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Semigroups of moment functions

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1. Introduction

If φ and ω are moment functions on a *-semigroup S (see definitions in Section 2) then the pointwise product $\varphi\omega$ is again a moment function. It is natural to ask for a characterization of families $(\varphi_t)_{t>0}$ of moment functions such that $\varphi_{r+t}=\varphi_r\varphi_t$ for all $r,t>0$. Restricting the problem a little bit, we inquire what functions $\psi\colon S\to\mathbf{C}$ are such that $e^{-t\psi}$ is a moment function for all $t>0$.

Functions $\psi\colon S\to\mathbf{C}$ such that $e^{-t\psi}$ is a *bounded* moment function for each $t>0$ were characterized by Berg, Christensen, and Ressel ([2, 4.3.19]), assuming the existence of a so-called Lévy function, which was shown by Buchwalter [10]. We quote the result of Berg, Christensen, and Ressel in Section 9.

Buchwalter [8], in the case where S is either $(\mathbf{N}_0^p,+)$ with the identical involution or $(\mathbf{N}_0^p\times\mathbf{N}_0^p,+)$ with the involution $(m,n)^*=(n,m)$, characterized those $\psi\colon S\to\mathbf{C}$ such that $e^{-t\psi}$ is a moment function for all $t>0$. In the case $S=\mathbf{N}_0^p$, his result (Théorème 2) states that the following conditions are equivalent (with notation as in Section 2):

(i) $e^{-t\psi}$ is a moment function for all $t>0$;

(ii) $\langle\psi,R\rangle\le 0$ for all $R\in(\mathcal{E}_0)_+$;

(iii) $-R\psi$ is a moment function for all $R\in(\mathcal{E}_0)_+$;

(iv) there exist $a\in\mathbf{R}$, $b\in\mathbf{R}^p$, a nonnegative quadratic form q on $\mathbf{R}^p$, and a measure λ on $\mathbf{R}^p\setminus\{1\}$ such that $\int[\sum_k(X_k-1)^2](1+r^2)^m\,d\lambda<\infty$ for all $m\in\mathbf{N}_0$ (with $r=\sqrt{\sum X_k^2}$) and

$$-\psi(n)=a+\langle b,n\rangle+\tfrac{1}{2}q(n)+\int(X^n-1-\langle n,X-1\rangle)\,d\lambda,\quad n\in\mathbf{N}_0^p.$$

(Here $X^n=X_1^{n_1}\dots X_p^{n_p}$ and $\langle n,X\rangle=n_1X_1+\dots+n_pX_p$ for $X=(X_1,\dots,X_p)\in\mathbf{R}^p$ and $n=(n_1,\dots,n_p)\in\mathbf{N}_0^p$.)

We shall show how this result generalizes to arbitrary *-semigroups. For semiperfect semigroups (where every positive definite function is a moment function) it turns out that (ii) can be replaced by the condition that ψ be negative definite. The corresponding result for perfect semigroups (where each positive definite function is the moment function of a unique measure) will be used in the proof of the general case.

2. Preliminaries

A **-semigroup* is a commutative semigroup with neutral element, equipped with an involution written $s \mapsto s^*$. In general we write the semigroup operation as multiplication and call the neutral element a unit, denoted by 1. Let S be a *-semigroup. A function $\varphi: S \to \mathbf{C}$ is *positive definite* if

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(s_j s_k^*) \geq 0$$

for every choice of $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbf{C}$. The set of all positive definite functions on S is denoted by $\mathcal{P}(S)$. Every positive definite function φ is *hermitian* in the sense that $\varphi(s^*) = \bar{\varphi}(s)$ for all $s \in S$. A function $\sigma: S \to \mathbf{C}$ is a *character* if $\sigma(1) = 1$, $\sigma(s^*) = \bar{\sigma}(s)$, and $\sigma(st) = \sigma(s)\sigma(t)$ for all $s, t \in S$. The set of all characters on S is denoted by S^*. We denote by $\mathcal{A}(S^*)$ the smallest sigma field in S^* rendering the evaluation $\sigma \mapsto \sigma(s): S^* \to \mathbf{C}$ measurable for each $s \in S$. Let $F_+(S^*)$ denote the set of all measures μ on $\mathcal{A}(S^*)$ such that $\int |\sigma(s)| \, d\mu(\sigma) < \infty$ for all $s \in S$. Measures in $F_+(S^*)$ are bounded because $1 \in S$. For $\mu \in F_+(S^*)$ we define $\mathcal{L}\mu: S \to \mathbf{C}$ by

$$\mathcal{L}\mu(s) = \int \sigma(s) \, d\mu(\sigma), \quad s \in S. \tag{2.1}$$

A function $\varphi: S \to \mathbf{C}$ is a *moment function* if there is some $\mu \in F_+(S^*)$ such that $\mathcal{L}\mu = \varphi$, and a moment function φ is *determinate* if there is only one such μ. Every bounded positive definite function is a determinate moment function ([2, 4.2.5]). The set of all moment functions on S is denoted by $\mathcal{H}(S)$. The *-semigroup S is *perfect* if every positive definite function on S is a determinate moment function. A *-semigroup S is perfect if it is **-divisible* in the sense that for each $s \in S$ there exist $t \in S$ and $m, n \in \mathbf{N}_0$ with $m + n \geq 2$ such that $s = t^m t^{*n}$ ([7, Theorem 4]).

Consider S^* with the topology of pointwise convergence and let $\mathcal{B}(S^*)$ denote the Borel σ-field. A *Radon measure* on S^* is a measure μ defined on $\mathcal{B}(S^*)$, finite on compact sets and such that $\mu(B) = \sup\{\mu(C) \mid C \text{ compact}, C \subset B\}$ for all $B \in \mathcal{B}(S^*)$. Let $E_+(S^*)$ be the set of Radon measures μ on S^* for which $\mu|_{\mathcal{A}(S^*)}$ belongs to $F_+(S^*)$. If S is countable then S^* is a Polish space, $\mathcal{A}(S^*) = \mathcal{B}(S^*)$, every bounded measure on $\mathcal{B}(S^*)$ is a Radon measure, and $F_+(S^*) = E_+(S^*)$. The terms *Radon* *moment function*, *Radon determinate*, and *Radon perfect* are defined in analogy with their counterparts without the qualification "Radon", using $E_+(S^*)$ instead of $F_+(S^*)$. Radon perfect semigroups were called "perfect" in [2] and in [7] where perfect semigroups were introduced under the name of "quasi-perfect".

A *-*subsemigroup* of S is a subsemigroup T, stable under the involution and such that $1\in T$. For a *-subsemigroup T of S we define $p_{S,T}\colon S^*\to T^*$ by $p_{S,T}(\sigma)=\sigma|_T$, $\sigma\in S^*$. Then

$$\mathcal{A}(S^*)=\bigcup p_{S,T}^{-1}(\mathcal{B}(T^*)),$$

the union extending over all countable *-subsemigroups T of S. If T is a *-subsemigroup of S then $p_{S,T}$ is measurable with respect to $\mathcal{A}(S^*)$ and $\mathcal{A}(T^*)$, and if $\mu\in F_+(S^*)$ then $\mu^{p_{S,T}}$ (the image of μ under $p_{S,T}$) is in $F_+(T^*)$ and $\mathcal{L}(\mu^{p_{S,T}})=(\mathcal{L}\mu)|_T$.

A function $\psi\colon S\to\mathbf{C}$ is *negative definite* if ψ is hermitian and

$$\sum_{j=1}^{n} c_j\bar{c}_k\psi(s_js_k^*)\leq 0$$

for every choice of $s_1,\dots,s_n\in S$ and $c_1,\dots,c_n\in\mathbf{C}$ satisfying $\sum_{j=1}^{n}c_j=0$. A function $\psi\colon S\to\mathbf{C}$ is negative definite if and only if $e^{-t\psi}\in\mathcal{P}(S)$ for all $t>0$ ([2, 3.2.2]). The set of all negative definite functions on S is denoted by $\mathcal{N}(S)$.

For $s\in S$ we define a linear operator E_s in $\mathbf{C}^S$ by

$$E_s\varphi(t)=\varphi(st),\quad t\in S.$$

Since $E_aE_b=E_{ab}$, the complex linear span $\mathcal{E}$ of the operators E_s, $s\in S$, is an algebra containing the identity operator $I=E_1$. We consider $\mathcal{E}$ with the involution $P\mapsto P^*$ defined by $E_s^*=E_{s^*}$, $s\in S$. Let $\mathcal{E}_{\mathrm{sa}}=\{P\in\mathcal{E}\,|\,P^*=P\}$ (similarly for every complex vector space with involution) and note that $\mathcal{E}=\mathcal{E}_{\mathrm{sa}}\oplus i\mathcal{E}_{\mathrm{sa}}$. Define a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbf{C}^S\times\mathcal{E}$ by

$$\langle\varphi,P\rangle=P\varphi(0),\quad\varphi\in\mathbf{C}^S,\quad P\in\mathcal{E}$$

and note that if $\varphi\in\mathbf{C}^S$ is hermitian then $\langle\varphi,P^*\rangle=\overline{\langle\varphi,P\rangle}$ for all $P\in\mathcal{E}$. Write

$$\mathcal{E}_+=\{P\in\mathcal{E}\mid\langle\sigma,P\rangle\geq 0\text{ for all }\sigma\in S^*\},$$

define an ideal $\mathcal{E}_0$ in $\mathcal{E}$ by

$$\mathcal{E}_0=\{P\in\mathcal{E}\mid\langle 1,P\rangle=0\}=\operatorname{span}\{E_s-I\mid s\in S\},$$

write

$$\mathcal{E}_0^2=\operatorname{span}\{PQ\mid P,Q\in\mathcal{E}_0\}=\operatorname{span}\{(E_s-I)(E_t-I)\mid s,t\in S\},$$

and define

$$(\mathcal{E}_0^2)_+=\mathcal{E}_0^2\cap\mathcal{E}_+.$$

If $\varphi\in\mathcal{H}(S)$ then $\langle\varphi,P\rangle\geq 0$ for all $P\in\mathcal{E}_+$. A function $\varphi\in\mathbf{C}^S$ is in $\mathcal{P}(S)$ if and only if $\langle\varphi,PP^*\rangle\geq 0$ for all $P\in\mathcal{E}$. A function $\psi\in\mathbf{C}^S$ is in $\mathcal{N}(S)$ if and only if ψ is hermitian and $\langle\psi,PP^*\rangle\leq 0$ for all $P\in\mathcal{E}_0$. The latter condition implies that $-PP^*\psi\in\mathcal{P}(S)$ for all $P\in\mathcal{E}_0$.

Lemma 2.1. *The operators PP^*, $P\in\mathcal{E}_0$, span $\mathcal{E}_0^2$.*

Proof. For $Q,R\in\mathcal{E}_0^2$ we have $QR=\frac{1}{4}\sum_{n=0}^{3}i^n(Q+i^nR^*)(Q^*+i^{-n}R)$. $\square$

Lemma 2.2. *We have $\mathcal{E}_+\subset\mathcal{E}_{\mathrm{sa}}$ if and only if S^* separates points in S.*

Proof. If S^* does not separate points in S, choose $a,b\in S$ with $a\neq b$ such that $\sigma(a)=\sigma(b)$ for all $\sigma\in S^*$. Then $z(E_a-E_b)\in\mathcal{E}_+\setminus\mathcal{E}_{\mathrm{sa}}$ for $z\in\mathbf{C}\setminus(\mathbf{R}\cup i\mathbf{R})$.

Now suppose that S^* does separate points. For $s\in S$ define $\chi_s\colon S^*\to\mathbf{C}$ by $\chi_s(\sigma)=\sigma(s)$ for $\sigma\in S^*$. By [2, 6.1.8], the functions χ_s, $s\in S$, are linearly independent. It follows that if $Q\in\mathcal{E}$ and $\langle\sigma,Q\rangle=0$ for all $\sigma\in S^*$ then $Q=0$. (If $Q=\sum_{j=1}^{n}c_jE_{s_j}$ then $\langle\sigma,Q\rangle=0$ for all $\sigma\in S^*$ if and only if $\sum_{j=1}^{n}c_j\chi_{s_j}=0$, which forces $c_j=0$ for all j, that is, $Q=0$.) Now if $P\in\mathcal{E}_+$ then $\langle\sigma,P^*\rangle=\overline{\langle\sigma,P\rangle}=\langle\sigma,P\rangle$, that is, $\langle\sigma,P^*-P\rangle=0$, for all $\sigma\in S^*$, whence $P^*-P=0$. $\square$

3. Continuity and convolution in $F_+(S^*)$

On the set of bounded Radon measures on S^*, the *weak topology* is defined by the condition that a net (μ_i) converges to a measure μ if and only if $\int h\,d\mu_i\to\int h\,d\mu$ for every bounded continuous function h on S^*. The *inverse limit topology* (also called the $\mathcal{J}$ topology) on the set of bounded measures on $\mathcal{A}(S^*)$ is defined by the condition that a net (μ_i) converges to a measure μ if and only if $\mu_i^{p_{S,T}}\to\mu^{p_{S,T}}$ weakly for every countable *-subsemigroup T of S.

If $\mu\in F_+(S^*)$ and $P\in\mathcal{E}_+$ then $\langle\,\cdot\,,P\rangle\mu$ (the measure with density $\langle\,\cdot\,,P\rangle$ with respect to μ, where $\langle\,\cdot\,,P\rangle$ is the function $\sigma\mapsto\langle\sigma,P\rangle\colon S^*\to\mathbf{C}$) is again in $F_+(S^*)$, and $\mathcal{L}(\langle\,\cdot\,,P\rangle\mu)=P\mathcal{L}\mu$. The *$\mathcal{L}$-topology* on $F_+(S^*)$ is defined by the condition that a net (μ_i) in $F_+(S^*)$ converges to $\mu\in F_+(S^*)$ if and only if $\langle\,\cdot\,,PP^*\rangle\mu_i\to\langle\,\cdot\,,PP^*\rangle\mu$ in the inverse limit topology for all $P\in\mathcal{E}$.

Lemma 3.1. *A net (μ_i) in $F_+(S^*)$ converges to $\mu\in F_+(S^*)$ in the $\mathcal{L}$-topology if and only if $\langle\,\cdot\,,E_{ss^*}\rangle\mu_i\to\langle\,\cdot\,,E_{ss^*}\rangle\mu$ in the inverse limit topology for all $s\in S$.*

Proof. Assume that the condition holds and let $P\in\mathcal{E}$. Choose $s_1,\dots,s_n\in S$ and $c_1,\dots,c_n\in\mathbf{C}$ such that $P=\sum_{j=1}^{n}c_jE_{s_j}$. If T is a countable *-subsemigroup of

S containing $\{s_1, \dots, s_n\}$ then $\sum_{j=1}^n \langle \cdot, E_{s_j s_j^*}\rangle \mu_i^{ps,T} \to \sum_{j=1}^n \langle \cdot, E_{s_j s_j^*}\rangle \mu^{ps,T}$ in the weak topology on $E_+(T^*)$, and since $\langle \cdot, PP^*\rangle / \sum_{j=1}^n \langle \cdot, E_{s_j s_j^*}\rangle$ is a bounded continuous function on T^* it follows that $\langle \cdot, PP^*\rangle \mu_i^{ps,T} \to \langle \cdot, PP^*\rangle \mu^{ps,T}$ weakly. This being so for all such T, we have $\langle \cdot, PP^*\rangle \mu_i \to \langle \cdot, PP^*\rangle \mu$ in the inverse limit topology. $\square$

Proposition 3.1. *If (μ_i) is a net in $F_+(S^*)$ such that $(\mathcal{L}\mu_i)$ converges pointwise to some $\varphi \in \mathbf{C}^S$ then some subnet of (μ_i) converges in the $\mathcal{L}$-topology to some $\mu \in F_+(S^*)$ such that $\mathcal{L}\mu = \varphi$.*

Proof. With no restriction, assume that (μ_i) is a universal net. For $P \in \mathcal{E}$ we have $\mathcal{L}(\langle \cdot, PP^*\rangle \mu_i) = PP^* \mathcal{L}\mu_i \to PP^*\varphi$, so by [6, Proposition 3] , $(\langle \cdot, PP^*\rangle \mu_i)$ converges in the inverse limit topology to some $\mu_P \in F_+(S^*)$ such that $\mathcal{L}\mu_P = PP^*\varphi$. Hence $(\langle \cdot, I+PP^*\rangle \mu_i)$ converges in the inverse limit topology to $\mu_I + \mu_P$. If $P = \sum_{j=1}^n c_j E_{s_j}$ and if T is a countable *-subsemigroup of S containing $\{s_1, \dots, s_n\}$ then $\langle \cdot, I+PP^*\rangle \mu_i^{ps,T} \to (\mu_I + \mu_P)^{ps,T}$ in the weak topology on $E_+(T^*)$, and since $(1+\langle \cdot, PP^*\rangle)^{-1}$ is a bounded continuous function on T^*, it follows that $\mu_i^{ps,T} \to (1+\langle \cdot, PP^*\rangle)^{-1}(\mu_I + \mu_P)^{ps,T}$ weakly. This being so for all such T, we have $\mu_i \to (1+\langle \cdot, PP^*\rangle)^{-1}(\mu_I + \mu_P)$ in the inverse limit topology, so

$$\mu_I = (1+\langle \cdot, PP^*\rangle)^{-1}(\mu_I + \mu_P),$$

whence $\mu_P = \langle \cdot, PP^*\rangle \mu_I$. Now $\mu_i \to \mu_I$ in the $\mathcal{L}$-topology and $\mathcal{L}\mu_I = II^*\varphi = \varphi$. $\square$

Proposition 3.2. *The mapping $\mathcal{L}$ is continuous with respect to the $\mathcal{L}$-topology on $F_+(S^*)$ and the topology of pointwise convergence on $\mathbf{C}^S$.*

Proof. If $\mu_i \to \mu$ in the $\mathcal{L}$-topology on $F_+(S^*)$ then for all $P \in \mathcal{E}$ we have that $\langle \cdot, PP^*\rangle \mu_i \to \langle \cdot, PP^*\rangle \mu$ in the inverse limit topology, hence

$$\langle \mathcal{L}\mu_i, PP^*\rangle = (\langle \cdot, PP^*\rangle \mu_i)(S^*) \to (\langle \cdot, PP^*\rangle \mu)(S^*) = \langle \mathcal{L}\mu, PP^*\rangle.$$

That $\mathcal{L}\mu_i(s) = \langle \mathcal{L}\mu_i, E_s\rangle \to \langle \mathcal{L}\mu, E_s\rangle = \mathcal{L}\mu(s)$ for $s \in S$ now follows from the fact that E_s is a linear combination of operators of the form PP^*, namely,

$$E_s = \tfrac{1}{4} \sum_{n=0}^{3} i^{-n}(I + i^n E_s)(I + i^{-n} E_{s^*}). \quad \square$$

To define convolution in $F_+(S^*)$, let $m\colon S^* \times S^* \to S^*$ denote pointwise multiplication, that is, $m(\sigma, \tau) = \sigma\tau$ for $\sigma, \tau \in S^*$. Then m is measurable with respect to the σ-fields $\mathcal{A}(S^*) \otimes \mathcal{A}(S^*)$ in $S^* \times S^*$ and $\mathcal{A}(S^*)$ in S^*. To see this, note that by the definition of $\mathcal{A}(S^*)$ it suffices to show that $(\sigma,\tau)\mapsto\sigma\tau(s)$ is measurable for each $s\in S$. But this function is the product of the measurable functions $(\sigma,\tau)\mapsto\sigma(s)$ and $(\sigma,\tau)\mapsto\tau(s)$.

For any two bounded measures μ and ν on $\mathcal{A}(S^*)$ we can now define a third, $\mu*\nu$, by $\mu*\nu=(\mu\otimes\nu)^m$. Commutativity and associativity of this convolution follows from the corresponding properties of multiplication in S^*. There is a neutral element, ε_1, where 1 is the unit of S^*.

If $\mu,\nu\in F_+(S^*)$ then

$$\int |\sigma(s)|\,d(\mu*\nu)(\sigma)=\int |\sigma(s)|\,d\mu(\sigma)\int |\sigma(s)|\,d\nu(\sigma)<\infty,\quad s\in S,$$

so $\mu*\nu\in F_+(S^*)$. Repeating the computation without the absolute signs we get $\mathcal{L}(\mu*\nu)(s)=\mathcal{L}\mu(s)\mathcal{L}\nu(s)$, or

$$\mathcal{L}(\mu*\nu)=\mathcal{L}\mu\cdot\mathcal{L}\nu.$$

Note that $\mathcal{L}\varepsilon_1=1$, and that this moment function is determinate since it is bounded.

If U is another *-semigroup and if $f\colon S\to U$ is a **-homomorphism*, that is, a homomorphism satisfying $f(1)=1$ and $f(s^*)=f(s)^*$ for all $s\in S$, then the mapping $f^*\colon U^*\to S^*$ defined by $f^*(\omega)=\omega\circ f$ for $\omega\in U^*$ is a homomorphism, and it is easily seen that $\{\mu^{f^*}\,|\,\mu\in F_+(U^*)\}\subset F_+(S^*)$ and

$$(\mu*\nu)^{f^*}=\mu^{f^*}*\nu^{f^*}$$

for $\mu,\nu\in F_+(U^*)$. If T is a *-subsemigroup of S then $(\mu*\nu)^{p_{S,T}}=\mu^{p_{S,T}}*\nu^{p_{S,T}}$ for all $\mu,\nu\in F_+(S^*)$.

Lemma 3.2. *Convolution in $F_+(S^*)$ is continuous in the $\mathcal{J}$ topology.*

Proof. Suppose $\mu_i\to\mu$ and $\nu_i\to\nu$ in the inverse limit topology on $F_+(S^*)$. If T is a countable *-subsemigroup of S then $\mu_i^{p_{S,T}}\to\mu^{p_{S,T}}$ and $\nu_i^{p_{S,T}}\to\nu^{p_{S,T}}$ weakly in $E_+(T^*)$, so $(\mu_i*\nu_i)^{p_{S,T}}=\mu_i^{p_{S,T}}*\nu_i^{p_{S,T}}\to\mu^{p_{S,T}}*\nu^{p_{S,T}}=(\mu*\nu)^{p_{S,T}}$ weakly since convolution in $E_+(T^*)$ is continuous in the weak topology ([2, 2.3.4]). This being so for all such T, it follows that $\mu_i*\nu_i\to\mu*\nu$ in the inverse limit topology. □

Proposition 3.3. *Convolution in $F_+(S^*)$ is continuous in the $\mathcal{L}$-topology.*

Proof. Suppose $\mu_i\to\mu$ and $\nu_i\to\nu$ in the $\mathcal{L}$-topology on $F_+(S^*)$; we have to show that $\mu_i*\nu_i\to\mu*\nu$ in the $\mathcal{L}$-topology. By Lemma 3.1 it suffices to show that $\langle\,\cdot\,,E_{ss^*}\rangle\mu_i*\nu_i\to\langle\,\cdot\,,E_{ss^*}\rangle\mu*\nu$ in the inverse limit topology for each $s\in S$. Since $\langle\,\cdot\,,E_{ss^*}\rangle\mu_i*\nu_i=(\langle\,\cdot\,,E_{ss^*}\rangle\mu_i)*(\langle\,\cdot\,,E_{ss^*}\rangle\nu_i)$ and similarly for μ and ν, the result follows from Lemma 3.2. □

A *convolution semigroup* in $F_+(S^*)$ is a family $(\mu_t)_{t\geq 0}$ in $F_+(S^*)$ such that $\mu_0=\varepsilon_1$ and $\mu_{t+u}=\mu_t*\mu_u$ for all $t,u\geq 0$.

Proposition 3.4. *A convolution semigroup* (μ_t) *in* $F_+(S^*)$ *is continuous in the* $\mathcal{L}$*-topology if and only if there is some* $\psi\in\mathcal{N}(S)$ *such that* $\mathcal{L}\mu_t=e^{-t\psi}$ *for all* $t\geq 0$.

Proof. First suppose $\mathcal{L}\mu_t=e^{-t\psi}$, $t\geq 0$, for some $\psi\in\mathcal{N}(S)$. For $t\to 0$ we have $\mathcal{L}\mu_t\to 1$ pointwise. Since 1 is a determinate moment function and $1=\mathcal{L}\varepsilon_1$, by Proposition 3.1 it follows that $\mu_t\to\varepsilon_1$ in the $\mathcal{L}$-topology as $t\to 0$. By Lemma 3.1 it suffices to show that $(\langle\cdot, E_{ss^*}\rangle\mu_t)$ is continuous in the inverse limit topology for each $s\in S$. Note that $(\langle\cdot, E_{ss^*}\rangle\mu_t)$ is again a convolution semigroup, continuous at 0 in the inverse limit topology. For any countable *-subsemigroup T of S, containing s, we have to show that $(\langle\cdot, E_{ss^*}\rangle\mu_t^{p_{S,T}})$, which is a convolution semigroup, weakly continuous at 0, is weakly continuous. But this follows from [4, Remark (5) and Corollary].

Now assume that (μ_t) is continuous. By Proposition 3.2 it follows that for $s\in S$ the function $t\mapsto\mathcal{L}\mu_t(s)$ is a continuous multiplicative function on $(\mathbf{R}_+,+)$, mapping 0 to 1, hence of the form $t\mapsto e^{-t\psi(s)}$ for some $\psi(s)\in\mathbf{C}$. The function ψ is in $\mathcal{N}(S)$ since $e^{-t\psi}=\mathcal{L}\mu_t\in\mathcal{P}(S)$ for all $t>0$. $\square$

4. Quadratic forms

A complex function h on a *-semigroup S is **-additive* if h is a *-homomorphism of S into $\mathbf{C}$ considered with addition and complex conjugation.

Lemma 4.1. *For* $P\in\mathcal{E}$ *we have* $P\in\mathcal{E}_0^2$ *if and only if* $\langle a+h, P\rangle=0$ *for every* $a\in\mathbf{R}$ *and every* **-additive function* $h: S\to\mathbf{C}$.

Proof. The condition is necessary since $\mathcal{E}_0^2$ is spanned by operators of the form $(E_s-I)(E_t-I)$ with $s,t\in S$, and for those we have, if $a\in\mathbf{C}$ and if $h: S\to\mathbf{C}$ is additive,

$$\begin{aligned}\langle a+h, (E_s-I)(E_t-I)\rangle &= \langle a+h, E_{st}-E_s-E_t+I\rangle\\ &= a+h(st)-a-h(s)-a-h(t)+a+h(1)=0.\end{aligned}$$

Now suppose the condition holds, and write $P=\sum_{j=1}^n c_jE_{s_j}$ with $s_j\in S$ and $c_j\in\mathbf{C}$. Since $\langle 1, P\rangle=0$ then $\sum_{j=1}^n c_j=0$, so $P=\sum_{j=1}^n c_j(E_{s_j}-I)\in\mathcal{E}_0$. Write $X=\mathcal{E}_0/\mathcal{E}_0^2$ and let $Q\mapsto\widetilde{Q}: \mathcal{E}_0\to X$ be the quotient mapping. Since $\mathcal{E}_0^2$ is *-stable, there is a unique involution on X such that $(Q^*)^\sim=(\widetilde{Q})^*$ for $Q\in\mathcal{E}_0$. Define $\pi_1,\pi_2: X\to X_{\mathrm{sa}}$ by $x=\pi_1(x)+i\pi_2(x)$, $x\in X$, and note that $\pi_1(x^*)=\pi_1(x)$, $\pi_2(x^*)=-\pi_2(x)$ for $x\in X$. If ξ is a real linear form on X_{sa} then $h(s)=\xi(\pi_1([E_s-I]^\sim))$ defines a *-additive function h on S since $h(s^*)=h(s)=\bar{h}(s)$ and

$$h(st)-h(s)-h(t)=\xi(\pi_1([(E_s-I)(E_t-I)]^\sim))=0$$

as $(E_s-I)(E_t-I)\in\mathcal{E}_0^2$. By the assumption it follows that $0=\langle h,P\rangle=\sum_{j=1}^n c_j h(s_j)$. Writing $c_j=a_j+ib_j$ with $a_j,b_j\in\mathbf{R}$, we thus have $0=\sum_{j=1}^n a_j h(s_j)=\xi(\pi_1(\widetilde{Q}))$ where $Q=\sum_{j=1}^n a_j E_{s_j}=\sum_{j=1}^n a_j(E_{s_j}-I)$. In the same way we get $\xi(\pi_1(\widetilde{R}))=0$ where $R=\sum_{j=1}^n b_j E_{s_j}$; so $\xi(\pi_1(\widetilde{P}))=0$. This being so for every linear form ξ on X_{sa}, it follows that $\pi_1(\widetilde{P})=0$. It can be shown similarly that $\xi(\pi_2(\widetilde{P}))=0$ for every linear form ξ on X_{sa}; in this case, use the fact that $h(s)=i\xi(\pi_2([E_s-I]^\sim))$ defines a *-additive function. Thus $\pi_2(\widetilde{P})=0$, hence $\widetilde{P}=0$, that is, $P\in\mathcal{E}_0^2$. □

A *quadratic form* on a *-semigroup S is a hermitian function $q\colon S\to\mathbf{C}$ satisfying the homogeneity property $q(s^2)=4q(s)$, $s\in S$, and the functional equation

$$(E_r-I)(E_s-I)(E_t-I)q=0, \quad r,s,t\in S.$$

Note that this equation is equivalent to $PQRq=0$ for all $P,Q,R\in\mathcal{E}_0$. If q is a quadratic form, then by [11, Theorem 3], from the functional equation it follows that there is a unique triple (a,h,B) such that $a\in\mathbf{C}$, $h\colon S\to\mathbf{C}$ is additive, $B\colon S\times S\to\mathbf{C}$ is symmetric and biadditive, and $q(s)=a+h(s)+B(s,s)$ for $s\in S$. From the homogeneity of q it easily follows that $a=0$ and $h=0$, so

$$q(s)=B(s,s), \quad s\in S.$$

From the fact that q is hermitian, by the uniqueness statement in [11, Theorem 3] it follows that $B(s^*,t^*)=\overline{B}(s,t)$ for all $s,t\in S$. Quadratic forms as defined here are the same as (2-homogeneous) quadratic forms in the sense of [9].

Proposition 4.1. *Let S be a *-semigroup, write $X=\mathcal{E}_0/\mathcal{E}_0^2$, let $P\mapsto\widetilde{P}\colon\mathcal{E}_0\to X$ be the quotient mapping, and define $j\colon S\to X$ by $j(s)=(E_s-I)^\sim$ for $s\in S$. Then j is *-additive and*

(i) *for each additive function $h\colon S\to\mathbf{C}$ there is a unique linear form ξ on X such that $h=\xi\circ j$;*

(ii) *for each quadratic form q on S there is a unique sesquilinear form $\langle\cdot,\cdot\rangle$ on X such that*

$$q(s)=-\langle j(s),j(s^*)\rangle, \quad s\in S.$$

Moreover, $\langle x^*,y^*\rangle=\langle y,x\rangle=\overline{\langle x,y\rangle}$ for all $x,y\in X$. Finally, q is negative definite if and only if $\langle x,x\rangle\geq 0$ for all $x\in X$.

Proof. We consider X with the unique involution that makes the mapping $P\mapsto\widetilde{P}$ *-preserving. Then j is *-preserving; and j is additive since

$$j(st)-j(s)-j(t)=((E_s-I)(E_t-I))^\sim=0$$

because of $(E_s-I)(E_t-I)\in\mathcal{E}_0^2$.

(i): If $h\colon S\to\mathbf{C}$ is additive then $P\mapsto\langle h,P\rangle$ is a linear form on $\mathcal{E}_0$ which vanishes on $\mathcal{E}_0^2$. Hence there is a unique linear form ξ on X such that $\langle h,P\rangle=\xi(\widetilde{P})$ for all $P\in\mathcal{E}_0$, whence $h(s)=\langle h,E_s-I\rangle=\xi(j(s))$ for $s\in S$.

(ii): Let B be the unique symmetric biadditive function on S such that $q(s)=B(s,s)$, $s\in S$. For $t\in S$ the function $B(\cdot,t)$ is additive, so by (i) there is a unique linear form ξ_t on X such that

$$B(s,t)=\xi_t(j(s)),\quad s\in S.$$

For $r,t\in S$ the linear form $\xi_r+\xi_t$ on X satisfies $(\xi_r+\xi_t)(j(s))=B(s,r)+B(s,t)=B(s,rt)$ for all $s\in S$, and by the uniqueness of ξ_{rt} it follows that $\xi_{rt}=\xi_r+\xi_t$. Thus, for $x\in X$ the mapping $t\mapsto\xi_t(x)\colon S\to\mathbf{C}$ is additive, so there is a unique linear form η_x on X such that

$$\xi_t(x)=\eta_x(j(t)),\quad t\in S.$$

For $x,y\in X$ and $\alpha,\beta\in\mathbf{C}$, the linear form $\alpha\eta_x+\beta\eta_y$ on X satisfies $(\alpha\eta_x+\beta\eta_y)(j(t))=\alpha\xi_t(x)+\beta\xi_t(y)=\xi_t(\alpha x+\beta y)$ for all $t\in S$, and by the uniqueness of $\eta_{\alpha x+\beta y}$ it follows that $\eta_{\alpha x+\beta y}=\alpha\eta_x+\beta\eta_y$. Thus $(x,y)\mapsto\eta_x(y)$ is a bilinear form on X. Define

$$\langle x,y\rangle=-\eta_x(y^*),\quad x,y\in X.$$

Then $\langle\cdot,\cdot\rangle$ is a sesquilinear form and

$$B(s,t)=\xi_t(j(s))=\eta_{j(s)}(j(t))=-\langle j(s),j(t^*)\rangle,\quad s,t\in S.$$

In particular, $q(s)=-\langle j(s),j(s^*)\rangle$ for $s\in S$. Since $B(s,t)=B(t,s)$, and since $j(S)$ spans X, then $\langle x^*,y^*\rangle=\langle y,x\rangle$; since $B(s^*,t^*)=\overline{B}(s,t)$ then $\langle x^*,y^*\rangle=\overline{\langle x,y\rangle}$. For $P=\sum_{i=1}^n c_i(E_{s_i}-I)\in\mathcal{E}_0$,

$$\begin{aligned}\langle q,PP^*\rangle&=\sum_{i,k=1}^n c_i\bar{c}_k(q(s_is_k^*)-q(s_i)-q(s_k^*)-q(1))\\&=\sum_{i,k=1}^n c_i\bar{c}_k(B(s_is_k^*,s_is_k^*)-B(s_i,s_i)-B(s_k^*,s_k^*)+B(1,1))\\&=2\sum_{i,k=1}^n c_i\bar{c}_kB(s_i,s_k^*)=-2\sum_{i,k=1}^n c_i\bar{c}_k\langle j(s_i),j(s_k)\rangle=-2\langle x,x\rangle\end{aligned}$$

where $x=\sum_{i=1}^n c_ij(s_i)=\widetilde{P}$. This shows that q is negative definite if and only if $\langle x,x\rangle\geq 0$ for all $x\in X$. $\square$

Proposition 4.2. *If q is a negative definite quadratic form on a *-semigroup S then $e^{-tq}\in\mathcal{H}(S)$.*

Proof. By Proposition 4.1 there is a nonnegative sesquilinear form $\langle\cdot,\cdot\rangle$ on $X=\mathcal{E}_0/\mathcal{E}_0^2$ such that $\langle x^*,y^*\rangle=\langle y,x\rangle$ for all $x,y\in X$ and $q(s)=-\langle j(s),j(s^*)\rangle$, $s\in S$, with $j\colon S\to X$ as in Proposition 4.1. Since j is a *-homomorphism, it suffices to show that $x\mapsto e^{\langle x,x^*\rangle}\in\mathcal{H}(X)$. Since X is *-divisible, hence perfect, it suffices to show $x\mapsto e^{\langle x,x^*\rangle}\in\mathcal{P}(X)$. It suffices to show that this function is positive definite on each finite-dimensional linear subspace of X, so we may as well assume that X is of finite dimension. Write $Z=X/\{x\in X\,|\,\langle x,x\rangle=0\}$ and consider Z with the unique involution rendering the quotient mapping *-preserving. (The space $\{x\in X\,|\,\langle x,x\rangle=0\}$ is *-stable because $\langle x^*,x^*\rangle=\langle x,x\rangle$.) Since the quotient mapping is a *-homomorphism, it suffices to show that the function $z\mapsto e^{\langle z,z^*\rangle}$ is positive definite on Z, where $\langle\cdot,\cdot\rangle$ is the inner product on Z canonically associated with the sesquilinear form on X denoted by the same symbol. Choose a basis $(e_1,\dots,e_n)$ of Z_{sa} over $\mathbf{R}$. Then (e_i) is basis of Z over $\mathbf{C}$, and for $z=\sum_{k=1}^n z_k e_k\in Z$ we have $z^*=\sum_{k=1}^n \bar{z}_k e_k$, hence $\langle z,z^*\rangle=\sum_{k=1}^n z_k^2$, so we have to show that the function $(z_1,\dots,z_n)\mapsto\prod_{k=1}^n e^{z_k^2}$ is positive definite on $(\mathbf{C}^n,+)$. Since a pointwise product of positive definite functions is again positive definite ([2, 3.1.12]), it suffices to show that $z\mapsto e^{z^2}$ is positive definite on $(\mathbf{C},+)$. This amounts to showing that $z\mapsto -z^2\in\mathcal{N}(\mathbf{C})$. But if $z_1,\dots,z_m\in\mathbf{C}$ and $c_1,\dots,c_m\in\mathbf{C}$ with $\sum_{i=1}^m c_i=0$ then $\sum_{i,k=1}^m c_i\bar{c}_k(z_i+\bar{z}_k)^2=\sum_{i,k=1}^m c_i\bar{c}_k(z_i^2+2z_i\bar{z}_k+\bar{z}_k^2)=2\left|\sum_{i=1}^m c_iz_i\right|^2\ge 0$. $\square$

5. The complex Lévy function

If S is a *-semigroup, define a σ-ring $\mathcal{A}(S^*\setminus\{1\})$ in $S^*\setminus\{1\}$ by

$$\mathcal{A}(S^*\setminus\{1\})=\{A\in\mathcal{A}(S^*)\,|\,1\notin A\}.$$

A *complex Lévy function* for S is a function $H\colon S\times S^*\to\mathbf{C}$ satisfying

(i) $H(\cdot,\sigma)$ is *-additive for each $\sigma\in S^*$;

(ii) $H(s,\cdot)$ is $\mathcal{A}(S^*)$-measurable for each $s\in S$;

(iii) if μ is a measure on $\mathcal{A}(S^*\setminus\{1\})$ satisfying

$$\int\langle\sigma,PP^*\rangle\,d\mu(\sigma)<\infty,\quad P\in\mathcal{E}_0,\tag{5.1}$$

then

$$\int|1-\sigma(s)+H(s,\sigma)|\,d\mu(\sigma)<\infty,\quad s\in S.\tag{5.2}$$

Since the operators $E_s - I$, $s \in S$, span $\mathcal{E}_0$ then (5.1) is equivalent to

$$\int |1-\sigma(s)|^2 \, d\mu(\sigma) < \infty, \quad s \in S.$$

Proposition 5.1. *For every *-semigroup there is a complex Lévy function.*

Proof. Choose a linear complement $\mathcal{D}$ of $(\mathcal{E}_0^2)_{\mathrm{sa}}$ in the real vector space $\mathcal{E}_{\mathrm{sa}}$. Then $\mathcal{C} = \mathcal{D} + i\mathcal{D}$ is a *-stable linear complement of $\mathcal{E}_0^2$ in $\mathcal{E}$. Define $\pi: \mathcal{E} \to \mathcal{C}$ by

$$\pi(P+Q) = Q, \quad P \in \mathcal{E}_0^2, \; Q \in \mathcal{C}$$

and note that π is *-preserving. Now let

$$H(s,\sigma) = \langle \sigma, \pi(E_s - I) \rangle, \quad s \in S, \; \sigma \in S^*.$$

For $\sigma \in S^*$ the function $H(\,\cdot\,,\sigma)$ is clearly hermitian, and

$$H(st,\sigma) - H(s,\sigma) - H(t,\sigma) = \langle \sigma, \pi(E_{st} - E_s - E_t + I) \rangle = 0$$

since $(E_s - I)(E_t - I) \in \mathcal{E}_0^2 = \ker \pi$. Property (ii) follows from the fact that for all $Q \in \mathcal{E}$ the function $\langle\,\cdot\,, Q\rangle$ is measurable, being a linear combination of functions of the form $\sigma \mapsto \sigma(s)$ with $s \in S$. Finally, if μ satisfies (5.1) and $s \in S$ then

$$\int |1-\sigma(s) + H(s,\sigma)| \, d\mu(\sigma) = \int |\langle \sigma, I - E_s + \pi(E_s - I)\rangle| \, d\mu < \infty$$

since $\pi(R) - R \in \mathcal{E}_0^2$ for all $R \in \mathcal{E}$, hence in particular for $R = E_s - I$. □

6. The case of perfect semigroups

Theorem 6.1. *Suppose S is a perfect semigroup with a complex Lévy function H. For a function $\psi: S \to \mathbf{C}$, the following conditions are equivalent:*

(i) *There is a convolution semigroup (μ_t) in $F_+(S^*)$ such that $\mathcal{L}\mu_t = e^{-t\psi}$ for all $t > 0$;*

(ii) *$e^{-t\psi} \in \mathcal{P}(S)$ for all $t > 0$;*

(iii) *$\psi \in \mathcal{N}(S)$;*

(iv) *there exist $a \in \mathbf{R}$, a *-additive function $h: S \to \mathbf{C}$, a negative definite quadratic form q on S, and a measure μ on $\mathcal{A}(S^* \setminus \{1\})$ such that*

$$\psi(s) = a + h(s) + q(s) + \int (1 - \sigma(s) + H(s,\sigma)) \, d\mu(\sigma), \quad s \in S. \tag{6.1}$$

Assuming that these conditions hold, the convolution semigroup (μ_t) *in* (i) *is uniquely determined by* ψ *and is continuous in the* $\mathcal{L}$*-topology, and the objects* a, h, q, μ *in* (iv) *are uniquely determined by* ψ. *(We call* μ *the* Lévy measure *of* ψ.*)*

Proof. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (i): For $t \geq 0$, since S is perfect, there is a unique $\mu_t \in F_+(S^*)$ such that $\mathcal{L}\mu_t = e^{-t\psi}$. For $t, u \geq 0$ we have $\mathcal{L}(\mu_t * \mu_u) = \mathcal{L}\mu_t \cdot \mathcal{L}\mu_u = e^{-t\psi} e^{-u\psi} = e^{-(t+u)\psi}$, and by the uniqueness of μ_{t+u} it follows that $\mu_{t+u} = \mu_t * \mu_u$. So (μ_t) is a convolution semigroup. The continuity of (μ_t) follows from Proposition 3.4.

(ii) $\Leftrightarrow$ (iii): Already observed.

(iii) $\Rightarrow$ (iv): For $P \in \mathcal{E}_0$ we have $-PP^*\psi \in \mathcal{P}(S)$, and since S is perfect, there is a unique $\lambda_P \in F_+(S^*)$ such that $-PP^*\psi = \mathcal{L}\lambda_P$. For $P, Q \in \mathcal{E}_0$ we have

$$\mathcal{L}(\langle \cdot, QQ^* \rangle \lambda_P) = QQ^* \mathcal{L}\lambda_P = -PP^*QQ^*\psi = PP^*\mathcal{L}\lambda_Q = \mathcal{L}(\langle \cdot, PP^* \rangle \lambda_Q),$$

and since S is perfect, it follows that

$$\langle \cdot, QQ^* \rangle \lambda_P = \langle \cdot, PP^* \rangle \lambda_Q. \tag{6.2}$$

Write $G_P = \{\sigma \in S^* \mid \langle \sigma, P \rangle \neq 0\}$ for $P \in \mathcal{E}_0$ and define a measure μ_P on $\mathcal{A}(G_P) = \{A \in \mathcal{A}(S^*) \mid A \subset G_P\}$ by

$$\mu_P = \langle \cdot, PP^* \rangle^{-1} (\lambda_P|_{G_P}). \tag{6.3}$$

By (6.2),

$$\mu_P|_{G_P \cap G_Q} = \mu_Q|_{G_P \cap G_Q}, \quad P, Q \in \mathcal{E}_0.$$

We wish to define a measure μ on $\mathcal{A}(S^* \setminus \{1\})$ such that

$$\langle \cdot, PP^* \rangle \mu = \lambda_P|_{\mathcal{A}(S^* \setminus \{1\})}, \quad P \in \mathcal{E}_0. \tag{6.4}$$

For $A \in \mathcal{A}(S^* \setminus \{1\})$ there is a countable *-subsemigroup $T = \{t_1, t_2, \ldots\}$ of S, such that each character on T extends to a character on S ([3, Theorem 3]), and some $B \in \mathcal{B}(T^*)$ such that $A = p_{S,T}^{-1}(B)$. Since $1 \notin A$ then $1 \notin B$, so $A \subset \bigcup_{n=1}^{\infty} G_{P_n}$ where $P_n = E_{t_n} - I \in \mathcal{E}_0$. It is now clear that μ can be defined by

$$\mu|_{G_P} = \mu_P, \quad P \in \mathcal{E}_0.$$

We now have $\mu|_{G_P} = \mu_P = \langle \cdot, PP^* \rangle^{-1} (\lambda_P|_{G_P})$ for $P \in \mathcal{E}_0$. To get (6.4), it remains to be shown that $\lambda_P(A \setminus G_P) = 0$ for $A \in \mathcal{A}(S^* \setminus \{1\})$. Since A is contained in the union of countably many G_Q, it suffices to show $\lambda_P(G_Q \setminus G_P) = 0$ for $Q \in \mathcal{E}_0$. But this follows from (6.2).

Since $\int\langle\sigma, PP^*\rangle\, d\mu(\sigma)<\infty$ for all $P\in\mathcal{E}_0$, we can define $\psi_\mu: S\to\mathbf{C}$ by

$$\psi_\mu(s)=\int(1-\sigma(s)+H(s,\sigma))\,d\mu(\sigma),\quad s\in S.$$

For $P\in\mathcal{E}_0$ we have

$$PP^*(\psi-\psi_\mu)(s)=-\int\sigma(s)\,d\lambda_P(\sigma)+\int\langle\sigma,PP^*\rangle\sigma(s)\,d\mu(\sigma)=-\lambda_P^*(\{1\}),$$

a constant. Hence $PP^*R(\psi-\psi_\mu)=0$ for all $P,R\in\mathcal{E}_0$, and since $\{PP^*\,|\,P\in\mathcal{E}_0\}$ spans $\mathcal{E}_0^2$ (Lemma 2.1), it follows that $PQR(\psi-\psi_\mu)=0$ for all $P,Q,R\in\mathcal{E}_0$. By [11, Theorem 3] it follows that

$$\psi-\psi_\mu=a+h+q$$

where $a\in\mathbf{C}$, $h: S\to\mathbf{C}$ is additive, and q is the diagonal of a symmetric biadditive function on S. Since $\psi-\psi_\mu$ is hermitian then $a\in\mathbf{R}$, h is *-additive, and q is a quadratic form. Since $PP^*q=PP^*(\psi-\psi_\mu)=-\lambda_P^*(\{1\})\le 0$ for $P\in\mathcal{E}_0$ then q is negative definite. The triple (a,h,q) is uniquely determined according to [11, Theorem 3].

(iv) $\Rightarrow$ (iii): $\langle\psi,PP^*\rangle=\langle q,PP^*\rangle-\int\langle\sigma,PP^*\rangle\,d\mu(\sigma)\le 0$ for $P\in\mathcal{E}_0$. $\square$

7. The case of arbitrary semigroups

Lemma 7.1. *For each $n\in\{1,3,5,7,\dots\}$ there exist constants C_1, C_2, and C_3 such that for all $z\in\mathbf{C}$,*

(i) $|1-z|\le C_1\left|1-z|z|^{n-1}\right|$;

(ii) $\left|n-1-n\operatorname{Re} z+\operatorname{Re} z|z|^{n-1}\right|\le C_2\left|1-z|z|^{n-1}\right|^2$;

(iii) $\left|\operatorname{Im} z-\operatorname{Im} z|z|^{n-1}\right|\le C_3\left|1-z|z|^{n-1}\right|^2$.

Proof. (i): If $C_1\ge 1$ then the inequality holds for all z with $|z|$ large enough. The function $1-z|z|^{n-1}$ has no other zero than 1 since if $z|z|^{n-1}=1$ then $z>0$, hence $z^n=1$, so $z=1$. It therefore suffices to show

$$\liminf_{z\to 1}\left|\frac{1-z|z|^{n-1}}{1-z}\right|>0.$$

With $z=1+x+iy$ we have

$$1-z|z|^{n-1}=1-(1+x+iy)\sum_{k=0}^{(n-1)/2}\binom{(n-1)/2}{k}(1+x)^{n-1-2k}y^{2k},$$

and since $y^2/|1-z|\to 0$, it suffices to show

$$\liminf_{(x,y)\to(0,0)}\left|\frac{1-(1+x+iy)(1+x)^{n-1}}{x+iy}\right|>0.$$

Since $(1+x)^{n-1}=\sum_{k=0}^{n-1}\binom{n-1}{k}x^k$ and $x^2/(x+iy)\to 0$, it suffices to show

$$\begin{aligned}0<\liminf_{(x,y)\to(0,0)}\left|\frac{1-(1+x+iy)(1+(n-1)x)}{x+iy}\right| &= \liminf_{(x,y)\to(0,0)}\left|\frac{nx+iy+(n-1)ixy}{x+iy}\right|\\ =\liminf_{(x,y)\to(0,0)}\left|\frac{nx+iy}{x+iy}\right|,\end{aligned}$$

which is true since $n^2x^2+y^2\geq x^2+y^2$.

(ii): For any $C_2>0$, the inequality holds for all z with $|z|$ large enough. It therefore suffices to show

$$\limsup_{z\to 1}\left|\frac{n-1-n\operatorname{Re}z+\operatorname{Re}z|z|^{n-1}}{(1-z|z|^{n-1})^2}\right|<\infty.$$

By (i) it suffices to show

$$\limsup_{z\to 1}\left|\frac{n-1-n\operatorname{Re}z+\operatorname{Re}z|z|^{n-1}}{(1-z)^2}\right|<\infty.$$

With $z=1+x+iy$ we have

$$n-1-n\operatorname{Re}z+\operatorname{Re}z|z|^{n-1}=-1-nx+(1+x)\sum_{k=0}^{(n-1)/2}\binom{(n-1)/2}{k}(1+x)^{n-1-2k}y^{2k},$$

and since $y^2\leq|1-z|^2$, it suffices to show

$$\limsup_{(x,y)\to(0,0)}\left|\frac{(1+x)^n-1-nx}{x^2+y^2}\right|<\infty;$$

but this follows from $(1+x)^n-1-nx=\sum_{k=2}^{n}\binom{n}{k}x^k$.

(iii): For any $C_3>0$, the inequality holds for all z with $|z|$ large enough. It therefore suffices to show

$$\limsup_{z\to 1}\left|\frac{\operatorname{Im}z-\operatorname{Im}z|z|^{n-1}}{(1-z|z|^{n-1})^2}\right|<\infty.$$

By (i) it suffices to show

$$\limsup_{z\to 1}\left|\frac{\operatorname{Im} z-\operatorname{Im} z|z|^{n-1}}{(1-z)^2}\right|<\infty.$$

With $z=1+x+iy$ we have

$$\operatorname{Im} z|z|^{n-1}-\operatorname{Im} z=y\sum_{k=0}^{(n-1)/2}\binom{(n-1)/2}{k}(1+x)^{n-1-2k}y^{2k}-y,$$

and since $y^2\le|1-z|^2$, it suffices to show

$$\limsup_{(x,y)\to(0,0)}\left|\frac{y(1+x)^{n-1}-y}{x^2+y^2}\right|<\infty.$$

But this follows from $y(1+x)^{n-1}-y=\sum_{k=1}^{n-1}\binom{n-1}{k}x^k y$. $\square$

Proposition 7.1. *For every *-semigroup S there are a *-semigroup U and a *-homomorphism $f\colon S\to U$ such that, denoting by $\mathcal{F}$, $\mathcal{F}_0$, $\mathcal{F}_0^2$, and $(\mathcal{F}_0^2)_+$ the objects that are to U what $\mathcal{E}$, $\mathcal{E}_0$, $\mathcal{E}_0^2$, and $(\mathcal{E}_0^2)_+$ are to S, and defining a linear mapping $F\colon\mathcal{E}\to\mathcal{F}$ by $F(E_s)=E_{f(s)}$ for $s\in S$,*

(i) *U is perfect;*

(ii) *$f^*\colon U^*\to S^*$ is an isomorphism between the measurable spaces $(U^*,\mathcal{A}(U^*))$ and $(S^*,\mathcal{A}(S^*))$;*

(iii) *$\nu\mapsto\nu^{f^*}\colon F_+(U^*)\to F_+(S^*)$ is a homeomorphism with respect to the $\mathcal{L}$-topologies;*

(iv) *for each *-additive function $h\colon S\to\mathbf{C}$ there is a unique *-additive function $\tilde{h}\colon U\to\mathbf{C}$ such that $h=\tilde{h}\circ f$;*

(v) *for $P\in\ker F$ we have $\{P,-P\}\subset(\mathcal{E}_0^2)_+$;*

(vi) $F((\mathcal{E}_0^2)_+)=(\mathcal{F}_0^2)_+\cap F(\mathcal{E})$;

(vii) $(\mathcal{F}_0^2)_{\mathrm{sa}}=F((\mathcal{E}_0^2)_{\mathrm{sa}})+(\mathcal{F}_0^2)_+$;

(viii) $\mathcal{F}_0=F(\mathcal{E}_0)+\mathcal{F}_0^2$;

(ix) $\mathcal{F}=F(\mathcal{E})+\mathcal{F}_0^2$;

(x) *if H is a complex Lévy function for S, there is a unique complex Lévy function K for U such that $K(f(s),\omega)=H(s,f^*(\omega))$ for all $s\in S$ and $\omega\in U^*$;*

(xi) *for each negative definite quadratic form q on S there is a unique negative definite quadratic form $\tilde{q}$ on U such that $q=\tilde{q}\circ f$.*

Proof. For $n\in\{1,3,5,7,\ldots\}$, define $f_n\colon S\to\mathbf{C}^{S^*}$ by

$$f_n(s)(\sigma)=\sigma(s)|\sigma(s)|^{1/n-1},\quad s\in S,\ \sigma\in S^*$$

with $0|0|^{1/n-1}=0$ by definition. It is shown in the proof of [6, Proposition 6], that the set

$$U=\bigcup_{n\in\{1,3,5,7,\dots\}} f_n(S),$$

with pointwise multiplication and complex conjugation, is a perfect semigroup, that the mapping $j\colon S^*\to U^*$ defined by

$$j(\sigma)(u)=u(\sigma),\quad \sigma\in S^*,\ u\in U$$

is measurable with respect to $\mathcal{A}(S^*)$ and $\mathcal{A}(U^*)$, that $\mu\mapsto\mu^j$ maps $F_+(S^*)$ into $F_+(U^*)$, that the mapping $f=f_1$ is a *-homomorphism, and that $f^*\circ j$ is the identical mapping on S^*.

(ii): It remains to be shown that $j\circ f^*$ is the identical mapping on U^*, that is,

$$u(f^*(\omega))=\omega(u),\quad \omega\in U^*,\ u\in U. \tag{7.1}$$

We first note that for $s\in S$ and $n\in\{1,3,5,7,\dots\}$,

$$\text{if } u=f_n(s) \text{ then } \sigma(s)=u(\sigma)|u(\sigma)|^{n-1} \text{ for } \sigma\in S^*; \tag{7.2}$$

this is an immediate consequence of the definition of f_n. Concerning (7.1), choosing $s\in S$ and $n\in\{1,3,5,7,\dots\}$ such that $u=f_n(s)$, we have $u(u\bar{u})^{(n-1)/2}=f(s)$ by (7.2), hence, using (7.2) again,

$$\omega(u)|\omega(u)|^{n-1}=\omega(f(s))=f^*(\omega)(s)=u(f^*(\omega))|u(f^*(\omega))|^{n-1},$$

and (7.1) follows.

(iii): Let us show that $\mu\mapsto\mu^j\colon F_+(S^*)\to F_+(U^*)$ is continuous with respect to the $\mathcal{L}$-topologies. Suppose (μ_i) is a net in $F_+(S^*)$ converging in the $\mathcal{L}$-topology to some $\mu\in F_+(S^*)$; we have to show that $\mu_i^j\to\mu^j$ in the $\mathcal{L}$-topology on $F_+(U^*)$. By Lemma 3.1 it suffices to show $\langle\cdot\,,E_{u\bar{u}}\rangle\mu_i^j\to\langle\cdot\,,E_{u\bar{u}}\rangle\mu^j$ in the inverse limit topology for each $u\in U$, and this amounts to showing

$$\langle\cdot\,,E_{u\bar{u}}\rangle(\mu_i^j)^{p_{U,V}}\to\langle\cdot\,,E_{u\bar{u}}\rangle(\mu^j)^{p_{U,V}}\quad\text{weakly} \tag{7.3}$$

for every countable *-subsemigroup V of U containing u.

Choose $s\in S$ and $n\in\{1,3,5,7,\dots\}$ such that $u=f_n(s)$, then choose a countable *-subsemigroup T of S such that $s\in T$ and

$$V\subset\bigcup_{m\in\{1,3,5,7,\dots\}} f_m(T)$$

and furthermore such that every character on T extends to a character on S (which is possible by [3, Theorem 3]). Since $\mu_i \to \mu$ in the $\mathcal{L}$-topology then $(1+\langle\cdot, E_{ss^*}\rangle)\mu_i \to (1+\langle\cdot, E_{ss^*}\rangle)\mu$ in the inverse limit topology, hence

$$(1+\langle\cdot, E_{ss^*}\rangle)\mu_i^{p_{S,T}} \to (1+\langle\cdot, E_{ss^*}\rangle)\mu^{p_{S,T}} \quad \text{weakly.} \tag{7.4}$$

We wish to define a mapping $k: T^* \to V^*$ such that

$$k \circ p_{S,T} = p_{U,V} \circ j, \tag{7.5}$$

that is, $k(\sigma|_T)(v) = v(\sigma)$ for $\sigma \in S^*$ and $v \in V$. Given $\tau \in T^*$, we can choose $\sigma \in S^*$ such that $\tau = \sigma|_T$, and then we must have

$$k(\tau)(v) = v(\sigma), \quad v \in V.$$

The right-hand side is independent of the choice of σ since, choosing $t \in T$ and $m \in \{1, 3, 5, 7, \dots\}$ such that $v = f_m(t)$, we have

$$v(\sigma) = \sigma(t)|\sigma(t)|^{1/n-1} = \tau(t)|\tau(t)|^{1/n-1}.$$

This shows that k is well-defined and also that k is continuous. From (7.4) it now follows, using the fact that $\langle\cdot, E_{ss^*}\rangle = \langle\cdot, E_{f(ss^*)}\rangle \circ k$, that

$$(1+\langle\cdot, E_{f(ss^*)}\rangle)(\mu_i^{p_{S,T}})^k \to (1+\langle\cdot, E_{f(ss^*)}\rangle)(\mu^{p_{S,T}})^k \quad \text{weakly,}$$

which by (7.5) is equivalent to

$$(1+\langle\cdot, E_{f(ss^*)}\rangle)(\mu_i^j)^{p_{U,V}} \to (1+\langle\cdot, E_{f(ss^*)}\rangle)(\mu^j)^{p_{U,V}} \quad \text{weakly,}$$

and then (7.3) follows from the fact that $\langle\cdot, E_{u\bar{u}}\rangle/(1+\langle\cdot, E_{f(ss^*)}\rangle)$ is a bounded continuous function on V^*.

(iv): If $u \in U$ has the form $u = f_n(s)$ with $s \in S$ and $n \in \{1, 3, 5, 7, \dots\}$ then, since $f(s) = u^{(n+1)/2}\bar{u}^{(n-1)/2}$, we must have $h(s) = \frac{1}{2}(n+1)\tilde{h}(u) + \frac{1}{2}(n-1)\tilde{h}(\bar{u})$, that is,

$$\operatorname{Re}\tilde{h}(u) = \frac{1}{n}\operatorname{Re} h(s), \quad \operatorname{Im}\tilde{h}(u) = \operatorname{Im} h(s). \tag{7.6}$$

We have to show that this is independent of the choice of s and n. Suppose $r \in S$ and $m \in \{1, 3, 5, 7, \dots\}$ are another pair such that $u = f_m(r)$. For $t > 0$ we have $e^{th} \in S^*$, so using (7.2),

$$\begin{aligned} e^{t((m+1)h(s)/2+(m-1)\bar{h}(s)/2)} &= e^{th(s)}|e^{th(s)}|^{m-1} \\ &= u(e^{th})|u(e^{th})|^{n-1}\big|u(e^{th})|u(e^{th})|^{n-1}\big|^{m-1} \\ &= u(e^{th})|u(e^{th})|^{mn-1} \\ &= u(e^{th})|u(e^{th})|^{m-1}\big|u(e^{th})|u(e^{th})|^{m-1}\big|^{n-1} \\ &= e^{th(r)}|e^{th(r)}|^{n-1} = e^{t((n+1)h(r)/2+(n-1)\bar{h}(r)/2)}. \end{aligned}$$

This being so for all $t>0$, it follows that

$$\tfrac{1}{2}(m+1)h(s)+\tfrac{1}{2}(m-1)\bar{h}(s)=\tfrac{1}{2}(n+1)h(r)+\tfrac{1}{2}(n-1)\bar{h}(r),$$

whence $m\operatorname{Re}h(s)=n\operatorname{Re}h(r)$ and $\operatorname{Im}h(s)=\operatorname{Im}h(r)$, as desired. Thus $\tilde{h}$ is well-defined. Since $u^*=f_n(s^*)$, then $n\operatorname{Re}\tilde{h}(u^*)=\operatorname{Re}h(s^*)=\operatorname{Re}h(s)=n\operatorname{Re}\tilde{h}(u)$ and (similarly) $\operatorname{Im}\tilde{h}(u^*)=-\operatorname{Im}\tilde{h}(u)$, so $\tilde{h}$ is *-preserving.

Given $u,v\in U$, choose $s,r\in S$ and $n,m\in\{1,3,5,7,...\}$ such that $u=f_n(s)$ and $v=f_m(r)$. Then $uv=f_{nm}(q)$ where $q=sr(ss^*)^{(m-1)/2}(rr^*)^{(n-1)/2}$, so

$$\operatorname{Re}\tilde{h}(uv)=\frac{1}{nm}\operatorname{Re}h(q)=\frac{1}{nm}(m\operatorname{Re}h(s)+n\operatorname{Re}h(r))=\operatorname{Re}(\tilde{h}(u)+\tilde{h}(v))$$

and (similarly) $\operatorname{Im}\tilde{h}(uv)=\operatorname{Im}(\tilde{h}(u)+\tilde{h}(v))$, which shows that $\tilde{h}$ is additive.

(v): Suppose $P\in\ker F$. For $\omega\in U^*$ we have $\langle f^*(\omega),P\rangle=\langle\omega,F(P)\rangle=0$, and since f^* maps U^* onto S^*, this shows $\langle\sigma,P\rangle=0$ for all $\sigma\in S^*$, that is, $\{P,-P\}\subset\mathcal{E}_+$. If $a\in\mathbf{R}$ and if $h\colon S\to\mathbf{C}$ is *-additive, choosing $\tilde{h}$ as in (iv) we have $\langle a+h,P\rangle=\langle(a+\tilde{h})\circ f,P\rangle=\langle a+\tilde{h},F(P)\rangle=0$. This being so for all such a and h, by Lemma 4.1 it follows that $P\in\mathcal{E}_0^2$, hence also $-P\in\mathcal{E}_0^2$.

(vi): The left-hand side is clearly contained in the right-hand side. Now suppose $Q\in(\mathcal{F}_0^2)_+\cap F(\mathcal{E})$ and choose $P\in\mathcal{E}$ such that $Q=F(P)$. If $a\in\mathbf{R}$ and if $h\colon S\to\mathbf{C}$ is *-additive, choosing $\tilde{h}$ as in (iv) we have

$$\langle a+h,P\rangle=\langle(a+\tilde{h})\circ f,P\rangle=\langle a+\tilde{h},F(P)\rangle=\langle a+\tilde{h},Q\rangle=0$$

since $Q\in\mathcal{F}_0^2$. This being so for all such a and h, by Lemma 4.1 it follows that $P\in\mathcal{E}_0^2$. For $\omega\in U^*$ we have $\langle f^*(\omega),P\rangle=\langle\omega,F(P)\rangle=\langle\omega,Q\rangle\geq 0$, and since f^* maps U^* onto S^*, it follows that $P\in\mathcal{E}_+$. Thus $P\in(\mathcal{E}_0^2)_+$, which shows $Q\in F((\mathcal{E}_0^2)_+)$.

(vii): We note that U^* separates points in U. Indeed, if $u,v\in U$ are such that $\omega(u)=\omega(v)$ for all $\omega\in U^*$ then, using the identity (7.1) and the fact that f^* maps U^* onto S^*, we get $u(\sigma)=v(\sigma)$ for all $\sigma\in S^*$, that is, $u=v$. By Lemma 2.2 it follows that $\mathcal{F}_+\subset\mathcal{F}_{\mathrm{sa}}$, hence $(\mathcal{F}_0^2)_+\subset(\mathcal{F}_0^2)_{\mathrm{sa}}$, which shows that the right-hand side in (vii) is contained in the left-hand side. To show the converse inclusion, it suffices to show that for each $Q\in\mathcal{F}_0^2$ there is some $P\in F((\mathcal{E}_0^2)_{\mathrm{sa}})$ such that $|\langle\omega,Q\rangle|\leq\langle\omega,P\rangle$ for all $\omega\in U^*$, since for $Q\in(\mathcal{F}_0^2)_{\mathrm{sa}}$ we then have $Q=-P+(Q+P)$ with $-P\in F((\mathcal{E}_0^2)_{\mathrm{sa}})$ and $Q+P\in(\mathcal{F}_0^2)_+$. It suffices to consider the case $Q=(E_u-I)(E_v-I)$ with $u,v\in U$ since such Q span $\mathcal{F}_0^2$. Since

$$|\langle\omega,(E_u-I)(E_v-I)\rangle|=|\langle\omega,E_u-I\rangle\langle\omega,E_v-I\rangle|\leq\tfrac{1}{2}(|\langle\omega,E_u-I\rangle|^2+|\langle\omega,E_v-I\rangle|^2),$$

it suffices to consider the case $v=u^*$. Choose $s\in S$ and $n\in\{1,3,5,7,\ldots\}$ such that $u=f_n(s)$. By Lemma 7.1(i) applied to $z=\sigma(s)|\sigma(s)|^{1/n-1}$, for $\omega\in U^*$ and $\sigma=f^*(\omega)\in S^*$ we have

$$\begin{aligned}|\langle\omega,E_u-I\rangle|^2&=|\omega(u)-1|^2=|u(\sigma)-1|^2=\left|\sigma(s)|\sigma(s)|^{1/n-1}-1\right|^2\\&\le C_1|\sigma(s)-1|^2=C_1|\langle\sigma,E_s-I\rangle|^2=C_1|\langle\omega,F(E_s-I)\rangle|^2\end{aligned}$$

which shows that $P=F((E_s-I)(E_{s^*}-I))$ has the required property.

(viii): It suffices to show $E_u-I\in F(\mathcal{E}_0)+\mathcal{F}_0^2$ for $u\in U$, since such operators span $\mathcal{F}_0$. Let us show

$$(n-1)I+\tfrac12(n+1)E_{f(s)}-\tfrac12(n-1)E_{f(s^*)}-nE_u\in\mathcal{F}_0^2 \tag{7.7}$$

if $s\in S$, $n\in\{1,3,5,7,\ldots\}$, and $u=f_n(s)$. Let P be the operator in (7.7). The sum of the coefficients is zero, so $\langle a,P\rangle=0$ for $a\in\mathbf{R}$. If $\tilde h\colon U\to\mathbf{C}$ is *-additive then

$$\begin{aligned}\langle\tilde h,P\rangle&=(n-1)\tilde h(1)+\tfrac12(n+1)\tilde h(f(s))-\tfrac12(n-1)\tilde h(f(s^*))-n\tilde h(u)\\&=\operatorname{Re}\tilde h(f(s))+n\operatorname{Im}\tilde h(f(s))-n\tilde h(u)=0\end{aligned}$$

by (7.6). By Lemma 4.1 it follows that $P\in\mathcal{F}_0^2$. It thus suffices to show

$$-I+\tfrac12(n+1)E_{f(s)}-\tfrac12(n-1)E_{f(s^*)}\in F(\mathcal{E}_0);$$

but this operator is $F(Q)$ where $Q=-I+\frac12(n+1)E_s-\frac12(n-1)E_{s^*}\in\mathcal{E}_0$.

(ix): It suffices to show $E_u\in F(\mathcal{E})+\mathcal{F}_0^2$ for $u\in U$. But this follows from (7.7).

(x): For $\sigma\in S^*$ let h_σ be the *-additive function $s\mapsto H(s,\sigma)$ on S and let $\tilde h_\sigma$ be the unique *-additive function on U such that $h_\sigma=\tilde h_\sigma\circ f$. Since $K(\cdot,\omega)$ has to be *-additive for $\omega\in U^*$, we must have

$$K(u,\omega)=\tilde h_{f^*(\omega)},\quad u\in U,\ \omega\in U^*.$$

To see that $\omega\mapsto K(u,\omega)$ is measurable for $u\in U$, since f^* is measurable it suffices to show that $\sigma\mapsto\tilde h_\sigma(u)$ is measurable on S^*. Choosing $s\in S$ and $n\in\{1,3,5,7,\ldots\}$ such that $u=f_n(s)$, by (7.6) we have

$$\tilde h_\sigma(u)=\frac1n\operatorname{Re}h_\sigma(s)+i\operatorname{Im}h_\sigma(s)=\frac1n\operatorname{Re}H(s,\sigma)+i\operatorname{Im}H(s,\sigma),$$

so the measurability follows from that of $\sigma\mapsto H(s,\sigma)$.

Now let ν be a measure on U^* such that

$$\int\langle\omega,PP^*\rangle\,d\nu(\omega)<\infty,\quad P\in\mathcal{F}_0; \tag{7.8}$$

we have to show

$$\int |1-\omega(u)+K(u,\omega)|\, d\nu(\omega) < \infty$$

for $u\in U$. Choose $s\in S$ and $n\in\{1,3,5,7,\dots\}$ such that $u=f_n(s)$. We then have to show

$$\int \left|1-\sigma(s)|\sigma(s)|^{1/n-1}+\frac{1}{n}\operatorname{Re}H(s,\sigma)+i\operatorname{Im}H(s,\sigma)\right| d\mu(\sigma) < \infty \tag{7.9}$$

where $\mu=\nu^{f^*}$. Since $F(\mathcal{E}_0)\subset\mathcal{F}_0$, by (7.8) we have $\int\langle\sigma,QQ^*\rangle\, d\mu(\sigma)<\infty$ for $Q\in\mathcal{E}_0$, so

$$\int |1-\sigma(s)+H(s,\sigma)|\, d\mu(\sigma) < \infty. \tag{7.10}$$

To show that the real part of the integrand in (7.9) is integrable, since the real part of the integrand in (7.10) is integrable, it suffices to show

$$\int \left|n-1-n\operatorname{Re}\sigma(s)|\sigma(s)|^{1/n-1}+\operatorname{Re}\sigma(s)\right| d\mu(\sigma) < \infty.$$

Since $E_s-I\in\mathcal{E}_0$, we have

$$\int |1-\sigma(s)|^2\, d\mu(\sigma) < \infty, \tag{7.11}$$

so it suffices to show $\left|n-1-n\operatorname{Re}w|w|^{1/n-1}-\operatorname{Re}w\right|\le C|1-w|^2$ for some $C>0$ and all $w\in\mathbf{C}$, which is equivalent to Lemma 7.1(ii).

To show that the imaginary part of the integrand in (7.9) is integrable, since the imaginary part of the integrand in (7.10) is integrable, it suffices to show

$$\int \left|\operatorname{Im}\sigma(s)-\operatorname{Im}\sigma(s)|\sigma(s)|^{1/n-1}\right| d\mu(\sigma) < \infty.$$

By (7.11) it suffices to show $\left|\operatorname{Im}w-\operatorname{Im}w|w|^{1/n-1}\right|\le C|1-w|^2$ for some $C>0$ and all $w\in\mathbf{C}$, which is equivalent to Lemma 7.1(iii).

(xi): Write $X=\mathcal{E}_0/\mathcal{E}_0^2$, let $P\mapsto\widetilde{P}\colon\mathcal{E}_0\to X$ be the quotient mapping, and define $j\colon S\to X$ by $j(s)=(E_s-I)^\sim$, $s\in S$. By Proposition 4.1 there is a unique nonnegative sesquilinear form $\langle\cdot,\cdot\rangle$ on X such that $q(s)=-\langle j(s),j(s^*)\rangle$ for all $s\in S$. Write $Z=\mathcal{F}_0/\mathcal{F}_0^2$, let $Q\mapsto\widetilde{Q}\colon\mathcal{F}_0\to Z$ be the quotient mapping, and define $k\colon U\to Z$ by $k(u)=(E_u-I)^\sim$, $u\in U$. By Proposition 4.1, existence and uniqueness of $\tilde{q}$ as claimed is equivalent to the existence and uniqueness of a sesquilinear form $\langle\cdot,\cdot\rangle$ on Z such that

$$\langle j(s),j(s^*)\rangle=\langle k(f(s)),k(f(s^*))\rangle. \tag{7.12}$$

Since $P\mapsto F(P)^\sim\colon \mathcal{E}_0\to Z$ vanishes on $\mathcal{E}_0^2$, there is a unique linear mapping $m\colon X\to Z$ (clearly *-preserving) such that

$$F(P)^\sim=m(\widetilde{P}),\quad P\in\mathcal{E}_0.$$

If (7.12) holds then, because of

$$k(f(s))=(E_{f(s)}-I)^\sim=(F(E_s-I))^\sim=m((E_s-I)^\sim)=m(j(s)),$$

it follows that

$$\langle j(s),j(s^*)\rangle=\langle m(j(s)),m(j(s^*))\rangle,\quad s\in S.$$

Then $[x,y]=\langle m(x),m(y)\rangle-\langle x,y\rangle$ defines a sesquilinear form $[\cdot,\cdot]$ on X satisfying $[j(s),j(s^*)]=0$ for all $s\in S$, and by the uniqueness statement of Proposition 4.1 it follows that $[\cdot,\cdot]$ is identically zero, that is,

$$\langle x,y\rangle=\langle m(x),m(y)\rangle,\quad x,y\in X. \tag{7.13}$$

Since $m(X)=Z$ (by (viii)), this shows the uniqueness in (7.12). If only we have (7.13) then (7.12) follows. Since the sesquilinear form $\langle\cdot,\cdot\rangle$ on X is nonnegative, it suffices to show $\langle x,x\rangle=0$ for $x\in\ker m$. If $x\in\ker m$ then $x=\widetilde{P}$ for some $P\in\mathcal{E}_0$ with $F(P)^\sim=0$, that is, $F(P)\in F(\mathcal{E}_0)\cap\mathcal{F}_0^2$. But this implies $P\in\mathcal{E}_0^2$ (so $x=0$) since we saw in the proof of (vi) that $F^{-1}(\mathcal{F}_0^2)\subset\mathcal{E}_0^2$. □

Theorem 7.1. *Suppose S is a *-semigroup with a complex Lévy function H. For a function $\psi\colon S\to\mathbf{C}$, the following conditions are equivalent*:

(i) *There is a convolution semigroup (μ_t) in $F_+(S^*)$ such that $\mathcal{L}\mu_t=e^{-t\psi}$ for all $t\geq0$;*

(ii) *$e^{-t\psi}\in\mathcal{H}(S)$ for all $t\geq0$;*

(iii) *ψ is hermitian and $\langle\psi,P\rangle\leq0$ for all $P\in(\mathcal{E}_0^2)_+$;*

(iv) *there exist $a\in\mathbf{R}$, a *-additive function $h\colon S\to\mathbf{C}$, a negative definite quadratic form q on S, and a measure μ on $\mathcal{A}(S^*\setminus\{1\})$ such that*

$$\psi(s)=a+h(s)+q(s)+\int(1-\sigma(s)+H(s,\sigma))\,d\mu(s),\quad s\in S. \tag{7.14}$$

The convolution semigroups occurring in (i) *are all continuous in the $\mathcal{L}$-topology. There is a one-to-one correspondence between the convolution semigroups* (μ_t) *occurring in* (i) *and the measures* μ *occurring in* (iv), *each set being in a one-to-one correspondence with the set of those* $\Psi\in\mathcal{N}(U)$ *such that* $\psi=\Psi\circ f$, *where* U *and* f *are as in Proposition* 7.1.

Proof. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): For $s\in S$ and $t>0$ we have $e^{-t\psi(s^*)}=e^{-t\bar{\psi}(s)}$ since $e^{-t\psi}$, being a moment function, is hermitian. This being so for all $t>0$, it follows that $\psi(s^*)=\bar{\psi}(s)$. For $P\in(\mathcal{E}_0^2)_+$ we have $\langle 1,P\rangle=0$ since $P\in\mathcal{E}_0$ and $\langle e^{-t\psi},P\rangle\geq 0$ since $P\in\mathcal{E}_+$ and $e^{-t\psi}\in\mathcal{H}(S)$. It follows that

$$\langle\psi,P\rangle=\lim_{t\to 0}\frac{1}{t}\langle 1-e^{-t\psi},P\rangle\leq 0.$$

(iii) $\Rightarrow$ (iv): Let U, f, $\mathcal{F}$, etc., and F be as in Proposition 7.1. If $P\in\ker F$ then by Proposition 7.1(v) we have $\{P,-P\}\subset(\mathcal{E}_0^2)_+$, and by (iii) it follows that $\langle\psi,P\rangle=0$. Hence there is a unique linear form L_0 on $F(\mathcal{E})$, clearly hermitian, such that

$$\langle\psi,P\rangle=L_0(F(P)),\quad P\in\mathcal{E}.$$

Evidently $L_0\leq 0$ on $F((\mathcal{E}_0^2)_+)$. By Proposition 7.1(vi), this means that $L_0\leq 0$ on $(\mathcal{F}_0^2)_+\cap F(\mathcal{E})$.

We wish to extend $L_0|_{(\mathcal{F}_0^2\cap F(\mathcal{E}))_{\mathrm{sa}}}$ to a real linear form on $(\mathcal{F}_0^2)_{\mathrm{sa}}$ which is ≤ 0 on $(\mathcal{F}_0^2)_+$. By [2, 1.2.7], this can be done if $(\mathcal{F}_0^2\cap F(\mathcal{E}))_{\mathrm{sa}}+(\mathcal{F}_0^2)_+=(\mathcal{F}_0^2)_{\mathrm{sa}}$. The proof of Proposition 7.1(vi) shows that $\mathcal{F}_0^2\cap F(\mathcal{E})=F(\mathcal{E}_0^2)$, so the condition follows from $F((\mathcal{E}_0^2)_{\mathrm{sa}})+(\mathcal{F}_0^2)_+=(\mathcal{F}_0^2)_{\mathrm{sa}}$, which is true by Proposition 7.1(vii).

Thus there is a real linear form L' on $(\mathcal{F}_0^2)_{\mathrm{sa}}$ which extends $L_0|_{(\mathcal{F}_0^2\cap F(\mathcal{E}))_{\mathrm{sa}}}$ and is ≤ 0 on $(\mathcal{F}_0^2)_+$. Let L_1 be the unique hermitian complex linear form on $\mathcal{F}_0^2$ which extends L'. Since $L_0|_{\mathcal{F}_0^2\cap F(\mathcal{E})}$ and $L_1|_{\mathcal{F}_0^2\cap F(\mathcal{E})}$ are both hermitian and coincide on $(\mathcal{F}_0^2\cap F(\mathcal{E}))_{\mathrm{sa}}$, they are equal. It follows that there is a unique linear form L on $F(\mathcal{E})+\mathcal{F}_0^2$ which extends both L_0 and L_1. Clearly L is hermitian and $L\leq 0$ on $(\mathcal{F}_0^2)_+$. By Proposition 7.1(ix), L is defined on all of $\mathcal{F}$. Defining $\Psi\colon U\to\mathbf{C}$ by $\Psi(u)=L(E_u)$, $u\in U$, we have $\psi=\Psi\circ f$, Ψ is hermitian, and $\langle\Psi,PP^*\rangle=L(PP^*)\leq 0$ for $P\in\mathcal{F}_0$, that is, $\Psi\in\mathcal{N}(U)$.

By Theorem 6.1 it now follows that there exist unique a, $\tilde{h}$, $\tilde{q}$, and ν such that $a\in\mathbf{R}$, $\tilde{h}$ is a *-additive function on U, $\tilde{q}$ is a negative definite quadratic form on U, ν is a measure on $\mathcal{A}(U^*\setminus\{1\})$, and

$$\Psi(u)=a+\tilde{h}(u)+\tilde{q}(u)+\int(1-\omega(u)+K(u,\omega))\,d\nu(\omega),\quad u\in U \tag{7.15}$$

where K is the complex Lévy function for U determined in Proposition 7.1(x). From the latter equation we get (iv) with $h=\tilde{h}\circ f$, $q=\tilde{q}\circ f$, and $\mu=\nu^{f^*}$.

(iv) $\Rightarrow$ (iii): For $P\in(\mathcal{E}_0^2)_+$ we have $\langle\psi,P\rangle=\langle q,P\rangle-\int\langle\sigma,P\rangle\,d\mu(\sigma)$, so it suffices to show $\langle q,P\rangle\leq 0$. Since q is negative definite, $e^{-tq}\in\mathcal{H}(S)$ for $t>0$ by Proposition 4.2, so

$$\langle q,P\rangle=\lim_{t\to 0}\frac{1}{t}\langle 1-e^{-tq},P\rangle\leq 0.$$

(iii) $\Rightarrow$ (i): As in the proof of (iii) $\Rightarrow$ (iv) there is some $\Psi\in\mathcal{N}(U)$ such that $\psi=\Psi\circ f$. By Theorem 6.1 there is a unique convolution semigroup (ν_t) in $F_+(U^*)$ such that $\mathcal{L}\nu_t=e^{-t\Psi}$ for all $t>0$. With $\mu_t=\nu_t^{f^*}$ we then have $\mu_t\in F_+(S^*)$, (μ_t) is a convolution semigroup, and $\mathcal{L}\mu_t=e^{-t\psi}$ for all $t>0$.

The convolution semigroups occurring in (i) are continuous in the $\mathcal{L}$-topology by Proposition 3.4.

To see that the convolution semigroups (μ_t) occurring in (i) are in a one-to-one correspondence with those $\Psi\in\mathcal{N}(U)$ such that $\psi=\Psi\circ f$, first let such a Ψ be given. Let (ν_t) be the unique convolution semigroup in $F_+(U^*)$ such that $\mathcal{L}\nu_t=e^{-t\Psi}$ for all $t\geq 0$ (cf. Theorem 6.1). Define $\mu_t=\nu_t^{f^*}$ for $t\geq 0$. Then (μ_t) is a convolution semigroup in $F_+(S^*)$ and $\mathcal{L}\mu_t=e^{-t\Psi}\circ f=e^{-t\psi}$ for all $t\geq 0$.

Conversely, let (μ_t) be a convolution semigroup in $F_+(S^*)$ such that $\mathcal{L}\mu_t=e^{-t\psi}$ for all $t\geq 0$. By Proposition 3.4, (μ_t) is continuous in the $\mathcal{L}$-topology. With $\nu_t=\mu_t^{f^{*-1}}$, the family (ν_t) is a convolution semigroup in $F_+(U^*)$, continuous in the $\mathcal{L}$-topology by Proposition 7.1(iii), so by Proposition 3.4 there is some $\Psi\in\mathcal{N}(U)$ such that $\mathcal{L}\nu_t=e^{-t\Psi}$ for all $t\geq 0$. Since $e^{-t\Psi}\circ f=(\mathcal{L}\nu_t)\circ f=\mathcal{L}(\nu_t^{f^*})=\mathcal{L}\mu_t=e^{-t\psi}$ for all $t>0$ then $\Psi\circ f=\psi$.

The mappings $\Psi\mapsto(\mu_t)$ and $(\mu_t)\mapsto\Psi$ that we have defined are clearly inverses of each other.

To see that the measures μ occurring in (iv) are in a one-to-one correspondence with those $\Psi\in\mathcal{N}(U)$ such that $\psi=\Psi\circ f$, first consider any such Ψ. Let ν be the Lévy measure of Ψ. By Theorem 6.1, (7.15) holds for some $a\in\mathbf{R}$, some *-additive function $\tilde{h}\colon U\to\mathbf{C}$, and some negative definite quadratic form $\tilde{q}$ on U, where K is the complex Lévy function on U determined in Proposition 7.1(x). Hence we get (iv) with $h=\tilde{h}\circ f$, $q=\tilde{q}\circ f$, and $\mu=\nu^{f^*}$.

Conversely, suppose (iv) holds and define $\Psi\colon U\to\mathbf{C}$ by (7.15), where $\tilde{h}$ is the unique *-additive function on U such that $h=\tilde{h}\circ f$, $\tilde{q}$ is the unique negative definite quadratic form on U such that $q=\tilde{q}$, $\nu=\mu^{f^{*-1}}$, and K is the complex Lévy function on U determined by Proposition 7.1(x). It follows that $\Psi\in\mathcal{N}(U)$ and that ν is the Lévy measure of Ψ.

The mappings $\Psi\mapsto\mu$ and $\mu\mapsto\Psi$ that we have defined are clearly the inverses of each other. $\square$

8. The case of semiperfect semigroups

A *-semigroup S is *semiperfect* if every positive definite function on S is a moment function.

Theorem 8.1. *Suppose S is a semiperfect semigroup with a complex Lévy function H. For a function $\psi\colon S\to\mathbf{C}$ the following conditions are equivalent:*

(i) *There is a convolution semigroup (μ_t) in $F_+(S^*)$ such that $\mathcal{L}\mu_t=e^{-t\psi}$ for all $t\geq 0$;*

(ii) *$e^{-t\psi}\in\mathcal{P}(S)$ for all $t>0$;*

(iii) *$\psi\in\mathcal{N}(S)$;*

(iv) *there exist $a\in\mathbf{R}$, a *-additive function $h\colon S\to\mathbf{C}$, a negative definite quadratic form q on S, and a measure μ on $\mathcal{A}(S^*\setminus\{1\})$ such that*

$$\psi(s)=a+h(s)+q(s)+\int(1-\sigma(s)+H(s,\sigma))\,d\mu(\sigma),\quad s\in S. \tag{8.1}$$

Proof. (i) $\Rightarrow$ (iv): By Proposition 3.4, (μ_t) is continuous in the $\mathcal{L}$-topology. Let U and f be as in Proposition 7.1. With $\nu_t=\mu_t^{f^{*-1}}$ for $t\geq 0$, the family (ν_t) is a convolution semigroup in $F_+(U^*)$, continuous in the $\mathcal{L}$-topology, so by Proposition 3.4 there exists $\Psi\in\mathcal{N}(U)$ such that $\mathcal{L}\nu_t=e^{-t\Psi}$ for all t. Since $e^{-t\Psi}\circ f=(\mathcal{L}\nu_t)\circ f=\mathcal{L}(\nu_t^{f^*})=\mathcal{L}\mu_t=e^{-t\psi}$ for all $t>0$ then $\Psi\circ f=\psi$. Now use (iv) from Theorem 6.1, with K the Lévy function for U determined in Proposition 7.1(x), and transfer the result to S using $\psi=\Psi\circ f$.

(iv) $\Rightarrow$ (iii): $\langle\psi,PP^*\rangle=\langle q,PP^*\rangle-\int\langle\sigma,PP^*\rangle\,d\mu(\sigma)\leq 0$ for $P\in\mathcal{E}_0$.

(iii) $\Rightarrow$ (ii): Already observed.

(ii) $\Rightarrow$ (i): For $t>0$, since S is semiperfect, we can choose $\lambda_t\in F_+(S^*)$ such that $\mathcal{L}\lambda_t=e^{-t\psi}$. Let (t_i) be a universal subnet of the identical net on $(]0,\infty[,\geq)$. For $t>0$ we have, denoting by $\lfloor t/t_i\rfloor$ the largest integer not exceeding t/t_i,

$$\mathcal{L}(\lambda_{t_i}^{*\lfloor t/t_i\rfloor})=(\mathcal{L}\lambda_{t_i})^{\lfloor t/t_i\rfloor}=e^{-\lfloor t/t_i\rfloor t_i\psi}\to e^{-t\psi},$$

so by Proposition 3.1, $(\lambda_{t_i}^{*\lfloor t/t_i\rfloor})$ converges in the $\mathcal{L}$-topology to some $\mu_t\in F_+(S^*)$ such that $\mathcal{L}\mu_t=e^{-t\psi}$. We note that $\mathcal{L}\lambda_{t_i}=e^{-t_i\psi}\to 1$, and since $1=\mathcal{L}\varepsilon_1$ is determinate, by Proposition 3.1 it follows that $\lambda_{t_i}\to\varepsilon_1$ in the $\mathcal{L}$-topology. For $t,u>0$ and each i there is some $n_i\in\{0,1\}$ such that $\lfloor(t+u)/t_i\rfloor=\lfloor t/t_i\rfloor+\lfloor u/t_i\rfloor+n_i$, hence

$$\mu_{t+u}=\lim_i\lambda_{t_i}^{*\lfloor(t+u)/t_i\rfloor}=\lim_i\lambda_{t_i}^{*\lfloor t/t_i\rfloor}*\lambda_{t_i}^{*\lfloor u/t_i\rfloor}*\lambda_{t_i}^{*n_i}=\mu_t*\mu_u.\quad\square$$

9. Comparison with the classical case

For a *-semigroup S, let $\widehat{S}$ denote the set of all bounded characters on S. Note that $|\sigma(s)|\le 1$ for $\sigma\in\widehat{S}$ and $s\in S$. A *Lévy function* for S is a mapping $L\colon S\times\widehat{S}\to\mathbf{R}$ satisfying the following requirements:

(i) $L(\,\cdot\,,\sigma)$ is a *-homomorphism of S into $(\mathbf{R},+,-\mathrm{id})$ for each $\sigma\in\widehat{S}$;

(ii) $L(s,\cdot\,)$ is $\mathcal{B}(\widehat{S})$-measurable for each $s\in S$;

(iii) whenever μ is a Radon measure on $\widehat{S}\setminus\{1\}$ satisfying

$$\int (1-\operatorname{Re}\sigma(s))\,d\mu(\sigma)<\infty,\quad s\in S \tag{9.1}$$

then

$$\int |1-\sigma(s)+iL(s,\sigma)|\,d\mu(\sigma)<\infty,\quad s\in S. \tag{9.2}$$

(Here $(\mathbf{R},+,-\mathrm{id})$ denotes $(\mathbf{R},+)$ with the involution $x^*=-x$.) That a Lévy function exists for every *-semigroup S was shown in [10].

Proposition 9.1. *If a complex Lévy function H for S is constructed as in the proof of Proposition* 5.1 *then $L=\operatorname{Im}H|_{S\times\widehat{S}}$ is a Lévy function for S.*

Proof. Condition (i) is clearly fulfilled. To see that (ii) holds, note that for $s\in S$, with H as in the proof of Proposition 5.1, we have $H(s,\sigma)=\langle\sigma,Q\rangle$ for some $Q\in\mathcal{E}$ and all $\sigma\in S^*$, and this function of σ, being a linear combination of functions of the form $\sigma\mapsto\sigma(t)$ with $t\in S$, is in fact continuous on S^*. To see that (iii) holds, suppose μ satisfies (9.1). For $z\in\mathbf{C}$ with $|z|\le 1$ we have

$$(\operatorname{Im}z)^2\le 1-(\operatorname{Re}z)^2=(1+\operatorname{Re}z)(1-\operatorname{Re}z)\le 2(1-\operatorname{Re}z)$$

and $(1-\operatorname{Re}z)^2\le 2(1-\operatorname{Re}z)$, hence $|1-z|^2\le 4(1-\operatorname{Re}z)$. It follows that

$$\int |1-\sigma(s)|^2\,d\mu(\sigma)\le 4\int(1-\operatorname{Re}\sigma(s))\,d\mu(\sigma)<\infty,\quad s\in S,$$

and since H is a complex Lévy function,

$$\int |1-\sigma(s)+H(s,\sigma)|\,d\mu(\sigma)<\infty,\quad s\in S.$$

The integrability of the imaginary part shows that the imaginary part in (9.2) is integrable, and the integrability of the real part in (9.2) is equivalent to (9.1). $\square$

Let $\mathcal{P}^b(S)$ denote the set of all bounded positive definite functions on S, and let $\mathcal{N}^l(S)$ denote the set of all $\psi\in\mathcal{N}(S)$ such that $\operatorname{Re}\psi$ is bounded below. For $f\in\mathbf{C}^S$ and $a\in S$, define $\Gamma_a f\in\mathbf{C}^S$ by

$$\Gamma_a f(s)=\tfrac{1}{2}(f(s+a)+f(s+a^*)),\quad s\in S.$$

In [2, 4.3.7, 4.3.11, and 4.3.19], it was shown that for a *-semigroup S with a Lévy function L and a function $\psi\colon S\to\mathbf{C}$, the following conditions are equivalent:

(i) There is a convolution semigroup (μ_t) of Radon measures on $\widehat{S}$, continuous in the weak topology, such that $\mathcal{L}\mu_t=e^{-t\psi}$ for all $t\geq 0$;

(ii) $e^{-t\psi}\in\mathcal{P}^b(S)$ for all $t>0$;

(iii) $\psi\in\mathcal{N}^l(S)$;

(iv) ψ is hermitian and $(\Gamma_a-I)\psi\in\mathcal{P}^b(S)$ for all $a\in S$;

(v) there exist $a\in\mathbf{R}$, a *-homomorphism $l\colon S\to(\mathbf{R},+,-\operatorname{id})$, a function $q\colon S\to\mathbf{R}_+$ satisfying $2q(s)+2q(t)=q(s+t)+q(s+t^*)$ for $s,t\in S$, and a Radon measure μ on $\widehat{S}\setminus\{1\}$, satisfying (9.1), such that

$$\psi(s)=a+il(s)+q(s)+\int(1-\sigma(s)+iL(s,\sigma))\,d\mu(\sigma),\quad s\in S.$$

(Functions q as in (v) are called nonnegative Maserick quadratic forms in the terminology of [9].) The convolution semigroup (μ_t) in (i) and the quadruple (a,l,q,μ) in (v) are uniquely determined by ψ; the measure μ is called the *Lévy measure* of ψ.

It is not really difficult to see that if $\psi\in\mathcal{N}^l(S)$ has Lévy measure λ then the measure μ on $\mathcal{A}(S^*\setminus\{1\})$ defined by

$$\mu(A)=\lambda(A\cap\widehat{S}),\quad A\in\mathcal{A}(S^*\setminus\{1\}) \tag{9.3}$$

is the only measure that can occur in (iv) in Theorem 7.1. In this sense, our use of the term "Lévy measure" in Theorem 6.1 is in agreement with the use of that term in the result quoted above.

Condition (iv) might lead one to wonder whether in Theorem 7.1 condition (iii) could be replaced with the condition

$$\psi \text{ is hermitian} \quad\text{and}\quad -PP^*\psi\in\mathcal{H}(S) \text{ for all } P\in\mathcal{E}_0.$$

The answer is "no" for the semigroup $S=\mathbf{N}_0^2$ with the identical involution. We omit the proof.

Proposition 9.2. *If $\psi\colon S\to\mathbf{C}$ satisfies the conditions of Theorem* 7.1 *and if some measure μ occurring in* (iv) *satisfies $\mu_*(S^*\setminus\widehat{S})=0$ then that is the only measure that can occur in* (iv). (*We then say that μ is the* Lévy measure *of ψ.*)

Proof. If a, h, q, and μ are as in Theorem 7.1(iv) then for $P\in\mathcal{E}_0$,

$$-PP^*\psi(s)=-\langle q,PP^*\rangle+\int\sigma(s)\langle\sigma,PP^*\rangle\,d\mu(\sigma),\quad s\in S,$$

that is, $-PP^*\psi=\mathcal{L}\lambda_P$ where $\lambda_P\in F_+(S^*)$ is given by

$$\lambda_P|_{\mathcal{A}(S^*\setminus\{1\})}=\langle\,\cdot\,,PP^*\rangle\mu,\quad \lambda_P^*(\{1\})=-\langle q,PP^*\rangle.$$

If for one such μ we have $\mu_*(S^*\setminus\widehat{S})=0$ then each λ_P is supported by $\widehat{S}$, so the moment function $-PP^*\psi$ is bounded and therefore determinate. For an arbitrary μ occurring in (iv) we thus get the same family $(\lambda_P)_{P\in\mathcal{E}_0}$, and we saw in the proof of Theorem 6.1 (without using the perfectness of S) that the family (λ_P) uniquely determines μ. $\square$

We shall see in an example in the next section that it may happen that a measure μ occurring in (iv) of Theorem 7.1 satisfies $\mu_*(S^*\setminus\widehat{S})=0$, yet is not the Lévy measure of any $\psi\in\mathcal{N}^l(S)$.

10. Examples

Consider the semigroup $S=(\mathbf{N}_0,+)$ with its unique involution, $n^*=n$. The dual semigroup S^* is identified with $\mathbf{R}$ by identifying $x\in\mathbf{R}$ with the character $n\mapsto x^n$ on S. Thus a measure on $\mathbf{R}$ is in $F_+(\mathbf{R})$ if and only if it has moments of all orders, and for $\mu\in F_+(\mathbf{R})$ we have

$$\mathcal{L}\mu(n)=\int x^n\,d\mu(x),\quad n\in\mathbf{N}_0.$$

The function $H\colon\mathbf{N}_0\times\mathbf{R}\to\mathbf{R}$ given by $H(n,x)=n(x-1)$ is a complex Lévy function, and since $\mathbf{N}_0$ is semiperfect ([2, 6.2.2]), it follows that a function $\psi\colon\mathbf{N}_0\to\mathbf{R}$ is negative definite if and only if

$$\psi(n)=a+bn-cn^2+\int(1-x^n+n(x-1))\,d\mu(x),\quad n\in\mathbf{N}_0 \tag{10.1}$$

for some $a,b\in\mathbf{R}$, some $c\geq 0$, and some measure μ on $\mathbf{R}\setminus\{1\}$ satisfying

$$\int(x-1)^2\,d\mu(x)<\infty. \tag{10.2}$$

For $0<\alpha<1$ the function $s\mapsto s^\alpha$ is negative definite on $(\mathbf{R}_+,+)$ ([2, 6.5.15]) and

$$s^\alpha=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty(1-e^{-st})\frac{dt}{t^{1+\alpha}},\quad s\in\mathbf{R}_+$$

by a formula in the proof of [2, 3.2.10]. For $n\in\mathbf{N}_0$ we thus have

$$n^\alpha=\frac{\alpha}{\Gamma(1-\alpha)}\int_0^1(1-x^n)\frac{dx}{x(-\log x)^{1+\alpha}}$$

which shows that the negative definite function $n\mapsto n^\alpha$ on $\mathbf{N}_0$ has Lévy measure $dx/x(-\log x)^{1+\alpha}|_{]0,1[}$.

For $1<\alpha<2$ the function $s\mapsto -s^\alpha$ is negative definite on $\mathbf{R}_+$ ([2, 6.5.15]) and

$$-s^\alpha=\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_0^\infty(1-e^{-st}-st)\frac{dt}{t^{1+\alpha}},\quad s\in\mathbf{R}_+$$

by a formula in the proof of [2, 3.2.11]. For $n\in\mathbf{N}_0$ we thus have

$$-n^\alpha=\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\int_0^1(1-x^n+n\log x)\frac{dx}{x(-\log x)^{1+\alpha}}$$

which shows that the negative definite function $n\mapsto -n^\alpha$ on $\mathbf{N}_0$ has Lévy measure $dx/x(-\log x)^{1+\alpha}|_{]0,1[}$. Since

$$\int_0^1(1-x)\frac{dx}{x(-\log x)^{1+\alpha}}=\infty,$$

the condition (9.1) is not fulfilled, so this measure is not the Lévy measure of any $\psi\in\mathcal{N}^l(\mathbf{N}_0)$.

The function $s\mapsto -s\log s$ (with $0\log 0=0$ by definition) is negative definite on $\mathbf{R}_+$ ([2, 6.5.16]). We have

$$-s\log s=\int_0^1(1-x^s-s(1-x))\frac{dx}{x(\log x)^2},\quad s\in\mathbf{R}_+.$$

To see this, verify that the two sides coincide at $s=0$ and $s=1$ and have the same second derivative. In particular,

$$-n\log n=\int_0^1(1-x^n-n(1-x))\frac{dx}{x(\log x)^2},\quad n\in\mathbf{N}_0$$

which shows that the negative definite function $n \mapsto -n\log n$ on $\mathbf{N}_0$ has Lévy measure $dx/x(\log x)^2|_{]0,1[}$. This is another example of a Lévy measure μ, satisfying $\mu_*(S^* \setminus \widehat{S})=0$, which is not the Lévy measure of any $\psi \in \mathcal{N}^l(S)$.

Another example is

$$-n\sum_{k=2}^{n}\frac{1}{k}=\int_0^1(1-x^n-n(1-x))\frac{dx}{(1-x)^2}, \quad n\in\mathbf{N}_0$$

which can be verified by a computation that exploits the fact that the function $x \mapsto (1-x^n-n(1-x))/(1-x)^2$ is a polynomial.

Suppose $\psi\colon \mathbf{N}_0 \to \mathbf{R}$ is such that $-\psi \in \mathcal{P}(\mathbf{N}_0)$. If $\lambda \in F_+(\mathbf{R})$ is such that $\mathcal{L}\lambda = -\psi$ then $\lambda|_{\mathbf{R}\setminus\{1\}}$ is one of those measures μ that can occur in (10.1). Hence, if only one measure μ can occur in (10.1) then $-\psi$ is determinate. It can happen, however, that $-\psi$ is determinate, yet several distinct measures μ can occur in (10.1).

To see this, let σ and τ be distinct Nevanlinna extremal measures (see [1]) with $\mathcal{L}\sigma = \mathcal{L}\tau$. With no restriction, assume $\sigma(\{1\})=0$. Write

$$\mu=(1-x)^{-2}\sigma|_{\mathbf{R}\setminus\{1\}}, \quad \nu=(1-x)^{-2}\tau|_{\mathbf{R}\setminus\{1\}},$$

and $\psi=-\mathcal{L}\mu$. Then $\mu\neq\nu$, and both can occur in (10.1). For μ this is clear, and for ν it follows from the fact that

$$\begin{aligned}(I-E_1)^2\psi+\mathcal{L}((1-x)^2\nu) &= -\mathcal{L}((1-x)^2\mu)+\mathcal{L}((1-x)^2\nu)\\ &= -\mathcal{L}\sigma+\sigma(\{1\})+\mathcal{L}\tau-\tau(\{1\}) = -\tau(\{1\}),\end{aligned}$$

a constant ≤ 0. Nevertheless, $-\psi$ is a determinate moment function. Indeed, since σ is Nevanlinna extremal then $(1+x^2)^{-1}\sigma$ is determinate (by a theorem of Riesz, cf. [1]), and so is μ, which has a bounded density with respect to $(1+x^2)^{-1}\sigma$ because of $1\notin \operatorname{supp}\sigma$.

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