}_1^l(\mathring{\mathbf{R}}^d)$. It is essential that *the class* $C_0^\infty(\mathring{\mathbf{R}}^d)$ *is dense in* $\mathcal{H}_1^l(\mathring{\mathbf{R}}^d)$.

If d is odd then the Hardy inequality holds true

$$\int\frac{|u|^2}{|x|^{2l}}\,dx\le C_{2.6}\mathcal{J}_l[u],\quad u\in\mathcal{H}_1^l(\mathring{\mathbf{R}}^d),\ d\text{ odd}. \tag{2.6}$$

(The sharp value of the constant in (2.6) is well known, but we do not need it.) The inequality (2.6) implies that *the form* $\mathcal{J}_l$ *given by* (2.1) *defines an equivalent metric on* $\mathcal{H}_1^l$ *for odd* d*'s.*

2.2. The estimate (2.6) fails for an even d. Moreover, any weighted L_2-estimate by $\mathcal{J}_l$ is impossible on $\mathcal{H}_1^l$ for a non-trivial weight. To obtain such an estimate, one has to restrict the function class. In order to do this we must first discuss a convenient representation for $\mathcal{J}_l$ in spherical coordinates. This was found by Yu. Egorov and V. Kondrat'ev [EK].

Let (r,ω) be the spherical coordinates in $\mathbf{R}^d$ and $\mathcal{P}(d,k)$, $k\in\mathbf{Z}_+$ be the space of all spherical $(d-1)$-dimensional harmonics of the order k. As is well known (see, e.g., [SW, Section IV.2]),

$$\mu(d,k):=\dim\mathcal{P}(d,k)=\binom{d+k-1}{k}-\binom{d+k-3}{k-2}. \tag{2.7}$$

Let $\Phi_{k\nu}(\omega)$, $1\le\nu\le\mu(d,k)$ be an orthonormal basis in $\mathcal{P}(d,k)$ with respect to the scalar product in $L_2(\mathbf{S}^{d-1})$. (The choice of a basis in $\mathcal{P}(d,k)$ is arbitrary and a coordinate-free presentation would also be possible.) With any function $u\in H^l_{\mathrm{loc}}(\mathbf{R}^d)$ we associate its Fourier series with respect to the system $\{\Phi_{k\nu}(\omega)\}$, $k\in\mathbf{Z}_+$, $1\le\nu\le\mu(d,k)$:

$$u(x)=u(r,\omega)=\sum_k\sum_\nu F_{k\nu}(r)\Phi_{k\nu}(\omega),$$

where

$$F_{k\nu}(r)=\int_{\mathbf{S}^{d-1}}u(r,\omega)\overline{\Phi_{k\nu}(\omega)}\,d\omega. \tag{2.8}$$

Let

$$r=e^t,\quad z_{k\nu}(t)=r^{(d/2)-l}F_{k\nu}(r). \tag{2.9}$$

Proposition 2.1. ([EK, Lemma 2]) *There is the following equality*

$$\mathcal{J}_l[u]=\sum_{k,\nu}\int_{\mathbf{R}}(L_k(D_t)z_{k\nu}(t))\,\overline{z_{k\nu}(t)}\,dt,\quad u\in C_0^\infty(\mathring{\mathbf{R}}^d), \tag{2.10}$$

where the functions $z_{k\nu}$ are defined in (2.8), (2.9) *and*

$$L_k(\tau)=\prod_{j=0}^{l-1}\left(\tau^2+\left(k-l+\tfrac{1}{2}d+2j\right)^2\right)=:\sum_{i=0}^{l}\varrho_{ki}\tau^{2i},\quad k\in\mathbf{Z}_+. \tag{2.11}$$

For the numbers ϱ_{ki} defined by (2.11) we have

$$\varrho_{k0}\geq 0,\quad \varrho_{ki}>0\quad \text{for } 1\leq i\leq l. \tag{2.12}$$

The equality (2.10) can obviously be written in a more convenient way

$$\mathcal{J}_l[u]=\sum_{k\in\mathbf{Z}_+}\sum_{\nu}\sum_{i=0}^{l}\varrho_{ki}\int_{\mathbf{R}}|z_{k\nu}^{(i)}(t)|^2\,dt,\quad u\in C_0^\infty(\mathring{\mathbf{R}}^d). \tag{2.13}$$

It is not difficult to see from the representation (2.13) that the equality of the coefficient ϱ_{k0} to zero (see (2.12)) for at least one of the values of $k\in\mathbf{Z}_+$, makes the Hardy inequality incorrect. Let us define

$$\Xi=\Xi(d,l)=\{k\in\mathbf{Z}_+:\varrho_{k0}=L_k(0)=0\}. \tag{2.14}$$

For an odd value d we always have $\Xi=\emptyset$. On the contrary

$$\Xi=\left\{1,3,\dots,l-\tfrac{1}{2}d\right\},\quad d \text{ even},\ l-\tfrac{1}{2}d \text{ odd},$$
$$\Xi=\left\{0,2,\dots,l-\tfrac{1}{2}d\right\},\quad d \text{ even},\ l-\tfrac{1}{2}d \text{ even}.$$

From the definition of Ξ and (2.7) we obtain

$$\mathfrak{N}(d,l):=\sum_{k\in\Xi}\mu(d,k)=\binom{l+\frac{1}{2}d-1}{d-1}. \tag{2.15}$$

2.3. We are now ready to describe the "right" version of the Hardy inequality (2.6) for even d found by M. Solomyak [S1]. *From now on, unless otherwise specified, we assume that d is even.*

For $u\in H^l_{\mathrm{loc}}(\mathbf{R}^d)$ we define

$$u_\Xi(r,\omega)=\sum_{k\in\Xi(d,l),\nu} F_{k\nu}(r)\Phi_{k\nu}(\omega) \tag{2.16}$$

and consider the integral

$$I_l[u]=\int\left(\frac{|u-u_\Xi|^2}{|x|^{2l}}+\frac{|u_\Xi|^2}{|x|^{2l}\log^2|x|}\right)dx. \tag{2.17}$$

If the integral in (2.17) is convergent, then (2.3) is certainly fulfilled and the equality

$$u_\Xi(r,\omega)|_{r=1}=0 \tag{2.18}$$

must also hold. The equality (2.18) is equivalent to the system of conditions

$$F_{k\nu}(1)=0,\quad k\in\Xi(d,l),\ 1\le\nu\le\mu(d,k) \tag{2.19}$$

which can also be rewritten as

$$z_{k\nu}(0)=0,\quad k\in\Xi(d,l),\ 1\le\nu\le\mu(d,k). \tag{2.20}$$

Put

$$\widetilde{\mathcal{J}}_{l,\gamma}[u]=\mathcal{J}_l[u]+\gamma I_l[u],\quad \gamma\ge 0$$

and introduce the function space

$$\mathcal{H}^l=\mathcal{H}^l(\mathring{\mathbf{R}}^d)=\{u\in H^l_{\mathrm{loc}}(\mathbf{R}^d):\widetilde{\mathcal{J}}_{l,1}[u]<\infty\}.$$

The conditions (2.3) *and* (2.18) *are fulfilled on $\mathcal{H}^l$ automatically.* The forms $\widetilde{\mathcal{J}}_{l,\gamma}$ with different $\gamma>0$ define equivalent metrics on $\mathcal{H}^l$. It is important that this is true even for $\gamma=0$. This is implied by the following statement:

Proposition 2.2. ([S1]) *Let d be even and $2l\ge d$. Then*

$$I_l[u]\le C_{2.21}\mathcal{J}_l[u],\quad u\in\mathcal{H}^l(\mathring{\mathbf{R}}^d), \tag{2.21}$$

with $C_{2.21}=C(d,l)$. The class

$$\{u\in C_0^\infty(\mathring{\mathbf{R}}^d):(2.18)\ \textit{is fulfilled}\} \tag{2.22}$$

is dense in $\mathcal{H}^l(\mathring{\mathbf{R}}^d)$.

2.4. *In what follows we consider $\mathcal{H}^l$ as a Hilbert space with respect to the metric form $\mathcal{J}_l$.* This space is obviously complete. It is also clear that the definition of $\mathcal{H}^l$ is equivalent to the following direct description:

$$\mathcal{H}^l(\mathring{\mathbf{R}}^d) = \{u \in H^l_{\mathrm{loc}}(\mathbf{R}^d) : \mathcal{J}_l[u] < \infty \text{ and (2.3) and (2.18) are fulfilled}\}.$$

Let us also notice that *the equality* (2.13) *can be extended by continuity to all* $u \in \mathcal{H}^l(\mathring{\mathbf{R}}^d)$.

We now introduce the following subspaces in $\mathcal{H}^l(\mathring{\mathbf{R}}^d)$:

(2.23) $$\mathcal{W} := \{w \in \mathcal{H}^l : w_\Xi = w\},$$

(2.24) $$\mathcal{Y} := \{y \in \mathcal{H}^l : y_\Xi = 0\}.$$

It follows directly from (2.13) that (with respect to the metric form $\mathcal{J}_l$)

(2.25) $$\mathcal{H}^l(\mathring{\mathbf{R}}^d) = \mathcal{W} \oplus \mathcal{Y}.$$

In particular,

(2.26) $$\mathcal{J}_l(u) = \mathcal{J}_l(y) + \mathcal{J}_l(w), \quad u \in \mathcal{H}^l, \ w = u_\Xi, \ y = u - w.$$

From (2.16) and (2.8) we see that the class $C_0^\infty(\mathring{\mathbf{R}}^d)$ (which is dense in $\mathcal{H}^l(\mathring{\mathbf{R}}^d)$) is invariant with respect to projections onto $\mathcal{W}$ and onto $\mathcal{Y}$.

2.5. If $2l = d$ we obviously have $\Xi = \{0\}$ and $\mathfrak{N}(d, l) = \mu(d, 0) = 1$. The function u_Ξ depends only on r:

$$u_\Xi(r) = \frac{1}{\operatorname{meas} \mathbf{S}^{d-1}} \int_{\mathbf{S}^{d-1}} u(r, \omega)\, d\omega.$$

For $y = u - u_\Xi$ we have

$$\int_{\mathbf{S}^{d-1}} y(r, \omega)\, d\omega = 0 \quad \text{for any } r > 0.$$

These facts give us some simplifications in the case $2l = d$. The corresponding considerations are similar to [BL] where $2l = d = 2$. When $2l > d$ the technical differences are more substantial.

2.6. We also need the following subspace of the Hilbert space $H^l(\mathbf{R}^d)$:

$$\widetilde{H}^l = \widetilde{H}^l(\mathbf{R}^d) = \{u \in H^l(\mathbf{R}^d) : \text{(2.3) and (2.18) are fulfilled}\}.$$

In view of (2.4) and (2.15)

(2.27) $$\dim(H^l(\mathbf{R}^d)/\widetilde{H}^l(\mathbf{R}^d)) = \binom{l + \frac{1}{2}d}{d}.$$

Notice also that the class (2.22) is dense in $\widetilde{H}^l(\mathbf{R}^d)$.

3. Statement of the problem. Reduction to a spectral problem for a compact operator

3.1. We begin with a precise definition of the operator $A(\alpha)$ given by (0.1). The basic Hilbert space is $L_2(\mathbf{R}^d)$. The quadratic form $\mathcal{J}_l$ considered on the domain $\mathfrak{d}[\mathcal{J}_l]=H^l(\mathbf{R}^d)$ is positive and closed in $L_2(\mathbf{R}^d)$. The selfadjoint operator in $L_2(\mathbf{R}^d)$ corresponding to this form coincides with the operator $(-\Delta)^l$ defined on $H^{2l}(\mathbf{R}^d)$. Let V be a real measurable function and

$$b_V[u] := \int V|u|^2\,dx. \tag{3.1}$$

Below we shall impose some conditions (see Subsection 3.4) which, in particular, provide the compactness of the form b_V in $H^l(\mathbf{R}^d)$. Then by Proposition 1.2 the form

$$a(\alpha)=\mathcal{J}_l-\alpha b_V, \quad \alpha>0,\ \mathfrak{d}[a(\alpha)]=H^l(\mathbf{R}^d), \tag{3.2}$$

is semibounded from below and closed in $L_2(\mathbf{R}^d)$. The selfadjoint operator $A(\alpha)$ in $L_2(\mathbf{R}^d)$ generated by this form can be naturally taken as a realization of the operator $(-\Delta)^l-\alpha V$ given by (0.1). It follows from the same Proposition 1.2 that the negative spectrum of $A(\alpha)$ is discrete for all $\alpha>0$. If it is finite, then we shall denote the number of its negative eigenvalues by $\mathcal{N}(\alpha)$.

Let us now consider the form $\mathcal{J}_l-\alpha b_V$ on the "poorer" domain:

$$\tilde{a}(\alpha)=\mathcal{J}_l-\alpha b_V, \quad \alpha>0,\ \mathfrak{d}[\tilde{a}(\alpha)]=\widetilde{H}^l(\mathbf{R}^d). \tag{3.3}$$

The operator, which is selfadjoint in $L_2(\mathbf{R}^d)$ and corresponds to the form (3.3) is denoted by $\tilde{A}(\alpha)$ and the number of its negative eigenvalues by $N(\alpha)$. From (2.27) and the variational principle we obtain the inequalities

$$N(\alpha)\le\mathcal{N}(\alpha)\le N(\alpha)+\binom{l+\frac{1}{2}d}{d}. \tag{3.4}$$

3.2. Let

$$T(V)=T(\mathcal{J}_l, b_V, \mathcal{H}^l) \tag{3.5}$$

now be the operator generated by the form (3.1) in the space $\mathcal{H}^l=\mathcal{H}^l(\mathring{\mathbf{R}}^d)$ (the last space is defined in Subsection 2.3). In other words, the operator (3.5) corresponds to the quotient of quadratic forms

$$b_V[u]/\mathcal{J}_l[u], \quad u\in\mathcal{H}^l(\mathring{\mathbf{R}}^d), \tag{3.6}$$

in the same sense as it was clarified in Subsection 1.4. If $T(V)$ is compact, then by (1.10)

$$N(\alpha)=n_+(\alpha^{-1},T(V))=n_+(\alpha^{-1},(3.6)). \tag{3.7}$$

In particular, if $T(V)\in\mathcal{C}(\mathcal{H}^l)$, then $N(\alpha)$ (and consequently $\mathcal{N}(\alpha)$) is finite for all $\alpha>0$. Futhermore, it follows from (3.4) and (3.7) that

$$n_+(\alpha^{-1},T(V))\leq\mathcal{N}(\alpha)\leq n_+(\alpha^{-1},T(V))+\binom{l+\frac{1}{2}d}{d}. \tag{3.8}$$

By using the two-sided estimate (3.8) *we reduce the study of the quantity* $\mathcal{N}(\alpha)=\operatorname{rank}E_{A(\alpha)}(\mathbf{R}_-)$ *to the study of spectral characteristics of the* (*compact*) *operator* (3.5), *or, equivalently, of the variational quotient* (3.6). Although we have only the function n_+ involved in (3.7) and (3.8), we shall for the sake of symmetry, study the behaviour of both functions $n_\pm(\,\cdot\,,(3.6))$.

3.3. For technical reasons we shall also need to consider the operator

$$T_\gamma(V):=T(\mathcal{J}_{l,\gamma},b_V,\mathcal{H}_1^l(\mathring{\mathbf{R}}^d)),\quad \gamma>0 \tag{3.9}$$

together with the operator $T(V)$. The operator (3.9) corresponds to the quotient of the quadratic forms

$$\frac{\int V|u|^2\,dx}{\int(|\nabla^l u|^2+\gamma|x|^{-2l}|u|^2)\,dx},\quad \gamma>0,\ u\in\mathcal{H}_1^l(\mathring{\mathbf{R}}^d), \tag{3.10$_\gamma$}$$

where the space $\mathcal{H}_1^l$ was introduced in Subsection 2.1. We use the estimates of the quasinorm $\|T_\gamma(V)\|_{\varkappa,\infty}$ for the semiclassical value

$$\varkappa=d/2l \tag{3.11}$$

by a quasinorm V in a suitable separable class. These estimates were obtained in [BS5] for $2l>d$ and in [S3] for $2l=d$. We give here not only these estimates, but also their proofs since they will be significantly used in the future.

Let us introduce some notation. We consider in $\mathbf{R}^d$ the domains

$$\Omega_0=B_1,\quad \Omega_j=B_{e^j}\setminus\operatorname{clos}B_{e^{j-1}}\quad \text{for } j\in\mathbf{N}. \tag{3.12}$$

With a measurable function V we associate the sequence

$$\boldsymbol{\eta}(V)=\boldsymbol{\eta}(V,d,l):=\{\eta_j(V,d,l)\}_{j\geq 0},$$

where

$$\eta_0(V,d,l)=\int_{\Omega_0}|V|\,dx, \tag{3.13}$$

$$\eta_j(V,d,l)=\int_{\Omega_j}|x|^{2l-d}|V|\,dx, \quad j\in\mathbf{N}, \text{ for } 2l>d, \tag{3.14}$$

and

$$\eta_j(V,d,d/2)=\|V\|^{(\mathrm{av})}_{L\log L(\Omega_j)}, \quad j\in\mathbf{Z}_+. \tag{3.15}$$

Recall that the Orlicz norm $\|\cdot\|^{(\mathrm{av})}_{L\log L(E)}$ is defined in (1.14).

Let us now introduce the function class $G(d,l)$

$$G(d,l)=\{V: V \text{ is measurable on } \mathbf{R}^d;\ \boldsymbol{\eta}(V,d,l)\in l_\varkappa\}. \tag{3.16}$$

Here (as always) $\varkappa$ is the exponential given in (3.11). The class $G(d,l)$ is a complete linear quasinormed separable space with respect to the natural quasinorm

$$\|V\|_{G(d,l)}=\|\boldsymbol{\eta}(V,d,l)\|_\varkappa.$$

Proposition 3.1. *Let $2l\ge d$, $V\in G(d,l)$ and $\gamma>0$. Then $T_\gamma(V)\in\mathcal{C}_{\varkappa,\infty}(\mathcal{H}^l_1)$ and*

$$\|T_\gamma(V)\|_{\varkappa,\infty}\le C_{3.17}\|V\|_{G(d,l)}, \quad C_{3.17}=C(d,l,\gamma), \tag{3.17}$$

or, equivalently,

$$n(\lambda,T_\gamma(V))\le C^\varkappa_{3.17}\lambda^{-\varkappa}\sum_{j\ge0}\eta^\varkappa_j(V,d,l). \tag{3.18}$$

Proof. We can take $\gamma=1$. For every domain Ω_j defined in (3.12) we consider the quotient of the quadratic forms

$$\frac{\int_{\Omega_j}V|u|^2\,dx}{\int_{\Omega_j}(|\nabla^l u|^2+e^{-2lj}|u|^2)\,dx}, \quad j\in\mathbf{Z}_+. \tag{3.19$_j$}$$

It follows from the variational principle that

$$n(\lambda,T_1(V))=n(\lambda,(3.10)_1)\le\sum_{j\ge0}n(\lambda,(3.19)_j).$$

Therefore it is sufficient to obtain the estimate

$$n(\lambda,(3.19)_j)\le C\lambda^{-\varkappa}\eta^\varkappa_j(V,d,l), \quad j\in\mathbf{Z}_+ \tag{3.20$_j$}$$

with a constant independent of j. The necessary estimates can easily be obtained from the well-known estimates for a fixed domain using scaling. Namely, for $2l>d$ according to Theorem 4.1(2) from [BS3]

$$n(\lambda,(3.19)_j)\leq C\lambda^{-\varkappa}\left(\int_{\Omega_j}|V|\,dx\right)^{\varkappa}\leq C\lambda^{-\varkappa}\eta_j^{\varkappa}(V,d,l),\quad j=0,1.$$

The estimate $(3.20)_j$ for $j>1$ is reduced to $(3.20)_1$ after the substitition $x\mapsto e^jx$; this does not change the value of the constant C. For $2l=d$ and $j=0,1$ the estimate $(3.20)_j$ is contained in Corollary 2.2 from [S3]. The change of variables $x\mapsto e^jx$ again reduces $(3.20)_j$ for $j>1$ to the case $j=1$. By (1.15) the value of the constant is not changed (cf. the proof of Corollary 2.4 in [S3]). Thus $(3.20)_j$ and therefore (3.18) is established.

Remark 3.2. For $2l=d$ we could have avoided the Orlicz norms by replacing $\boldsymbol{\eta}\left(V,d,\frac{1}{2}d\right)$ by the sequence $\tilde{\boldsymbol{\eta}}(V,d,\sigma)$ with some $\sigma>1$ where

$$\tilde{\eta}_0(V,d,\sigma)=\left(\int_{\Omega_0}|V|^\sigma\,dx\right)^{1/\sigma},$$

$$\tilde{\eta}_j(V,d,\sigma)=\left(\int_{\Omega_j}|x|^{d(\sigma-1)}|V|^\sigma\,dx\right)^{1/\sigma},\quad j\in\mathbf{N}.$$

In this case ($2l=d$) (3.17) can be replaced by the more rough estimate

$$\|T_\gamma(V)\|_{1,\infty}\leq C_{3.21}\|\tilde{\boldsymbol{\eta}}(V,d,\sigma)\|_1,\quad 2l=d,\ C_{3.21}=C(d,\sigma,\gamma). \tag{3.21}$$

The inequality (3.21), of course, follows from (3.17) when $2l=d$. This estimate, however, can easily be derived directly by using the same scaling as in [BL] (where $d=2$).

3.4. The estimate (3.17) gives the Weyl-type spectral asymptotic formula for the operator (3.9). The following statement is true.

Proposition 3.3. *Let $V\in G(d,l)$ and $T_\gamma(V)$, $\gamma>0$ be the operator* (3.9). *Then*

$$\Delta_\varkappa^\pm(T_\gamma(V))=\delta_\varkappa^\pm(T_\gamma(V))=c_d\int V_\pm^\varkappa\,dx, \tag{3.22}$$

where c_d is the constant (0.4).

Proof. By using Proposition 1.1 together with the estimate (3.17) it is enough to verify (3.22) for a set dense in $G(d,l)$. Thus we can assume that

$$\operatorname{supp}V\subset B_R\backslash\operatorname{clos}B_\varrho=:\Omega(\varrho,R),\quad 0<\varrho<R.$$

Let us consider the quotient of the type $(3.10)_\gamma$ with the integration over the domain $\Omega=\Omega(\varrho, R)$, on the class $H_0^l(\Omega)$ (problem $\mathcal{D}(\Omega)$) and on the class $H^l(\Omega)$ (problem $\mathcal{E}(\Omega)$). The standard variational arguments give

$$\delta_\varkappa^\pm(\mathcal{D}(\Omega)) \le \delta_\varkappa^\pm((3.10)_\gamma) \le \Delta_\varkappa^\pm((3.10)_\gamma) \le \Delta_\varkappa^\pm(\mathcal{E}(\Omega)). \tag{3.23}$$

On the other hand, the asymptotic formulae

$$\delta_\varkappa^\pm(\mathcal{D}(\Omega)) = \Delta_\varkappa^\pm(\mathcal{D}(\Omega)) = \delta_\varkappa^\pm(\mathcal{E}(\Omega)) = \Delta_\varkappa^\pm(\mathcal{E}(\Omega)) = c_d \int_\Omega V_\pm^\varkappa \, dx \tag{3.24}$$

are known. For $2l>d$, (3.24) is contained in [BS1] (see also [BS3, Theorem 4.6]). For $2l=d$ one can find (3.24) in [S3, Theorem 2.2] and the remark to this theorem. From (3.23) and (3.24) follows (3.22). □

Now let *d be odd.* Then $\Xi(d,l)=\emptyset$ and the classes $\mathcal{H}^l(\mathring{\mathbf{R}}^d)$ and $\mathcal{H}_1^l(\mathring{\mathbf{R}}^d)$ coincide. From the Hardy inequality (2.6) and Proposition 3.1 we get

$$\|T(V)\|_{\varkappa,\infty} \le C\|V\|_{G(d,l)}, \quad d \text{ odd}, \ 2l>d. \tag{3.25}$$

Using (3.25) we can obtain in turn the Weyl-type asymptotics

$$\Delta_\varkappa^\pm(T(V)) = \delta_\varkappa^\pm(T(V)) = c_d \int V_\pm^\varkappa \, dx, \quad d \text{ odd}, \ 2l>d.$$

(The argument is the same as in the proof of Proposition 3.3.)

If d is even, then the condition $V\in G(d,l)$ neither implies the inclusions $T(V)\in \mathcal{C}_{\varkappa,\infty}(\mathcal{H}^l)$ nor that $T(V)\in\mathcal{B}(\mathcal{H}^l)$. The analysis of these difficulties is the main content of this paper.

For a compactly supported V, the estimate of the type (3.25) remains valid for an even d, but the constant in the estimate depends on the size of the support. Namely, the following statement is true.

Proposition 3.4. *Let d be even, $2l\ge d$, and $V\in G(d,l)$,* $\operatorname{supp} V\subset B_R$, *$R\ge 1$. Then $T(V)\in\mathcal{C}_{\varkappa,\infty}$ and*

$$\|T(V)\|_{\varkappa,\infty} \le C_{3.26}(R)\|V\|_{G(d,l)}. \tag{3.26}$$

Proof. Using (2.21) the quotient (3.6) can be estimated from above (with a constant) by the quotient

$$\frac{b_V[u]}{(\mathcal{J}_l[u]+I_l[u])}, \quad u\in\mathcal{H}^l(\mathring{\mathbf{R}}^d). \tag{3.27}$$

The weights in the integral I_l (see (2.17)) are separated from zero in the ball B_R. Therefore (3.27) can be estimated from above (with a constant) by the quotient

$$\frac{\int_{B_R} V|u|^2\,dx}{\int_{B_R}(|\nabla^l u|^2+|u|^2)\,dx}, \quad u\in H^l(B_R). \tag{3.28}$$

At the same time we have withdrawn the conditions (2.3) and (2.18) which can only increase the value of $n_\pm$. The given variational arguments reduce the problem to one of estimating the quantity $\|(3.28)\|_{\varkappa,\infty}$. The latter is estimated by the norm V in $L(B_R)$ when $2l>d$ and by the norm V in $L\log L(B_R)$ when $2l=d$ (see Theorem 4.1(2) from [BS3] and Theorem 2.1 from [S3] respectively). The mentioned norms for V are estimated by the norm V in $G(d,l)$. Thus (3.26) is established. □

4. The main theorem

The main result of this paper, Theorem 4.2, is formulated and proved in this section. *In everything that follows we assume that the condition*

$$V\in G(d,l) \tag{4.1}$$

is fulfilled. The class $G(d,l)$ is defined in (3.13)–(3.16).

4.1. In the space $\mathcal{H}^l(\mathring{\mathbf{R}}^d)$ we single out a subspace

$$\mathcal{H}^l_e(R):=\{u\in\mathcal{H}^l: u(x)=0 \text{ for } |x|<R\}, \quad R\geq 1. \tag{4.2}$$

Define

$$B(R)=\mathbf{R}^d\backslash \operatorname{clos} B_R=\{x\in\mathbf{R}^d:|x|>R\}.$$

The subspace (4.2) is naturally identified with the Hilbert space

$$\mathcal{H}^l(R):=\left\{u\in H^l_{\mathrm{loc}}(\overline{B(R)}): \int_{B(R)}|\nabla^l u|^2\,dx<\infty,\ D^\beta u|_{|x|=R}=0 \text{ for } |\beta|<l\right\}, \tag{4.3}$$

whose metric form is $\int_{B(R)}|\nabla^l u|^2\,dx$. It is clear that the class $C_0^\infty(B(R))$ is dense in $\mathcal{H}^l(R)$.

It can be seen from (2.16) and (2.8) that the projections in $\mathcal{H}^l$ onto the subspaces $\mathcal{W}$, $\mathcal{Y}$ (defined in (2.23), (2.24)) and on the subspace $\mathcal{H}^l_e(R)$, commute. Let us introduce the subspace

$$\mathcal{W}(R):=\mathcal{W}\cap\mathcal{H}^l_e(R) \tag{4.4}$$

and consider the restriction of the quotient (3.6) to $\mathcal{W}(R)$

$$(4.5)_R \qquad \frac{\int_{B(R)} V|w|^2\,dx}{\int_{B(R)} |\nabla^l w|^2\,dx}, \quad w\in\mathcal{W}(R).$$

We shall see that the behaviour of the quotient (3.6) on the relatively weak subspace $\mathcal{W}(R)$ does not allow us to extend the asymptotics (3.22) to the case $\gamma=0$ if we only have the condition (4.1).

It is clear from (2.16) that $w=w_\Xi$ can be identified with the vector-valued function $\mathbf{F}=\{F_{k\nu}\}$, $k\in\Xi(d,l)$, $\nu=1,\dots,\mu(d,k)$. The vector dimension of $\mathbf{F}$ is $\mathfrak{N}(d,l)$ (recall that μ, Ξ and $\mathfrak{N}$ are defined in (2.7), (2.14) and (2.15)). The quotient $(4.5)_R$ can be written in terms of $\mathbf{F}$. Then we obtain a (vector) problem on the semiaxis (R,∞) for the quotient of two forms where the first is of order zero. The second form is of differential order l and strictly positive. The vector function $\mathbf{F}$ satisfies the Dirichlet boundary condition at the point $r=R$: $d^m\mathbf{F}/dr^m|_{r=R}=0$, $m=0,\dots,l-1$. The following statement holds.

Proposition 4.1. *Let us assume that for some $R\geq 1$*

$$(4.6) \qquad \Delta_q((4.5)_R)<\infty, \quad q>1/2l, \ R\geq 1.$$

Then (4.6) is fulfilled for any $R\geq 1$ and the functionals

$$(4.7) \qquad \Delta_q^{\pm}((4.5)_R), \ \delta_q^{\pm}((4.5)_R), \quad q>1/2l, \ R\geq 1$$

are independent of R.

Proof. Let $R_1>R$. For the quotient $(4.5)_R$ we set the additional Dirichlet boundary condition at the point R_1. This can only decrease the value of the function $n_\pm(\lambda,(4.5)_R)$ by the number $l\mathfrak{N}(l,d)$. The new problem is decomposed into the orthogonal sum of the problems on intervals (R,R_1) and (R_1,∞). The contribution to $n_\pm(\lambda)$ from the finite interval has the standard order $O(\lambda^{-1/2l})$. Therefore the functionals (4.7) for R and R_1 coincide. □

4.2. Let us agree to denote the value of the functionals (4.6) and (4.7), common for all $R\geq 1$, respectively by

$$(4.8) \qquad \hat\Delta_q(V) \text{ and } \hat\Delta_q^{\pm}(V), \ \hat\delta_q^{\pm}(V), \quad q>1/2l$$

and formulate the main theorem.

Theorem 4.2. *Let d be even, $2l\geq d$ and* (4.1) *be fulfilled. Then*
(a) *if $\hat{\Delta}_\varkappa(|V|)<\infty$, then $T(V)\in\mathcal{C}_{\varkappa,\infty}$ and*

$$\Delta_\varkappa^\pm(T(V))=c_d\int V_\pm^\varkappa\,dx+\hat{\Delta}_\varkappa^\pm(V), \tag{4.9}$$

$$\delta_\varkappa^\pm(T(V))=c_d\int V_\pm^\varkappa\,dx+\hat{\delta}_\varkappa^\pm(V); \tag{4.10}$$

(b) *if $\hat{\Delta}_q(|V|)<\infty$ for some $q>\varkappa$, then $T(V)\in\mathcal{C}_{q,\infty}$ and*

$$\Delta_q^\pm(T(V))=\hat{\Delta}_q^\pm(V),\quad \delta_q^\pm(T(V))=\hat{\delta}_q^\pm(V). \tag{4.11}$$

Remark 4.3. The result of Theorem 4.2 can be interpreted as follows: the contributions to $n(\lambda,T(V))$ from the subspaces $\mathcal{Y}$ and $\mathcal{W}$ are asymptotically independent as $\lambda\to0$. The restriction of the quotient (3.6) to $\mathcal{W}(R)$ generates an ordinary differential operator (system) on a semiaxis whose spectrum is responsible for the terms $\hat{\Delta}_q^\pm(V)$ and $\hat{\delta}_q^\pm(V)$, $q\geq\varkappa$ in (4.9)–(4.11); see Section 6 for details. In a different situation a similar effect was met in [JMS] when studying the eigenvalue asymptotics of the Neumann–Laplacian of regions with cusps. In the case of [JMS] the auxiliary operator was a one-dimensional Schrödinger operator on a semiaxis with an increasing potential.

Of course, the statements of the theorem are informative only when there exist potentials with $\hat{\Delta}_q^\pm(V)>0$, $q\geq\varkappa$. The detailed discussion of these questions is the content of Section 6, where we give a concrete meaning to the "conditional" statements of Theorem 4.2. The basic material is taken from Section 5, where we give some information about the spectrum of one-dimensional problems of the type $(4.5)_R$.

4.3. Theorem 4.2 is proved in several steps. The first one is very simple. *Let us denote by $T(V,\mathcal{Y})$ and $T(V,\mathcal{W})$ the operators corresponding to the quotient* (3.6) *restricted to $\mathcal{Y}$ and $\mathcal{W}$ respectively.*

Proposition 4.4. *Under the conditions of Theorem* 4.2 *we have*

$$T(V)\in\mathcal{C}_{p,\infty}(\mathcal{H}^l),\quad T(|V|)\in\mathcal{C}_{p,\infty}(\mathcal{H}^l), \tag{4.12}$$

where $p=\varkappa$ and $p=q$ for the statements (a) *and* (b) *of Theorem* 4.2 *respectively. The estimate*

$$\|T(|V|,\mathcal{Y})\|_{\varkappa,\infty}\leq C_{4.13}\|V\|_{G(d,l)} \tag{4.13}$$

holds true.

Proof. Of the two inclusions (4.12) it is enough to prove the second one. Let $u=y+w$ according to the decomposition (2.25). From (2.26) and the inequality

$$b_{|V|}[u]\leq 2(b_{|V|}[y]+b_{|V|}[w])$$

we see that it is sufficient to consider the operators $T(|V|,\mathcal{Y})$ and $T(|V|,\mathcal{W})$. The Hardy inequality (2.21) on $\mathcal{Y}$ takes the form

$$\int |y|^2|x|^{-2l}\,dx\leq C_{2.21}\mathcal{J}_l[y],\quad y\in\mathcal{Y}.$$

This allows us to reduce the estimate for $T(|V|,\mathcal{Y})$ to that of the operator $T_1(|V|)$ (defined according to (3.9)). The necessary estimate coincides with (3.17) where $\gamma=1$ and V is replaced by $|V|$. Therefore $T(|V|,\mathcal{Y})\in\mathcal{C}_{\varkappa,\infty}$ and (4.13) is thus fulfilled. It only remains for us to notice that the inclusion $T(|V|,\mathcal{W})\in\mathcal{C}_{p,\infty}$ is equivalent to the assumption $\hat{\Delta}_p(|V|)<\infty$. □

4.4. Along with $\mathcal{W}(R)$ (see (4.4)) we introduce the space

$$\mathcal{Y}(R):=\mathcal{Y}\cap\mathcal{H}^l_e(R)$$

and remark that

$$\mathcal{H}^l_e(R)=\mathcal{Y}(R)\oplus\mathcal{W}(R). \tag{4.14}$$

We denote by $T(V,\mathcal{Y}(R))$ *and* $T(V,\mathcal{H}^l_e(R))$ *the operators corresponding to the restrictions of the quotient* (3.6) *to* $\mathcal{Y}(R)$ *and* $\mathcal{H}^l_e(R)$ *respectively.* Let us now establish the following result.

Proposition 4.5. *Under the conditions of Theorem* 4.2 *the following relations are fulfilled*

$$\Delta^{\pm}_p(T(V,\mathcal{H}^l_e(R)))=\hat{\Delta}^{\pm}_p(V)+o(1),\quad R\to\infty, \tag{4.15}$$

$$\delta^{\pm}_p(T(V,\mathcal{H}^l_e(R)))=\hat{\delta}^{\pm}_p(V)+o(1),\quad R\to\infty, \tag{4.16}$$

where $p=\varkappa$ *and* $p=q$ *for the statements* (a) *and* (b) *of Theorem* 4.2 *respectively.*

Proof. The lower estimates follow directly from the variational principle

$$\Delta^{\pm}_p(T(V,\mathcal{H}^l_e(R)))\geq\hat{\Delta}^{\pm}_p(V),\quad \delta^{\pm}_p(T(V,\mathcal{H}^l_e(R)))\geq\hat{\delta}^{\pm}_p(V). \tag{4.17}$$

In order to obtain the upper estimates we put $u=y+w$ according to (4.14). For any $\varepsilon\in(0,1)$ we have

$$b_V[u]\leq\int_{B(R)}(V+\varepsilon^{-1}|V|)|y|^2\,dx+\int_{B(R)}(V+\varepsilon|V|)|w|^2\,dx,$$

hence

$$\Delta_p^{\pm}(T(V,\mathcal{H}_e^l(R)))\leq\Delta_p^{\pm}(T(V+\varepsilon^{-1}|V|,\mathcal{Y}(R)))+\hat{\Delta}_p^{\pm}(V+\varepsilon|V|), \tag{4.18}$$

$$\delta_p^{\pm}(T(V,\mathcal{H}_e^l(R)))\leq\Delta_p^{\pm}(T(V+\varepsilon^{-1}|V|,\mathcal{Y}(R)))+\hat{\delta}_p^{\pm}(V+\varepsilon|V|). \tag{4.19}$$

Let χ_R be the indicator function of $B(R)$. From the variational arguments and the estimate (4.13) we obtain

$$\begin{aligned}\Delta_\varkappa^{\pm}(T(V+\varepsilon^{-1}|V|,\mathcal{Y}(R)))&\leq\Delta_\varkappa^{\pm}(T((1+\varepsilon^{-1})\chi_R|V|,\mathcal{Y}))\\&\leq\|T((1+\varepsilon^{-1})\chi_R|V|,\mathcal{Y})\|_{\varkappa,\infty}^{\varkappa}\\&\leq C_{4.13}^{\varkappa}(1+\varepsilon^{-1})^{\varkappa}\|\chi_RV\|_{G(d,l)}^{\varkappa}.\end{aligned}\tag{4.20}$$

The right hand side of (4.20) tends to zero as $R\to\infty$. Besides, (4.20) implies $\Delta_p^{\pm}(T(V+\varepsilon^{-1}|V|,\mathcal{Y}(R)))=0$ when $p>\varkappa$. From (4.18) we now obtain

$$\limsup_{R\to\infty}\Delta_p^{\pm}(T(V,\mathcal{H}_e^l(R)))\leq\hat{\Delta}_p^{\pm}(V+\varepsilon|V|). \tag{4.21}$$

Since $\hat{\Delta}_p(\varepsilon|V|)=\varepsilon^p\hat{\Delta}_p(|V|)\to0$ as $\varepsilon\to0$, using Proposition 1.1 we can pass to the limit in (4.21) as $\varepsilon\to0$. Thus

$$\limsup_{R\to\infty}\Delta_p^{\pm}(T(V,\mathcal{H}_e^l(R)))\leq\hat{\Delta}_p^{\pm}(V). \tag{4.22}$$

By analogy we obtain from (4.19) that

$$\limsup_{R\to\infty}\delta_p^{\pm}(T(V,\mathcal{H}_e^l(R)))\leq\hat{\delta}_p^{\pm}(V). \tag{4.23}$$

By combining (4.17) with (4.22) and (4.23) we derive (4.15) and (4.16). □

4.5. Along with $\mathcal{H}_e^l(R)$ (see (4.2) and also (4.3)) we introduce the class

$$\mathcal{H}_i^l(R)=\{u\in\mathcal{H}^l:u(x)=0\text{ for }|x|>R\},\quad R\geq1. \tag{4.24}$$

Both classes $\mathcal{H}_e^l(R)$ and $\mathcal{H}_i^l(R)$ are subspaces in $\mathcal{H}^l$. They are orthogonal (with respect to the metric of the form $\mathcal{J}_l$) but do not generate the whole $\mathcal{H}^l$. The class (4.24) is naturally identified (algebraically and topologically) with the subspace in $H_0^l(B_R)$

$$\mathcal{H}_i^l(R)=\{u\in H_0^l(B_R):(2.3)\text{ and }(2.18)\text{ are fulfilled}\}. \tag{4.25}$$

The operator generated by the restriction of the quotient (3.6) to $\mathcal{H}_i^l(R)$ is denoted by $T(V,\mathcal{H}_i^l(R))$.

Proposition 4.6. *Let the assumptions of the statements* (a) *or* (b) *of Theorem* 4.2 *be satisfied. Then for an arbitrary* $R \geq 1$ *we respectively obtain,*

in case (a)

$$\Delta^{\pm}_{\varkappa}(T(V)) = c_d \int_{B_R} V_{\pm}^{\varkappa}\, dx + \Delta^{\pm}_{\varkappa}(T(V, \mathcal{H}^l_e(R))), \tag{4.26}$$

$$\delta^{\pm}_{\varkappa}(T(V)) = c_d \int_{B_R} V_{\pm}^{\varkappa}\, dx + \delta^{\pm}_{\varkappa}(T(V, \mathcal{H}^l_e(R))); \tag{4.27}$$

in case (b)

$$\Delta^{\pm}_{q}(T(V)) = \Delta^{\pm}_{q}(T(V, \mathcal{H}^l_e(R))), \quad q > \varkappa, \tag{4.28}$$

$$\delta^{\pm}_{\varkappa}(T(V)) = \delta^{\pm}_{\varkappa}(T(V, \mathcal{H}^l_e(R))), \quad q > \varkappa. \tag{4.29}$$

The proofs of the statements (a) and (b) will be parallel. We assume that $p = \varkappa$ and $p = q$ in the cases (a) and (b) respectively. The inclusions (4.12) are implied by Proposition 4.4.

We begin with the following remark. According to (4.25)

$$\dim H^l_0(B_R)/\mathcal{H}^l_i(R) = \binom{l + \frac{1}{2}d}{d} < \infty.$$

Therefore the spectral asymptotic formulae for $T(V, \mathcal{H}^l_i(R))$ are the same as for the quotient

$$\frac{\int_{B_R} V|u|^2\, dx}{\int_{B_R} |\nabla^l u|^2\, dx}, \quad u \in H^l_0(B_R). \tag{4.30}$$

Under the condition (4.1) the standard Weyl-type formula is valid for (4.30) (see references concerning (3.24)). Thus

$$\Delta^{\pm}_{\varkappa}(T(V, \mathcal{H}^l_i(R))) = \delta^{\pm}_{\varkappa}(T(V, \mathcal{H}^l_i(R))) = c_d \int_{B_R} V_{\pm}^{\varkappa}\, dx. \tag{4.31}$$

Since $\mathcal{H}^l_i(R) \oplus \mathcal{H}^l_e(R) \subset \mathcal{H}^l$ the variational principle leads us to

$$n_{\pm}(\lambda, T(V)) \geq n_{\pm}(\lambda, T(V, \mathcal{H}^l_i(R))) + n_{\pm}(\lambda, T(V, \mathcal{H}^l_e(R))).$$

The latter inequality together with (4.31) immediately imply the relations (4.26)–(4.29) with "=" replaced by "≥". We must now obtain the inequality "≤" which is more complicated. This is done in the next subsection.

4.6. From now on we denote by $D_p^{\pm}$ any of the functionals $\Delta_p^{\pm}$ or $\delta_p^{\pm}$. Let us first assume that for some $\varrho>0$

$$V(x)=0 \quad \text{for } R-\varrho<|x|<R+\varrho. \tag{4.32}$$

Fix a function $\zeta=\bar{\zeta}\in C^\infty(\mathbf{R}^d)$ such that $\zeta(x)=1$ for $\big||x|-R\big|>\varrho$ and $\zeta(x)=0$ near the sphere $|x|=R$. Given $\varepsilon\in(0,1)$ we consider the quadratic forms

$$\mathcal{J}_l^{(\varepsilon)}[u]=\varepsilon\mathcal{J}_l[u]+(1-\varepsilon)\int\zeta^2|\nabla^l u|^2\,dx,$$

and

$$\widehat{\mathcal{J}}_l^{(\varepsilon)}[u]=\varepsilon\mathcal{J}_l[u]+(1-\varepsilon)\mathcal{J}_l[\zeta u].$$

All the three forms $\mathcal{J}_l$, $\mathcal{J}_l^{(\varepsilon)}$ and $\widehat{\mathcal{J}}_l^{(\varepsilon)}$ define equivalent metrics in $\mathcal{H}^l$. Moreover

$$\mathcal{J}_l[u]\geq\mathcal{J}_l^{(\varepsilon)}[u], \quad u\in\mathcal{H}^l. \tag{4.33}$$

The form $\mathcal{J}_l^{(\varepsilon)}-\widehat{\mathcal{J}}_l^{(\varepsilon)}$ is of differential order $\left(l-\frac{1}{2}\right)$ and its coefficients vanish except when $\big||x|-R\big|<\varrho$. Hence this form is compact in $\mathcal{H}^l$ and thus we can apply Proposition 1.5. Together with (4.33) this gives

$$D_p^{\pm}(T(V))\leq D_p^{\pm}(\mathcal{J}_l^{(\varepsilon)},b_V,\mathcal{H}_l)=D_p^{\pm}(\widehat{\mathcal{J}}_l^{(\varepsilon)},b_V,\mathcal{H}_l). \tag{4.34}$$

Let us clarify that we have here used the notation mentioned in Subsection 1.5, namely, an operator is replaced by its respective variational triple. Let us use Proposition 1.4 with $\mathcal{H}=\mathcal{H}^l$, $\mathcal{H}_1=\mathcal{H}_i^l(R)\oplus\mathcal{H}_e^l(R)$; $a=\widehat{\mathcal{J}}_l^{(\varepsilon)}$, $a_1=\mathcal{J}_l$; $b=b_1=b_V$ and $\Gamma\colon u\to\zeta u$. The condition (1.11) is fulfilled in view of (4.32) and the choice of ζ; (1.12) also holds true with $t=1-\varepsilon$. Therefore from (1.13) we obtain

$$n_{\pm}(\lambda,\widehat{\mathcal{J}}_l^{(\varepsilon)},b_V,\mathcal{H}^l)\leq n_{\pm}((1-\varepsilon)\lambda,T(V,\mathcal{H}_i^l(R)))+n_{\pm}((1-\varepsilon)\lambda,T(V,\mathcal{H}_e^l(R))).$$

Let us multiply the last inequality by λ^p and pass to the upper and lower limits as $\lambda\to 0$. Using (4.34) we find

$$D_p^{\pm}(T(V))\leq(1-\varepsilon)^{-p}\Delta_p^{\pm}(T(V,\mathcal{H}_i^l(R)))+(1-\varepsilon)^{-p}D_p^{\pm}(T(V,\mathcal{H}_e^l(R))).$$

From here and (4.31) we obtain (4.26)–(4.29) with "$\leq$" instead of "$=$", as $\varepsilon\to 0$. Thus under the additional restriction (4.32) the equalities (4.26)–(4.29) are established.

The restriction (4.32) now remains to be eliminated. Let V satisfy the conditions of Theorem 4.2. For $\varrho\le 1$ we introduce

$$V_\varrho(x)=\begin{cases} V(x), & \text{if } \big||x|-R\big|>\varrho\\ 0, & \text{if } \big||x|-R\big|<\varrho.\end{cases}$$

Then $T(V)=T(V_\varrho)+T(V-V_\varrho)$ and $\hat\Delta_p(|V_\varrho|)\le\hat\Delta_p(|V|)$. Thus the conditions of Theorem 4.2 and also (4.32) are satisfied for V_ϱ. Besides, the estimate (3.26) can be applied to $T(V-V_\varrho)$ with the same constant $C_{3.26}(R+1)$ (independent of ϱ) and consequently

$$\|T(V)-T(V_\varrho)\|_{\varkappa,\infty}\to 0 \quad \text{as } \varrho\to 0.$$

This, together with Proposition 1.1, implies

$$D^\pm_\varkappa(T(V_\varrho))\to D^\pm_\varkappa(T(V)) \quad \text{as } \varrho\to 0$$

and $D^\pm_p(T(V_\varrho))=D^\pm_p(T(V))$ for $p=q$ $(>\varkappa)$. The same is obviously true for the functionals $D^\pm_p(T(V_\varrho,\mathcal{H}^l_e(R)))$. These facts allow us to pass to the limit (as $\varrho\to 0$) in the formulae (4.26)–(4.29) written for V_ϱ. □

4.7. The proof of Theorem 4.2. One only needs to pass to the limit as $R\to\infty$ in formulae (4.26)–(4.29). This, together with (4.15) and (4.16), gives the relations (4.9)–(4.11). □

4.8. Using the inequalities (3.8) we reformulate Theorem 4.2 in terms of the function $\mathcal{N}(\alpha)$, i.e. the number of negative eigenvalues of the operator (0.1) generated by the form (3.2) in $L_2(\mathbf{R}^d)$. Notice that the relations (4.9)–(4.11) from Theorem 4.2 are only needed with the sign "+". Let us recall that the functionals (4.8) involved in the statement of this theorem, correspond to the spectrum of the quotient $(4.5)_R$ and are independent of R.

Theorem 4.7. *Let $2l\ge d$, d be even and* (4.1) *be fulfilled.*
(a) *If $\hat\Delta_\varkappa(|V|)<\infty$, then*

$$\limsup_{\alpha\to\infty}\alpha^{-\varkappa}\mathcal{N}(\alpha)=c_d\int V_+^\varkappa\,dx+\hat\Delta^+_\varkappa(V),$$

$$\liminf_{\alpha\to\infty}\alpha^{-\varkappa}\mathcal{N}(\alpha)=c_d\int V_+^\varkappa\,dx+\hat\delta^+_\varkappa(V).$$

If in this case $\hat\Delta^+_\varkappa(V)=0$, then the Weyl-type asymptotic formula holds

$$\lim_{\alpha\to\infty}\alpha^{-\varkappa}\mathcal{N}(\alpha)=c_d\int V_+^\varkappa\,dx.$$

(b) *If* $\hat{\Delta}_q(|V|) < \infty$ *and* $q > \varkappa$, *then*

$$\limsup_{\alpha\to\infty} \alpha^{-q} \mathcal{N}(\alpha) = \hat{\Delta}_q^+(V),$$
$$\liminf_{\alpha\to\infty} \alpha^{-q} \mathcal{N}(\alpha) = \hat{\delta}_q^+(V).$$

We discuss the properties of the functionals (4.8) in Section 5. After this we give some concrete versions of the "semieffective" Theorem 4.7 in Section 6.

5. An auxiliary spectral problem on the semiaxis

The material given in this section represents an extraction from the paper [BLS] containing the proofs and further references. It concerns a vector spectral problem on the semiaxis $\mathbf{R}_+$. We formulate this problem as a problem for a variational triple. Our purpose is to give some facts which will allow us to study the spectral properties of the quotient (4.5).

5.1. Denote by $\mathcal{H}_0^{l,1} = \mathcal{H}_0^{l,1}(\mathbf{R}_+)$, $l \geq 1$, the completion of the class $C_0^\infty(\mathbf{R}_+)$ with respect to the metric form

$$\mathcal{J}_{l,1}[f] = \int_{\mathbf{R}_+} (|f^{(l)}|^2 + |f'|^2)\, dt. \tag{5.1}$$

The direct description of this Hilbert space is the following

$$\mathcal{H}_0^{l,1}(\mathbf{R}_+) = \{f \in H^l_{\mathrm{loc}}(\overline{\mathbf{R}}_+) : \mathcal{J}_{l,1}[f] < \infty,\ f(0) = \ldots = f^{(l-1)}(0) = 0\}. \tag{5.2}$$

Let $1 \leq \mathfrak{n} < \infty$. We shall consider a vector version of the space $\mathcal{H}_0^{l,1}$,

$$\mathcal{H}_{0,\mathfrak{n}}^{l,1} := \mathcal{H}_0^{l,1}(\mathbf{R}_+; \mathbf{C}^{\mathfrak{n}}) = (\mathcal{H}_0^{l,1}(\mathbf{R}_+))^{\mathfrak{n}}. \tag{5.3}$$

In what follows the standard scalar product and the norm in $\mathbf{C}^{\mathfrak{n}}$ are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$. The operator norm in $\mathbf{C}^{\mathfrak{n}}$ of an $(\mathfrak{n}\times\mathfrak{n})$-matrix is also denoted by $|\cdot|$.

Let us consider a set of Hermitian matrices $m_1, \ldots, m_l$ where m_1 *and* m_l *are positive definite matrices* in $\mathbf{C}^{\mathfrak{n}}$ and where m_j, $1 < j < l$, *are nonnegative.* Introduce the quadratic form

$$\mathcal{M}[z] = \sum_{1 \leq j \leq l} \int_{\mathbf{R}_+} \langle m_j z^{(j)}, z^{(j)} \rangle\, dt, \quad z \in \mathcal{H}_{0,\mathfrak{n}}^{l,1}. \tag{5.4}$$

It is clear that the form (5.4) defines a metric in $\mathcal{H}^{l,1}_{0,\mathfrak{n}}$ which is equivalent to the original one.

Futhermore, let $Q(t)$ be an Hermitian $(\mathfrak{n}\times\mathfrak{n})$-matrix-function with entries in $L_{1,\mathrm{loc}}(\overline{\mathbf{R}}_+)$. We define the quadratic form

$$\mathcal{Q}[z]=\int_{\mathbf{R}_+}\langle Q(t)z,z\rangle\,dt \tag{5.5}$$

and consider the operator $T(\mathcal{M},\mathcal{Q})$ generated by the variational triple $(\mathcal{M},\mathcal{Q},\mathcal{H}^{l,1}_{0,\mathfrak{n}})$. The decreasing rate of the eigenvalues of $T(\mathcal{M},\mathcal{Q})$ (when it is compact) is defined by the behaviour of $Q(t)$ as $t\to\infty$. To some extent this problem is nonstandard due to the absence of the term of degree zero in (5.4) and because there exists a competition between the influence of the lowest order and the highest order terms in the quadratic form $\mathcal{M}[\,\cdot\,]$. The same arguments concern the form (5.1).

5.2. The behaviour of $Q(t)$ for large t's will be described by means of the sequences

$$\boldsymbol{\xi}(Q,p)=\{\xi_k(Q,p)\},\quad k\in\mathbf{Z}_+,$$

$$\xi_0(Q,p)=\int_0^1|Q(t)|\,dt,\quad \xi_k(Q,p)=\int_{2^{k-1}}^{2^k}t^{2p-1}|Q(t)|\,dt,\quad k\in\mathbf{N}, \tag{5.6}$$

where $p\geq 1$ is an additional numerical parameter. We write $\boldsymbol{\xi}(Q)$ instead of $\boldsymbol{\xi}(Q,1)$. By $\boldsymbol{\xi}_*$ we denote the above sequence omitting the first term: $\boldsymbol{\xi}_*=\{\xi_k(Q)\}$, $k\in\mathbf{N}$. We shall employ the same convention to other sequences.

The following statement contains the main estimates for the operator $T(\mathcal{M},\mathcal{Q})$. *The constants in these estimates depend on* $\mathfrak{n}$, l *and on the lower bounds of the matrices* m_1 *and* m_l. This will not be indicated in future notation.

Proposition 5.1. *Let* $T=T(\mathcal{M},\mathcal{Q})$ *be the operator generated by the triple* $(\mathcal{M},\mathcal{Q},\mathcal{H}^{l,1}_{0,\mathfrak{n}})$, *where* $\mathcal{M}$ *is the form* (5.4), $\mathcal{Q}$ *is the form* (5.5) *and* $\mathcal{H}^{l,1}_{0,\mathfrak{n}}$ *is the Hilbert space defined in* (5.2) *and* (5.3). *Then the following estimates are fulfilled provided that their right hand sides are finite.*

(1) *If* $q=\infty$ *then*

$$\|T\|\leq C\|\boldsymbol{\xi}(Q)\|_\infty; \tag{5.7}$$

if $\boldsymbol{\xi}(Q)\in c_0$, *then* $T\in\mathcal{C}$.

(2) *For* $\frac{1}{2}<q<\infty$ *we have*

$$\|T\|_{q,\infty}\leq C\|\boldsymbol{\xi}(Q)\|_{q,\infty},\quad C=C(q), \tag{5.8}$$

$$\Delta_q(T)\leq C\Delta_q(\boldsymbol{\xi}(Q)),\quad C=C(q). \tag{5.9}$$

(3) *If* $q=\frac{1}{2}$ *then*

$$\|T\|_{1/2,\infty} \leq C\|\boldsymbol{\xi}(Q)\|_{1/2}. \tag{5.10}$$

If $l>1$ *and the right hand side in* (5.10) *is finite, then*

$$\Delta_{1/2}(T)=0. \tag{5.11}$$

(4) *If* $l>1$ *and* $1/2l<q<\frac{1}{2}$, *then*

$$\|T\|_{q,\infty} \leq C\|\boldsymbol{\xi}(Q,(2q)^{-1})\|_{\infty}, \qquad C=C(q), \tag{5.12}$$

$$\Delta_q(T) \leq C \limsup_{k\to\infty} \xi_k(Q,(2q)^{-1}), \quad C=C(q). \tag{5.13}$$

(5) *If* $q=\frac{1}{2}l$ *then*

$$\|T\|_{1/2l,\infty} \leq C\|\boldsymbol{\xi}(Q,l)\|_{1/2l}. \tag{5.14}$$

Proposition 5.1 is part of a more general Theorem 2.4 from [BLS] where all the proofs are given. Here we only notice that the value $q=\frac{1}{2}$ is critical. If $q\geq\frac{1}{2}$, then the lowest order term of the form $\mathcal{M}$ is most essential. This corresponds to the value $p=1$ (see (5.6)) in the estimates (5.7)–(5.10). For $q<\frac{1}{2}$ and $l>1$ (estimates (5.12)–(5.14)) we have $p=p(q)>1$. This in turn, is responsible for the fact that the highest order term has an influence in the form $\mathcal{M}$.

5.3. The statements (1) and (2) in Proposition 5.1 are reversible for Q of a fixed sign.

Proposition 5.2. *Under the conditions of Proposition* 5.1, *let* $\pm Q(t)\geq 0$.

(1) *If* $T=T(\mathcal{M},\mathcal{Q})\in\mathcal{B}$, *then* $\boldsymbol{\xi}(Q)\in l_\infty$ *and*

$$\|\boldsymbol{\xi}_*(Q)\|_\infty \leq C\|T\|. \tag{5.15}$$

Moreover, if $T\in\mathcal{C}$, *then* $\boldsymbol{\xi}(Q)\in c_0$.

(2) *There exists a constant* $\varrho>0$, *such that*

$$n(\varrho\lambda,\boldsymbol{\xi}_*(Q)) \leq 2n(\lambda,T), \quad \lambda>0. \tag{5.16}$$

Remark 5.3. The sequence $\boldsymbol{\xi}_*$, not $\boldsymbol{\xi}$, is involved in the estimates (5.15) and (5.16). This is connected with the impossibility of estimating the integral $\xi_0(Q)$ by $n(\lambda,T)$.

Let us now compare Proposition 5.1(1)–(2) with Proposition 5.2.

Proposition 5.4. *Under the conditions of Proposition* 5.1, *let* $\pm Q(t)\geq 0$.

(1) *The inclusions* $T\in\mathcal{B}$ *and* $\boldsymbol{\xi}(Q)\in l_\infty$ *and also the inclusions* $T\in\mathcal{C}$ *and* $\boldsymbol{\xi}(Q)\in c_0$ *are equivalent. The estimates* (5.7) *and* (5.15) *are fulfilled.*

(2) *For* $\frac{1}{2}<q<\infty$ *the inclusions* $T\in\mathcal{C}_{q,\infty}$ *and* $\boldsymbol{\xi}(Q)\in l_{q,\infty}$ *are equivalent. Along with* (5.8) *the estimate*

$$\|\boldsymbol{\xi}_*(Q)\|_{q,\infty}\leq C\|T\|_{q,\infty} \tag{5.17}$$

holds true. Besides,

$$\Delta_q(T)\asymp\Delta_q(\boldsymbol{\xi}(Q)), \tag{5.18}$$

$$\delta_q(\boldsymbol{\xi}(Q))\leq C\delta_q(T). \tag{5.19}$$

The estimate (5.16) does not give us anything to be able to reverse the statements (3)–(5) in Proposition 5.1. Indeed, if $q=\frac{1}{2}$, then there is a "gap" between $l_{1/2}$ and $l_{1/2,\infty}$ in these estimates. (In particular, this gap gives (5.11)). If $q<\frac{1}{2}$, then the upper bounds contain $\boldsymbol{\xi}(Q,p)$ with $p=p(q)>1$, while $p=1$ in (5.16).

In fact, the statement (4) of Proposition 5.1 is also reversible, though for a much narrower class of Q. Say, this is the case if $|Q(t)|\asymp\phi(t)$ where $\phi\searrow$, see [BLS, Subsection 6.6]. However, this gives nothing for our operator (3.5), in view of Proposition 6.4 below.

5.4. Along with $\mathcal{M}[z]$ we also consider the quadratic form $\mathcal{M}_1[z]$ which is responsible for the lowest term of the form $\mathcal{M}$

$$\mathcal{M}_1[z]=\int_0^\infty\langle m_1z',z'\rangle\,dt,\quad z\in\mathcal{H}^1_{0,\mathrm{n}}, \tag{5.20}$$

where

$$\mathcal{H}^1_{0,\mathrm{n}}:=\mathcal{H}^{1,1}_{0,\mathrm{n}}=\left\{z\in H^1_{\mathrm{loc}}(\overline{\mathbf{R}}_+;\mathbf{C}^{\mathrm{n}}):\int_{\mathbf{R}_+}|z'|^2\,dt<\infty,\ z(0)=0\right\}. \tag{5.21}$$

Let us introduce the operator $T_1=T(\mathcal{M}_1,\mathcal{Q})$ generated by the variational triple $(\mathcal{M}_1,\mathcal{Q},\mathcal{H}^1_{0,\mathrm{n}})$. The results of Proposition 5.1(1)–(3) and Proposition 5.4 can, of course, be applied to this operator. In particular, it can be seen that the estimate $n(\lambda,\cdot)=O(\lambda^{-q})$, $q>\frac{1}{2}$ is provided for T and T_1 by the same conditions on Q. Moreover, for $\pm Q\geq 0$ these conditions are necessary. As already noticed, this means that the terms with derivatives of the order $j>1$ appearing in (5.4), do not have any influence on the estimates in the classes $\mathcal{C}_{q,\infty}$ with $q>\frac{1}{2}$. The statement given below, illustrates this effect in a much sharper form.

Proposition 5.5. *Let the operator $T_1=T(\mathcal{M}_1,\mathcal{Q})$ be generated by the triple $(\mathcal{M}_1,\mathcal{Q},\mathcal{H}^1_{0,\mathfrak{n}})$ defined in* (5.20), (5.5) *and* (5.21). *Then, under the conditions of Proposition* 5.1(2), *i.e. if $\boldsymbol{\xi}(Q)\in l_{q,\infty}$, $q>\frac{1}{2}$, we have*

$$\Delta_q^{\pm}(T)=\Delta_q^{\pm}(T_1),\ \ \delta_q^{\pm}(T)=\delta_q^{\pm}(T_1),\quad q>\tfrac{1}{2}. \tag{5.22}$$

The last proposition implies that the spectral asymptotics of the order λ^{-q}, $q>\frac{1}{2}$, for the operators $T(\mathcal{M},\mathcal{Q})$ and $T(\mathcal{M}_1,\mathcal{Q})$ coincide.

The proofs of Propositions 5.4 and 5.5 are contained in more general statements (in Corollary 2.9 and in Theorem 2.10) from [BLS].

Let us now assume that $Q(t)$ in (5.5) is of the form

$$Q(t)=\beta(t)X, \tag{5.23}$$

where $\beta=\bar{\beta}$ is a scalar-valued function and X is an Hermitian matrix which is independent of t. Let us fix the m_1-orthonormal basis in $\mathbf{C}^{\mathfrak{n}}$ consisting of eigenvectors of the pencil $X-\mu m_1$. Then X transforms to a diagonal matrix and m_1 becomes the unit matrix. As a result, the problem on the variational quotient for the triple $(\mathcal{M}_1,\mathcal{Q},\mathcal{H}^1_{0,\mathfrak{n}})$ splits into an orthogonal sum of $\mathfrak{n}$ scalar problems. More precisely, let $\{\mu_i^+\}$ and $\{-\mu_j^-\}$ be the positive and negative eigenvalues of the pencil $X-\mu m_1$. Consider the quotient

$$\frac{\int_{\mathbf{R}_+}\beta(t)|f|^2\,dt}{\int_{\mathbf{R}_+}|f'|^2\,dt},\quad f\in\mathcal{H}^1_0:=\mathcal{H}^1_{0,1}. \tag{5.24}$$

Proposition 5.6. *Let us assume that $T_1=T(\mathcal{M}_1,\mathcal{Q},\mathcal{H}^1_{0,\mathfrak{n}})$ and $Q(t)$ in* (5.5) *be given by* (5.23). *Let $T_0(\beta)$ be the operator generated by the quotient* (5.24). *Then*

$$n_\pm(\lambda,T_1)=\sum_i n_\pm(\lambda/\mu_i^+,T_0(\beta))+\sum_j n_\mp(\lambda/\mu_j^-,T_0(\beta)). \tag{5.25}$$

The proof is obvious.

Let us give an example. Put

$$\beta_q(t)=\begin{cases}0 & \text{for } 0<t<e,\\ t^{-2}(\log t)^{1/q} & \text{for } t\geq e,\end{cases}\qquad q>\tfrac{1}{2}, \tag{5.26}$$

and let $T_0(\beta_q)$ be the operator generated by the quotient (5.24) where $\beta=\beta_q$. Then (see [BL])

$$\Delta_q(T_0(\beta_q))=\delta_q(T_0(\beta_q))=\frac{2^{2(q-1)}\Gamma\left(q-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(q)}=:K(q),\quad q>\tfrac{1}{2}. \tag{5.27}$$

If now in (5.23) $Q(t)=\beta_q(t)X$, then (5.25) implies

$$\delta_q^{\pm}(T_1)=\Delta_q^{\pm}(T_1)=K(q)\sum_j(\mu_j^{\pm})^q,\quad q>\tfrac{1}{2}. \tag{5.28}$$

Using (5.22) we can carry over the asymptotics (5.28) onto the corresponding operator T.

5.5. Let us finish this section by a simple technical remark. Denote the sum of the singular numbers of a matrix Q by $|Q|_1$; in other words $|\cdot|_1$ is the C_1-norm. If $\pm Q\geq 0$, then $|Q|_1=\pm \operatorname{Tr} Q$. Along with the sequence $\boldsymbol{\xi}(Q,p)$ defined in (5.6) we also consider the sequence $\hat{\boldsymbol{\xi}}(Q,p)$, $p\geq 1$ setting

$$(5.29)\qquad \hat{\xi}_0(Q,p)=\int_0^1 |Q(t)|_1\,dt,\quad \hat{\xi}_k(Q,p)=\int_{2^{k-1}}^{2^k} t^{2p-1}|Q(t)|_1\,dt,\ k\in\mathbf{N}.$$

We shall write $\hat{\boldsymbol{\xi}}(Q)$ instead of $\hat{\boldsymbol{\xi}}(Q,1)$. The inequalities

$$\xi_k(Q,p)\leq\hat{\xi}_k(Q,p)\leq\mathfrak{n}\,\xi_k(Q,p),\quad k\in\mathbf{Z}_+,$$

imply the following result.

Proposition 5.7. *The statements in this section remain valid if* $\boldsymbol{\xi}(Q,p)$ *is replaced by* $\hat{\boldsymbol{\xi}}(Q,p)$.

The sequence $\hat{\boldsymbol{\xi}}(Q,p)$ is convenient because if $\pm Q(t)\geq 0$, then we can change the order of Tr and $\int$ in (5.29).

6. Analysis of the main theorem

We take up the study of the behaviour of the functionals (4.8) using the results from Section 5. This will allow us to give a more concrete meaning to the statements of Theorem 4.2 (and thus of Theorem 4.7). We shall find a certain difference between the cases $\frac{1}{2}d\leq l<d$ and $l\geq d$.

The functionals (4.6) and (4.7) generated by the quotient $(4.5)_R$ are independent of $R\geq 1$ (Proposition 4.1). This allowed us to introduce the notation (4.8) for these functionals. It is convenient here to associate this notation with the quotient $(4.5)_R$ where $R=1$. Then, passing to the variable t (see (2.9)), we obtain a vector problem on the semiaxis $\mathbf{R}_+$.

6.1. According to (2.16) we have the following representation for a function $w\in\mathcal{W}(1)$ (see the definition of $\mathcal{W}(R)$ in (4.4))

$$w=\sum_{k,\nu} F_{k\nu}(r)\Phi_{k\nu}(\omega),\quad k\in\Xi(d,l),\ 1\leq\nu\leq\mu(d,k),$$

where $d^m F_{k\nu}(r)/dr^m|_{r=1}=0$, $0\leq m\leq l-1$. It is convenient to pass to the variable t and identify $w\in\mathcal{W}(1)$ with the correspondig vector-valued function $z=\{z_{k\nu}\}$ using (2.9). The vector dimension of z is equal to $\mathfrak{n}=\mathfrak{N}(d,l)$ (see (2.15)). According to (2.13) we find

$$\mathcal{J}_l[w]=\sum_{k\in\Xi}\sum_{\nu}\sum_{i=1}^{l}\varrho_{ki}\int_{\mathbf{R}_+}|z_{k\nu}^{(i)}(t)|^2\,dt=:\mathcal{M}_{d,l}[z], \tag{6.1}$$

where z satisfies the Dirichlet boundary conditions at $t=0$.

Thus the denominator in the quotient $(4.5)_1$ turns into the quadratic form of the type (5.4) and we see that $z\in\mathcal{H}^{l,1}_{0,\mathfrak{n}}$. In this case all the matrices m_j in (5.4) are diagonal and every diagonal splits into $\#\Xi=\left[\frac{1}{2}\left(l-\frac{1}{2}d+1\right)\right]$ intervals of the length $\mu(d,k)$, where all the entries are equal to each other and coincide with ϱ_{kj}. According to (2.12), all the matrices m_j, $1\leq j\leq l$, are positive by definition.

Let us now transform the numerator in $(4.5)_1$. It is easy to see that

$$\int_{|x|\geq 1}V(x)|w|^2\,dx=\int_{\mathbf{R}_+}\langle Q_V(t)z,z\rangle\,dt=:\mathcal{Q}_V[z], \tag{6.2}$$

where $Q_V(t)$ is an Hermitian matrix with the entries

$$q_{k_1\nu_1,k_2\nu_2}(t)=e^{2lt}\int_{\mathbf{S}^{d-1}}V(e^t,\omega)\Phi_{k_1\nu_1}(\omega)\overline{\Phi_{k_2\nu_2}(\omega)}\,d\omega \tag{6.3}$$

for $k_1,k_2\in\Xi(d,l)$, $1\leq\nu_1\leq\mu(d,k_1)$, $1\leq\nu_2\leq\mu(d,k_2)$. Hence the variational quotient $(4.5)_1$ transforms to

$$\frac{\mathcal{Q}_V[z]}{\mathcal{M}_{d,l}[z]},\quad z\in\mathcal{H}^{l,1}_{0,\mathfrak{n}}, \tag{6.4}$$

where the forms $\mathcal{M}_{d,l}$ and $\mathcal{Q}_V$ are defined in (6.1) and (6.2), and the space $\mathcal{H}^{l,1}_{0,\mathfrak{n}}$ is given by (5.3). Thus we have the following result.

Proposition 6.1. *Let $\mathcal{T}(V)=T(\mathcal{M}_{d,l},\mathcal{Q}_V)$ be the operator generated by the variational triple $(\mathcal{M}_{d,l},\mathcal{Q}_V,\mathcal{H}^{l,1}_{0,\mathfrak{n}})$ (i.e. the quotient (6.4)). Then the functionals (4.8) coincide with the respective functionals for the operator $\mathcal{T}(V)$.*

6.2. Let us mention several cases where the description of the quotient (6.4) can be considerably simplified.

(1) *The case $2l=d$.* Then $\Xi=\{0\}$, $\mathfrak{n}=1$, i.e. (6.4) is a scalar problem. In this case

$$Q_V(t)=\frac{e^{2lt}}{\operatorname{meas}\mathbf{S}^{d-1}}\int_{\mathbf{S}^{d-1}}V(e^t,\omega)\,d\omega.$$

(2) *The case of spherical symmetry*: $V = V(r)$. It can be seen from (6.3) that in this case Q_V is a factor of the unit matrix and the form (6.2) is equal to

$$\mathcal{Q}_V[z] = \int_{\mathbf{R}_+} e^{2lt} V(e^t) |z|^2 \, dt.$$

The corresponding quotient (6.4) decomposes into the orthogonal sum of scalar problems.

(3) *Potentials of the type*

$$V(x) = F(r) g(\omega). \tag{6.5}$$

In this case

$$Q_V(t) = e^{2lt} F(e^t) X(g), \tag{6.6}$$

where $X(g)$ is a constant matrix. The matrix (6.6) is of the form (5.23) and everything mentioned in Section 5.4 holds true for this matrix.

Let us remark that if $V(x) \geq 0$, then $Q_V(t) \geq 0$.

6.3. In order to apply the results of Section 5 we have to express, or at least estimate, $\boldsymbol{\xi}(Q_V, p)$ in terms of the original potential V. In fact, it is more convenient to deal with the sequence $\hat{\boldsymbol{\xi}}(Q_V, p)$ defined in (5.29). We now obtain an explicit expression for $\hat{\boldsymbol{\xi}}(Q_V, p)$ assuming $V \geq 0$ and a simple majorant in the general case.

If $V \geq 0$, then according to (6.3) we find

$$|Q_V(t)|_1 = \operatorname{Tr} Q_V(t) = e^{2lt} \int_{\mathbf{S}^{d-1}} V(e^t, \omega) \sum_{k \in \Xi} \sum_{1 \leq \nu \leq \mu(d,k)} |\Phi_{k\nu}(\omega)|^2 \, d\omega.$$

Since

$$(\operatorname{meas} \mathbf{S}^{d-1}) \sum_{1 \leq \nu \leq \mu(d,k)} |\Phi_{k\nu}(\omega)|^2 = \mu(d, k)$$

we obtain

$$(\operatorname{meas} \mathbf{S}^{d-1}) \operatorname{Tr} Q_V(t) = \mathfrak{N}(d, l) e^{2lt} \int_{\mathbf{S}^{d-1}} V(e^t, \omega) \, d\omega, \quad V \geq 0. \tag{6.7}$$

This implies

$$\hat{\boldsymbol{\xi}}(Q_V, p) = \mathfrak{N}(d, l) (\operatorname{meas} \mathbf{S}^{d-1})^{-1} \boldsymbol{\theta}(V, p), \quad \pm V \geq 0, \tag{6.8}$$

where the sequence $\boldsymbol{\theta}(V,p)$ is defined by the formulae

$$
\begin{aligned}
\theta_0(V,p)&=\int_{1<|x|<e}|x|^{2l-d}|V(x)|\,dx,\\
\theta_j(V,p)&=\int_{2^{j-1}<\log|x|<2^j}|x|^{2l-d}(\log|x|)^{2p-1}|V(x)|\,dx,\quad j\in\mathbf{N}.
\end{aligned}
\tag{6.9}
$$

For an alternating potential $V=V_+-V_-$ it is obvious that $Q_V(t)=Q_{V_+}(t)-Q_{V_-}(t)$ and $|Q_V(t)|_1\leq\operatorname{Tr}(Q_{V_+}(t)+Q_{V_-}(t))=\operatorname{Tr}Q_{|V|}(t)$. For $\operatorname{Tr}Q_{|V|}(t)$ we use (6.7) and after that arrive at the estimate

$$
\hat{\boldsymbol{\xi}}(Q_V,p)\leq\mathfrak{N}(d,l)(\operatorname{meas}\mathbf{S}^{d-1})^{-1}\boldsymbol{\theta}(V,p).
\tag{6.10}
$$

Remark 6.2. The relations (6.8) and (6.10) allow us to substitute $\hat{\boldsymbol{\xi}}$ (and consequently $\boldsymbol{\xi}$) by the sequence $\boldsymbol{\theta}$ in all that follows.

As usual we put $\boldsymbol{\theta}(V,1)=:\boldsymbol{\theta}(V)$.

6.4. The case $\frac{1}{2}d\leq l<d$. In this case $\varkappa=d/2l$ implies $\frac{1}{2}<\varkappa\leq 1$ and the parameter $q\geq\varkappa$ in the statement of Theorem 4.2 also satisfies the condition $q>\frac{1}{2}$. This allows us to use the most advanced part of the results of Section 5 while interpreting Theorem 4.2. Let us give the corresponding statement.

Theorem 6.3. *Under the conditions of Theorem* 4.2, *let* $\frac{1}{2}d\leq l<d$.

(1) *The following hold*

$$
\{\boldsymbol{\theta}(V)\in l_{q,\infty}\}\Longrightarrow\{T(V)\in\mathcal{C}_{q,\infty}\},\quad q\geq\varkappa,
\tag{6.11}
$$

$$
\{\boldsymbol{\theta}(V)\in l^0_{q,\infty}\}\Longrightarrow\{T(V)\in\mathcal{C}^0_{q,\infty}\},\quad q\geq\varkappa.
\tag{6.12}
$$

(2) *If V is of a fixed sign, then the implications* (6.11) *and* (6.12) *are reversible. Besides, in this case*

$$
\begin{aligned}
\hat{\Delta}_q(V)&\asymp\Delta_q(\boldsymbol{\theta}(V)),\quad q\geq\varkappa,\\
\hat{\delta}_q(V)&\geq C\delta_q(\boldsymbol{\theta}(V)),\quad q\geq\varkappa.
\end{aligned}
\tag{6.13}
$$

(3) *The condition $\Delta_\varkappa(\boldsymbol{\theta}(V))=0$ is sufficient for the Weyl-type asymptotics*

$$
\Delta^{\pm}_\varkappa(T(V))=\delta^{\pm}_\varkappa(T(V))=c_d\int V^\varkappa_\pm\,dx
\tag{6.14}
$$

to be true. For a potential V of a fixed sign this condition is also necessary.

(4) *The functionals involved in the statement of Theorem* 4.2 *coincide with the corresponding functionals for the quotient*

$$
\frac{\int_{\mathbf{R}_+}\langle Q_Vz,z\rangle\,dt}{\int_{\mathbf{R}_+}\left(\sum_{k\in\Xi}\varrho_{k1}\sum_{1\leq\nu\leq\mu(d,k)}|z'_{k\nu}(t)|^2\right)dt},\quad z\in\mathcal{H}^1_{0,\mathrm{n}}.
\tag{6.15}
$$

Proof. Let us recall that $\mathcal{H}^1_{0,\mathbf{n}}$ is defined in (5.21) and the denominator in (6.15) corresponds to the terms with $i=1$ in the sum (6.1). Using Remark 6.2 in all the references in Section 5, we replace $\boldsymbol{\xi}(Q_V)$ by the sequence $\boldsymbol{\theta}(V)$ defined in (6.9). Then (1) follows from (5.8) and (5.9), (2) follows from Proposition 5.4(2), (3) follows from (4.9), (4.10) and (6.13) with $q=\varkappa$ and finally (4) is a direct corollary of Proposition 5.5. □

We omit a reformulation of Theorem 6.3 in terms of the operator (0.1). This can be done automatically.

Let us show that all the possibilities mentioned in Theorem 6.3 are realizable. We consider the potential $V_q(|x|)$, such that

$$V_q(r)=\begin{cases} r^{-2l}(\log r)^{-2}(\log\log r)^{-1/q} & \text{for } \log r\geq e,\\ 0 & \text{for } \log r<e.\end{cases}$$

It is easy to see that $V_q\in G(d,l)$ for any $q\in\mathbf{R}\setminus\{0\}$. However, we are interested here in the values $q\geq\varkappa$. The corresponding potential Q_{V_q} is exactly equal to

$$Q_{V_q}=\beta_q(t)\mathbf{1},\quad q\geq\varkappa, \tag{6.16}$$

where β_q is the functon (5.26) and $\mathbf{1}$ is the unit matrix in $\mathbf{C}^{\mathfrak{N}(d,l)}$. It immediately follows from (5.27) that $\boldsymbol{\theta}(V_q)\in l_{q,\infty}$ and $\hat{\Delta}_q(V_q)=\hat{\delta}_q(V_q)>0$. Thus if $q=\varkappa$, then (4.9) and (4.10) give us an asymptotic formula of the Weyl-type order $\varkappa$ but with an additional (non-Weyl-type) term which is proportional to the constant $K(\varkappa)$ defined in (5.27). We shall not present the final formulae since they can be obtained quite elementarily. For an arbitrary $q>\varkappa$ the potential (6.16) gives finite values of $\Delta_q(T(V_q))=\delta_q(T(V_q))>0$ in (4.11). Finally we obtain that the corresponding operator $T(V_q)$ is unbounded for $q<0$. In terms of the negative spectrum of the operator (0.1) this means that $\mathcal{N}(\alpha)=\infty$ for any $\alpha>0$.

If V is of the form (6.5) with $F=V_q$, $q\geq\varkappa$, then the asymptotic coefficients $\hat{\Delta}^{\pm}_q(V)=\hat{\delta}^{\pm}_q(V)$ can also be calculated explicitly by means of the formulae (5.28). The values of these coefficients are the same if $V(x)=g(\omega)(V_q(r)+o(1))$ as $r\to\infty$.

6.5. The case $l\geq d$. In this case it is also possible that $V\in G(d,l)$ but the Weyl-type asymptotics for $n(\lambda,T(V))$ (and therefore for $\mathcal{N}(\alpha)$) is violated. However, the corresponding examples are a bit more complicated and not as exhaustive as for $\frac{1}{2}d\leq l<d$.

Let us consider a radial potential

$$V(x)=\begin{cases} c_j|x|^{-2l}, & \text{if } 2^j-1<\log|x|<2^j,\ j\in\mathbf{N},\\ 0, & \text{otherwise.}\end{cases} \tag{6.17}$$

The coefficients c_j will be chosen later.

According to (3.12)–(3.14), we see that for the potential given by (6.17)

$$V\in G(d,l) \quad \text{if and only if} \quad \{c_j\}\in l_\varkappa.$$

Now (6.9) implies

$$\theta_j(V,p)\asymp 2^{j(2p-1)}c_j, \quad j\in\mathbf{N},\ p\geq 1.$$

In particular,

$$\theta_j(V):=\theta_j(V,1)\asymp 2^j c_j, \quad j\in\mathbf{N}.$$

Let now $c_j=2^{-j}j^{-1/q}$, $q>\frac{1}{2}$. Then $V\in G(d,l)$ and $\boldsymbol{\theta}(V)\in l_{q,\infty}\setminus l^0_{q,\infty}$. It follows from Proposition 5.4(2) and Theorem 4.2 that $T(V)\in\mathcal{C}_{q,\infty}\setminus\mathcal{C}^0_{q,\infty}$. Moreover, here $\delta_q(T(V))>0$.

We were unable to construct similar examples for $\varkappa\leq q\leq\frac{1}{2}$. The main obstacle for such a construction is that generally the statement (4) in Proposition 5.1 is not reversible. It is not difficult from the above example to derive that for any $q\in\left[\varkappa,\frac{1}{2}\right]$ there exists a potential V, such that the operator associated with the variational triple (6.4) belongs to $\mathcal{C}_{q,\infty}$ but does not belong to $\mathcal{C}_{q',\infty}$ for any $q'<q$. However, it is unclear whether this operator belongs to $\mathcal{C}_{q,\infty}\setminus\mathcal{C}^0_{q,\infty}$.

In the above construction $\boldsymbol{\eta}(V)$ (see (3.13) and (3.14)) was a lacunary sequence. On the contrary, if the sequence $\boldsymbol{\eta}(V)$ behaves sufficiently regularly (for example, monotonically), then it is easy to show that using only condition (4.1) we obtain the Weyl-type asymptotics (6.14). To this end, note first that from (3.14) and (6.9) follows

$$\theta_j:=\theta_j(V,(2\varkappa)^{-1})\asymp 2^{j(\varkappa^{-1}-1)}\sum_{2^{j-1}+1}^{2^j}\eta_j, \quad j\geq 1. \tag{6.18}$$

The two-sided estimate (6.18) implies a simple criteria for the inclusion $\boldsymbol{\theta}\in c_0$ in terms of the sequence $\boldsymbol{\eta}$: $\boldsymbol{\theta}\in c_0$ *if and only if*

$$n^{-1}\sum_{i=n+1}^{2n}\eta_i=o(n^{-1/\varkappa}). \tag{6.19}$$

It is easy to verify that, generally speaking, $\boldsymbol{\eta}\in l_\varkappa$ does not imply (6.19). The following statement imposes such conditions on $\boldsymbol{\eta}$ that both inclusions $\boldsymbol{\eta}\in l_\varkappa$ and $\boldsymbol{\theta}\in c_0$ hold true and thus guarantee the asymptotics (6.14).

Proposition 6.4. *Let $l\geq d$ and let the sequence $\boldsymbol{\eta}(V)$ admit the factorization*

$$\eta_j:=\eta_j(V)=\beta_j^{\varkappa^{-1}-1}\gamma_j, \quad j\in\mathbf{Z}_+, \tag{6.20}$$

$$\beta_j\geq 0, \quad \beta_j\searrow 0, \quad \gamma_j\geq 0, \quad \sum_j\beta_j<\infty, \quad \sum_j\gamma_j<\infty. \tag{6.21}$$

Then $V \in G(d,l)$ *and* $\boldsymbol{\theta}(V,(2\varkappa)^{-1}) \in c_0$.

Proof. By using the Hölder inequality, we obtain from (6.20)

$$\sum_j \eta_j^{\varkappa} \le \Big(\sum_j \beta_j\Big)^{1-\varkappa} \Big(\sum_j \gamma_j\Big)^{\varkappa},$$

so that $V \in G(d,l)$ follows from (6.21). The conditions on the sequence $\boldsymbol{\beta}$ imply that $\beta_j = o(j^{-1})$. Hence (again using (6.20)) we have

$$n^{-1} \sum_{i=n+1}^{2n} \eta_i = o(n^{-1/\varkappa}) \sum_{i=n+1}^{2n} \gamma_i.$$

The latter yields (6.19). □

Remark 6.5. For a *monotone* sequence $\boldsymbol{\eta}(V) \in l_\varkappa$ one can take $\beta_j = \gamma_j = \eta_j^{\varkappa}$. Then the conditions (6.20) and (6.21) are satisfied and therefore $\boldsymbol{\theta}(V,(2\varkappa^{-1})) \in c_0$.

It now remains for us to give the following result in the case when $l \ge d$ and the sequence $\boldsymbol{\eta}(V)$ behaves regularly.

Theorem 6.6. *Let* $l \ge d$. *Suppose that for a given* V *the sequence* $\boldsymbol{\eta}(V)$ *defined in* (3.13) *and* (3.14), *admits the factorization* (6.20), (6.21). *Then* $V \in G(d,l)$, $\hat{\Delta}_\varkappa(|V|) = 0$ *and the Weyl-type asymptotic formula* (6.14) *holds true.*

Proof. Using Proposition 6.4 we obtain $V \in G(d,l)$ and $\boldsymbol{\theta}(V,(2\varkappa^{-1})) \in c_0$. From (5.13) when $q = \varkappa$ we now derive that $\hat{\Delta}_\varkappa(|V|) = 0$. Finally (6.14) follows from (4.9) and (4.10). □

The examples given in Subsections 6.4, 6.5 and Theorem 6.6 show that the case $l \ge d$ is somewhat more complicated in comparison with $\frac{1}{2}d \le l < d$. In particular, we have no examples of V such that $n(\lambda, T(V))$ has the order $O(\lambda^{-\varkappa})$ as $\lambda \to 0$, but the asymptotic formula (6.14) fails.

References

[B] Birman, M. Sh., On the spectrum of singular boundary-value problems, *Mat. Sb.* **55** (1961), 125–174 (Russian). English transl.: in *Eleven Papers on Analysis*, Amer. Math. Soc. Transl. **53**, pp. 23–80, Amer. Math. Soc., Providence, R. I., 1966.

[BL] Birman, M. Sh. and Laptev, A., The negative discrete spectrum of a two-dimensional Schrödinger operator, *Comm. Pure Appl. Math.* **49** (1996), 967–997.

[BLS] Birman, M. Sh., Laptev, A. and Solomyak, M., On the eigenvalue behaviour for a class of differential operators on the semiaxis, to appear in *Math. Nachr.*

[BS1] Birman, M. Sh. and Solomyak, M. Z., The leading term in the asymptotic spectral formula for "non-smooth" elliptic problems, *Funktsional Anal. i Prilozhen.* **4** (1970), 1–13 (Russian). English transl.: *Functional Anal. Appl.* **4** (1970), 265–275.

[BS2] Birman, M. Sh. and Solomyak, M. Z., Spectral asymptotics of nonsmooth elliptic operators. I, *Trudy Moskov. Mat. Obshch.* **27** (1972), 3–52 (Russian). English transl.: *Trans. Moscow Math. Soc.* **27** (1972), 1–52.

[BS3] Birman, M. Sh. and Solomyak, M. Z., Quantitive analysis in Sobolev imbedding theorems and applications to spectral theory, in *Tenth Mathematical School* (Mitropol'skiĭ, Yu. O. and Shestopal, A. F., eds.), pp. 5–189, Izdanie Inst. Mat. Akad. Nauk Ukrain. SSSR, Kiev, 1974 (Russian). English transl.: in *Qualitative Analysis in Sobolev Imbedding Theorems and Applications to Spectral Theory*, Amer. Math. Soc. Transl. **114**, pp. 1–132, Amer. Math. Soc., Providence, R. I., 1980.

[BS4] Birman, M. Sh. and Solomyak, M. Z., *Spectral Theory of Selfadjoint Operators in Hilbert Space*, Leningrad Univ., Leningrad, 1980 (Russian). English transl.: D. Reidel Publ. Co., Dordrecht, 1987.

[BS5] Birman, M. Sh. and Solomyak, M. Z., Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations, in *Estimates and Asymptotics for Discrete Spectra of Integral and Differential Equations* (Birman, M. Sh., ed.), Adv. Soviet Math. **7**, pp. 1–55, Amer. Math. Soc., Providence, R. I., 1991.

[BS6] Birman, M. Sh. and Solomyak, M. Z., Schrödinger operator. Estimates of the number of the bound states as a function-theoretic problem, Lectures of the XIV School on Operator Theory on Functional Spaces I, Novgorod State Pedagogical Institute, Novgorod, 1990 (Russian). English transl.: in *Spectral Theory of Operators* (Gindikin, S. G., ed.), Amer. Math. Soc. Transl. **150**, pp. 1–54, Amer. Math. Soc., Providence, R. I., 1992.

[C] Cwikel, M., Weak type estimates for singular values and the number of bound states of Schrödinger operators, *Ann. of Math.* **106** (1977), 93–100.

[EK] Egorov, Yu. V. and Kondrat'ev, V. A., On the negative spectrum of an elliptic operator, *Mat. Sb.* **181**:2 (1990), 147–166 (Russian). English transl.: *Math. USSR-Sb.* **69** (1991), 155–177.

[JMS] Jaksić, V., Molcanov, S. and Simon, B., Eigenvalue asymptotics of the Neumann laplacian of regions and manifolds with cusps, *J. Funct. Anal.* **106** (1992), 59–79.

[KR] Krasnosel'skii, M. A. and Rutickii, Ya. B., *Convex Functions and Orlicz Spaces*, Fizmatgiz, Moscow, 1958 (Russian). English transl.: P. Nordhoff, Groningen, 1961.

[L] Lieb, E. H., Bounds on the eigenvalues of the Laplace and Schrödinger operators, *Bull. Amer. Math. Soc.* **82** (1976), 751–753.

[NS] Naimark, K. and Solomyak, M., Regular and patological eigenvalue behaviour for the equation $-lu = Vu$ on the semiaxis, Preprint.

[RS] REED, M. and SIMON, B., *Methods of Modern Mathematical Physics,* 4, Academic Press, New York–San Francisco–London, 1978.

[R1] ROSENBLUM, G. V., The distribution of the discrete spectrum for singular differential operators, *Dokl. Akad. Nauk SSSR* **202** (1972), 1012–1015 (Russian). English transl.: *Soviet Math. Dokl.* **13** (1972).

[R2] ROSENBLUM, G. V., Distribution of the discrete spectrum of singular differential operators, *Izv. Vyssh. Uchebn. Zaved. Mat.* **1** (1976), 75–86 (Russian). English transl.: *Soviet Math.* (*Iz. VUZ*) **20** (1976), 63–71.

[S1] SOLOMYAK, M., A remark on the Hardy inequalities, *Integral Equations Operator Theory* **19** (1994), 120–124.

[S2] SOLOMYAK, M., Piecewise-polynomial approximation of functions from $H^l((0,1)^d)$, $2l=d$, and applications to the spectral theory of Schrödinger operator, *Israel J. Math.* **86** (1994), 253–276.

[S3] SOLOMYAK, M., Spectral problems related to the critical exponent in the Sobolev embedding theorem, *Proc. London Math. Soc.* **71** (1995), 53–75.

[SW] STEIN, E. M. and WEISS, G., *Introduction to Fourier Analysis on Euclidean Spaces,* Princeton Univ. Press, Princeton, N. J., 1971.

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