Ark. Mat., **35** (1997), 171–184


# Ergodicity of the hard-core model on $\mathbf{Z}^2$ with parity-dependent activities

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To the memory of my grandparents Ingrid and Arne Broman, 1913–1995.

## 1. Introduction

In this paper we study the hard-core model (short for "hard-core lattice gas model") which originally was introduced in statistical mechanics as a crude model for a gas whose particles have non-negligible radii and cannot overlap (see [4], [6], [7]). Later, it was independently discovered in operations research where it arises in the modelling of certain communications networks (see [11], [12], [17]). A third field where the model has attracted interest is ergodic theory, where it is called "the golden mean subshift" (see [16]; the curious name comes for the fact that $\log\big((1+\sqrt{5}\,)/2\big)$ arises as the topological entropy of the model on a one-dimensional lattice).

Suppose first that $G$ is a finite graph with vertex set $V$. For $v,w\in V$, write $v\sim w$ if $v$ and $w$ are adjacent in $G$, i.e. if there is an edge between them. Let $\Omega=\{0,1\}^V$ be the set of all configurations of 0's and 1's on $V$, and call $\omega\in\Omega$ *feasible* if no two 1's are adjacent in $\omega$, i.e. if $\omega_v\omega_w=0$ whenever $v\sim w$. Let $a_v, v\in V$, be strictly positive real numbers, and define the *hard-core measure on $G$ with activities $\{a_v\}_{v\in V}$* to be the measure $\mu$ on $\Omega$ for which

$$\mu(\omega)=\begin{cases} Z^{-1}\prod_{v\in V} a_v^{\omega_v} & \text{if } \omega \text{ is feasible,}\\ 0 & \text{otherwise}\end{cases}$$

where $Z$ is the appropriate normalizing constant making $\mu$ a probability measure. $Z$ will always denote normalizing constants whose value may change from appearance to appearance.

Our main concern will be with the case when $G$ is infinite. In this case we call a probability measure $\mu$ on $\{0,1\}^V$ a hard-core measure on $G$ with activities

$\{a_v\}_{v\in V}$ if for all finite subsets $\Lambda$ of $V$, all $\eta\in\{0,1\}^\Lambda$ and $\mu$-a.e. $\eta'\in\{0,1\}^{V\setminus\Lambda}$ we have

$$\mu(\omega\equiv\eta \text{ on } \Lambda \,|\, \omega\equiv\eta' \text{ on } V\setminus\Lambda)=\begin{cases} Z^{-1}\prod_{v\in V} a_v^{\omega_v} & \text{if } \eta\sqcup\eta' \text{ is feasible,}\\ 0 & \text{otherwise}\end{cases} \tag{1}$$

where $\eta\sqcup\eta'$ is the configuration on $V$ which agrees with $\eta$ on $\Lambda$ and with $\eta'$ on $V\setminus\Lambda$. It is easily checked that this set of conditional distributions is consistent and also that the hard-core measure on a finite graph satisfies the same type of conditional distributions (making this extension to infinite graphs quite natural). Furthermore, hard-core measures are Markov random fields in the sense that the conditional distribution (1) only depends on $\eta'$ through the values of $\eta'$ on the set $\{y\in V\setminus\Lambda:\exists x\in\Lambda \text{ such that } x\sim y\}$ of sites adjacent to $\Lambda$. A standard compactness argument now shows that for any infinite but locally finite graph $G$ with activities $\{a_v\}_{v\in V}$ there exists at least one hard-core measure.

The canonical case of a hard-core model on an infinite graph is when $G$ is the nearest-neighbour graph on the integer lattice $\mathbf{Z}^d$ (i.e. $V=\mathbf{Z}^d$ with edges between each pair of vertices whose Euclidean distance is 1) with constant activities $a_x=a$ for all $x\in\mathbf{Z}^d$. The most important result on this model is due to Dobrushin [4] and says that when $d\geq 2$ and $a$ is sufficiently large, then there exists more than one measure $\mu$ satisfying (1). This phenomenon is referred to as a phase transition, and does not occur for $d=1$ (see [4] or [17]). Absence of phase transition (i.e. existence of only one measure with the prescribed conditional probabilities) is referred to as ergodicity. For any $d$ and sufficiently small $a$ we have ergodicity (see e.g. [1]).

A generalization of this setup was studied by van den Berg and Steif [1]. They allowed the activities of the vertices $x\in\mathbf{Z}^d$ to take two different values $a_e$ and $a_o$:

$$a_x=\begin{cases} a_e & \text{if } x \text{ is even,}\\ a_o & \text{if } x \text{ is odd}\end{cases} \tag{2}$$

calling $x\in\mathbf{Z}^d$ even (odd) if the sum of its coordinates is even (odd). They conjectured that for any $d$ and any $a_e\neq a_o$ there is a unique hard-core measure (in contrast to the case $a_e=a_o$). For $d=1$ this follows as in the $a_e=a_o$ case. The main achievement of the present paper is a proof of the conjecture for $d=2$:

**Theorem 1.1.** *The hard-core model on $\mathbf{Z}^2$ with activities given by (2) has a unique hard-core measure whenever $a_e\neq a_o$.*

One of the motivations for studying the hard-core model with parity-dependent activities as in (2) is to obtain a better understanding of the nature of the phase transition in the $a_e=a_o$ case. We now give a brief explanation of this, referring to [1] for further discussion. Let $d=2$ and let $a_e=a_o=a$ be sufficiently large to get a phase transition. Then there exist two particular hard-core measures $\mu^e$ and $\mu^o$ with the following properties. With $\mu^e$-probability 1, "most" of $\mathbf{Z}^2$ forms a checkerboard pattern, with $\omega_x=1$ when $x$ is even and $\omega_x=0$ when $x$ is odd. Only in small regions ("islands") of $\mathbf{Z}^2$ do we see the opposite situation with $\omega_x=1$ when $x$ is odd and $\omega_x=0$ when $x$ is even. The other measure $\mu^o$ has the opposite behaviour, i.e. it is concentrated on the event that there is a predominant checkerboard pattern with $\omega_x=1$ when $x$ is odd and $\omega_x=0$ when $x$ is even. There is a strong symmetry between $\mu^e$ and $\mu^o$; if we pick a configuration according to $\mu^e$ and shift it one step, then the induced probability measure (for the shifted configuration) is precisely $\mu^o$. Intuitively, it is this symmetry which makes the phase transition possible. Theorem 1.1 provides support for this intuition, in that when $a_e\neq a_o$ the symmetry between the even lattice $\{x\in\mathbf{Z}^2 : x \text{ is even}\}$ and the odd lattice $\{x\in\mathbf{Z}^2 : x \text{ is odd}\}$ is broken, and the phase transition disappears. This is analogous to the ferromagnetic Ising model in dimension 2 or higher, where a phase transition occurs with zero external field and sufficiently low temperature, while with non-zero external field the $\pm1$ symmetry of the model is broken and there is no phase transition.

We mention that our proof of Theorem 1.1 can be easily adapted to give an alternative proof in 2 dimensions to the classical result [14] that non-zero external field in the Ising model implies ergodicity; in fact some parts of the proof become slightly easier in that setting than in the hard-core case.

A feature of the hard-core model which we shall make use of (and which is central to the applications in operations research) is that a hard-core measure on a finite graph arises as the unique stationary distribution of a certain continuous time reversible Markov chain. For a finite graph $G$ with vertex set $V$, and positive numbers $\{a_v\}_{v\in V}$, define a continuous time Markov chain $M$ with state space $\{\omega\in\{0,1\}^V : \omega \text{ feasible}\}$ and transition rates

$$\gamma(\omega,\omega')=\begin{cases} a_v & \text{if } \omega=\omega' \text{ except at } v \text{ where } \omega_v=0,\ \omega'_v=1,\\ 1 & \text{if } \omega=\omega' \text{ except at } v \text{ where } \omega_v=1,\ \omega'_v=0,\\ 0 & \text{otherwise.}\end{cases}$$

The following result is well known and easily verified by checking irreducibility and the detailed equilibria $\mu(\omega)\gamma(\omega,\omega')=\mu(\omega')\gamma(\omega',\omega)$ for all $\omega,\omega'$.

**Proposition 1.2.** *The hard-core measure $\mu$ for $G$ with activities $\{a_v\}_{v\in V}$ is the unique stationary distribution for the Markov chain $M$.*

The rest of this paper is organized as follows. In Section 2, we quote and discuss some preliminary results from [1]. In Section 3 (resp. 4), we prove results about stochastic domination (resp. percolation) properties of the hard-core model on $\mathbf{Z}^2$. Finally, in Section 5, we put all these ingredients together and prove Theorem 1.1.

We mention that it is only in the proof of Proposition 4.1 in Section 4 that we use essentially 2-dimensional techniques. This, however, is a key step of the proof of Theorem 1.1, so we believe that different ideas will be needed in order to prove the theorem for general $d$. See [1] for a promising approach.

## 2. Some results of van den Berg and Steif

In this section, we recall some of the progress that was made in [1] towards proving ergodicity for the hard-core model on $\mathbf{Z}^d$ with $a_e \neq a_o$. All of the results of this section are valid for any $d$, but we will for convenience and concreteness give them for $d=2$ only.

Let $\Lambda_n = \{-n, \dots, n\}^2$ and also write $\Lambda_n$ for the nearest neighbour graph on this vertex set. Define the partial order $\preceq$ on $\Omega_n = \{0,1\}^{\Lambda_n}$ by

$$\omega \preceq \omega' \quad \text{if and only if} \quad \begin{cases} \omega_x \leq \omega'_x & \text{for all even } x, \\ \omega_x \geq \omega'_x & \text{for all odd } x. \end{cases}$$

This may seem like a somewhat unusual partial order e.g. to readers used to the monotone particle systems in [15]. However, the hard-core model on a bipartite lattice possesses a certain "antimonotonicity" between the even and the odd lattice which makes $\preceq$ the most useful choice of partial order. Fix $\{a_x\}_{x\in\Lambda_n}$, and write $\mu_n$ for the hard-core measure on $\Lambda_n$ with these activities. It is easy to check that

$$\mu_n(\omega \wedge \omega')\mu_n(\omega \vee \omega') \geq \mu_n(\omega)\mu_n(\omega') \tag{3}$$

for all $\omega, \omega' \in \Omega_n$, where $\omega \wedge \omega'$ is defined by $(\omega\wedge\omega')_x = \min(\omega_x, \omega'_x)$ for even $x$ and $(\omega\wedge\omega')_x = \max(\omega_x, \omega'_x)$ for odd $x$, and $\omega \vee \omega'$ is defined by $(\omega\vee\omega')_x = \max(\omega_x, \omega'_x)$ for even $x$ and $(\omega\vee\omega')_x = \min(\omega_x, \omega'_x)$ for odd $x$. This is the FKG condition, which implies, by the well-known FKG theorem (see e.g. [15]),

$$\mu_n(A \cap B) \geq \mu_n(A)\mu_n(B) \quad \text{for all increasing events } A, B \subseteq \Omega_n \tag{4}$$

(an event $A$ is called increasing if $\omega \in A$, $\omega \preceq \omega'$ implies $\omega' \in A$). It is an immediate consequence that $\mu_n(A\cap B) \geq \mu_n(A)\mu_n(B)$ also in the case when $A$ and $B$ both are decreasing events (where decreasing is defined in the obvious way).

Now define the boundary $\partial\Lambda_n = \{y \in \mathbf{Z}^d \setminus \Lambda_n : \exists x \in \Lambda_n$ such that $x$ and $y$ are nearest neighbours$\}$ and let $\delta \in \{0,1\}^{\partial\Lambda_n}$ be a feasible configuration on $\partial\Lambda_n$. Let $\mu_n^\delta$ be the probability measure on $\Omega_n$ for which

$$\mu_n^\delta(\omega) = \begin{cases} Z^{-1}\prod_{x\in\Lambda_n} a_x^{\omega_x} & \text{if } \omega \sqcup \delta \text{ is feasible}, \\ 0 & \text{otherwise.} \end{cases}$$

This is tantamount to conditioning $\mu$ on the event that $\omega_x=0$ for all $x\in\Lambda_n$ which are adjacent to some $y\in\partial\Lambda_n$ with $\delta(y)=1$. The inequalities (3) and (4) hold also for $\mu_n^\delta$.

If $\nu$ and $\nu'$ are probability measures on $\Omega_n$, we say that $\nu'$ *dominates* $\nu$, and write $\nu\preceq\nu'$ if $\nu(A)\le\nu'(A)$ for every increasing event $A\in\Omega_n$. The following result is a consequence of the FKG condition (3):

**Lemma 2.1.** *Let $\delta,\delta'\in\partial\Lambda_n$ be such that $\delta\preceq\delta'$. Then $\mu_n^\delta\preceq\mu_n^{\delta'}$.*

We are interested in two particular boundary configurations $\delta^e,\delta^o\in\{0,1\}^{\partial\Lambda_n}$, where $\delta^e$ is defined by

$$\delta_x^e=\begin{cases}1 & \text{if } x \text{ is even,}\\ 0 & \text{if } x \text{ is odd}\end{cases}$$

and $\delta^o$ is defined analogously. These boundary configurations are extremal in the sense that $\delta^o\preceq\delta\preceq\delta^e$ for any $\delta\in\{0,1\}^{\partial\Lambda_n}$, whence, by Lemma 2.1,

$$\mu_n^o\preceq\mu_n^\delta\preceq\mu_n^e \tag{5}$$

where $\mu_n^o$ is short for $\mu_n^{\delta^o}$ and $\mu_n^e$ is short for $\mu_n^{\delta^e}$. The corresponding statement holds when $\Lambda_n$ is replaced by any finite subset of $\mathbf{Z}^2$. If we define activities $a_x$ for all $x\in\mathbf{Z}^2$ we get, as another consequence of Lemma 2.1 that the limits $\lim_{n\to\infty}\mu_n^o$ and $\lim_{n\to\infty}\mu_n^e$ exist and are monotone. Let $\mu^o$ and $\mu^e$ denote the limiting measures on $\{0,1\}^{\mathbf{Z}^2}$. The conditional probabilities in (1) are preserved in the limit, so $\mu^o$ and $\mu^e$ are hard-core measures. From (5) it is possible to deduce the next result:

**Lemma 2.2.** *For any hard-core measure $\mu$ on $\mathbf{Z}^2$ with activities $\{a_x\}_{x\in\mathbf{Z}^2}$ we have*

$$\mu^o\preceq\mu\preceq\mu^e.$$

*Hence the hard-core model on $\mathbf{Z}^2$ with activities $\{a_x\}_{x\in\mathbf{Z}^2}$ is ergodic if and only if $\mu^o=\mu^e$.*

Note also that the FKG inequality (4) is inherited from $\mu_n^e$ and $\mu_n^o$ to $\mu^e$ and $\mu^o$.

We now specialize to the case with activities $a_e$ and $a_o$ as in (2). In a sense, van den Berg and Steif came very close to proving Theorem 1.1, in that they demonstrated ergodicity for Lebesgue-a.e. $(a_e,a_o)$. More precisely, they proved the following.

**Proposition 2.3.** *For fixed $c$, the hard-core model on $\mathbf{Z}^2$ with activities $a_e=\exp(c+h)$ and $a_o=\exp(c-h)$ is ergodic for all but at most countably many $h$.*

Very briefly, their proof can be described as follows. They considered the so called pressure function

$$P_c(h)=\lim_{n\to\infty}\frac{Z(n,h,c)}{|\Lambda_n|}$$

where $Z(n,h,c)$ is the normalizing constant for the hard-core model on $\Lambda_n$ with activities $a_e = \exp(c+h)$ and $a_o = \exp(c-h)$, and $|\Lambda_n| = (2n+1)^2$ is the cardinality of $\Lambda_n$. They showed that for any $c$, $P_c(h)$ is a convex function of $h$, and furthermore that differentiability of $P_c(h)$ at $h=h_0$ implies ergodicity of the hard-core model on $\mathbf{Z}^2$ with activities $a_e = \exp(c+h_0)$ and $a_o = \exp(c-h_0)$. The proposition now follows from the fact that a convex function from $\mathbf{R}$ to $\mathbf{R}$ is differentiable at all but at most countably many points.

## 3. Stochastic domination

The task in this section is to obtain comparison results for hard-core measures with different activities. Write $\mu^e_{n,(a_e,a_o)}$ (resp. $\mu^o_{n,(a_e,a_o)}$) for the hard-core measure on $\Lambda_n$ with activities $a_e$ and $a_o$ (as in (2)) and boundary condition $\delta^e$ (resp. $\delta^o$).

**Lemma 3.1.** *Pick $a_e$, $a_e^*$, $a_o$ and $a_o^*$ in such a way that $a_e \le a_e^*$ and $a_o \ge a_o^*$. Then*

$$\mu^e_{n,(a_e,a_o)} \preceq \mu^e_{n,(a_e^*,a_o^*)} \tag{6}$$

*and*

$$\mu^o_{n,(a_e,a_o)} \preceq \mu^o_{n,(a_e^*,a_o^*)}. \tag{7}$$

Our proof of this lemma is based on the following result (or rather its proof) due to Holley [10].

**Proposition 3.2.** *Suppose $\mu$ and $\mu'$ are two probability measures on $\{0,1\}^{\Lambda_n}$ which assign strictly positive probability to each $\omega \in \{0,1\}^{\Lambda_n}$. If for each $\omega$, $\omega' \in \{0,1\}^{\Lambda_n}$ we have*

$$\mu(\omega \wedge \omega')\mu'(\omega \vee \omega') \ge \mu(\omega)\mu'(\omega') \tag{8}$$

*then*

$$\mu \preceq \mu'.$$

*Proof of Lemma* 3.1. We prove (6) only, as (7) is completely analogous. Note first that Proposition 3.2 cannot be applied directly to hard-core measures, because the assumption that each element of $\{0,1\}^{\Lambda_n}$ has strictly positive probability is obviously violated. This problem is solved as follows. Holley's proof (which also appears in [15]) is based on couplings of two Markov chains, both of which in our case become essentially the Markov chain in Proposition 1.2 (with the appropriate activities and definitions of 'feasible'). An inspection of that proof shows that the strict positivity condition can be replaced by the assumption that the Markov chains are irreducible. Our Markov chains are irreducible, which is easy to see because each feasible state communicates with the state of all 0's. Hence, all we need in order to prove the lemma is to check that (8) holds with $\mu=\mu^e_{n,(a_e,a_o)}$ and $\mu'=\mu^e_{n,(a_e^*,a_o^*)}$. Define $k^e_\omega$ (resp. $k^o_\omega$) to be the number of 1's on the even (resp. odd) lattice in $\omega$, and define $k^e_{\omega'}$ etc. analogously. We have

$$k^e_{\omega\wedge\omega'}-k^e_\omega=k^e_{\omega'}-k^e_{\omega\vee\omega'}$$

and

$$k^o_{\omega\wedge\omega'}-k^o_\omega=k^o_{\omega'}-k^o_{\omega\vee\omega'}.$$

It follows that

$$\begin{aligned}\frac{\mu^e_{n,(a_e,a_o)}(\omega\wedge\omega')\mu^e_{n,(a_e^*,a_o^*)}(\omega\vee\omega')}{\mu^e_{n,(a_e,a_o)}(\omega)\mu^e_{n,(a_e^*,a_o^*)}(\omega')}&=\frac{a_e^{k^e_{\omega\wedge\omega'}}a_o^{k^o_{\omega\wedge\omega'}}(a_e^*)^{k^e_{\omega\vee\omega'}}(a_o^*)^{k^o_{\omega\vee\omega'}}}{a_e^{k^e_\omega}a_o^{k^o_\omega}(a_e^*)^{k^e_{\omega'}}(a_o^*)^{k^o_{\omega'}}}\\&=\left(\frac{a_e^*}{a_e}\right)^{k^e_{\omega\vee\omega'}-k^e_{\omega'}}\left(\frac{a_o}{a_o^*}\right)^{k^o_{\omega\wedge\omega'}-k^o_\omega}\\&\geq 1\end{aligned}$$

where the inequality holds because $a_e^*/a_e$ and $a_o/a_o^*$ both are greater than 1 while the exponents $(k^e_{\omega\vee\omega'}-k^e_{\omega'})$ and $(k^o_{\omega\wedge\omega'}-k^o_\omega)$ both are nonnegative. Hence

$$\mu^e_{n,(a_e,a_o)}(\omega\wedge\omega')\mu^e_{n,(a_e^*,a_o^*)}(\omega\vee\omega')\geq\mu^e_{n,(a_e,a_o)}(\omega)\mu^e_{n,(a_e^*,a_o^*)}(\omega')$$

so we are done. □

Now write $\mu^e_{(a_e,a_o)}$ and $\mu^o_{(a_e,a_o)}$ for the limiting measures $\lim_{n\to\infty}\mu^e_{n,(a_e,a_o)}$ and $\lim_{n\to\infty}\mu^o_{n,(a_e,a_o)}$ on $\{0,1\}^{\mathbf{Z}^2}$. It is clear that the stochastic domination in Lemma 3.1 is preserved in the limit as $n\to\infty$, so we have, similarly as in Lemma 2.2,

**Lemma 3.3.** *Pick $a_e$, $a_e^*$, $a_o$ and $a_o^*$ in such a way that $a_e\leq a_e^*$ and $a_o\geq a_o^*$. Then*

$$\mu^e_{(a_e,a_o)}\preceq\mu^e_{(a_e^*,a_o^*)}$$

*and*

$$\mu^o_{(a_e,a_o)}\preceq\mu^o_{(a_e^*,a_o^*)}.$$

## 4. Percolation properties of hard-core measures

An element of $\{0,1\}^{\mathbf{Z}^2}$ chosen at random according to $\mu^e$ (or $\mu^o$) has certain a.s. geometric properties, and in this section we study some of these. To this end, it is convenient to transform $\omega\in\{0,1\}^{\mathbf{Z}^2}$ by "inverting" the odd lattice. Define the mapping $\psi\colon\{0,1\}^{\mathbf{Z}^2}\to\{0,1\}^{\mathbf{Z}^2}$ coordinatewise by

$$(\psi(\omega))_x=\begin{cases}0 & \text{if } x \text{ is even and } \omega_x=0, \text{ or if } x \text{ is odd and } \omega_x=1,\\ 1 & \text{if } x \text{ is even and } \omega_x=1, \text{ or if } x \text{ is odd and } \omega_x=0.\end{cases}$$

It is clear from the definition that

$$(\psi(\omega))_x\le(\psi(\omega'))_x \text{ for all } x\in\mathbf{Z}^2 \quad\text{if and only if}\quad \omega\preceq\omega'. \tag{9}$$

We are interested in connected components of the nearest-neighbour graphs on the random vertex-sets $V_0=\{x\in\mathbf{Z}^2:(\psi(\omega))_x=0\}$ and $V_1=\{x\in\mathbf{Z}^2:(\psi(\omega))_x=1\}$ (write $G_0$ and $G_1$ for the corresponding random graphs). In particular, we ask the basic percolation-theoretic question of whether $G_0$ and $G_1$ contain infinite connected components. The main result of this section is the following.

**Proposition 4.1.** *For any choice of activity parameters $a_e$ and $a_o$, we have that the hard-core measure $\mu^e$ on $\mathbf{Z}^2$ with these activities satisfies either*

$$\mu^e(G_0 \textit{ contains an infinite connected component})=0 \tag{10}$$

*or*

$$\mu^e(G_1 \textit{ contains an infinite connected component})=0 \tag{11}$$

*or possibly both. The same thing holds for $\mu^o$.*

Hence, infinite connected components of $G_0$ and $G_1$ can a.s. not coexist. Our proof of this result is an adaptation of Yu Zhang's elegant proof of the $\ge$ half of the famous Harris–Kesten theorem ([9], [13]) which states that the critical value $p_c$ for standard two-dimensional bond percolation satisfies $p_c=\frac{1}{2}$. Zhang never published his proof, but it appears in [8, p. 195–196].

A result analogous to Proposition 4.1 for the two-dimensional Ising model appears in [3], where it was proved using Harris' [9] original approach. More general results along these lines appear in [5]. It is possible to adapt the methods of [3] and [5] in order to prove Proposition 4.1, but we prefer to stick with Zhang's method. This is just a matter of taste.

In order to prove Proposition 4.1, we first need the following lemma, whose proof requires familiarity with the paper [2] by Burton and Keane (this may seem annoying, but the paper is only five pages long and very instructive, so we are almost inclined to think we are doing the reader a favour in asking for this familiarity).

**Lemma 4.2.** *For any choice of activity parameters $a_e$ and $a_o$, we have that the hard-core measure $\mu^e$ on $\mathbf{Z}^2$ with these activities satisfies*

$$\mu^e(G_0 \textit{ contains at most one infinite connected component}) = 1$$

*and*

$$\mu^e(G_1 \textit{ contains at most one infinite connected component}) = 1.$$

*The same things hold with $\mu^e$ replaced by $\mu^o$.*

*Proof.* We prove uniqueness of the infinite connected component for $G_1$ under $\mu^e$, the other three cases being completely analogous. Burton and Keane [2] have a quite general uniqueness theorem which, however, we cannot apply directly, because the following two assumptions of their theorem are violated for $\mu^e$:

(i) translation invariance, and

(ii) finite energy, i.e. for any $n$, any $\eta \in \{0,1\}^{\Lambda_n}$ and $\mu^e$-a.e. $\eta' \in \{0,1\}^{\mathbf{Z}^2 \setminus \Lambda_n}$ we should have $\mu^e(\eta|\eta') > 0$.

Condition (i) fails in the phase transition region of the parameter space, because shifting $\omega$ one step will move the predominant checkerboard pattern from the even lattice to the odd lattice. However, standard monotonicity arguments (see e.g. Section IV.1 in [15]) show that

(i′) $\mu^e$ is invariant under translation by arbitrary $x \in 2\mathbf{Z}^2$

because translation by $x \in 2\mathbf{Z}^2$ takes the even lattice to the even lattice and the odd to the odd. As to condition (ii), this fails for the simple reason that certain elements of $\{0,1\}^{\Lambda_n}$ are disallowed (a 1 must not sit next to a 1 in $\omega$, so that e.g. a 1 in the even lattice must not sit next to a 0 in the odd lattice in $\psi(\omega)$). However, $\mu^e$ satisfies

(ii′) given any $n$ and ($\mu^e$-almost) any $\omega \in \{0,1\}^{\mathbf{Z}^2}$ with the property that two or more infinite connected components of $G_0$ intersect $\Lambda_n$, we may modify the configuration in $\Lambda_n$ in such a way that

(a) all the infinite connected components of $G_0$ that intersected $\Lambda_n$ become connected to each other, and

(b) the resulting configuration on $\Lambda_n$ has positive conditional probability given the configuration on $\mathbf{Z}^2 \setminus \Lambda_n$.

To see that (ii′) holds, consider the following modification of $\omega$ on $\Lambda_n$. On $\Lambda_{n-1}$ we place the "even checkerboard pattern", so that $\psi(\omega) \equiv 1$ on $\Lambda_{n-1}$. For $x \in \Lambda_n \setminus \Lambda_{n-1}$ we keep $\omega_x$ as it is if $x$ is even and let $\omega_x = 0$ if $x$ is odd. That (a) holds for this modified configuration is immediate while (b) follows from the definition of a hard-core measure together with the observation that the modified configuration is feasible.

An inspection of the proof of the uniqueness theorem in [2] shows that it works (essentially unchanged) when conditions (i) and (ii) are replaced by (i′) and (ii′). Hence it can be applied to show that $\mu^e$-a.s. $G_1$ has at most one infinite connected component. □

*Proof of Proposition* 4.1. We give the proof for $\mu^e$; it is exactly the same for $\mu^o$. The tail $\sigma$-field of $\mu^e$ is trivial (this follows from the fact that $\mu^e$ is extremal in the set of all hard-core measures, which in turn follows from an application of Corollary 2.8 in Chapter II of [15]) whence events concerning the existence of infinite connected components of $G_0$ and $G_1$ have probability 0 or 1. Assume, for contradiction, that (10) and (11) fail for $\mu^e$, so that

$$\mu^e(G_0 \text{ and } G_1 \text{ both contain infinite connected components}) = 1. \tag{12}$$

We can then pick $n$ so large that

$$\mu^e(A_n) > 0.999$$

and

$$\mu^e(B_n) > 0.999$$

where $A_n$ (resp. $B_n$) is the event that $G_0$ (resp. $G_1$) contains an infinite cluster which intersects $\Lambda_n$. Let $A_n^L$ (resp. $A_n^R$, $A_n^T$ and $A_n^B$) be the event that some vertex in the left (resp. right, top and bottom) side of the square-shaped vertex set $\Lambda_n \setminus \Lambda_{n-1}$ is in some infinite path of $G_0$ which uses no other vertex of $\Lambda_n$, and define $B_n^L$, $B_n^R$, $B_n^T$ and $B_n^B$ analogously. Write $\neg E$ for the complement of an event $E$. We have that $A_n = A_n^L \cup A_n^R \cup A_n^T \cup A_n^B$ and that all five events are decreasing with respect to the partial order $\preceq$. Hence, by the FKG inequality (4),

$$\begin{aligned}\mu^e(A_n) &= \mu^e(A_n^L \cup A_n^R \cup A_n^T \cup A_n^B)\\ &= 1 - \mu^e(\neg A_n^L \cap \neg A_n^R \cap \neg A_n^T \cap \neg A_n^B)\\ &\le 1 - \mu^e(\neg A_n^L)\mu^e(\neg A_n^R)\mu^e(\neg A_n^T)\mu^e(\neg A_n^B)\end{aligned}$$

and since by symmetry $A_n^L$, $A_n^R$, $A_n^T$ and $A_n^B$ all have the same $\mu^e$-probability, we have

$$\mu^e(\neg A_n^L) \le (1 - \mu^e(A_n))^{1/4}$$

whence

$$\mu^e(A_n^L) = \mu^e(A_n^R) \geq 1-(1-\mu^e(A_n))^{1/4} > 1-0.001^{1/4} > 0.82. \tag{13}$$

In the same way, we get

$$\mu^e(B_n^T) = \mu^e(B_n^B) > 0.82. \tag{14}$$

Now define the event $E = A_n^L \cap A_n^R \cap B_n^T \cap B_n^B$. From (13) and (14), we obtain

$$\mu^e(E) \geq 1-4(1-0.82) = 0.28 > 0.$$

When $E$ occurs, both the left-hand side and the right-hand side of $\Lambda_n$ are intersected by some infinite connected component of $G_0$. By Lemma 4.2, they must in fact belong to the same infinite connected component. This connected component now separates its complement into (at least) two pieces, thus preventing the infinite connected components of $G_1$ intersecting the top and bottom sides of $\Lambda_n$ from reaching each other (see the picture on p. 196 of [8]). This contradicts Lemma 4.2, so the assumption (12) must be false. □

## 5. Proof of the main result

Collecting the results in Sections 2–4, it now takes just a few more steps in order to prove Theorem 1.1.

**Lemma 5.1.** *Fix $h>0$ and $c$, and let $a_e^* = \exp(c+h)$, $a_o^* = \exp(c-h)$ and $a_e = a_o = \exp(c)$. Then*

$$\mu^e_{(a_e,a_o)} \preceq \mu^o_{(a_e^*,a_o^*)}.$$

*Proof.* By Proposition 2.3, we can find $h' \in (0,h)$ such that the hard-core model with parameters $a_e' = \exp(c+h')$ and $a_o' = \exp(c-h')$ is ergodic, so that by Lemma 2.2 we have

$$\mu^e_{(a_e',a_o')} = \mu^o_{(a_e',a_o')}.$$

Applying Lemma 3.3 twice we get

$$\mu^e_{(a_e,a_o)} \preceq \mu^e_{(a_e',a_o')} = \mu^o_{(a_e',a_o')} \preceq \mu^o_{(a_e^*,a_o^*)}$$

as desired. □

**Lemma 5.2.** *For any choice of activity parameters $a_e^*$ and $a_o^*$ such that $a_e^* > a_o^*$ we have*

$$\mu^o_{(a_e^*, a_o^*)}(G_0 \text{ contains an infinite connected component}) = 0.$$

*Proof.* Let $c = \frac{1}{2}\log(a_e^* a_o^*)$ and $h = \frac{1}{2}\log(a_e^*/a_o^*)$, so that $a_e^* = \exp(c+h)$ and $a_o^* = \exp(c-h)$. For $i = 0, 1$, let $E_i$ denote the event that $G_i$ contains an infinite connected component. $E_0$ is decreasing with respect to $\preceq$, whence by Lemma 5.1 it is sufficient to show that

$$\mu^e_{(a_e, a_o)}(E_0) = 0 \tag{15}$$

where $a_e = a_o = \exp(c)$. Symmetry implies

$$\mu^e_{(a_e, a_o)}(E_0) = \mu^o_{(a_e, a_o)}(E_1)$$

and

$$\mu^e_{(a_e, a_o)}(E_1) = \mu^o_{(a_e, a_o)}(E_0).$$

By tail triviality, one of the following four cases must then be true:

1. $\mu^e_{(a_e, a_o)}(E_0 \cap E_1) = 1$ and $\mu^o_{(a_e, a_o)}(E_0 \cap E_1) = 1$,
2. $\mu^e_{(a_e, a_o)}(E_0 \cap \neg E_1) = 1$ and $\mu^o_{(a_e, a_o)}(\neg E_0 \cap E_1) = 1$,
3. $\mu^e_{(a_e, a_o)}(\neg E_0 \cap E_1) = 1$ and $\mu^o_{(a_e, a_o)}(E_0 \cap \neg E_1) = 1$,
4. $\mu^e_{(a_e, a_o)}(\neg E_0 \cap \neg E_1) = 1$ and $\mu^o_{(a_e, a_o)}(\neg E_0 \cap \neg E_1) = 1$.

Case 1 contradicts Proposition 4.1, and case 2 contradicts Lemma 2.2. Hence case 3 or case 4 holds, and in either case (15) follows. □

*Proof of Theorem* 1.1. By symmetry, it is sufficient to prove the theorem for $a_e > a_o$. By Lemma 2.2 we have $\mu^o_{(a_e, a_o)} \preceq \mu^e_{(a_e, a_o)}$ and need to show $\mu^o_{(a_e, a_o)} = \mu^e_{(a_e, a_o)}$, whence it is sufficient to show that $\mu^e_{(a_e, a_o)} \preceq \mu^o_{(a_e, a_o)}$. For this we need to show that $\mu^e_{(a_e, a_o)}(A) \leq \mu^o_{(a_e, a_o)}(A)$ for any increasing event $A$, and it is easy to see that it suffices to consider cylinder events (i.e. events that depend only on the values of $\omega$ at finitely many $x \in \mathbf{Z}^2$). Given any $\varepsilon > 0$ and any increasing cylinder event $A$, we will now give a proof that

$$\mu^e_{(a_e, a_o)}(A) \leq \mu^o_{(a_e, a_o)}(A) + \varepsilon, \tag{16}$$

and once this is done, the theorem follows.

Pick $n$ so large that $A$ depends on the values in $\Lambda_n$ only. Then pick $N>n$ so large that

(17) $\mu^o_{(a_e,a_o)}(G_0$ has a connected component intersecting both $\Lambda_n$ and $\mathbf{Z}^2\backslash\Lambda_N)<\varepsilon$

which is possible by Lemma 5.2. We now construct a random configuration $X\in\{0,1\}^{\mathbf{Z}^2}$ distributed according to $\mu^o_{(a_e,a_o)}$. First assign values to $X$ on $\mathbf{Z}^2\backslash\Lambda_N$ according to the correct distribution. Then assign values to $X$ on $\Lambda_N$ sequentially as follows. Let $x_1,\dots,x_{(2N+1)^2}$ be an arbitrary ordering of the sites in $\Lambda_N$. At each step, pick the largest $i$ such that

(a) the value of $X$ at $x_i$ has not yet been decided, and

(b) for some $y\sim x_i$, $X(y)$ has been decided in such a way that $(\psi(X))_y=0$,

and assign $x_i$ a value according to the correct conditional distribution. When no such $i$ can be found, the sequential procedure stops. Write $D$ for the event that this happens before any $x\in\Lambda_n$ has been assigned a value. It follows from (17) that

$$\mu^o_{(a_o,a_e)}(D)>1-\varepsilon. \tag{18}$$

Write $L$ for the random set of sites that have not yet been assigned values when the sequential procedure stops. Note that $\psi(X)\equiv 1$ on the boundary $\partial L=\{y\in\mathbf{Z}^2\backslash L:\exists x\in L$ such that $x\sim y\}$. By the Markov property of the hard-core model, the conditional distribution of the values on $L$ is then given by the hard-core model on $L$ with even boundary condition. Hence, by the remark following (5), we have

$$\mu^o_{(a_e,a_o)}(A\,|\,D)\geq\mu_{(a_e,a_o)}(A)$$

for any hard-core measure $\mu_{(a_e,a_o)}$, so that in particular

$$\mu^o_{(a_e,a_o)}(A\,|\,D)\geq\mu^e_{(a_e,a_o)}(A)$$

whence, by (18),

$$(1-\varepsilon)\mu^o_{(a_e,a_o)}(A)\geq\mu^e_{(a_e,a_o)}(A)$$

and (16) follows. $\square$

*Acknowledgement.* I am grateful to the organizers and participants (especially Jeff Steif) of the Institute for Elementary Studies 1995 workshop in Pinecrest, California. Some of the lively discussions taking place there improved my understanding of the hard-core model, thus making this paper possible. I would also like to thank Rob van den Berg for valuable comments. This research has been supported by the Swedish Natural Science Research Council.

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*Received September 15, 1995*

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