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On removable sets for quasiconformal mappings

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Given a closed set $E \subseteq \mathbf{R}^n$, let

$$\mathrm{QCH}(E) = \{\text{homeomorphisms of } \mathbf{R}^n \text{ to } \mathbf{R}^n \text{ which are quasiconformal off } E\};$$

we call E removable for QCH if every f in QCH(E) is quasiconformal in $\mathbf{R}^n$.

In $\mathbf{R}^2$, the following theorems ([C], [G], [K]) are known:

Theorem A. (Carleson, Gehring) *If $S \subseteq \mathbf{R}^1$ is compact, then $S \times [0,1]$ is removable for* QCH *if and only if S is countable.*

Theorem B. (Kaufman) *If $S \subseteq \mathbf{R}^1$ is uncountable, then $S \times [0,1]$ contains a graph E which is not removable for* QCH.

In $\mathbf{R}^n$, $n \geq 2$, Cantor type sets are removable ([HK]):

Theorem C. (Heinonen and Koskela) *Let E be a closed set in $\mathbf{R}^n$. Suppose that there exist $a > 1$, and $\{r_j\}$, $r_j \to 0$ as $j \to \infty$, such that at each $x \in E$, the annular regions $\{y : a^{-1} r_j < |y-x| < a r_j\}$ do not meet E. Then E is removable for* QCH. *In fact, if f is quasiconformal off E, then f has a quasiconformal extension on $\mathbf{R}^n$.*

In this note, we prove the following extension of Theorems A and C.

Theorem 1. *If $n \geq m \geq 2$ and S is an $\{\alpha_k\}$-porous set in $\mathbf{R}^m$ with $\sum \alpha_k = \infty$, then $S \times \mathbf{R}^{n-m}$ is removable for* QCH *in $\mathbf{R}^n$. If S is closed in $\mathbf{R}^1$, then $S \times \mathbf{R}^{n-1}$ is removable for* QCH *in $\mathbf{R}^n$ if and only if S is countable.*

Given $\{\alpha_k\}$, $0 < \alpha_k < 1$, a set $S \subseteq \mathbf{R}^n$ is called *$\{\alpha_k\}$-porous* if there exists a sequence of coverings $\mathcal{C}_k = \{B_{k,j} \equiv B(x_{k,j}, r_{k,j})\}$ of S, by balls with mutually disjoint interiors, so that each $B_{k,j} \setminus S$ contains a ring $R_{k,j} \equiv \{x : (1-\alpha_k) r_{k,j} < |x - x_{k,j}| < r_{k,j}\}$,

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$\bigcup_{\mathcal{C}_{k+1}} B_{k+1,j} \subseteq \bigcup_{\mathcal{C}_k} (B_{k,j} \setminus R_{k,j})$, and $\sup_j r_{k,j} \to 0$ as $k \to \infty$. Here $B(x,r) = \{y \in \mathbf{R}^n : |y-x| < r\}$.

An $\{\alpha_k\}$-porous set has zero n-dimensional Hausdorff measure provided that

$$\sum \alpha_k = \infty.$$

Remarks on Theorem 1 are given in Sections 1 and 2.

The next theorem is related to the previous one, but uses special consideration restricted to the plane.

Theorem 2. *Let Ω be an open set in $\mathbf{R}^2$, and E be an $\{\alpha_k\}$-porous set in Ω with $\sum \alpha_k = \infty$. Let f be meromorphic in $\Omega \setminus E$ with the spherical derivative $f^* = |f'|(1+|f^2|)^{-1}$ in $L^2(\Omega \setminus E)$. Then f can be extended by continuity to be meromorphic in Ω.*

The hypotheses are fulfilled if f is schlicht on $\Omega \setminus E$; the extension will be schlicht (possibly taking the value ∞) on Ω.

Koebe has proved that every planar domain can be mapped conformally onto a slit domain whose complement has area zero ([AS, Theorem III.11C]). In [B], Bishop raised the question: if E is closed, has positive area and no interior, is there a $G \in \mathrm{CH}(E)$, so that $G(E)$ has zero area? Here

$$\mathrm{CH}(E) = \{\text{homeomorphisms on } \mathbf{R}^2, \text{ conformal off } E\}.$$

We provide a partial answer.

Theorem 3. *Given any $E \subseteq \mathbf{R}^2$ of positive area, there exists $G \in \mathrm{CH}(E)$ that maps a subset F (of E) of positive area to a set $G(F)$ of zero area.*

We conjecture that in this theorem, F can be chosen to have full measure, and that the answer to Bishop's question in its full generality is negative.

1. Proof and remarks of Theorem 1

We shall prove the first statement in Theorem 1 only. The second statement regarding the removability of $S \times \mathbf{R}^1$ follows from minor modifications of the proof of Theorem A.

Let $\sum \alpha_k = \infty$ and S be an $\{\alpha_k\}$-porous set in $\mathbf{R}^m$ with $n \geq m \geq 2$, and let $f \in \mathrm{QCH}(S \times \mathbf{R}^{n-m})$. In order to show that f is quasiconformal in $\mathbf{R}^n$, we only need to verify that f is ACL—absolutely continuous on lines ([V, Theorem 34.6]).

Because S has zero m-dimensional Hausdorff measure, it suffices to show that f is ACL on almost every hyperplane $\mathbf{R}^m \times \{a\}$, where $a \in \mathbf{R}^{n-m}$.

When $m=n$, the hyperplanes are $\mathbf{R}^n$, and the set A below should be disregarded.

Let Q be a ball in $\mathbf{R}^m$ whose interior meets S but whose boundary does not intersect S. After deleting a set $A\subseteq\mathbf{R}^{n-m}$ of Hausdorff Λ^{n-m}-measure zero, we may assume that f is in $W^{1,n}$ on $(Q\backslash S)\times\{a\}$ $(a\notin A)$,

$$\int_{(Q\backslash S)\times\{a\}} |f'(x)|^n \, dx_1 \, dx_2 \, \dots \, dx_m < \infty.$$

Fix $a\notin A$, let $B_{k,j}$, $x_{k,j}$, $r_{k,j}$, $R_{k,j}$ be retained from the definition of porosity for S, and let

$$\lambda_{k,j} = \int_{R_{k,j}\times\{a\}} |f'(x)|^n \, dx_1 \, \dots \, dx_m,$$

$\gamma_{k,j}$ be a sphere in $R_{k,j}$ concentric to $R_{k,j}$ of radius $r'_{k,j} > \frac{1}{2} r_{k,j}$, on which

$$\int_{\gamma_{k,j}\times\{a\}} |f'(x)|^n \, d\sigma(x) < C\lambda_{k,j}\alpha_k^{-1} r_{k,j}^{-1}, \tag{1.1}$$

where σ is the Hausdorff Λ^{m-1}-measure, and C is a constant depending on m and n only. This is possible because f is quasiconformal in $R_{k,j}\times\mathbf{R}^{n-m}$.

Denote by $D_{k,j}$ the ball in $\mathbf{R}^m$ bounded by $\gamma_{k,j}$, by $\bar{x}$ the radial projection of the point $x\in D_{k,j}\times\{a\}$ to the sphere $\gamma_{k,j}\times\{a\}$, with $\overline{(x_{k,j}, a)} = (x_{k,j}, a)$, and by $m_{k,j}$ the average of f on $\gamma_{k,j}\times\{a\}$.

Define

$$f_k(x) = \begin{cases} f(x) & \text{on } \left(\mathbf{R}^m \backslash \bigcup_j D_{k,j}\right)\times\{a\}, \\ m_{k,j} + \dfrac{|x-(x_{k,j},a)|}{r'_{k,j}}(f(\bar{x}) - m_{k,j}) & \text{on } D_{k,j}\times\{a\}, \text{ for each } j. \end{cases}$$

Clearly $f_k\to f$ everywhere, and in L^n on $\mathbf{R}^m\times\{a\}$. We note that the derivative of f_k on $\mathbf{R}^m\times\{a\}$ satisfies

$$\begin{aligned} \int_{D_{kj}\times\{a\}} |f'_k(x)|^n \, dx_1 \, \dots \, dx_m &\le C \int_{D_{k,j}\times\{a\}} |f'(\bar{x})|^n + r_{k,j}^{-n} |f(\bar{x}) - m_{k,j}|^n \, dx_1 \, \dots \, dx_m \\ &\le C\lambda_{k,j}\alpha_k^{-1} + C r_{k,j}^{-n+1} \int_{\gamma_{k,j}\times\{a\}} |f(x) - m_{k,j}|^n \, d\sigma(x) \\ &\le C\lambda_{k,j}\alpha_k^{-1} + C r_{k,j} \int_{\gamma_{k,j}\times\{a\}} |f'(x)|^n \, d\sigma(x) \\ &\le C\lambda_{k,j}\alpha_k^{-1}, \end{aligned}$$

in view of (1.1) and the Poincaré inequality. Here C denotes constants depending on m and n only, with values varying from line to line. Hence for $l>k$,

$$\int_{Q\times\{a\}}|(f_k-f_l)'|^n\,dx_1\,...\,dx_m\le C\alpha_k^{-1}\sum_j\lambda_{k,j}+C\alpha_l^{-1}\sum_i\lambda_{l,i}$$
$$+\sum_j\int_{(D_{k,j}\setminus S)\times\{a\}}|f'|^n\,dx_1\,...\,dx_m.$$

Since $\sum\alpha_k=\infty$ and $\sum_k\sum_j\lambda_{k,j}<\infty$, $\liminf_{k\to\infty}\alpha_k^{-1}\big(\sum_j\lambda_{k,j}\big)=0$. Taking a subsequence of the f_k's we find that $f\in W^{1,n}(Q\times\{a\})$. Thus f is ACL on $Q\times\{a\}$ because f is continuous. Hence f is ACL on $\mathbf{R}^m\times\{a\}$, and f is quasiconformal on $\mathbf{R}^n$. This completes the proof of Theorem 1.

The proof is about extension of Sobolev functions, and is still valid for the class $W^{1,1}$; but it becomes more technical (as is natural) if the hypothesis of continuity is omitted.

In view of the successful application of the concept of a ring-like porous set, we may ask the following.

Question. Can porous sets be defined in a conceptually different way to allow more removable sets? Can the porosity be defined to depend on the exponent of integrability of $|f'|$?

The total disconnectedness of S in Theorem 1 is not essential. We note from the proof that in some situations, the removability for QCH is about the extendibility of Sobolev functions. For example, it follows from the Sobolev extension theorems of Jones [J] that *if S is a quasicircle in $\mathbf{R}^2$ then $S\times\mathbf{R}^{n-2}$ is removable for* QCH *in* $\mathbf{R}^n$.

Question. Are the sets $S\times\mathbf{R}^{n-m}$ in Theorem 1 removable for the class of functions quasiconformal on $\mathbf{R}^n\setminus S\times\mathbf{R}^{n-m}$?

This is a question on homeomorphic continuation of quasiconformal mappings; we have no conjecture in general. Martio and Näkki [MN] have proved a theorem on homeomorphic continuation of quasiconformal mappings defined on domains with totally disconnected boundary sets. First applying their theorem to obtain a homeomorphic extension, then applying Theorem 1 to show quasiconformality, we arrive at an answer for $m=n$.

Corollary. *If $n\geq 2$, S is an $\{\alpha_k\}$-porous set in $\mathbf{R}^n$ with $\sum\alpha_k=\infty$, and f is quasiconformal on $\mathbf{R}^n\setminus S$, then f can be extended to be quasiconformal on $\mathbf{R}^n$.*

When $m=n-1$, we conjecture that the answer is again positive. This is based on the modulus estimation of a certain curve family in the next section.

2. Modulus of a certain curve family

Given a family Γ of curves in $\overline{\mathbf{R}}^n$, let $\operatorname{adm}(\Gamma)$ denote the family of Borel measurable functions $\varrho: \mathbf{R}^n \to [0,\infty]$ such that $\int_\gamma \varrho\, d|x| \geq 1$ for all locally rectifiable $\gamma \in \Gamma$. We recall that the modulus of Γ is defined to be

$$\operatorname{mod}(\Gamma) = \inf_{\varrho \in \operatorname{adm}(\Gamma)} \int_{\mathbf{R}^n} \varrho^n \, dx.$$

Given $E, F, G \subseteq \mathbf{R}^n$, denote by $\triangle(E, F; G)$ the family of curves with interior in G and endpoints on $\overline{E}$ and $\overline{F}$ respectively.

Proposition 1. *Let $n \geq 3$, S be an $\{\alpha_k\}$-porous set on $\mathbf{R}^{n-1} \cap \{|x| < \frac{1}{2}\}$ with $\sum \alpha_k = \infty$, D be the infinite cylinder $\{x \in \mathbf{R}^{n-1} : |x| < 1\} \times \mathbf{R}^1$, and Ω be the set $D \setminus (S \times \mathbf{R}^1)$. Suppose that $I = \{a\} \times [0, h]$ is a line segment on $S \times \mathbf{R}^1$ with $a \in S$ and $h > 0$. Then*

$$\operatorname{mod}(\triangle(I, \partial D, \Omega)) > C_n h,$$

where C_n is a positive constant depending only on n.

Let S be the set in Proposition 1, and f be a quasiconformal mapping on $\mathbf{R}^n \setminus (S \times \mathbf{R}^1)$. In view of the modulus estimation, the cluster set $f(I)$ of a line segment $I \subseteq S \times \mathbf{R}^1$ consists of more than one point. The conjecture at the end of Section 1 is based on this observation.

The proposition may be derived from the interpretation of $\operatorname{mod}(\Gamma)$ in terms of variational integrals, and the fact that $S \times \mathbf{R}^1$ is removable for $W^{1,n}$ in $\mathbf{R}^n$, which we prefer not to elaborate. Instead, we present a direct proof.

Proof. We retain $B_{k,j}$, $x_{k,j}$, $r_{k,j}$, $R_{k,j}$ from the definition of the porosity for S, and assume as we may that $\sup \alpha_k \leq \frac{1}{2}$.

After a piecewise linear transformation, we may assume that a is the origin. Let

$$\Gamma = \triangle(I, \partial D, \Omega),$$

$$\Gamma_1 = \{\gamma \in \Gamma : \gamma \text{ lies on a hyperplane } x_n = c \text{ for some } c \in \mathbf{R}^1\},$$

and note that $\operatorname{mod}(\Gamma) \geq \operatorname{mod}(\Gamma_1)$.

Let $\varrho \in \operatorname{adm}(\Gamma_1)$ with $\varrho \equiv 0$ off $D \cap \{0 \leq x_n \leq h\}$. We shall modify ϱ so that the new $\tilde{\varrho}$ is in $\operatorname{adm}(\Gamma_2)$, where

$$\Gamma_2 = \{\gamma \in \triangle(I, \partial D, D) : \gamma \text{ is a line segment on which } x_n \equiv c \text{ for some } c \in \mathbf{R}^1\}.$$

We denote points in $\mathbf{R}^n$ by x, points in $\mathbf{R}^{n-1}$ by x' and write $x = (x', x_n)$.

Fix a, $0 \le a \le h$, and define

$$\lambda_{k,j} = \int_{R_{k,j}\times\{a\}} \varrho(x)\,dx',$$
$$m_{k,j} = \int_{R_{k,j}\times\{a\}} \varrho^n(x)\,dx'.$$

In each $R_{k,j}$, there is a sphere $\gamma_{k,j}$ concentric with $R_{k,j}$ such that

$$\int_{\gamma_{k,j}\times\{a\}} \varrho(x)\,d\sigma(x) \le C\alpha_k^{-1} r_{k,j}^{-1}\lambda_{k,j}, \tag{2.1}$$

and

$$\int_{\gamma_{k,j}\times\{a\}} \varrho^n(x)\,d\sigma(x) \le C\alpha_k^{-1} r_{k,j}^{-1} m_{k,j}, \tag{2.2}$$

where σ is the Hausdorff Λ^{n-2}-measure, and C is a constant depending on n only. Denote by $D_{k,j}$ the ball in $\mathbf{R}^{n-1}$ bounded by $\gamma_{k,j}$.

Given s, a line segment with endpoints on $\gamma_{k,j}=\partial D_{k,j}$, we shall denote by ζ the great circle of $\gamma_{k,j}$ lying on the hyperplane containing $x_{k,j}$ and the endpoints p,q of s. We shall let s_1 be the shorter arc of $\zeta\setminus\{p,q\}$, or either arc if both have the same length.

Consider, from now on, only those $B_{k,j}$ on which

$$\operatorname{dist}(0, B_{k,j}) > 5 \operatorname{diam} B_{k,j}.$$

Let $x' \in D_{k,j}$ and s be the segment on the line through 0 and x', with endpoints on $\gamma_{k,j}$. Let $w(x')$ be the unique point on s_1 satisfying $|w(x')|=|x'|$. And define on $x_n=a$,

$$\phi_{k,j}(x) = \begin{cases} \varrho(w(x'),a), & x' \in D_{k,j}, \\ 0, & \text{elsewhere}; \end{cases} \tag{2.3}$$

and

$$\psi_{k,j}(x) = \begin{cases} \alpha_k^{-1} r_{k,j}^{-n+1}\lambda_{k,j}, & x' \in D_{k,j}, \\ 0, & \text{elsewhere}. \end{cases} \tag{2.4}$$

Let $L\in\Gamma_2\cap\{x_n=a\}$ and assume that $s\equiv L\cap D_{k,j}\times\{a\}\neq\emptyset$. If $s\cap\left(\frac14 B_{k,j}\right)\times\{a\}\neq\emptyset$, we may find an arc s_2 on $\gamma_{k,j}$ joining the endpoints p and q of s, such that

$$\int_{s_2} \varrho(x)\,|dx| \le C\alpha_k^{-1} r_{k,j}^{-n+2}\lambda_{k,j}, \tag{2.5}$$

for some properly chosen constant C depending only on n. Define

$$s^{\sim}=\begin{cases} s_1, & \text{if } s\cap\left(\frac{1}{4}B_{k,j}\right)\times\{a\}=\emptyset,\\ s_2, & \text{if } s\cap\left(\frac{1}{4}B_{k,j}\right)\times\{a\}\neq\emptyset.\end{cases} \tag{2.6}$$

Here cB is the ball concentric to B of radius c times that of B. Then

$$\int_s(\varrho(x)+\phi_{k,j}(x)+\psi_{k,j}(x))\,|dx|\geq C\int_{\tilde{s}}\varrho(x)\,|dx|. \tag{2.7}$$

To see this, we consider the case $s\cap\left(\frac{1}{4}B_{k,j}\right)\times\{a\}=\emptyset$ and the case $s\cap\left(\frac{1}{4}B_{k,j}\right)\times\{a\}\neq\emptyset$ separately, and use (2.3), and (2.4) and (2.5) respectively. Also we deduce from (2.2) and the Hölder inequality that

$$\begin{aligned}\int_{D_{k,j}\times\{a\}}&(\varrho(x)+\phi_{k,j}(x)+\psi_{k,j}(x))^n\,dx'\\ &\leq C\int_{D_{k,j}\times\{a\}}\varrho^n(x)\,dx'+C\alpha_k^{-1}\int_{R_{k,j}\times\{a\}}\varrho^n(x)\,dx'.\end{aligned} \tag{2.8}$$

We are now ready to modify ϱ on $x_n=a$. If $\int_{\{x_n=a\}}\varrho^n(x)\,dx'=\infty$, we leave $\varrho(x)$ unchanged and let $\tilde{\varrho}_a(x)=\varrho(x)$ on $x_n=a$.

If $\int_{\{x_n=a\}}\varrho^n(x)\,dx'<\infty$, we denote by $M_k=\sum_j m_{k,j}$, thus $\sum_k M_k<\infty$. Since $\sum\alpha_k=\infty$, there exists a sequence $\{k_q\}$ so that

$$M_{k_q}\alpha_{k_q}^{-1}\leq 2^{-q}\int_{\{x_n=a\}}\varrho^n(x)\,dx'.$$

The measurability of the modified ϱ may be insured by choosing k_q the smallest possible value exceeding k_{q-1}.

Let $\mathcal{C}$ be a covering of $S\setminus\{0\}$, consisting of mutually disjoint balls from the set $\bigcup_{q\geq 1}\bigcup_j\{D_{k_q,j}\}$ satisfying $\operatorname{dist}(0,B_{k_q,j})>5\operatorname{diam}B_{k_q,j}$.

Define on $x_n=a$,

$$\tilde{\varrho}_a(x)=\begin{cases}\varrho(x)+\phi_{k,j}(x)+\psi_{k,j}(x), & \text{if } x'\in D_{k,j} \text{ and } D_{k,j}\in\mathcal{C},\\ \varrho(x), & \text{if } x'\notin\bigcup_{\mathcal{C}}D_{k,j}.\end{cases}$$

Given $L\in\Gamma_2\cap\{x_n=a\}$, define a new curve $\mathcal{L}$ in $\triangle(I,\partial D,\Omega)$ by

$$\mathcal{L}=\bigcup_{\mathcal{C}}((L\cap D_{k,j}\times\{a\})^{\sim})\cup\left(L\setminus\bigcup_{\mathcal{C}}D_{k,j}\times\{a\}\right),$$

where $\sim$ of a segment is defined by (2.6). It follows from (2.7) and (2.8) that

$$\int_L \tilde{\varrho}_a(x)\,|dx| \geq C \int_{\mathcal{L}} \varrho(x)\,|dx| \geq C,$$

and

$$\begin{aligned}\int_{\{x_n=a\}} \tilde{\varrho}_a^n(x)\,dx' &\leq C \int_{\{x_n=a\}} \varrho^n(x)\,dx' + C\sum_{q\geq 1} M_{k_q}\alpha_{k_q}^{-1} \\ &\leq C \int_{\{x_n=a\}} \varrho^n(x)\,dx'.\end{aligned} \tag{2.9}$$

Now let

$$\tilde{\varrho}(x) = \begin{cases} \tilde{\varrho}_a(x), & \text{if } x_n = a \text{ and } 0 \leq a \leq h, \\ 0, & \text{elsewhere in } \mathbf{R}^n. \end{cases}$$

Thus $C\tilde{\varrho} \in \operatorname{adm}(\Gamma_2)$ for some $C>0$. A standard application of Hölder's inequality shows that $\int_{\mathbf{R}^n} \tilde{\varrho}^n\,dx \geq Ch$. Therefore $\int_{\mathbf{R}^n} \varrho^n(x)\,dx \geq Ch$ in virtue of (2.9). This completes the proof of the proposition.

3. Proof of Theorem 2

Let $z_0 \in \Omega$ and let $r>0$ be so small that $B(z_0,r) \subseteq \Omega$ and the integral of f^{*2} over $\Omega_1 \equiv B(z_0,r)\backslash E$ is less than the surface area of the Riemann sphere. Then $f(\Omega_1)$ omits a set of positive area in the plane. We choose a compact subset K of positive area $m(K)$, contained in the omitted set.

By a theorem of Nguyen Xuen Uy [N], there is a function G of class Lip^1 in $\mathbf{R}^2$, analytic off K, whose expansion near ∞ is

$$G(w) = w^{-1} + a_2 w^{-2} + a_3 w^{-3} + \ldots, \quad |w| > R.$$

Since G is of class Lip^1,

$$|G'(w)| \leq C(1+|w|)^{-2} \quad \text{for } w \in \mathbf{R}^2 \backslash K.$$

Thus $F = G \circ f$ is analytic on Ω_1. Since $|F'| \leq C|f^*|$, we see that $F' \in L^2(\Omega_1)$.

Following the argument in the proof of Theorem 1, with $m=n=2$, we can conclude that F is the restriction to Ω_1 of a function in $W^{1,2}(B(z_0,r))$. If the boundary of $B(z_0,r)$ intersects E, we need to approximate the boundary by a smooth curve which avoids E; this is possible because E is totally disconnected. Since F satisfies the Cauchy–Riemann equations in Ω_1, by Weyl's lemma [AS], F

coincides a.e. in $B(z_0,r)$ with an analytic function $\widetilde{F}$ in $B(z_0,r)$. However, F is continuous on the open subset Ω_1 of $B(z_0,r)$, whence $\widetilde{F}$ actually extends F from Ω_1 to B.

In consequence, $F=G\circ f$ admits a limit ζ_0 at z_0. There is a number $w_0\in\mathbf{R}^2\setminus K$ such that $G(w_0)\neq\zeta_0$. It follows that w_0 is not a cluster value of f at z_0; that is, $h=(f-w_0)^{-1}$ is bounded on $B(z_0,r_1)\setminus E$ for some $0<r_1<r$. This implies that the usual derivative h' is in $L^2(B(z_0,r_1)\setminus E)$. The previous steps show that h can be extended to be analytic on $B(z_0,r_1)$. Therefore f can be extended to be meromorphic. This completes the proof of Theorem 2.

4. Proof of Theorem 3

We construct by induction a sequence of quasiconformal mappings g_n on $\mathbf{R}^2$ so that each g_n maps a large portion of $g_{n-1}\circ g_{n-2}\circ\ldots\circ g_1(E)$ to a small portion of $g_n\circ g_{n-1}\circ\ldots\circ g_1(E)$, and the limit function $g=\lim_{n\to\infty} g_n\circ\ldots\circ g_1$ is in QCH(E). The mappings g_n are approximately independent, therefore elementary ideas from probability can be used. In passing to the limit g, there is a certain technical point which unfortunately slows the exposition.

The construction uses the measurable Riemann mapping theorem repeatedly. For literature and proofs, see [AB] and [LV].

Fix $0<\alpha<1$, let $K=(1+\alpha)/(1-\alpha)$, and let $\mathcal{M}$ be the family of complex valued measurable functions on $\mathbf{R}^2$, bounded by α and supported in $B(0,1)$.

Theorem D. *There exists $p=p(\alpha)>2$, such that given any $\mu\in\mathcal{M}$, one may find a unique K-quasiconformal mapping f on $\mathbf{R}^2$, with the normalization $f(z)=z+O(1/z)$ near ∞, that solves the Beltrami equation $\bar\partial f=\mu\partial f$ on $\mathbf{R}^2$ a.e., and whose partials satisfy $\bar\partial f\in L^p$ and $\partial f-1\in L^p$.*

Moreover, there exists $A=A(\alpha,p)>1$ so that the areas $m(X)$ and $m(f(X))$ satisfy

$$m(f(X))\le A(m(X))^{1-2/p}, \tag{4.1}$$

$$m(X)\le A(m(f(X)))^{1-2/p}, \tag{4.2}$$

for any $X\subset B(0,2)$, and that

$$|f(z_1)-f(z_2)|\le A|z_1-z_2|^{1-2/p}, \tag{4.3}$$

$$|f^{-1}(z_1)-f^{-1}(z_2)|\le A|z_1-z_2|^{1-2/p}, \tag{4.4}$$

for any $z_1,z_2\in B(0,2)$.

Furthermore, suppose that f_n and f are the normalized solutions of the Beltrami equation with Beltrami coefficients μ_n and $\mu \in \mathcal{M}$ respectively. If $\mu_n \to \mu$ a.e., then $f_n \to f$ uniformly and $f_n^{-1} \to f^{-1}$ uniformly, and $\partial f_n - 1 \to \partial f - 1$ and $\bar\partial f_n \to \bar\partial f$ in L^p, and $\partial f_n^{-1} - 1 \to \partial f^{-1} - 1$ and $\bar\partial f_n^{-1} \to \bar\partial f^{-1}$ in L^p.

Proposition 2. *Let μ and ν in $\mathcal{M}$, and f and g be the normalized solutions of the Beltrami equation with coefficients μ and ν respectively. Then for any $X \subseteq B(0,1)$,*

$$|m(f(X)) - m(g(X))| \le 16\sqrt{K} A(\|\partial f - \partial g\|_p + \|\bar\partial f - \bar\partial g\|_p), \tag{4.5}$$

where p and A are as in Theorem D.

Proof. Let $f(z) = u(x,y) + iv(x,y)$ and $g(z) = s(x,y) + it(x,y)$, and J_f and J_g be the Jacobians of f and g respectively. Then

$$\begin{aligned}
|m(f(X)) - m(g(X))| &\le \int_X |J_g(z) - J_f(z)|\, dx\, dy \\
&\le \int_X |u_x||v_y - t_y| + |t_y||u_x - s_x| + |u_y||v_x - t_x| + |t_x||u_y - s_y|\, dx\, dy \\
&\le 2\sqrt{K}(m(f(X))^{1/2} + m(g(X))^{1/2}) m(X)^{(1/2)-(1/p)} (\|\partial f - \partial g\|_p + \|\bar\partial f - \bar\partial g\|_p),
\end{aligned}$$

in view of the quasiconformality of f and g. The estimate (4.5) follows from (4.1).

The building block in our construction is an elementary function defined in the following lemma, whose proof we shall omit.

Lemma 1. *Let D be a disk with center a and radius b, and let ϕ be a homeomorphism on D defined by*

$$\phi(z) = \begin{cases} a + \frac{1}{9}(z-a) & \text{for } z \in \frac{9}{10}D, \\ a + (9 - 8b/|z-a|)(z-a) & \text{for } z \in D \setminus \frac{9}{10}D. \end{cases}$$

Then ϕ is 81-quasiconformal on D, conformal on $\frac{9}{10}D$, and the identity map on ∂D; moreover the Jacobian J_ϕ is bounded between $\frac{1}{81}$ and 9.

We shall refer to ϕ by ϕ_D in the context.

Here cD is the disk concentric to D of radius c times that of D.

Denote by p the number $p\left(\frac{40}{41}\right)$ and A the constant $A\left(\frac{40}{41}, p\right)$ in Theorem D. Let $\varepsilon_k = 10^{-5k}$ for $k \ge 1$.

By a point of density argument, we may assume that E is compact, contained in $B\left(0, \frac{1}{2}\right)$ and having area $m(E) > 10^{-2}$.

Let $\{D_j^1\}$ be a finite collection of mutually disjoint closed disks, so that $\operatorname{diam} D_j^1 < \varepsilon_1$, $m(E \setminus \bigcup_j D_j^1) < \varepsilon_1$ and

$$m(E \cap D_j^1) > (1-\varepsilon_1) m(D_j^1) \quad \text{for each } j.$$

Thus

$$m\big(E \cap \tfrac{9}{10} D_j^1\big) > (0.81-\varepsilon_1) m(E \cap D_j^1) \quad \text{for each } j.$$

Let f_1 be the 81-quasiconformal mappings on $\mathbf{R}^2$ defined by

$$f_1(z) = \begin{cases} z, & z \in \mathbf{R}^2 \setminus \bigcup D_j^1, \\ \phi_{D_j^1}(z), & z \in D_j^1 \text{ for some } j; \end{cases}$$

and let μ_1 be the complex dilatation $\bar\partial f_1 / \partial f_1$ defined a.e. on $\mathbf{R}^2$. Note that $|\mu_1| \le \frac{40}{41}$,

$$\|f_1 - z\|_\infty, \ \|f_1^{-1} - z\|_\infty < \varepsilon_1,$$

and that

$$0.001 m(D_j^1) < m\big(f_1\big(E \cap \tfrac{9}{10} D_j^1\big)\big) < (0.01+\varepsilon_1) m(f_1(E \cap D_j^1)) \tag{4.6}$$

for each j.

Let $S_0 = E$, $S_1 = E \cap \bigcup_j D_j^1$ and $U_1 = \{z : 0 < \operatorname{dist}(z, S_1) < \delta_1\}$ be a narrow open band around S_1 (δ_1 will be made more precise later), and let

$$\nu_1 = \mu_1 \mathcal{X}_{\mathbf{R}^2 \setminus U_1}.$$

Since S_1 is closed, $m(U_1) \to 0$ and $\nu_1 \to \mu_1$ a.e., as $\delta_1 \to 0$.

Denote by g_1 the unique 81-quasiconformal mapping on $\mathbf{R}^2$ normalized by $g_1(z) = z + O(1/z)$ near ∞, which solves

$$\bar\partial g_1 = \nu_1 \partial g_1, \quad \text{a.e.},$$

and satisfies $\bar\partial g_1 \in L^p(\mathbf{R}^2)$ and $\partial g_1 - 1 \in L^p(\mathbf{R}^2)$. Note that g_1 is conformal on $\big(\mathbf{R}^2 \setminus \bigcup D_j^1\big) \cup U_1$.

Assume that δ_1 has been chosen small enough so that

$$\|g_1 - f_1\|_\infty, \ \|g_1^{-1} - f_1^{-1}\|_\infty < \varepsilon_1,$$

and that

$$\|\partial f_1 - \partial g_1\|_p + \|\bar\partial f_1 - \bar\partial g_1\|_p < 10^{-10} \varepsilon_1 A^{-1} \min_j m(D_j^1).$$

Applying (4.5) to f_1, g_1 and $X=E\cap\frac{9}{10}D_j^1$ and $E\cap D_j^1$, we conclude from (4.6) that

$$m\big(g_1\big(E\cap\tfrac{9}{10}D_j^1\big)\big)<(0.01+2\varepsilon_1)m(g_1(E\cap D_j^1))\quad\text{for each } j.$$

We observe for the future reference that $E\cap\frac{9}{10}D_j^1=S_1\cap\frac{9}{10}D_j^1$ and $E\cap D_j^1=S_1\cap D_j^1$.

Let g_0 be the identity map, $E_1=T_1=S_1$, $\eta_0=1$, $\eta_1=\frac{1}{2}$, $\alpha_0=1$ and

$$\alpha_1=\inf\{|g_1(z)-g_1(w)|:z,w\in B(0,1),|z-w|\ge\varepsilon_1\},$$

which is positive in view of the Hölder continuity of g_1^{-1}.

Assume that for each k $(1\le k\le n-1)$, numbers δ_k, α_k, η_k, 81-quasiconformal mappings g_k, collection of disks $\{D_j^k\}_j$, sets S_k, U_k, T_k and E_k have been constructed so that the properties A through H below are satisfied. It would be convenient to regard D_j^k, S_k, U_k and g_k as sets and function on the kth copy of $\mathbf{R}^2$, and T_k and E_k as subsets of E on the first copy of $\mathbf{R}^2$.

A. $\{D_j^k\}$ is a finite collection of mutually disjoint closed disks that nearly covers $g_{k-1}(S_{k-1})$:

$$\max_j \operatorname{diam} D_j^k<\min\{\varepsilon_k,\alpha_{k-1},\eta_{k-1}\},\tag{4.7}$$

$$m\Big(g_{k-1}(S_{k-1})\setminus\bigcup_j D_j^k\Big)<\varepsilon_k,\tag{4.8}$$

$$m(g_{k-1}(S_{k-1})\cap D_j^k)>(1-\varepsilon_k)m(D_j^k)\quad\text{for each } j.\tag{4.9}$$

Denote by

$$S_k=g_{k-1}(S_{k-1})\cap\bigcup_j D_j^k,\tag{4.10}$$

$$T_k=(g_{k-1}\circ\ldots\circ g_0)^{-1}(S_k),\tag{4.11}$$

the pull-back of S_k in E, and

$$U_k=\{z:\ 0<\operatorname{dist}(z,S_k)<\delta_k\},\tag{4.12}$$

an open band around S_k. Note that $T_k\subseteq T_{k-1}$.

B. $\{D_j^k\}$ are contained in a small neighborhood of $g_{k-1}(S_{k-1})$ and do not meet the boundaries of $g_{k-1}(D_i^{k-1})$ and $g_{k-1}\big(\frac{9}{10}D_i^{k-1}\big)$ for any i:

$$\bigcup_j D_j^k\subseteq g_{k-1}(S_{k-1}\cup U_{k-1})\setminus\bigcup_i g_{k-1}\big(\partial D_i^{k-1}\cup\partial\tfrac{9}{10}D_i^{k-1}\big),\tag{4.13}$$

when $k\geq 2$. The first inclusion is needed in proving that the composition $g_k\circ\ldots\circ g_1$ is 81-quasiconformal off S instead of 81^k-quasiconformal; the second inclusion is used to show that certain events are nearly independent.

C. The pull-back of $g_{k-1}(S_{k-1})\setminus\bigcup_j D_j^k$ is also small:

(4.14)
$$\begin{aligned}m(T_{k-1}\setminus T_k)&=m(g_{k-1}\circ\ldots\circ g_0)^{-1}\Big(g_{k-1}(S_{k-1})\setminus\bigcup_j D_j^k\Big)\\&\leq 10^{-5k}\min\Big(\min_j\{m(g_{k-2}\circ\ldots\circ g_0)^{-1}(S_{k-1}\cap\tfrac{9}{10}D_j^{k-1})\},\\&\qquad\min_j\{m(g_{k-2}\circ\ldots\circ g_0)^{-1}(S_{k-1}\cap D_j^{k-1}\setminus\tfrac{9}{10}D_j^{k-1})\}\Big),\end{aligned}$$

when $k\geq 2$; and
$$m(T_k)>10^{-3}.$$

D. g_k is 81-quasiconformal on $\mathbf{R}^2$, conformal off $\bigcup D_j^k\setminus U_k$, and satisfies

$$\|g_k-z\|_\infty<\min\{2\varepsilon_k,2\eta_{k-1}\},\tag{4.15}$$
$$\|(g_k\circ\ldots\circ g_0)^{-1}-(g_{k-1}\circ\ldots\circ g_0)^{-1}\|_\infty<2\varepsilon_k.\tag{4.16}$$

E. g_k maps a large portion of $S_k\cap D_j^k$ with respect to $m(g_{k-1}\circ\ldots\circ g_0)^{-1}$, to a small portion with respect to m:

(4.17)
$$m\big((g_{k-1}\circ\ldots\circ g_0)^{-1}(S_k\cap\tfrac{9}{10}D_j^k)\big)>(0.81-\varepsilon_k)m((g_{k-1}\circ\ldots\circ g_0)^{-1}(S_k\cap D_j^k)),$$
$$m\big(g_k(S_k\cap\tfrac{9}{10}D_j^k)\big)<(0.01+2\varepsilon_k)m(g_k(S_k\cap D_j^k)).\tag{4.18}$$

This is the property on which our probabilistic idea is based.

F. E_k is a closed subset of T_k satisfying

$$m(T_k\setminus E_k)\leq(0.985)^k,\tag{4.19}$$

and

$$m(g_k\circ\ldots\circ g_1)(E_k)\leq 5^{-k+5}.\tag{4.20}$$

G. $0<\eta_k\leq\frac{1}{2}\eta_{k-1}$ and

$$m(g_k\circ\ldots\circ g_1(E_k)+B(0,4\eta_k))<5^{-k+10}.$$

H. The number

$$\alpha_k \equiv \inf\{|g_k\circ\dots\circ g_1(z)-g_k\circ\dots\circ g_1(w)| : z,w\in B(0,1),\ |z-w|\geq\varepsilon_k\} \tag{4.21}$$

is positive.

We need to choose $\{D_j^n\}$, S_n, U_n, T_n, E_n, g_n, δ_n, η_n and α_n so that properties A through H hold for $k=n$.

Using the fact that quasiconformal mappings map sets of area zero to sets of area zero, and applying the Lebesgue differentiation theorems, and the n-fold version of (4.2), we may find finitely many mutually disjoint closed disks $\{D_j^n\}$ in $\mathbf{R}^2$ so that (4.7), (4.8), (4.9), (4.13) and (4.14) hold for $k=n$, and that for each j,

(4.22)
$$m\big((g_{n-1}\circ\dots\circ g_1)^{-1}\big(g_{n-1}(S_{n-1})\cap\tfrac{9}{10}D_j^n\big)\big) > (0.81-\varepsilon_n)m\big((g_{n-1}\circ\dots\circ g_1)^{-1}(D_j^n)\big).$$

Choose $\delta_n>0$ small, and define S_n, T_n and U_n as in (4.10), (4.11) and (4.12) with k replaced by n.

Since $m(E)>10^{-2}$, $m(E\backslash T_1)\leq 10^{-5}$ and $m(T_{k-1}\backslash T_k)\leq 10^{-5k}$ for $2\leq k\leq n$, we have $m(T_n)>10^{-3}$.

Let f_n be the 81-quasiconformal mapping on $\mathbf{R}^2$, defined by

$$f_n(z)=\begin{cases} z & \text{on } \mathbf{R}^2\backslash\bigcup D_j^n,\\ \phi_{D_j^n}(z) & \text{on } D_j^n \text{ for each } j,\end{cases}$$

and note that

$$\|f_n(z)-z\|_\infty \quad\text{and}\quad \|f_n^{-1}(z)-z\|_\infty<\min\{\varepsilon_n,\alpha_{n-1},\eta_{n-1}\}. \tag{4.23}$$

Let $\mu_n(z)=\bar\partial f_n(z)/\partial f_n(z)$ a.e. on $\mathbf{R}^2$, and

$$\nu_n=\mu_n\chi_{\mathbf{R}^2\backslash U_n}.$$

And let g_n be the unique quasiconformal mapping on $\mathbf{R}^2$, which solves $\bar\partial g_n=\nu_n\partial g_n$ on $\mathbf{R}^2$ a.e., with $g_n(z)=z+O(1/z)$ near ∞, and $\bar\partial g_n$ and $\partial g_n-1\in L^p(\mathbf{R}^2)$. Note that g_n is conformal in $(\mathbf{R}^2\backslash\bigcup_j D_j^n)\cup U_n$.

Because of Theorem D, δ_n can be chosen small enough so that

$$\|f_n-g_n\|_\infty<\min\{\varepsilon_n,\eta_{n-1}\}, \tag{4.24}$$

$$\|f_n^{-1}-g_n^{-1}\|_\infty<\alpha_{n-1}, \tag{4.25}$$

and

$$\|\partial f_n-\partial g_n\|_p+\|\bar\partial f_n-\bar\partial g_n\|_p<10^{-10}\varepsilon_n A^{-1}\min_j m(D_j^n). \tag{4.26}$$

It remains to verify properties D through H.

Let $k=n$. The inequality (4.15) follows from (4.23) and (4.24); and (4.16) follows from (4.23), (4.25) and the definition of α_{n-1}.

The inequality (4.17) follows from (4.22) and from the fact that $S_n \cap D_j^n = g_{n-1}(S_{n-1}) \cap D_j^n$ and that $S_n \cap \frac{9}{10} D_j^n = g_{n-1}(S_{n-1}) \cap \frac{9}{10} D_j^n$.

Recall that the Jacobian of f_n is bounded between $\frac{1}{81}$ and 9. Thus it follows from (4.9) with $k=n$ that

$$0.001 m(D_j^n) < m\big(f_n\big(S_n \cap \tfrac{9}{10} D_j^n\big)\big) < (0.01+\varepsilon_n) m(f_n(S_n \cap D_j^n))$$

for each j. Applying (4.5) to f_n, g_n and $X = S_n \cap \frac{9}{10} D_j^n$ and $S_n \cap D_j^n$, we deduce (4.18) from (4.26).

Property E plays a central role in the proof of F. First we employ partitions of T_h $(1 \le h \le n)$ defined as follows. Let $1 \le q \le h$, and let

$$A_q^h = (g_{q-1} \circ \dots \circ g_0)^{-1} \Big(\bigcup_j \tfrac{9}{10} D_j^q \Big) \cap T_h,$$

and

$$B_q^h = T_h \setminus A_q^h.$$

Because of (4.13), $B_q^h = (g_{q-1} \circ \dots \circ g_0)^{-1} \big(\bigcup_j \big(D_j^q \setminus \frac{9}{10} D_j^q \big) \big) \cap T_h$.

The sets $A_1^h, A_2^h, \dots, A_q^h$ generate a finite Boolean algebra of subsets of T_h, say $\mathcal{F}_q^h$. When $q=0$, the algebra is the trivial one.

We need to show that for every set X in $\mathcal{F}_q^h$, $0 \le q \le h-1$,

$$m(X \cap A_{q+1}^h) > \tfrac{3}{4} m(X). \tag{4.27}$$

Note from (4.13) and (4.17) that

$$m(X \cap A_{q+1}^{q+1}) > \tfrac{4}{5} m(X) \quad \text{for each } X \in \mathcal{F}_q^{q+1}.$$

Then for $h > q+1$ and $X \in \mathcal{F}_q^{q+1}$,

$$m(X \cap A_{q+1}^h) \ge m(X \cap A_{q+1}^{q+1}) - m(T_{q+1} \setminus T_h) \ge \tfrac{4}{5} m(X) - \sum_{l=q+1}^{h-1} m(T_l \setminus T_{l+1}).$$

Note that

$$m(X) \ge \min\Big(\min_j \big\{ m(g_q \circ \dots \circ g_0)^{-1} \big(S_{q+1} \cap \tfrac{9}{10} D_j^{q+1}\big) \big\}, \\ \min_j \big\{ m(g_q \circ \dots \circ g_0)^{-1} \big(S_{q+1} \cap \big(D_j^{q+1} \setminus \tfrac{9}{10} D_j^{q+1}\big)\big) \big\} \Big)$$

It follows from (4.14) that $m(T_l\backslash T_{l+1})\leq 10^{-5l}m(X)$ for $l\geq q+1$. This gives

$$m(X\cap A^h_{q+1})\geq \tfrac{31}{40}m(X)\quad \text{for } X\in\mathcal{F}^{q+1}_q,$$

which is a stronger assertion than (4.27).

To select the sets E_n, let $r_q=1$ on A^n_q and $r_q=0$ on B^n_q. Let E_n be the subset of T_n on which

$$r_1+\ldots+r_{n-1}>\tfrac{2}{3}(n-1).$$

We need to verify (4.19) and (4.20).

Let $\overline{m}$ be the relative Lebesgue measure on T_n, i.e., $\overline{m}(X)=m(X\cap T_n)/m(T_n)$, and $\mathcal{E}$ the integral with respect to $\overline{m}$ (expectation). Let $u_q=r_1+\ldots+r_q$, $1\leq q\leq n-1$. In view of (4.27), we obtain

$$\mathcal{E}(e^{-u_1/2})\leq \tfrac{3}{4}e^{-1/2}+\tfrac{1}{4}\equiv a_1,$$

and

$$\mathcal{E}(e^{-u_{n-1}/2})\leq a_1\mathcal{E}(e^{-u_{n-2}/2})\leq a_1^{n-1}.$$

Hence

$$\overline{m}(T_n\backslash E_n)=\overline{m}\{u_{n-1}\leq \tfrac{2}{3}(n-1)\}\leq a_1^{n-1}e^{(n-1)/3}.$$

Since $a_1e^{1/3}<0.985$, (4.19) follows.

As for the measure of $g_n\circ\ldots\circ g_1(E_n)$, let us consider the algebra $\mathcal{F}^n_{n-1}$ of subsets of T_n. It has 2^{n-1} atoms. Because of (4.18), each atom contained in E_n is mapped to a set of measure $<4\pi(0.02)^{2(n-1)/3}$. Since $(0.02)^{2/3}<\frac{1}{10}$ and there are 2^{n-1} atoms, we obtain the estimate (4.20).

Since $g_n\circ\ldots\circ g_1(E_n)$ is closed, we may choose η_n to satisfy property G. Because $(g_n\circ\ldots\circ g_1)^{-1}$ is Hölder continuous in $B(0,\frac{3}{2})$, the number α_n defined by (4.21) with $k=n$ is positive.

This completes the nth step of the induction.

We note from (4.15) and (4.16) that

$$g=\lim_{n\to\infty} g_n\circ\ldots\circ g_1$$

and

$$h=\lim_{n\to\infty}(g_n\circ\ldots\circ g_1)^{-1}$$

are uniform limits of uniformly continuous functions. Thus $h=g^{-1}$, and g is a homeomorphism on $\mathbf{R}^2$.

Each g_n is 81-quasiconformal on $\mathbf{R}^2$ and conformal off $\bigcup_j D_j^n \setminus U_n$. In view of property B,

$$(g_{n-1}\circ\ldots\circ g_0)^{-1}\Big(\bigcup_j D_j^n \setminus U_n\Big) \subseteq E \cup (g_{n-2}\circ\ldots\circ g_0)^{-1}(U_{n-1}) \setminus (g_{n-1}\circ\ldots\circ g_0)^{-1}(U_n).$$

Since $(g_{n-2}\circ\ldots\circ g_0)^{-1}(U_{n-1}) \setminus (g_{n-1}\circ\ldots\circ g_0)^{-1}(U_n)$ are mutually disjoint, $g_n\circ\ldots\circ g_1$ is 81-quasiconformal in $\mathbf{R}^2\setminus E$. Therefore g is 81-quasiconformal on $\mathbf{R}^2\setminus E$.

Let $T=\bigcap T_n$ and $F=\limsup E_n=\bigcap_{q=1}^{\infty}\bigcup_{n=q}^{\infty} E_n$. We recall from property C that $m(T)\geq 10^{-3}$, from (4.19) that $m(T\setminus E_n)<(0.985)^n$ whence $\sum m(T\setminus E_n)<\infty$. Thus $m(F)=m(T)>0$.

The estimate (4.20), our choice of η_k and the inequality

$$\|g-g_n\circ\ldots\circ g_1\|_\infty < 4\eta_n$$

derived from (4.15) show that $\sum m(g(E_n))<\infty$. Hence g maps F to a set of measure zero.

Finally, let $\mu_{g^{-1}}$ be the complex dilatation for g^{-1} and let Φ be the quasiconformal mapping that has dilatation $\mu_\Phi=0$ on $E=\mu_{g^{-1}}$ on $\mathbf{R}^2\setminus E$.

Then $G=\Phi\circ g\in \mathrm{CH}(E)$ and it maps F to a set of measure zero.

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