

Toric residues

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The Grothendieck local residue symbol

$$(1) \quad \text{Res}_0 \left(\frac{g \, dx_0 \wedge \dots \wedge dx_n}{f_0 \dots f_n} \right) = \frac{1}{(2\pi i)^{n+1}} \int_{|f_i|=\varepsilon} \frac{g \, dx_0 \wedge \dots \wedge dx_n}{f_0 \dots f_n}$$

(see [13, Chapter 5]) is defined whenever g, f_0, \dots, f_n are holomorphic in a neighborhood of $0 \in \mathbf{C}^{n+1}$ and f_0, \dots, f_n do not vanish simultaneously except at 0. In [19, 12.10], it was observed that when f_0, \dots, f_n are homogeneous of degree d and g is homogeneous of degree $\varrho = (n+1)(d-1)$, the residue symbol has the following nice properties.

Quotient property. *The map*

$$g \mapsto \text{Res}_0 \left(\frac{g \, dx_0 \wedge \dots \wedge dx_n}{f_0 \dots f_n} \right)$$

induces an isomorphism

$$\mathbf{C}[x_0, \dots, x_n]_{\varrho} / \langle f_0, \dots, f_n \rangle_{\varrho} \simeq \mathbf{C}$$

(the subscript refers to the graded piece in degree ϱ) uniquely characterized by the fact that the Jacobian determinant $J = \det(\partial f_i / \partial x_j)$ maps to d^{n+1} .

Trace property. Čech cohomology gives a naturally defined cohomology class $[\omega_g] \in H^n(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^n)$ such that under the trace map $\text{Tr}_{\mathbf{P}^n} : H^n(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^n) \simeq \mathbf{C}$, we have

$$\text{Res}_0 \left(\frac{g \, dx_0 \wedge \dots \wedge dx_n}{f_0 \dots f_n} \right) = \text{Tr}_{\mathbf{P}^n}([\omega_g]).$$

(We define $[\omega_g]$ in §1.)

In this paper, we will show how these properties of residues can be generalized to an arbitrary projective toric variety. The paper is organized into six sections

as follows. In §1, we define the cohomology class $[\omega_g] \in H^n(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^n)$, and then §2 generalizes this to define toric residues in terms of a toric analog of the trace property. We recall some commutative algebra associated with toric varieties in §3, and §4 introduces a toric version of the Jacobian. In §5, we show that the toric residue is uniquely characterized using a toric analog of the quotient property. Then §6 explores different ways of representing the toric residue as an integral, and an appendix discusses the relation between the trace map and the Dolbeault isomorphism.

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1. The Definition of ω_g for \mathbf{P}^n

Suppose that f_0, \dots, f_n are homogeneous polynomials of degree d which do not vanish simultaneously on \mathbf{C}^{n+1} except at the origin, and let g be homogeneous of degree $\varrho = (n+1)(d-1)$. Then consider the n -form

$$(2) \quad \Omega = \sum_{i=0}^n (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n.$$

As is well-known (see [12, §2]), our assumptions on g and f_0, \dots, f_n imply that

$$\omega_g = \frac{g\Omega}{f_0 \dots f_n}$$

descends to a meromorphic n -form on \mathbf{P}^n , also denoted ω_g . However, the affine open sets

$$U_i = \{x \in \mathbf{P}^n : f_i(x) \neq 0\}$$

form an open cover \mathcal{U} of \mathbf{P}^n . Then ω_g is holomorphic on $U_0 \cap \dots \cap U_n$, so it is a Čech cocycle in $C^n(\mathcal{U}, \Omega_{\mathbf{P}^n}^n)$. Further, since \mathcal{U} has $n+1$ elements, ω_g is a Čech cocycle and thus gives a class $[\omega_g] \in H^n(\mathcal{U}, \Omega_{\mathbf{P}^n}^n) = H^n(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^n)$. This is the class $[\omega_g]$ mentioned in the introduction.

2. Residues on toric varieties

We will work with an n -dimensional projective toric variety X over the complex numbers \mathbf{C} . Thus X is determined by a complete fan Σ in $N_{\mathbf{R}} = \mathbf{R}^n$. As usual, M

denotes the dual lattice of $N=\mathbf{Z}^n$ and $\Sigma(1)$ denotes the set of 1-dimensional cones in Σ . Each $\rho\in\Sigma(1)$ determines a divisor D_ρ on X and a generator $n_\rho\in N\cap\rho$. Standard references for toric varieties are [8], [9] and [18]. As explained in [7], X also has the homogeneous coordinate ring $S=\mathbf{C}[x_\rho]$, which is graded by the Chow group $A_{n-1}(X)$ so that a monomial $\prod_\rho x_\rho^{a_\rho}$ has degree given by the class $[\sum_\rho a_\rho D_\rho]\in A_{n-1}(X)$. Given a class $\alpha\in A_{n-1}(X)$, we let S_α denote the graded piece of S in degree α , and we write $\deg(f)=\alpha$ when $f\in S_\alpha$.

Our strategy for defining toric residues is inspired by the trace property of the introduction. Thus we need, first, a trace map $H^n(X, \Omega_X^n)\simeq\mathbf{C}$ and, second, a method that uses polynomials $g\in S_\rho$ (for some $\rho\in A_{n-1}(X)$) to create Čech cohomology classes $[\omega_g]\in H^n(X, \Omega_X^n)$. Then the toric residue will be easy to define.

We begin with the trace map. Since X need not be smooth, we cannot use the usual sheaf on n -forms on X . Instead, we use the sheaf of Zariski n -forms on X , which by abuse of notation will be written Ω_X^n (thus $\Omega_X^n=j_*\Omega_U^n$, where $j:U\rightarrow X$ is the inclusion of the smooth part of X). Since the toric variety X is Cohen–Macaulay with Ω_X^n as dualizing sheaf, we have a trace map $\text{Tr}_X:H^n(X, \Omega_X^n)\simeq\mathbf{C}$. The duality theory used here can be found in [18, §3.2].

Given an ample class $\beta\in A_{n-1}(X)$, there is a line bundle $\mathcal{O}_X(\beta)$ on X and a canonical isomorphism $S_\beta\simeq H^0(X, \mathcal{O}_X(\beta))$ (see [7, §3]). Regarding f as a section of $\mathcal{O}_X(\beta)$, we can talk about what it means for f to vanish at point of X . For the remainder of the paper, we make the following assumption:

- (3) $\beta\in A_{n-1}(X)$ is ample; $f_0, \dots, f_n\in S_\beta$ do not vanish simultaneously on X .

Given f_0, \dots, f_n as above, we set $U_i=\{x\in X:f_i(x)\neq 0\}$. Assumption (3) implies that the U_i form an affine open cover \mathcal{U} of X . As in §1, we can use this open cover to compute $H^n(X, \Omega_X^n)$ by Čech cohomology, so that every section of Ω_X^n over $U_0\cap\dots\cap U_n$ is a Čech cocycle in $C^n(\mathcal{U}, \Omega_X^n)$. Thus every $\omega\in\Omega_X^n(U_0\cap\dots\cap U_n)$ gives $[\omega]\in H^n(X, \Omega_X^n)$.

It remains to study sections of Ω_X^n over $U_0\cap\dots\cap U_n$. We begin by constructing an analog of the form (2). Fix an integer basis m_1, \dots, m_n for the lattice M . Then, given a subset $I=\{\rho_1, \dots, \rho_n\}\subset\Sigma(1)$ consisting of n elements, define

$$\det(n_I)=\det(\langle m_i, n_{\rho_j} \rangle_{1\leq i, j\leq n}).$$

Also set $dx_I=dx_{\rho_1}\wedge\dots\wedge dx_{\rho_n}$ and $\hat{x}_I=\prod_{\rho\notin I}x_\rho$. Note that $\det(n_I)$ and dx_I depend on how the $\rho\in I$ are ordered, while their product $\det(n_I)dx_I$ does not. Then we define the n -form Ω by the formula

$$(4) \quad \Omega = \sum_{|I|=n} \det(n_I)\hat{x}_I dx_I,$$

where the sum is over all n -element subsets $I \subset \Sigma(1)$. This form is well-defined up to ± 1 .

Now consider the graded S -module $\widehat{\Omega}_S^n = S \cdot \Omega$, where Ω is considered to have degree

$$\beta_0 = \sum_{\varrho} \deg(x_{\varrho}) = \left[\sum_{\varrho} D_{\varrho} \right] \in A_{n-1}(X).$$

Thus $\widehat{\Omega}_S^n \simeq S(-\beta_0)$ as graded S -modules. By [7, §3], every graded S -module gives rise to a sheaf on X , and by [4, §9], the sheaf associated to $\widehat{\Omega}_S^n$ is exactly Ω_X^n . Furthermore, we can describe sections of Ω_X^n with prescribed poles as follows.

Proposition 2.1. *Let $\alpha \in A_{n-1}(X)$ be a Cartier class, and let $Y \subset X$ be defined by the vanishing of $f \in S_{\alpha}$. Then*

$$H^0(X, \Omega_X^n(Y)) = \left\{ \frac{g\Omega}{f} : g \in S_{\alpha-\beta_0} \right\} \simeq S_{\alpha-\beta_0}.$$

Proof. This follows from Proposition 9.7 of [4]. (Although [4] assumes that X is simplicial, the results of §8 and §9 of [4] apply to all complete toric varieties.) \square

If we apply this proposition to $f = f_0 \dots f_n \in S_{(n+1)\beta}$, we get an n -form

$$\omega_g = \frac{g\Omega}{f_0 \dots f_n} \in \Omega_X^n(U_0 \cap \dots \cap U_n)$$

for all $g \in S_{(n+1)\beta-\beta_0}$. To simplify notation, we set

$$\varrho = (n+1)\beta - \beta_0.$$

Hence there are classes $[\omega_g] \in H^n(X, \Omega_X^n)$ for all $g \in S_{\varrho}$.

Now we can finally define the toric residue.

Definition 2.2. If $f_0, \dots, f_n \in S_{\beta}$ satisfy (3) and $g \in S_{\varrho}$, the toric residue is

$$\text{Res}(\omega_g) = \text{Tr}_X([\omega_g]).$$

The first properties of toric residues are easy to prove.

Proposition 2.3.

- (1) $\text{Res}(\omega_g)$ is \mathbf{C} -linear in g and antisymmetric in f_0, \dots, f_n .
- (2) $\text{Res}(\omega_g) = 0$ whenever $g \in \langle f_0, \dots, f_n \rangle_{\varrho}$.

Proof. The linearity of the toric residue is clear. The isomorphism $H^n(\mathcal{U}, \Omega_{\mathbf{P}^n}^n) \simeq H^n(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^n)$ comes from the differential in the Čech complex and hence depends antisymmetrically on how we order the open sets $U_i = \{f_i \neq 0\}$ of \mathcal{U} . Thus the toric residue is antisymmetric in the f_i . The second part of the proposition follows easily by considering the Čech coboundary $\delta: C^{n-1}(\mathcal{U}, \Omega_X^n) \rightarrow C^n(\mathcal{U}, \Omega_X^n)$. We omit the details. \square

As a corollary, we see that the toric residue induces a map

$$\text{Res}: S_\varrho / \langle f_0, \dots, f_n \rangle_\varrho \longrightarrow \mathbf{C}.$$

In the next section, we will see that the quotient on the right is one-dimensional, and in §5 we will show that the above map is in fact an isomorphism.

We should also mention that in the forthcoming paper [6], it will be shown that when X is simplicial, the toric residue $\text{Res}(\omega_g)$ can be expressed as a sum of local residues on X . Namely, given $f_0, \dots, f_n \in S_\beta$ which satisfy (3), let D_i be the divisor $f_i = 0$. Then for each $0 \leq k \leq n$, the intersection $D_0 \cap \dots \cap \widehat{D}_k \cap \dots \cap D_n$ is finite since it is contained in the affine variety $X - D_k$. This means for $g \in S_\varrho$, the meromorphic form

$$\omega_g = \frac{(g/f_k)\Omega}{f_0 \dots \widehat{f}_k \dots f_n}$$

has $D_0 \cup \dots \cup \widehat{D}_k \cup \dots \cup D_n$ as local polar divisor near $x \in D_0 \cap \dots \cap \widehat{D}_k \cap \dots \cap D_n$. Thus, for each such x , we can define the Grothendieck local residue symbol

$$\text{Res}_x \left(\frac{(g/f_k)\Omega}{f_0 \dots \widehat{f}_k \dots f_n} \right).$$

Then the following equality is a special case of the results of [6]:

$$\text{Res}(\omega_g) = (-1)^k \sum_{x \in D_0 \cap \dots \cap \widehat{D}_k \cap \dots \cap D_n} \text{Res}_x \left(\frac{(g/f_k)\Omega}{f_0 \dots \widehat{f}_k \dots f_n} \right).$$

3. Some commutative algebra

The basic commutative algebra for our situation is given in [3, Theorem 2.10 and Proposition 9.4]. In this section, we recast Batyrev's results in terms of the ring S and supply some of the details.

We first study the subring of S determined by an ample class $\beta \in A_{n-1}(X)$.

Proposition 3.1. *If $\beta \in A_{n-1}(X)$ is ample, then the ring $S_{*\beta} = \bigoplus_{k=0}^{\infty} S_{k\beta}$ is Cohen–Macaulay of dimension $n+1$, with canonical module given by $\omega_{S_{*\beta}} = \bigoplus_{k=0}^{\infty} S_{k\beta - \beta_0}$.*

Proof. Write β as $[\sum_{\varrho} a_{\varrho} D_{\varrho}]$. Then $\Delta = \{m \in M_{\mathbf{R}} : \langle m, n_{\varrho} \rangle \geq -a_{\varrho}\}$ is an n -dimensional convex polyhedron since β is ample. Let $\check{\sigma} \subset \mathbf{R} \times M_{\mathbf{R}}$ be the cone over $\{1\} \times \Delta$. The dual of $\check{\sigma}$ is a strongly convex rational polyhedral cone $\sigma \subset \mathbf{R} \times N_{\mathbf{R}}$. Then σ determines an $(n+1)$ -dimensional affine toric variety with coordinate ring

$$\begin{aligned} S_{\Delta} &= \mathbf{C}[\check{\sigma} \cap (\mathbf{Z} \times M)] = \mathbf{C}[t_0^k t^m : (k, m) \in \check{\sigma} \cap (\mathbf{Z} \times M)] \\ &= \mathbf{C}[t_0^k t^m : (1, m/k) \in \{1\} \times \Delta] \\ &= \mathbf{C}[t_0^k t^m : m \in k\Delta]. \end{aligned}$$

Hence S_{Δ} is Cohen–Macaulay by [8, Theorem 3.4] or [15]. But $S_{\Delta} \simeq S_{*\beta}$ follows from the proof of [4, Theorem 11.5], so that $S_{*\beta}$ is Cohen–Macaulay.

We next determine the canonical module of $S_{*\beta}$ (see [5] for more background on this topic). By [8, 4.6], the canonical module of the semigroup ring S_{Δ} is a certain ideal $I_{\Delta}^{(1)} \subset S_{\Delta}$ (this is the notation of [3, §2]). Then the proof of [4, Theorem 11.8] shows that under the isomorphism $S_{\Delta} \simeq S_{*\beta}$, the ideal $I_{\Delta}^{(1)} \subset S_{\Delta}$ maps to $\bigoplus_{k=0}^{\infty} \langle \Pi_{\varrho} x_{\varrho} \rangle_{k\beta} \subset S_{*\beta}$. This is isomorphic to $\bigoplus_{k=0}^{\infty} S_{k\beta - \beta_0}$ since $\Pi_{\varrho} x_{\varrho}$ has degree β_0 .

Alternatively, notice that $S_{*\beta} = \bigoplus_{k=0}^{\infty} S_{k\beta}$ has a natural grading and that $X = \text{Proj}(S_{*\beta})$ (see [3, §2]). Then the canonical module is $\bigoplus_{k=0}^{\infty} H^0(X, \Omega_X^n(k\beta))$ by [13, 5.1.8]. However, by Proposition 2.1, we can identify $H^0(X, \Omega_X^n(k\beta))$ with $S_{k\beta - \beta_0}$, which shows that $\bigoplus_{k=0}^{\infty} S_{k\beta - \beta_0}$ is the canonical module. \square

For a general Cohen–Macaulay variety X and ample β , $\bigoplus_{k=0}^{\infty} H^0(X, \mathcal{O}_X(k\beta))$ need not be Cohen–Macaulay, although some Veronese subring will be (see [13, 5.1.11]).

We next bring $f_0, \dots, f_n \in S_{\beta}$ into the picture.

Proposition 3.2. *If $f_0, \dots, f_n \in S_{\beta}$ satisfy (3), then:*

- (i) f_0, \dots, f_n is a regular sequence in $S_{*\beta} = \bigoplus_{k=0}^{\infty} S_{k\beta}$.
- (ii) $R = S_{*\beta} / \langle f_0, \dots, f_n \rangle$ is a zero-dimensional Cohen–Macaulay ring.
- (iii) The canonical module of R is $\omega_R = (\omega_{S_{*\beta}} / \langle f_0, \dots, f_n \rangle) \omega_{S_{*\beta}}[n+1]$ (where the $[n+1]$ indicates a shift in grading).

Proof. Since f_0, \dots, f_n define the empty subvariety of X , $R = S_{*\beta} / \langle f_0, \dots, f_n \rangle$ has dimension zero. Then (i) and (ii) follow from [5, Theorems 2.1.2 and 2.1.3], and (iii) follows from [5, Corollary 3.6.14]. \square

We can now determine $S_{\varrho} / \langle f_0, \dots, f_n \rangle_{\varrho}$, where as usual $\varrho = (n+1)\beta - \beta_0$.

Proposition 3.3. *There is a natural isomorphism $S_{\varrho}/\langle f_0, \dots, f_n \rangle_{\varrho} \simeq \mathbf{C}$.*

Proof. This follows from local duality for graded Cohen–Macaulay rings (see [5, 3.6]). In the zero-dimensional case, local duality is a natural isomorphism of graded R -modules

$$\omega_R \simeq \text{Hom}_{\mathbf{C}}(R, \mathbf{C}).$$

In particular, $(\omega_R)_0 \simeq \text{Hom}_{\mathbf{C}}(R_0, \mathbf{C}) \simeq \mathbf{C}$ since $R_0 = \mathbf{C}$. By Propositions 3.1 and 3.2, we have

$$(\omega_R)_0 = (\omega_{S_{*\beta}} / \langle f_0, \dots, f_n \rangle_{S_{*\beta}})_{n+1} = S_{(n+1)\beta - \beta_0} / \langle f_0, \dots, f_n \rangle_{(n+1)\beta - \beta_0},$$

and the corollary follows. \square

Note that when $\beta = \beta_0$ (so X is a Fano toric variety), we have $\omega_R \simeq R[n]$, so that $S_{*\beta_0}$ and R are Gorenstein (see [5, 3.6.11]). This case is of interest in mirror symmetry.

4. Toric Jacobians

This section will define a “Jacobian” of $f_0, \dots, f_n \in S_{\beta}$ which is closely related to the Jacobian $\det(\partial f_i / \partial x_j)$ when $X = \mathbf{P}^n$. Here is our main result.

Proposition 4.1. *If $f_0, \dots, f_n \in S_{\alpha}$, then there is $J \in S_{(n+1)\alpha - \beta_0}$ such that*

$$\sum_{i=0}^n (-1)^i f_i df_0 \wedge \dots \wedge \widehat{df}_i \wedge \dots \wedge df_n = J\Omega.$$

Furthermore, if $I = \{\varrho_1, \dots, \varrho_n\} \subset \Sigma(1)$ and $n_{\varrho_1}, \dots, n_{\varrho_n}$ are linearly independent, then

$$(5) \quad J = \det \begin{pmatrix} f_0 & \dots & f_n \\ \partial f_0 / \partial x_{\varrho_1} & \dots & \partial f_n / \partial x_{\varrho_1} \\ \vdots & \ddots & \vdots \\ \partial f_0 / \partial x_{\varrho_n} & \dots & \partial f_n / \partial x_{\varrho_n} \end{pmatrix} / \det(n_I) \hat{x}_I.$$

Proof. Note that we assume nothing about the ampleness of α or the vanishing of the f_i ’s. This proof was suggested by Eduardo Cattani and Alicia Dickenstein. We can assume $f_0 \neq 0$, so that

$$\sum_{i=0}^n (-1)^i f_i df_0 \wedge \dots \wedge \widehat{df}_i \wedge \dots \wedge df_n = f_0^{n+1} d(f_1/f_0) \wedge \dots \wedge d(f_n/f_0).$$

Let m_1, \dots, m_n be the basis of M used in the definition of Ω , and set

$$t_i = \Pi_{\varrho} x_{\varrho}^{\langle m_i, n_{\varrho} \rangle}.$$

Then t_1, \dots, t_n are coordinates for the torus $T \subset X$ (see [4, §9]), and each f_i/f_0 is a rational function of the t_i since f_i and f_0 have the same degree. Hence

$$d(f_1/f_0) \wedge \dots \wedge d(f_n/f_0) = J(t_1, \dots, t_n) dt_1 \wedge \dots \wedge dt_n$$

for some rational function $J(t_1, \dots, t_n)$. However, the proof of Proposition 9.5 in [4] shows that

$$\Omega = \Pi_{\varrho} x_{\varrho} \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_n}{t_n},$$

and it follows easily from the above equations that

$$\sum_{i=0}^n (-1)^i f_i df_0 \wedge \dots \wedge \widehat{df}_i \wedge \dots \wedge df_n = J\Omega,$$

where

$$J = f_0^{n+1} J(t_1, \dots, t_n) \frac{t_1 \dots t_n}{\Pi_{\varrho} x_{\varrho}}.$$

This equation shows that J has degree $(n+1)\alpha - \beta_0$ as a rational function of the x_{ϱ} 's.

If $I \subset \Sigma(1)$ has $|I|=n$, we let $D(f_I)$ denote the determinant in the numerator of (5). Then one easily computes that

$$\sum_{i=0}^n (-1)^i f_i df_0 \wedge \dots \wedge \widehat{df}_i \wedge \dots \wedge df_n = \sum_{|I|=n} D(f_I) dx_I.$$

Since the right hand side equals $J\Omega$, it follows that

$$J \det(n_I) \hat{x}_I = D(f_I)$$

for all I . This gives the desired formula (5) for J .

It remains to show that J is a polynomial in the x_{ϱ} 's. If we write J as a quotient of relatively prime polynomials in S , then (5) shows that the denominator divides \hat{x}_I for every I with $\det(n_I) \neq 0$. Since the fan of X is complete, every $\varrho \in \Sigma(1)$ is in some such I (ϱ lies in an n -dimensional cone σ , and I can be chosen to be an appropriate subset of $\sigma(1)$ containing ϱ). It follows that the \hat{x}_I 's are relatively prime, which forces the denominator of J to be a constant. Since we have already seen that J has degree $(n+1)\alpha - \beta_0$, it follows immediately that $J \in S_{(n+1)\alpha - \beta_0}$, and the proposition is proved. \square

In light of this proposition, we make the following definition.

Definition 4.2. Given $f_0, \dots, f_n \in S_\alpha$, the polynomial $J \in S_{(n+1)\alpha - \beta_0}$ satisfying

$$\sum_{i=0}^n (-1)^i f_i df_0 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n = J\Omega$$

is called the *toric Jacobian* of f_0, \dots, f_n .

For example, suppose $X = \mathbf{P}^n$ and f_0, \dots, f_n are homogeneous of degree d . If I is given by x_1, \dots, x_n , then applying the Euler formula $f_j = (1/d) \sum_{i=0}^n x_i \partial f_j / \partial x_i$ to (5) shows that the toric Jacobian is given by

$$\begin{aligned} J &= \det \begin{pmatrix} f_0 & \dots & f_n \\ \partial f_0 / \partial x_1 & \dots & \partial f_n / \partial x_1 \\ \vdots & \ddots & \vdots \\ \partial f_0 / \partial x_n & \dots & \partial f_n / \partial x_n \end{pmatrix} / x_0 \\ &= \det \begin{pmatrix} d^{-1} x_0 \partial f_0 / \partial x_0 & \dots & d^{-1} x_0 \partial f_n / \partial x_0 \\ \partial f_0 / \partial x_1 & \dots & \partial f_n / \partial x_1 \\ \vdots & \ddots & \vdots \\ \partial f_0 / \partial x_n & \dots & \partial f_n / \partial x_n \end{pmatrix} / x_0 = \frac{1}{d} \det(\partial f_i / \partial x_j). \end{aligned}$$

In [19, 12.10], it was assumed that $J = \det(\partial f_i / \partial x_j)$, which caused the residue formula given there to have an extra factor of d .

The toric Jacobian is also related to the hyperdeterminant of an $m \times (m+p-1) \times p$ matrix $A = (a_{ijk})$, as described in [10, §3 of Chapter 14]. From A , we get $m+p-1$ bilinear forms $f_j = \sum_{ik} a_{ijk} x_i y_k$ in variables $x_1, \dots, x_m, y_1, \dots, y_p$. Thus $X = \mathbf{P}^{m-1} \times \mathbf{P}^{p-1}$ and $f_j \in S_{1,1}$, where $S = \mathbf{C}[x_i; y_k]$ has the usual bigrading. The toric Jacobian J of f_1, \dots, f_{m+p-1} has degree $(m+p-1)(1,1) - (m,p) = (p-1, m-1)$. In [10, Chapter 14], J appears in equation (3.20), and in Theorem 3.19, the coefficients of J are used to compute the hyperdeterminant of A . Also, Proposition 3.21 gives an interesting combinatorial interpretation of the coefficients of J .

As our final example, let $f = x^2 z^2 + x^2 w^2 + y^2 z^2 + y^2 w^2 + \lambda x y z w \in \mathbf{C}[x, y, z, w]$. Then f has degree $(2,2)$, so that the toric Jacobian of $f, x f_x, z f_z$ has degree $3(2,2) - (2,2) = (4,4)$. Since $\Omega = (x dy - y dx) \wedge (z dw - w dz)$ for $\mathbf{P}^1 \times \mathbf{P}^1$, we know the $\det(n_I)$'s, and one computes that the toric Jacobian is given by

$$\begin{aligned} &4(\lambda x^4 z^2 w^2 + 4x^3 y z w^3 + 4x^3 y z^3 w + \lambda x^2 y^2 z^4 \\ &\quad + \lambda x^2 y^2 w^4 + 4x y^3 z^3 w + 4x y^3 z w^3 + \lambda y^4 z^2 w^2). \end{aligned}$$

Other examples are equally easy to compute.

5. Uniqueness of toric residues

Putting together what we proved in the previous sections, we can now state the main theorem of this paper.

Theorem 5.1. *Let X be a complete toric variety, and let $\beta \in A_{n-1}(X)$ be ample. If $f_0, \dots, f_n \in S_\beta$ do not vanish simultaneously on X , then:*

(i) *If $\varrho = (n+1)\beta - \beta_0$, the toric residue map $\text{Res}: S_\varrho / \langle f_0, \dots, f_n \rangle_\varrho \rightarrow \mathbf{C}$ from §2 is an isomorphism.*

(ii) *If $J \in S_\varrho$ is the toric Jacobian of f_0, \dots, f_n from §4, then*

$$\text{Res}(\omega_J) = n! \text{vol}(\Delta) = \text{deg}(F),$$

where $\text{vol}(\Delta)$ is the normalized volume of the convex polyhedron $\Delta \subset M_{\mathbf{R}}$ associated to β (see the proof of Proposition 3.1) and $F: X \rightarrow \mathbf{P}^n$ is the map defined by $F(x) = (f_0(x), \dots, f_n(x))$.

Proof. We know from Proposition 3.3 that $S_\varrho / \langle f_0, \dots, f_n \rangle_\varrho$ has dimension one. Hence (i) is an immediate consequence of (ii). To prove (ii), note that $F = (f_0, \dots, f_n): X \rightarrow \mathbf{P}^n$ comes from $n+1$ sections of an ample line bundle. Since the sections never vanish simultaneously and $\dim(X) = n$, F is defined everywhere and is finite and surjective.

We now proceed as in [19, 12.10]. By the definition of the toric Jacobian, we have

$$\begin{aligned} J\Omega &= \sum_{i=0}^n (-1)^i f_i df_0 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n \\ &= F^* \left(\sum_{i=0}^n (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n \right) = F^*(\Omega_{\mathbf{P}^n}). \end{aligned}$$

Thus

$$\omega_J = \frac{J\Omega}{f_0 \dots f_n} = F^* \left(\frac{\Omega_{\mathbf{P}^n}}{x_0 \dots x_n} \right).$$

Denote the n -form in parentheses by ω_1 . Then the map

$$F^*: H^n(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^n) \rightarrow H^n(X, \Omega_X^n)$$

satisfies $F^*([\omega_1]) = [\omega_J]$ (this is easy to see using Čech cohomology). Since F is finite and surjective, standard properties of the trace map imply that

$$\text{Res}(\omega_J) = \text{Tr}_X([\omega_J]) = \text{Tr}_X(F^*([\omega_1])) = \text{deg}(F) \text{Tr}_{\mathbf{P}^n}([\omega_1]) = \text{deg}(F)$$

(see [14, Chapter III]).

To complete the proof, we need to show that $\deg(F) = n! \operatorname{vol}(\Delta)$. If D is the divisor of a section of the line bundle $\mathcal{O}_X(\beta)$, then it is well-known that

$$D^n = n! \operatorname{vol}(\Delta)$$

(see [18, Proposition 2.10]). Note that when X is singular, we use the intersection number D^n as defined in [16, Chapter I] or [18, §2.2]. Since $F: X \rightarrow \mathbf{P}^n$ is finite and $F^*(\mathcal{O}_{\mathbf{P}^n}(1)) \simeq \mathcal{O}_X(\beta)$, it follows from [16, Proposition 6 of Chapter I, §2] that

$$D^n = \deg(F)H^n = \deg(F),$$

where $H \subset \mathbf{P}^n$ is a hyperplane. Thus $\deg(F) = n! \operatorname{vol}(V)$, and we are done. \square

Hence toric residues have both the quotient and trace properties mentioned in the introduction. Note also that the toric residue is uniquely characterized by these properties.

When $X = \mathbf{P}^n$ and f_0, \dots, f_n have degree d , we saw in §4 that the toric Jacobian is $J = (1/d) \det(\partial f_i / \partial x_j)$. Since $F = (f_0, \dots, f_n)$ has degree d^n , if we set $g = \det(\partial f_i / \partial x_j)$, then

$$\operatorname{Res}(\omega_g) = d \operatorname{Res}(\omega_J) = d \deg(F) = d^{n+1}.$$

However, for this choice of g , the Grothendieck residue symbol (1) is the local intersection number of the divisors in \mathbf{C}^{n+1} defined by $f_i = 0$, which is also d^{n+1} . Thus

$$\operatorname{Res}(\omega_g) = \operatorname{Res}_0 \left(\frac{g dx_0 \wedge \dots \wedge dx_n}{f_0 \dots f_n} \right),$$

and from here, one easily sees that equality holds for all $g \in S_\rho$ (see [19, 12.10]).

For another application of our theory, suppose that $f \in S_\beta$, where β is ample and f is nondegenerate in the sense of [4, Definition 4.13]. In this situation, f defines a hypersurface $Y \subset X$, and the ideal

$$J_0(f) = \langle x_\rho \partial f / \partial x_\rho \rangle \subset S$$

is closely related to the mixed Hodge structure of the affine hypersurface $Y \cap T$, where $T \subset X$ is the torus of X (see [3, §9] or [4, §11]). Then we get the following proposition.

Proposition 5.3. *Given $f \in S_\beta$, suppose that $n_{\varrho_1}, \dots, n_{\varrho_n}$ are linearly independent. Then $J_0(f) = \langle f, x_{\varrho_1} \partial f / \partial x_{\varrho_1}, \dots, x_{\varrho_n} \partial f / \partial x_{\varrho_n} \rangle$. Furthermore,*

$$\begin{aligned} f \text{ is nondegenerate} &\iff x_\varrho \partial f / \partial x_\varrho \text{ do not vanish simultaneously on } X \\ &\iff f, x_{\varrho_i} \partial f / \partial x_{\varrho_i} \text{ do not vanish simultaneously on } X. \end{aligned}$$

Finally, if β is ample and f is nondegenerate, then $S_\varrho / J_0(f)_\varrho \simeq \mathbf{C}$ ($\varrho = (n+1)\beta - \beta_0$), and the toric Jacobian of $f, x_{\varrho_1} \partial f / \partial x_{\varrho_1}, \dots, x_{\varrho_n} \partial f / \partial x_{\varrho_n}$ represents a nonzero element of $S_\varrho / J_0(f)_\varrho$.

Proof. We will use Euler formulas. Recall from [4, §3] that $\theta = \sum_\varrho b_\varrho x_\varrho \partial / \partial x_\varrho$ is an Euler vector field provided $\sum_\varrho b_\varrho n_\varrho = 0$. Further, such a θ determines a constant $\theta(\beta)$ such that the Euler formula

$$\theta(\beta) f = \sum_\varrho b_\varrho x_\varrho \partial f / \partial x_\varrho$$

holds for all $f \in S_\beta$.

Now, given any ϱ , there is a relation $n_\varrho + \sum_{i=1}^n b_i n_{\varrho_i} = 0$ since the n_{ϱ_i} are a basis over \mathbf{Q} . The resulting Euler formula shows that $x_\varrho \partial f / \partial x_\varrho \in \langle f, x_{\varrho_i} \partial f / \partial x_{\varrho_i} \rangle$. We also have $f \in J_0(f)$ (see the proof of [4, Lemma 10.5]), and it follows that f and the $x_{\varrho_i} \partial f / \partial x_{\varrho_i}$ generate $J_0(f)$.

The argument of [4, Proposition 3.5] adapts easily to show that f is nondegenerate if and only if $x_\varrho \partial f / \partial x_\varrho$ do not vanish simultaneously. This proves the first equivalence of the proposition, and the second equivalence is now trivial.

Finally, when β is ample and $f \in S_\beta$ is nondegenerate, we can apply Theorem 5.1 to $f, x_{\varrho_1} \partial f / \partial x_{\varrho_1}, \dots, x_{\varrho_n} \partial f / \partial x_{\varrho_n}$. This gives the final part of the proposition. \square

An example of this can be found in §4, where $X = \mathbf{P}^1 \times \mathbf{P}^1$ and $f = x^2 z^2 + x^2 w^2 + y^2 z^2 + y^2 w^2 + \lambda x y z w \in S = \mathbf{C}[x, y, z, w]$. One can check that f is nondegenerate provided $\lambda \neq 0, \pm 4$, so that the toric Jacobian of $f, x \partial f / \partial x$ and $z \partial f / \partial z$ (displayed at the end of §4) is a nonzero element of $S_{4,4} / J_0(f)_{4,4} \simeq \mathbf{C}$.

In general, the isomorphism $S_\varrho / J_0(f)_\varrho \simeq \mathbf{C}$ was previously known to Batyrev and is related to the cup-product pairing

$$H^{n-1}(Y \cap T, \mathbf{C}) \otimes H_c^{n-1}(Y \cap T, \mathbf{C}) \longrightarrow \mathbf{C}$$

(see [3, §9] or [4, §11]). If we assume in addition that X is simplicial, then the ideal quotient

$$J_1(f) = J_0(f) : \Pi_\varrho x_\varrho$$

also plays a role: it is related to the Hodge structure on the primitive cohomology (as defined in [4, 10.9]) of the hypersurface $Y \subset X$ defined by f , assuming $f \in S_\beta$ is nondegenerate and β ample. In this case, multiplication by $\Pi_\varrho x_\varrho$ induces a natural injection

$$S_{(n+1)\beta-2\beta_0}/J_1(f)_{(n+1)\beta-2\beta_0} \xrightarrow{\Pi_\varrho x_\varrho} S_\varrho/J_0(f)_\varrho,$$

and the composition

$$(6) \quad S_{(n+1)\beta-2\beta_0}/J_1(f)_{(n+1)\beta-2\beta_0} \xrightarrow{\Pi_\varrho x_\varrho} S_\varrho/J_0(f)_\varrho \xrightarrow{\text{Res}} \mathbf{C}$$

is closely related to cup product on the primitive cohomology of Y . In fact, if the primitive cohomology of Y is nontrivial, Poincaré duality implies that (6) is an isomorphism. This is used in [17, §5], where the toric residue is interpreted as an “expectation function” arising in mirror symmetry.

Is there an explicit formula for a polynomial that gives a nonzero element in the left-hand quotient of (6)? In the case of $X = \mathbf{P}^n$, the Hessian $\det(\partial^2 f / \partial x_i \partial x_j)$ is such an element. It would be very interesting to have a toric generalization of the Hessian.

Another question is what happens if $f_0, \dots, f_n \in S$ are homogeneous, do not vanish simultaneously on X , but have different degrees. One can still define the toric residue in this case, but none of the results of this paper apply. However, if X is simplicial and $\deg(f_i)$ is ample for all i , then a version of Theorem 5.1 still holds. This will be discussed in [6].

6. Integral representations

In this section, we will explore several ways of representing the toric residue as an integral. In order to do this, we will assume that X is a projective simplicial toric variety. Thus X is an orbifold (or V -manifold), so that by [2], we have a Dolbeault isomorphism $H_{\bar{\partial}}^{n,n}(X) \simeq H^n(X, \Omega_X^n)$.

Our first set of formulas use the Dolbeault isomorphism to express $\text{Res}(\omega_g)$ as an integral. We begin with $f_0, \dots, f_n \in S_\beta$ which satisfy (3). As in §2, we get the open covering \mathcal{U} of X and the Čech cocycle $\omega_g \in C^n(\mathcal{U}, \Omega_X^n)$ for $g \in S_\beta$. Now let

$$\varrho_i = \frac{|f_i|^2}{|f_0|^2 + \dots + |f_n|^2}$$

for $0 \leq i \leq n$. Since the f_i all have the same degree and do not vanish simultaneously, the ϱ_i are C^∞ -functions on X which sum to 1.

Proposition 6.1. *The class of the C^∞ (n, n) -form*

$$\eta_g = (-1)^{n(n+1)/2} n! \frac{g \sum_{i=0}^n (-1)^i \bar{f}_i d\bar{f}_0 \wedge \dots \wedge d\bar{f}_i \wedge \dots \wedge d\bar{f}_n \wedge \Omega}{(|f_0|^2 + \dots + |f_n|^2)^{n+1}}$$

maps to $[\omega_g]$ under the Dolbeault isomorphism $H_{\bar{\partial}}^{n,n}(X) \simeq H^n(X, \Omega_X^n)$.

Remark. This proposition gives a more precise version of a formula in [11, Chapter 5]. There, the factor $(-1)^{n(n+1)/2} n!$ appears as a constant C_n .

Proof. We first show that η_g lives on X . Since X is simplicial and complete, we can write X as a geometric quotient $(\mathbf{C}^{\Sigma(1)} - Z)/G$, where $Z \subset \mathbf{C}^{\Sigma(1)}$ is a proper closed subvariety and $G = \text{Hom}_{\mathbf{Z}}(A_{n-1}(X), \mathbf{C}^*)$ (see [7, §2]). This is also a quotient in the real analytic category, so that the form η_g on $\mathbf{C}^{\Sigma(1)} - Z$ descends to X if and only if it is invariant under G and is annihilated by all real vector fields to G . From [4, §3], we know that the complex Lie algebra of G , denoted $\text{Lie}(G)$, consists of the Euler vector fields $\theta = \sum_{\rho} b_{\rho} x_{\rho} \partial / \partial x_{\rho}$ described in the proof of Proposition 5.2.

The form η_g is clearly invariant under G since all f_i have the same degree. To see that it is annihilated by all real vector fields to G , it suffices to show that

$$\theta \lrcorner \eta_g = \bar{\theta} \lrcorner \eta_g = 0$$

for all $\theta \in \text{Lie}(G)$. However, Proposition 4.1 implies that η_g is a C^∞ multiple of $\bar{\Omega} \wedge \Omega$. Thus we need to prove that

$$\theta \lrcorner \Omega = \bar{\theta} \lrcorner \bar{\Omega} = \theta \lrcorner \bar{\Omega} = \bar{\theta} \lrcorner \Omega = 0.$$

The last two of these are trivially zero, and the vanishing of the second follows from the vanishing of the first. Hence it remains to show that $\theta \lrcorner \Omega = 0$ for any $\theta \in \text{Lie}(G)$. This will follow immediately from the following observation of Batyrev.

Lemma 6.2. *Let $\theta_1, \dots, \theta_r$ be an ordered basis of $\text{Lie}(G)$, and let $d\mathbf{x} = \bigwedge_{\rho} dx_{\rho}$ for some ordering of the x_{ρ} 's. Then there is a nonzero constant c such that*

$$(\theta_1 \wedge \dots \wedge \theta_r) \lrcorner d\mathbf{x} = c\Omega$$

(where \lrcorner denotes interior multiplication).

Proof. Consider the map $\bigoplus_{\rho} \mathbf{C} x_{\rho} \partial / \partial x_{\rho} \rightarrow N_{\mathbf{C}}$ which sends $x_{\rho} \partial / \partial x_{\rho}$ to n_{ρ} . This map is onto since X is complete, and the kernel consists of the Euler vector fields, which as above form the Lie algebra $\text{Lie}(G)$. Thus we have an exact sequence

$$(7) \quad 0 \longrightarrow \text{Lie}(G) \longrightarrow \bigoplus_{\rho} \mathbf{C} x_{\rho} \partial / \partial x_{\rho} \longrightarrow N_{\mathbf{C}} \longrightarrow 0.$$

We first compute $(\theta_1 \wedge \dots \wedge \theta_r) \lrcorner d\mathbf{x}$. Write $\theta_i = \sum_{\varrho} b_{\varrho}^i x_{\varrho} \partial / \partial x_{\varrho}$ for $1 \leq i \leq r$, and if $I \subset \Sigma(1)$ has cardinality n , let

$$\det(\hat{b}_I) = \det(b_{\varrho}^i : 1 \leq i \leq r, \varrho \notin I).$$

The matrix (b_{ϱ}^i) is square since $r = |\Sigma(1)| - n$ by (7). Then one computes that

$$(\theta_1 \wedge \dots \wedge \theta_r) \lrcorner d\mathbf{x} = \sum_{|I|=n} (-1)^I \det(\hat{b}_I) \hat{x}_I dx_I,$$

where $\hat{x}_I dx_I$ is as in §2 and $(-1)^I$ is the sign of the permutation of $\Sigma(1)$ which puts the $\varrho \in I$ at the end but otherwise preserves their order (see [20, p. 21]).

To relate this to Ω , we compute the determinant (as defined in [10, Appendix A]) of the exact sequence (7) with respect to the following ordered bases. First, $\theta_1, \dots, \theta_r$ give an ordered basis of $\text{Lie}(G)$, and the ordering of the x_{ϱ} gives an obvious ordered basis of the middle term of (7). Finally, recall that in §2 we picked an ordered basis m_1, \dots, m_n of M . So we let m_1^*, \dots, m_n^* be the dual basis of N . Then, by [10, Appendix A], the based exact sequence (7) has a determinant $c \in \mathbf{C}^*$. We claim that

$$(8) \quad c \det(n_I) = (-1)^I \det(\hat{b}_I)$$

for all $I \subset \Sigma(1)$ with $|I|=n$. Comparing the formula (4) for Ω to the above formula for $(\theta_1 \wedge \dots \wedge \theta_r) \lrcorner d\mathbf{x}$, the lemma will follow immediately once (8) is proved.

Let $I = \{\varrho_1, \dots, \varrho_n\}$. First note that (8) holds when $n_{\varrho_1}, \dots, n_{\varrho_n}$ are linearly dependent, since $\det(n_I) = 0$, and $\det(\hat{b}_I) = 0$ is also true as any relation $\sum_{i=0}^n b_i n_{\varrho_i} = 0$ must be a consequence of the relations $\sum_{\varrho} b_{\varrho}^i n_{\varrho} = 0$. On the other hand, if $n_{\varrho_1}, \dots, n_{\varrho_n}$ are linearly independent, consider the ordered basis of $\bigoplus \mathbf{C} x_{\varrho} \partial / \partial x_{\varrho}$ where the $x_{\varrho} \partial / \partial x_{\varrho}$ for $\varrho \in I$ all appear at the end (but otherwise we preserve their order). Using this basis and the bases θ_i and m_i^* for the other terms of (7), Proposition 11 of [10, Appendix A] expresses the determinant as a quotient of two determinants, where the numerator is the $r \times r$ determinant obtained using the θ_i and the $x_{\varrho} \partial / \partial x_{\varrho}$ for $\varrho \notin I$, and the denominator is the $n \times n$ determinant using $x_{\varrho} \partial / \partial x_{\varrho}$ for $\varrho \in I$ and the m_i^* . Hence the determinant of this based exact sequence is

$$c' = \frac{\det(\hat{b}_I)}{\det(n_I)}.$$

However, c and c' differ by the change of basis determinant for the middle term of (7) (see Proposition 9 of [10, Appendix A]), so that $c = (-1)^I c'$, and (8) is proved. \square

To continue the proof of Proposition 6.1, we next recall the Dolbeault isomorphism. As in [13, p. 45], we have the exact sequences

$$0 \longrightarrow \mathcal{Z}^{n,n-p-1} \longrightarrow \mathcal{A}^{n,n-p-1} \xrightarrow{\bar{\delta}} \mathcal{Z}^{n,n-p} \longrightarrow 0,$$

and the Dolbeault isomorphism $H_{\bar{\delta}}^{n,n}(X) \simeq H^n(X, \Omega_X^n)$ is obtained by composing the coboundary maps

$$\delta_p: H^p(X, \mathcal{Z}^{n,n-p}) \longrightarrow H^{p+1}(X, \mathcal{Z}^{n,n-p-1}).$$

To prove the proposition, it suffices to show that $\delta_{n-1} \circ \dots \circ \delta_0(\eta_g) = [\omega_g]$.

We will use the following notation. Given $0 \leq i_0 < \dots < i_p \leq n$, let $U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$. Also, let $j_1 < \dots < j_{n-p}$ be the complementary indices, so that $\{i_0, \dots, i_p\} \cup \{j_1, \dots, j_{n-p}\} = \{0, \dots, n\}$ is a disjoint union. Finally, let $\varepsilon(i_0, \dots, i_p)$ be the sign of the permutation sending $0, \dots, n$ to $i_0, \dots, i_p, j_1, \dots, j_{n-p}$ respectively.

We now define some Čech cochains. First, let $\gamma_p \in C^p(\mathcal{U}, \mathcal{A}^{n,n-p})$ be given by

$$(\gamma_p)_{i_0 \dots i_p} = \varepsilon(i_0, \dots, i_p) \bar{\delta} \varrho_{j_1} \wedge \dots \wedge \bar{\delta} \varrho_{j_{n-p}} \wedge \omega_g.$$

Note that each $\bar{\delta} \varrho_j$ is divisible by f_j , so that $\bar{\delta} \varrho_{j_1} \wedge \dots \wedge \bar{\delta} \varrho_{j_{n-p}}$ is divisible by $f_{j_1} \dots f_{j_{n-p}}$. This implies that $\gamma_p \in C^p(\mathcal{U}, \mathcal{A}^{n,n-p})$. Second, let $\tilde{\gamma}_p \in C^p(\mathcal{U}, \mathcal{A}^{n,n-p-1})$ be given by

$$(\tilde{\gamma}_p)_{i_0 \dots i_p} = \varepsilon(i_0, \dots, i_p) \sum_{s=1}^{n-p} (-1)^{s-1} \varrho_{j_s} \bar{\delta} \varrho_{j_1} \wedge \dots \wedge \widehat{\bar{\delta} \varrho_{j_s}} \wedge \dots \wedge \bar{\delta} \varrho_{j_{n-p}} \wedge \omega_g.$$

As above, we have $\tilde{\gamma}_p \in C^p(\mathcal{U}, \mathcal{A}^{n,n-p-1})$.

It is easy to see that

$$(9) \quad \bar{\delta} \tilde{\gamma}_p = (n-p) \gamma_p,$$

and, if δ is the coboundary in the Čech complex, then

$$(10) \quad \delta \tilde{\gamma}_p = (-1)^{p+1} \gamma_{p+1}$$

(we omit the straightforward but cumbersome proof).

From (9) and (10), we get a class $[\gamma_p] \in H^p(X, \mathcal{Z}^{n,n-p})$, and then a standard diagram chase, also using (9) and (10), tells us that

$$\delta_p((n-p)[\gamma_p]) = (-1)^{p+1} [\gamma_{p+1}] \in H^{p+1}(X, \mathcal{Z}^{n,n-p-1}).$$

It follows that $n![\gamma_0]$ maps to $(-1)^{n(n+1)/2}[\gamma_n]$ under the Dolbeault isomorphism. However, $\gamma_n=(\gamma_n)_{0\dots n}=\omega_g$, and using $\sum_{i=0}^n \varrho_i=1$, one can verify that γ_0 is given by

$$\gamma_0 = \bar{\partial}\varrho_1 \wedge \dots \wedge \bar{\partial}\varrho_n \wedge \omega_g = \left(\frac{f_0 \dots f_n \sum_{i=0}^n (-1)^i \bar{f}_i d\bar{f}_0 \wedge \dots \wedge \widehat{d\bar{f}_i} \wedge \dots \wedge d\bar{f}_n}{(|f_0|^2 + \dots + |f_n|^2)^{n+1}} \right) \wedge \omega_g,$$

where the second equality follows from [13, p. 655]. From here, the proposition follows immediately. \square

By Proposition A.1 in the appendix, we know that $\text{Tr}_X([\omega_g])=(-1/2\pi i)^n \int_X \eta_g$. Hence we obtain the following integral representations of the toric residue.

Theorem 6.3. *If X is simplicial and $f_0, \dots, f_n \in S_\beta$ satisfy (3), then for $g \in S_\varrho$, we have*

$$\text{Res}(\omega_g) = \frac{(-1)^{n(n-1)/2} n!}{(2\pi i)^n} \int_X \frac{g \sum_{i=0}^n (-1)^i \bar{f}_i d\bar{f}_0 \wedge \dots \wedge \widehat{d\bar{f}_i} \wedge \dots \wedge d\bar{f}_n \wedge \Omega}{(|f_0|^2 + \dots + |f_n|^2)^{n+1}}.$$

Furthermore, if J is the toric Jacobian of f_0, \dots, f_n , then

$$\text{Res}(\omega_g) = \frac{(-1)^{n(n-1)/2} n!}{(2\pi i)^n} \int_X \frac{g \bar{J} \bar{\Omega} \wedge \Omega}{(|f_0|^2 + \dots + |f_n|^2)^{n+1}}. \quad \square$$

Before giving our second set of integral formulas, we review some symplectic geometry. As above, X is a projective simplicial toric variety, so that X is a geometric quotient $(\mathbf{C}^{\Sigma(1)} - Z)/G$. Now let $G_{\mathbf{R}} = \text{Hom}_{\mathbf{Z}}(A_{n-1}(X), S^1)$ be the compact Lie group associated to G . This group has Lie algebra $\text{Lie}(G_{\mathbf{R}}) = \text{Hom}_{\mathbf{Z}}(A_{n-1}(X), \mathbf{R})$. The action of $G_{\mathbf{R}}$ on $\mathbf{C}^{\Sigma(1)}$ is Hamiltonian, which gives the *moment map*

$$\mu: \mathbf{C}^{\Sigma(1)} \longrightarrow \text{Lie}(G_{\mathbf{R}})^*$$

(see [1, Chapter VI, §3.1]). For us, the key property of μ is that if $\xi \in \text{Lie}(G_{\mathbf{R}})^* \simeq H^2(X, \mathbf{R})$ is in the Kähler cone of X , then $G_{\mathbf{R}}$ acts on $\mu^{-1}(\xi)$, and there is a natural isomorphism

$$\mu^{-1}(\xi)/G_{\mathbf{R}} \simeq X$$

(see [1, Chapter VI, Proposition 3.1.1]).

When $X = \mathbf{P}^n$, the moment map is $\mu(x_0, \dots, x_n) = \frac{1}{2}(|x_0|^2 + \dots + |x_n|^2)$, so that $\mu^{-1}(\xi)$ is the sphere S_r^{2n+1} of radius $r = \sqrt{2\xi}$, and the map $\mu^{-1}(\xi) \rightarrow \mathbf{P}^n$ is the Hopf fibration. Another example is $X = \mathbf{P}^1 \times \mathbf{P}^1$, where $\mu(x, y; z, w) =$

$\frac{1}{2}(|x|^2+|y|^2, |z|^2+|w|^2)$ and $\mu^{-1}(\xi_1, \xi_2) = S_{r_1}^3 \times S_{r_2}^3$, where $r_i = \sqrt{2\xi_i}$. In general, $\mu^{-1}(\xi)$ is a real manifold of dimension $|\Sigma(1)|+n$, and we call the map

$$\mu^{-1}(\xi) \longrightarrow X$$

the *generalized Hopf fibration* of X . When X is smooth, this map is a genuine fibration with fiber $G_{\mathbf{R}}$, but in the simplicial case, this is only true generically.

Using the generalized Hopf fibration, we get two more integral formulas for the toric residue.

Theorem 6.4. *If X is simplicial and $f_0, \dots, f_n \in S_\beta$ satisfy (3), let $d\mathbf{x} = \bigwedge_\rho dx_\rho$ for some ordering of the x_ρ 's. Then $\mu^{-1}(\xi)$ can be oriented so that for $g \in S_\rho$, we have*

$$\text{Res}(\omega_g) = \frac{n!}{(2\pi i)^{|\Sigma(1)|}} \int_{\mu^{-1}(\xi)} \frac{g \sum_{i=0}^n (-1)^i \bar{f}_i d\bar{f}_0 \wedge \dots \wedge \widehat{d\bar{f}_i} \wedge \dots \wedge d\bar{f}_n \wedge d\mathbf{x}}{(|f_0|^2 + \dots + |f_n|^2)^{n+1}}.$$

Furthermore, if J is the toric Jacobian of f_0, \dots, f_n , then

$$\text{Res}(\omega_g) = \frac{n!}{(2\pi i)^{|\Sigma(1)|}} \int_{\mu^{-1}(\xi)} \frac{g \bar{J} \bar{\Omega} \wedge d\mathbf{x}}{(|f_0|^2 + \dots + |f_n|^2)^{n+1}}.$$

Proof. The first step is to relate $\text{Lie}(G_{\mathbf{R}})$ to the Euler vector fields considered earlier. Each $\vartheta \in \text{Lie}(G_{\mathbf{R}})$ comes from a relation $\sum_\rho b_\rho n_\rho = 0$ where $b_\rho \in \mathbf{R}$. The Euler vector field $\theta = \sum_\rho b_\rho x_\rho \partial / \partial x_\rho$ is a holomorphic vector field tangent to G -orbits, and we also have a real vector field ϑ tangent to $G_{\mathbf{R}}$ orbits. These are connected by the formula

$$\vartheta = i(\theta - \bar{\theta}).$$

To see why, let t be a real parameter. If $x_\rho = u_\rho + iv_\rho$ gives real and imaginary parts, then at the point $(x_\rho) \in \mathbf{C}^{\Sigma(1)}$, we have

$$\vartheta = \frac{d}{dt} (e^{ib_\rho t} (u_\rho + iv_\rho))|_{t=0} = (b_\rho (-v_\rho + iu_\rho)).$$

Using real coordinates, $\vartheta = \sum_\rho b_\rho (-v_\rho \partial / \partial u_\rho + u_\rho \partial / \partial v_\rho)$, which implies $\vartheta = i(\theta - \bar{\theta})$.

Now suppose that $\vartheta_1, \dots, \vartheta_r$ form a basis of $\text{Lie}(G_{\mathbf{R}})$, where $r = |\Sigma(1)| - n$. The corresponding Euler vector fields are $\theta_1, \dots, \theta_r$, and we claim that

$$(11) \quad (\vartheta_1 \wedge \dots \wedge \vartheta_r) \lrcorner (\bar{\Omega} \wedge d\mathbf{x}) = c i^r (-1)^{nr} \bar{\Omega} \wedge \Omega,$$

where c satisfies $(\theta_1 \wedge \dots \wedge \theta_r) \lrcorner d\mathbf{x} = c \Omega$ as in Lemma 6.2. To prove this, first note that $\theta_j \lrcorner \Omega = 0$ by Lemma 6.2, and thus $\vartheta_j \lrcorner \bar{\Omega} = i(\theta_j - \bar{\theta}_j) \lrcorner \bar{\Omega} = 0$. Hence,

$$\vartheta_j \lrcorner (\bar{\Omega} \wedge d\mathbf{x}) = (-1)^n \bar{\Omega} \wedge (\vartheta_j \lrcorner d\mathbf{x}) = i(-1)^n \bar{\Omega} \wedge (\theta_j \lrcorner d\mathbf{x}),$$

and (11) now follows easily from Lemma 6.2.

We next express "integration over the fiber" in terms of interior multiplication.

Lemma 6.5. *Let η be a $G_{\mathbf{R}}$ -invariant form on $\mu^{-1}(\xi)$ of degree $|\Sigma(1)|+n$, and let $\vartheta_1, \dots, \vartheta_r$, $r=|\Sigma(1)|-n$, be a basis of $\text{Lie}(G_{\mathbf{R}})$. Then we have the product formula*

$$\int_{\mu^{-1}(\xi)} \eta = \int_X (\vartheta_1 \wedge \dots \wedge \vartheta_r) \lrcorner \eta \cdot \int_{G_{\mathbf{R}}} \vartheta_1^* \wedge \dots \wedge \vartheta_r^*,$$

where $\vartheta_1^*, \dots, \vartheta_r^*$ is the dual basis of $\vartheta_1, \dots, \vartheta_r$, regarded as invariant 1-forms on $G_{\mathbf{R}}$.

Proof. Since we can remove sets of measure zero, we can replace $\mu^{-1}(\xi)$ with a $G_{\mathbf{R}}$ -stable open set W where $G_{\mathbf{R}}$ acts freely with quotient $X_0 \subset X$. Then, by a partition of unity argument, we can replace X_0 with an open set U such that the fibration is diffeomorphic to a product over U . We can also assume that U has local coordinates u_1, \dots, u_{2n} . Since our product formula is invariant under diffeomorphism, we can replace the total space by $U \times G_{\mathbf{R}}$. For a small open set $V \subset G_{\mathbf{R}}$, we can find local coordinates t_1, \dots, t_r such that $\vartheta_i = \partial/\partial t_i$. Then

$$\eta = f(u_1, \dots, u_n) dt_1 \wedge \dots \wedge dt_r \wedge du_1 \wedge \dots \wedge du_{2n}$$

since η is invariant under $G_{\mathbf{R}}$. Hence the product formula holds on $U \times V$, and another partition of unity argument shows that it also holds on $U \times G_{\mathbf{R}}$. \square

Returning to the proof of Theorem 6.4, we can apply Lemma 6.5 to the G -invariant form

$$\eta = \frac{g\bar{J}\bar{\Omega} \wedge dx}{(|f_0|^2 + \dots + |f_n|^2)^{n+1}}$$

and use (11) to obtain

$$\int_{\mu^{-1}(\xi)} \eta = c i^r (-1)^{nr} \int_X \frac{g\bar{J}\bar{\Omega} \wedge \Omega}{(|f_0|^2 + \dots + |f_n|^2)^{n+1}} \cdot \int_{G_{\mathbf{R}}} \vartheta_1^* \wedge \dots \wedge \vartheta_r^*.$$

If we can prove that

$$(12) \quad c \int_{G_{\mathbf{R}}} \vartheta_1^* \wedge \dots \wedge \vartheta_r^* = \pm (2\pi)^r,$$

then the theorem will follow immediately from Theorem 6.3, provided we adjust the orientation of $\mu^{-1}(\xi)$ to make the sign disappear.

We will prove (12) using coordinates to compute the integral explicitly. First pick a subset $I = \{\varrho_1, \dots, \varrho_n\} \subset \Sigma(1)$ such that the n_{ϱ_i} are linearly independent. If \hat{I} is the complement of I in $\Sigma(1)$, then we have an exact sequence

$$(13) \quad 0 \longrightarrow \mathbf{Z} \otimes \hat{I} \longrightarrow A_{n-1}(X) \longrightarrow F \longrightarrow 0,$$

where F is a finite group. Applying $\text{Hom}_{\mathbf{Z}}(-, \mathbf{R})$, we get $\mathbf{R}^r = \text{Hom}_{\mathbf{Z}}(\mathbf{Z} \otimes \hat{I}, \mathbf{R}) \simeq \text{Lie}(G_{\mathbf{R}})$, and composing with the exponential map $\text{Lie}(G_{\mathbf{R}}) \rightarrow G_{\mathbf{R}}^0$, we get a covering map $\mathbf{R}^r \rightarrow G_{\mathbf{R}}^0$. Here, $G_{\mathbf{R}}^0 \subset G_{\mathbf{R}}$ is the connected component of the identity. For later purposes, note that

$$(14) \quad [G_{\mathbf{R}} : G_{\mathbf{R}}^0] = |A_{n-1}(X)_{\text{tor}}|.$$

Each vector field ϑ_i on $G_{\mathbf{R}}$ is determined by a relation $\sum_{\varrho} b_{\varrho}^i n_{\varrho} = 0$. If t_{ϱ} , $\varrho \notin I$, are the obvious coordinates on \mathbf{R}^r , the pull-back of ϑ_i to \mathbf{R}^r is $\sum_{\varrho \notin I} b_{\varrho}^i \partial / \partial t_{\varrho}$. Thus $\vartheta_1 \wedge \dots \wedge \vartheta_r$ pulls back to $\det(\hat{b}_I) \wedge_{\varrho \notin I} \partial / \partial t_{\varrho}$, where $\det(\hat{b}_I)$ is as in Lemma 6.2, and it follows that $\vartheta_1^* \wedge \dots \wedge \vartheta_r^*$ pulls back to $\det(\hat{b}_I)^{-1} dt$, where $dt = \wedge_{\varrho \notin I} dt_{\varrho}$.

We have an exact sequence

$$0 \longrightarrow \text{Hom}_{\mathbf{Z}}(A_{n-1}(X), 2\pi\mathbf{Z}) \longrightarrow \text{Lie}(G_{\mathbf{R}}) \longrightarrow G_{\mathbf{R}}^0 \longrightarrow 0,$$

and under the isomorphism $\text{Lie}(G_{\mathbf{R}}) \simeq \mathbf{R}^r$, the lattice $\text{Hom}_{\mathbf{Z}}(A_{n-1}(X), 2\pi\mathbf{Z})$ maps to a sublattice $L' \subset (2\pi\mathbf{Z})^r = \text{Hom}_{\mathbf{Z}}(\mathbf{Z} \otimes \hat{I}, 2\pi\mathbf{Z})$. Applying $\text{Hom}_{\mathbf{Z}}(-, \mathbf{Z})$ to (13) gives the exact sequence

$$0 \longrightarrow L' \longrightarrow (2\pi\mathbf{Z})^r \longrightarrow \text{Ext}_{\mathbf{Z}}^1(F, \mathbf{Z}) \longrightarrow \text{Ext}_{\mathbf{Z}}^1(A_{n-1}(X), \mathbf{Z}) \longrightarrow 0,$$

which implies

$$(15) \quad [(2\pi\mathbf{Z})^r : L'] = \frac{|F|}{|A_{n-1}(X)_{\text{tor}}|}.$$

Finally, the map $\mathbf{Z} \otimes \hat{I} \rightarrow A_{n-1}(X)$ of (13) embeds in a commutative diagram

$$\begin{array}{ccccccc} & & \mathbf{Z} \otimes \hat{I} & = & \mathbf{Z} \otimes \hat{I} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathbf{Z} \otimes \Sigma(1) & \longrightarrow & A_{n-1}(X) \longrightarrow 0 \end{array}$$

where the bottom row is exact (see [9, §3.4]). Since $\mathbf{Z} \otimes I = \text{coker}(\mathbf{Z} \otimes \hat{I} \rightarrow \mathbf{Z} \otimes \Sigma(1))$, the snake lemma and (13) imply

$$(16) \quad |F| = |\text{coker}(M \rightarrow \mathbf{Z} \otimes I)| = |\det(n_I)|,$$

where $\det(n_I)$ is as in the definition of Ω .

Using (14), (15) and (16), we see that $\int_{G_{\mathbf{R}}} \vartheta_1^* \wedge \dots \wedge \vartheta_r^*$ equals

$$\begin{aligned} |A_{n-1}(X)_{\text{tor}}| \int_{G_{\mathbf{R}}^0} \vartheta_1^* \wedge \dots \wedge \vartheta_r^* &= \pm |A_{n-1}(X)_{\text{tor}}| \int_{\mathbf{R}^r/L'} \det(\hat{b}_I)^{-1} dt \\ &= \pm \frac{|A_{n-1}(X)_{\text{tor}}|}{\det(\hat{b}_I)[(2\pi\mathbf{Z})^r : L']} \int_{\mathbf{R}^r/(2\pi\mathbf{Z})^r} dt \\ &= \pm \frac{(2\pi)^r |F|}{\det(\hat{b}_I)} = \pm \frac{(2\pi)^r |\det(n_I)|}{\det(\hat{b}_I)}, \end{aligned}$$

where the sign ± 1 depends on whether $\mathbf{R}^r \rightarrow G_{\mathbf{R}}^0$ is orientation preserving or not. Hence

$$c \int_{G_{\mathbf{R}}} \vartheta_1^* \wedge \dots \wedge \vartheta_r^* = \pm c \frac{(2\pi)^r |\det(n_I)|}{\det(\hat{b}_I)} = \pm (2\pi)^r \frac{c \det(n_I)}{\det(\hat{b}_I)} = \pm (2\pi)^r,$$

where the last equality holds by (8). This completes the proof of the theorem. \square

The simplest example is where $X = \mathbf{P}^n$. Here, we have seen that $\mu^{-1}(\xi)$ is a sphere S_r^{2n+1} , and then Theorem 6.4 tells us that $\text{Res}(\omega_g)$ is given by the integral

$$\frac{n!}{(2\pi i)^{n+1}} \int_{S_r^{2n+1}} \frac{g \sum_{i=0}^n (-1)^i \bar{f}_i d\bar{f}_0 \wedge \dots \wedge \widehat{d\bar{f}_i} \wedge \dots \wedge d\bar{f}_n \wedge dx_0 \wedge \dots \wedge dx_n}{(|f_0|^2 + \dots + |f_n|^2)^{n+1}}.$$

This formula appears in [11, Chapter 5] and [21, §5].

We should remark that the theory of toric residues, as given above, is not quite complete: one still needs to investigate whether the Grothendieck local residue, as defined in (1), generalizes to the toric case. This problem will be discussed in [6].

Appendix. The Dolbeault isomorphism and the trace map

In this appendix, let X be a variety which is a compact orbifold (or V -manifold). As in §6, this gives a Dolbeault isomorphism $H_{\bar{\partial}}^{n,n}(X) \simeq H^n(X, \Omega_X^n)$. Such a variety is also Cohen–Macaulay, so that we have maps

$$\begin{aligned} \text{Tr}_X: H^n(X, \Omega_X^n) &\rightarrow \mathbf{C}, \quad \text{the trace map,} \\ \int_X: H_{\bar{\partial}}^{n,n}(X) &\rightarrow \mathbf{C}, \quad \text{integration of } (n, n)\text{-forms.} \end{aligned}$$

These maps are related as follows.

Proposition A.1. *Let X be a compact orbifold variety, and suppose that the (n, n) -form η corresponds to $[\omega] \in H^n(X, \Omega_X^n)$ under the Dolbeault isomorphism. Then*

$$\mathrm{Tr}_X([\omega]) = \left(\frac{-1}{2\pi i}\right)^n \int_X \eta.$$

Proof. Consider the map $T: H^n(X, \Omega_X^n) \rightarrow \mathbf{C}$ defined by $T([\omega]) = \int_X \eta$, where η corresponds to $[\omega]$ under the Dolbeault isomorphism. This map has all of the formal properties of the trace map, except for the normalization condition, which says it takes the value 1 on the class

$$[\omega_1] = \left[\frac{\Omega_{\mathbf{P}^n}}{x_0 \dots x_n} \right] \in H^n(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^n).$$

It follows that T and Tr_X agree up to a constant, and furthermore, the normalization condition implies the constant is $T([\omega_1])$. By Proposition 6.1, $T([\omega_1]) = \int_{\mathbf{P}^n} \eta_1$, where

$$\eta_1 = (-1)^{n(n+1)/2} n! \frac{\bar{\Omega}_{\mathbf{P}^n} \wedge \Omega_{\mathbf{P}^n}}{(|x_0|^2 + \dots + |x_n|^2)^{n+1}}.$$

Hence, to prove the proposition, it suffices to show that $\int_{\mathbf{P}^n} \eta_1 = (-2\pi i)^n$.

If $\mathbf{C}^n \subset \mathbf{P}^n$ is the open set where $x_0 \neq 0$, then $\int_{\mathbf{P}^n} \eta_1 = \int_{\mathbf{C}^n} \eta_1$. On this open set, we can use $t_i = x_i/x_0$ as coordinates. Since $\Omega_{\mathbf{P}^n} = x_0^{n+1} dt_1 \wedge \dots \wedge dt_n$, we have

$$\int_{\mathbf{P}^n} \eta_1 = (-1)^{n(n+1)/2} n! \int_{\mathbf{C}^n} \frac{d\bar{t}_1 \wedge \dots \wedge d\bar{t}_n \wedge dt_1 \wedge \dots \wedge dt_n}{(1 + |t_1|^2 + \dots + |t_n|^2)^{n+1}}.$$

Denoting real and imaginary parts by $t_j = u_j + iv_j$, we have $d\bar{t}_j \wedge dt_j = 2i du_j \wedge dv_j$. Thus $d\bar{t}_1 \wedge \dots \wedge d\bar{t}_n \wedge dt_1 \wedge \dots \wedge dt_n = (2i)^n (-1)^{n(n-1)/2} du_1 \wedge dv_1 \wedge \dots \wedge du_n \wedge dv_n$, so that

$$\int_{\mathbf{P}^n} \eta_1 = (-2i)^n n! \int_{\mathbf{C}^n} \frac{du_1 \wedge dv_1 \wedge \dots \wedge du_n \wedge dv_n}{(1 + u_1^2 + v_1^2 + \dots + u_n^2 + v_n^2)^{n+1}}.$$

Since $du_1 \wedge dv_1 \wedge \dots \wedge du_n \wedge dv_n$ is the volume form on \mathbf{C}^n , the above integral is a standard multiple integral. Using polar coordinates for each u_j and v_j , the multiple integral equals $\pi^n/n!$, and it follows that $\int_{\mathbf{P}^n} \eta_1 = (-2\pi i)^n$. This completes the proof. \square

For the reader curious about the signs in Propositions 6.1 and A.1, we suggest checking $\mathrm{Tr}_{\mathbf{P}^1}([\omega_1]) = (-1/2\pi i) \int_{\mathbf{P}^1} \eta_1 = 1$ in detail.

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