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Fractal dimensions for Jarník limit sets of geometrically finite Kleinian groups; the semi-classical approach

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Abstract. We introduce and study the Jarník limit set $\mathcal{J}_\sigma$ of a geometrically finite Kleinian group with parabolic elements. The set $\mathcal{J}_\sigma$ is the dynamical equivalent of the classical set of well approximable limit points. By generalizing the method of Jarník in the theory of Diophantine approximations, we estimate the dimension of $\mathcal{J}_\sigma$ with respect to the Patterson measure. In the case in which the exponent of convergence of the group does not exceed the maximal rank of the parabolic fixed points, and hence in particular for all finitely generated Fuchsian groups, it is shown that this leads to a complete description of $\mathcal{J}_\sigma$ in terms of Hausdorff dimension. For the remaining case, we derive some estimates for the Hausdorff dimension and the packing dimension of $\mathcal{J}_\sigma$.

1. Statement and discussion of results

This paper continues the 'Diophantine analysis' (begun in [13], [14], [17]) of the limit set $L(G)$ of a non-elementary, geometrically finite Kleinian group G with parabolic elements. We assume that G acts discontinously on the $(N+1)$-dimensional unit ball D^{N+1} which is equipped with the hyperbolic metric d.

It is well known that G is of *δ-divergence type* ([18, Corollary 20]), i.e. the series

$$\sum_{g\in G} e^{-sd(0,g(0))}$$

diverges at its exponent of convergence $\delta=\delta(G)$, which is usually referred to as the *exponent of convergence of* G.

If μ denotes the Patterson measure on the limit set $L(G)$, then the δ-divergence type condition is equivalent to the fact that the geodesic flow is ergodic ([18, Theo-

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rem 14]). This implies, for μ-almost all ξ in $L(G)$, the 'ergodic law' ([18, Corollary 19])

$$\lim_{t\to\infty}\frac{\Delta(\xi_t)}{t}=0.$$

Here ξ_t is the unique point on the ray s_ξ from $0\in D^{N+1}$ to $\xi\in L(G)$, whose hyperbolic distance from 0 is equal to t; and further, Δ denotes the *ray excursion function*, which is defined, for $\xi\in L(G)$ and positive t, by

$$\Delta(\xi_t):=d(\xi_t,G(0)).$$

It is well known that the Hausdorff dimension of the set of points which obey this ergodic law is equal to δ ([18, Theorem 15]).

The aim of this paper is to determine fractal dimensions of certain sets of points which do not follow the above ergodic law. Throughout the paper let σ denote some positive number. We consider the *σ-Jarník limit set* $\mathcal{J}_\sigma(G)$ and the *strict σ-Jarník limit set* $\mathcal{J}_\sigma^+(G)$, which are defined by

$$\mathcal{J}_\sigma(G):=\left\{\xi\in L(G):\limsup_{t\to\infty}\frac{\Delta(\xi_t)}{t}\geq\frac{\sigma}{1+\sigma}\right\},$$
$$\mathcal{J}_\sigma^+(G):=\left\{\xi\in L(G):\limsup_{t\to\infty}\frac{\Delta(\xi_t)}{t}=\frac{\sigma}{1+\sigma}\right\}.$$

Now, $\mathcal{J}_\sigma(G)$ is a dense subset of the limit set $L(G)$. From this we deduce that $\dim_B(\mathcal{J}_\sigma(G))$, the box-counting dimension of $\mathcal{J}_\sigma(G)$, is equal to $\dim_H(L(G))$, the Hausdorff dimension of $L(G)$, and hence equal to δ (for $\dim_B(L(G))=\dim_H(L(G))=\delta$ ([15, Theorem 3]) and $\dim_B$ is invariant under taking the closure ([7, Proposition 3.4])). However, as we shall see in this paper, questions concerning other fractal dimensions of $\mathcal{J}_\sigma(G)$ and $\mathcal{J}_\sigma^+(G)$ are more subtle.

A first main result in this paper is the determination of $\dim_\mu(\mathcal{J}_\sigma(G))$, the dimension of $\mathcal{J}_\sigma(G)$ with respect to the Patterson measure μ (see Section 2 for the definition). We derive the following theorem, where $k_{\max}$ denotes the maximal rank of the parabolic elements of the underlying group.

Theorem A. *If G is a geometrically finite Kleinian group with parabolic elements, then*

$$\dim_\mu(\mathcal{J}_\sigma^+(G))=\dim_\mu(\mathcal{J}_\sigma(G))=\frac{\delta}{\delta+\sigma(2\delta-k_{\max})}.$$

For the proof of this theorem we shall construct and analyse probability measures which are supported on Cantor-like subsets of $\mathcal{J}_\sigma(G)$. Our proof generalizes the classical methods which were developed by Jarník ([8]) and later by Besicovitch ([2]) for their calculations of the Hausdorff dimension of the set of well approximable irrational numbers.

On the basis of Theorem A, we continue the investigation of $\mathcal{J}_\sigma(G)$ and $\mathcal{J}_\sigma^+(G)$, and derive an estimate for the Hausdorff dimension of $\mathcal{J}_\sigma(G)$ and $\mathcal{J}_\sigma^+(G)$ which is exact in the case '$\delta \le k_{\max}$'. More precisely, mainly by using *Billingsley's lemma* (see Section 2) and the *global measure formula* for the Patterson measure ([17, Theorem 2]), we derive the following theorem.

Theorem B. *If G is a geometrically finite Kleinian group with parabolic elements such that $\delta \le k_{\max}$, then*

$$\dim_H(\mathcal{J}_\sigma(G)) = \dim_H(\mathcal{J}_\sigma^+(G)) = \frac{\delta}{1+\sigma}.$$

We remark that this theorem covers in particular all finitely generated Fuchsian groups with parabolic elements. This follows, since for groups of this type we have that $k_{\max}=1$, and also that δ is less than or equal to 1. Hence Theorem B is applicable.

Also, if we let $G=\mathrm{PSL}_2(\mathbf{Z})$, Theorem B implies the classical number theoretical results of Jarník and Besicovitch ([8], [2]).

Now, in the remaining case, i.e. the case '$\delta > k_{\max}$', our semi-classical approach does not lead to an exact result for the Hausdorff dimension of the σ-Jarník limit set. However, our method still allows the derivation of the following approximations for the Hausdorff dimension $\dim_H(\mathcal{J}_\sigma(G))$ and the packing dimension $\dim_p(\mathcal{J}_\sigma(G))$.

Proposition C. *If G is a geometrically finite Kleinian group with parabolic elements such that $\delta > k_{\max}$, then*

$$\frac{\delta}{1+\sigma(2\delta - k_{\max})/\delta} \le \dim_H(\mathcal{J}_\sigma(G)) \le \frac{\delta}{1+\sigma} \le \dim_p(\mathcal{J}_\sigma(G)).$$

Also, we remark that the σ-Jarník limit set may also be expressed equivalently in terms of the set of standard horoballs $\{H_{g(p)}(r_g): p\in P,\ g\in \mathcal{T}_p\}$ (we refer to Section 2 for the definitions). For this, let $\mathcal{F}$ denote the set of functions $\phi: \mathbf{R}^+ \to \mathbf{R}^+$ such that $\lim_{x\to 0} \log\phi(x)/\log x = 0$. An elementary calculation in hyperbolic geometry shows that a necessary and sufficient condition for ξ to be an element of $\mathcal{J}_\sigma(G)$ is that there exists $\phi \in \mathcal{F}$ such that ξ is contained in the shadow at infinity $\mathcal{H}_{g(p)}(\phi(r_g) r_g^{1+\sigma})$ of infinitely many 'reduced' standard horoballs $H_{g(p)}(\phi(r_g) r_g^{1+\sigma})$. Hence, we have that

$$\mathcal{J}_\sigma(G) = \bigcup_{\phi\in\mathcal{F}} \bigcap_{n\in\mathbf{N}} \bigcup_{p\in P} \bigcup_{\substack{g\in\mathcal{T}_p \\ r_g \le 1/n}} \mathcal{H}_{g(p)}(\phi(r_g) r_g^{1+\sigma}).$$

If in this expression for $\mathcal{J}_\sigma(G)$ we take the first union only over those functions ϕ which are constant, then we derive the more classical *set of well approximable limit points*, which has already been investigated by Melián and Pestana for cofinite Kleinian groups ([9]). In this paper, we shall consider in particular the *set of simple well approximable limit points*, which is defined by

$$\mathcal{W}_\sigma := \bigcap_{n\in\mathbf{N}} \bigcup_{p\in P} \bigcup_{\substack{g\in\mathcal{T}_p \\ r_g\le 1/n}} \mathcal{H}_{g(p)}(r_g^{1+\sigma}).$$

We remark that the proofs in this paper immediately yield corresponding results for the fractal dimensions of the set of well approximable limit points.

Finally, we give a few applications of the results in this paper to related topics.

Convex cocompact groups

For a convex cocompact Kleinian group, the limit set and the Patterson measure form a 'δ-homogenous system' (see [13, Definition 0.1.1]). By replacing the parabolic fixed points in this paper by a finite set A of loxodromic fixed points and then using the geometrical techniques which were developed in [13] in combination with the semi-classical method of this paper, one derives the Hausdorff dimension of the set of limit points which are well approximable with respect to A.

Parabolic rational maps

Let $J(T)$ denote the Julia set of a parabolic rational map T. There exists a global measure formula for the $(\dim_H J(T))$-dimensional conformal measure on $J(T)$ which is similar to the formula for the Patterson measure ([6, Proposition 5.3]). Again, as in the Kleinian group case, one may derive a Dirichlet-type theorem which delivers economical coverings of $J(T)$ in terms of the dynamics of T (this Dirichlet-type theorem for $J(T)$ will appear in a joint paper with M. Urbański ([16])).

Since these two concepts are the main ingredients of our semi-classical method here, it is not difficult to see that this method also gives rise to the corresponding results for the set of points in $J(T)$ which are well approximable with respect to the rational indifferent periodic points of the parabolic rational map T.

Orbifolds

Let M_G be the cusped, geometrically finite Riemannian manifold corresponding to the Kleinian group G, which is of constant negative curvature and not necessarily of finite Riemannian volume. Let $\mathcal{S}(M_G)$ be the unit tangent bundle over M_G and let $\{\phi_t\}$ denote the geodesic flow on $\mathcal{S}(M_G)$. Further, let $\pi: \mathcal{S}(M_G) \to M_G$ be the canonical projection which maps each line element in $\mathcal{S}(M_G)$ to its base point in M_G. Analogous to the ray excursion function, the *geodesic excursion function* Δ_0 is defined, for $v \in \mathcal{S}(M_G)$ and positive t, by $\Delta_0(v,t) := d(\pi(v), \pi(\phi_t v))$.

For positive $\sigma_0 < 1$, let the *large deviation sets* $\widetilde{\mathcal{J}}_{\sigma_0}(M_G)$ be given by

$$\widetilde{\mathcal{J}}_{\sigma_0}(M_G) := \left\{ v \in \mathcal{S}(M_G) : \limsup_{t\to\infty} \frac{\Delta_0(v,t)}{t} = \sigma_0 \right\}.$$

Using the results in [15, Theorem 3], it is easily seen that Theorem B and Proposition C give rise to the following statements.

(i) *If* $\delta \le k_{\max}$, *then*

$$\dim_H(\widetilde{\mathcal{J}}_{\sigma_0}(M_G)) = \delta(2-\sigma_0)+1.$$

(ii) *If* $\delta > k_{\max}$, *then*

$$\delta\left(2-\sigma_0\left(\sigma_0+\frac{\delta(1-\sigma_0)}{2\delta-k_{\max}}\right)^{-1}\right)+1 \le \dim_H(\widetilde{\mathcal{J}}_{\sigma_0}(M_G)) \le \delta(2-\sigma_0)+1.$$

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2. Preliminaries

I. Conformal geometry and measure theory

As already stated in the introduction, we let G denote a non-elementary, geometrically finite Kleinian group with parabolic elements. Further, we let P be a complete set of inequivalent parabolic fixed points. For $p \in P$, the stabilizer G_p of p in G contains a maximal abelian subgroup of rank $k(p)$, and $k(p)$ is referred to as the *rank of* p. We choose a particular element $p_0 \in P$ such that $k(p_0) = k_{\max} := \max\{k(p) : p \in P\}$.

We assume that a set $\mathcal{T}_p \subset G$ of coset representatives of G_p in G is chosen such that if $g \in \mathcal{T}_p$, then $|g(0)| \le |h(0)|$ for all $h \in G_{g(p)} := gG_pg^{-1}$. Also let $\mathcal{T}_{\max} := \mathcal{T}_{p_0}$ and $\mathcal{T} := \bigcup_{p\in P} \mathcal{T}_p$.

It is known that the cusp regions of the manifold $M_G = D^{N+1}/G$ are represented in D^{N+1} by a set of pairwise disjoint horoballs. To be precise, to each $g \in \mathcal{T}_p$, we associate an $(N+1)$-ball $H_{g(p)}(r_g) \subset D^{N+1}$ which is tangential to $S^N := \partial D^{N+1}$ at $g(p)$ and whose Euclidean radius r_g is comparable to $(1-|g(0)|)$. The horoball $H_{g(p)}(r_g)$ is a $G_{g(p)}$-invariant subset of D^{N+1} and G permutes the set

$$\{H_{g(p)}(r_g) : g \in \mathcal{T}_p,\ p \in P\},$$

which we refer to as the *set of standard horoballs with top representation.*

We shall also require the notion of the *shadow at infinity* of a horoball $H_{g(p)}(r)$, which is defined by

$$\mathcal{H}_{g(p)}(r) := \{\xi \in S^N : s_\xi \cap H_{g(p)}(r) \neq \emptyset\}.$$

We now state two fundamental results concerning the geometry of standard horoballs and its relation to the limit set $L(G)$. These results are required in the following section.

If there is no risk of confusion, we shall use the notion $\ll$ and $\gg$ to indicate inequality with a positive constant factor, and if $x \ll y$ and $x \gg y$, i.e. if x and y are comparable, then we shall write $x \asymp y$.

Lemma 1 (Decoupling lemma). ([17, Proposition 2.3]) *For $p, q \in P$ and $g \in \mathcal{T}_p$, there exists $h \in \mathcal{T}_q$ such that*

$$h(q) \in \mathcal{H}_{g(p)}(2r_g) \quad \text{and} \quad r_h \asymp r_g.$$

Lemma 2 (Dirichlet-type theorem). ([17, Theorem 1]) *There exist positive constants k_D and $\varkappa$ such that, for all positive $\alpha < k_D$, the set*

$$\{\mathcal{H}_{g(p)}(r_g(\alpha)) : p \in P,\ g \in \mathcal{T}_p,\ r_g \geq \alpha\}$$

covers $L(G)$ with bounded multiplicity.

Here $r_g(\alpha) := \varkappa\sqrt{r_g \alpha}$ denotes the α-Dirichlet radius corresponding to r_g.

We assume that the reader is familiar with the construction and basic properties of the *Patterson measure* μ (we refer to [11], [12] and [10]).

For the purposes of the present paper it is sufficient to know that μ is a non-atomic probability measure which is supported on the limit set $L(G)$ and further, that there exists a uniform estimate for the μ-measure of N-balls in S^N which are centred around limit points. This latter estimate was derived in [17, Theorem 2], where we referred to it as the *global measure formula* (see also [19, §6]). In order to restate this formula here, we require the following notation.

For $\xi \in L(G)$ and positive t, we let $b(\xi_t)$ denote the intersection of S^N with the $(N+1)$-ball whose boundary is orthogonal to S^N and which intersects the ray s_ξ orthogonally at ξ_t. Hence, $b(\xi_t)$ is an N-ball in S^N whose radius is comparable to e^{-t}. Further, we define $k(\xi_t)$ to be equal to $k(p)$ if $\xi_t \in H_{g(p)}(r_g)$ for some $p \in P$ and $g \in \mathcal{T}_p$, and we let $k(\xi_t)$ be equal to δ otherwise.

Lemma 3 (Global measure formula). *If* $\xi \in L(G)$ *and* t *is positive, then*

$$\mu(b(\xi_t)) \asymp e^{-t\delta} e^{-(\delta - k(\xi_t))\Delta(\xi_t)}.$$

II. Fractal geometry

We recall a few results from the theory of fractal sets which are required in this paper and which cannot necessarily be found easily in the literature. We assume that the reader is familiar with the definition and basic results of Hausdorff measures and packing measures (see [7]).

Let Λ denote a compact subset of $\mathbf{R}^N$. Further, let m be a non-atomic Borel probability measure on Λ. For positive ε, a covering $\{U_i\}_{i \in \mathbf{N}}$ of $\Lambda' \subset \Lambda$ is an (m, ε)-*covering* of Λ' if $m(U_i) < \varepsilon$ for all $i \in \mathbf{N}$. If $\mathcal{U}_\varepsilon^m(\Lambda')$ denotes the set of all (m, ε)-coverings of Λ', then, for positive s, the *s-dimensional m-Hausdorff measure* $\mathcal{H}_s^m(\Lambda')$ is given by

$$\mathcal{H}_s^m(\Lambda') := \lim_{\varepsilon \to 0} \inf_{\{U_i\} \in \mathcal{U}_\varepsilon^m(\Lambda')} \sum_i m(U_i)^s.$$

Analogous to the Hausdorff dimension, we may define $\dim_m(\Lambda')$, the *m-dimension* of Λ', by

$$\dim_m(\Lambda') := \sup\{s : \mathcal{H}_s^m(\Lambda') = \infty\} = \inf\{s : \mathcal{H}_s^m(\Lambda') = 0\}.$$

A weaker version of the following lemma can be found in [3, Theorem 14.1]. The proof in [3] is easily extended to the slightly more general situation here.

By $B(z, r)$ we denote the spherical, Euclidean N-ball of radius r which is centred at z.

Lemma 4 (Billingsley's lemma). *Let* Λ *be a compact subset of* $\mathbf{R}^N$. *Further, let* m_1 *and* m_2 *be two Borel probability measures on* Λ. *If* E *is a Borel subset of* Λ *such that, for positive* s *and for each* $z \in E$, *we have*

$$\liminf_{r \to 0} \frac{\log m_1(B(z,r))}{\log m_2(B(z,r))} \geq s,$$

then it follows that

$$\dim_{m_2}(E) \geq s \dim_{m_1}(E).$$

For estimates concerning the s-dimensional Hausdorff measure $\mathcal{H}_s$ and packing measure $\mathcal{P}_s$, the following lemma turns out to be useful. For the proofs we refer to [19], [4] and [7] (see also [5, Theorem D, E]).

Lemma 5. *Let Λ be a compact subset of $\mathbf{R}^N$. Further, let m denote a Borel probability measure on Λ which is positive on open sets. For $z \in \Lambda$ and positive r, we define $\mathcal{D}_{m,s}(z,r) := m(B(z,r))/r^s$.*

Suppose that E is a Borel subset of Λ.

(i) *If, for all $z \in E$, we have $\limsup_{r\to 0} \mathcal{D}_{m,s}(z,r) \gg 1$, then $\mathcal{H}_s(F) \ll m(F)$ for all Borel subsets F of E.*

(ii) *If, for all $z \in E$, we have $\limsup_{r\to 0} \mathcal{D}_{m,s}(z,r) \ll 1$, then $\mathcal{H}_s(F) \gg m(F)$ for all Borel subsets F of E.*

(iii) *If, for all $z \in E$, we have $\liminf_{r\to 0} \mathcal{D}_{m,s}(z,r) \gg 1$, then $\mathcal{P}_s(F) \ll m(F)$ for all Borel subsets F of E.*

(iv) *If, for all $z \in E$, we have $\liminf_{r\to 0} \mathcal{D}_{m,s}(z,r) \ll 1$, then $\mathcal{P}_s(F) \gg m(F)$ for all Borel subsets F of E.*

3. Fractal dimensions of $\mathcal{J}_\sigma(G)$ and $\mathcal{J}_\sigma^+(G)$

I. Counting horoballs in the shade of a σ-reduced horoball

In this subsection we give a local counting estimate for standard horoballs. Roughly speaking, for a given horoball $H_{g(p_0)}(r_g)$, we calculate the cardinality of equally sized standard horoballs $H_{h(p_0)}(r_h)$ which are totally contained in the 'σ-shade' of $H_{g(p_0)}(r_g)$, i.e. for which we have that $\mathcal{H}_{h(p_0)}(r_h) \subset \mathcal{H}_{g(p_0)}(r_g^{1+\sigma})$. We show that the number of these horoballs is, as should be expected, comparable to $\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma}))/\mu(\mathcal{H}_{h(p_0)}(r_h))$.

We remark that for the purposes of this paper it is sufficient to give these estimates for the maximal rank parabolic fixed point p_0. Nevertheless, using the same arguments, similar estimates could be derived for arbitrary $p \in P$ as well.

We require the following notation. For positive τ, n in $\mathbf{N}$ and g in $\mathcal{T}_{\max}$, we order the elements in $\mathcal{T}_{\max}$ in the following way:

$$A_n(\tau) := \{h \in \mathcal{T}_{\max} : \tau^{n+1} \leq r_h < \tau^n\},$$
$$Q_n(g,\sigma,\tau) := \{h \in A_n(\tau) : \mathcal{H}_{h(p)}(r_h) \subset \mathcal{H}_{g(p)}(r_g^{1+\sigma})\}.$$

Now, we derive the following counting estimate.

Proposition 1. *There exists a positive number ϱ, positive constants k_0, k_1, k_2 and a rapidly increasing function $\iota: \mathbf{N} \to \mathbf{R}^+$ such that the following holds:*

If g is an element of $A_n(\varrho)$ for some n in $\mathbf{N}$ greater than k_0, then we have, for m in $\mathbf{N}$ greater than $\iota(n)$,

$$k_1 \varrho^{\delta(n-m)+\sigma n(2\delta - k_{\max})} \leq \operatorname{card}(Q_m(g,\sigma,\varrho)) \leq k_2 \varrho^{\delta(n-m)+\sigma n(2\delta-k_{\max})};$$

and, in particular,

$$\operatorname{card}(Q_m(g,\sigma,\varrho)) \geq 2.$$

Proof. Let g be in $\mathcal{T}_{\max}$ such that r_g^σ is less than $\min\{(4\varkappa)^{-1}, k_D^\sigma\}$. Further, for $q \in P$, let Q_q denote the set of $h \in \mathcal{T}_q$ such that $\mathcal{H}_{h(q)}(r_h) \subset \mathcal{H}_{g(p_0)}(r_g^{1+\sigma})$. If we define $c(g) := (4\varkappa)^{-1} r_g^{1+2\sigma}$, then we have for the Dirichlet radius $r_g(\theta)$, for θ less than $c(g)$,

$$r_g(\theta) < r_g^{1+\sigma}.$$

The Dirichlet-type theorem (Lemma 2) implies, for θ less than $c(g)$,

$$\mathcal{H}_{g(p_0)}(\varkappa r_g^{1+\sigma}) \cap L(G) \supset \left(\mathcal{H}_{g(p_0)}(r_g(\theta)) \cup \bigcup_{q\in P} \bigcup_{\substack{h \in Q_q \\ r_h \geq \theta}} \mathcal{H}_{h(q)}(r_h(\theta)) \right) \cap L(G).$$

Then, the finite multiplicity of the covering in the Dirichlet-type theorem and the fact that $\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma}))$ is comparable to $\mu(\mathcal{H}_{g(p_0)}(\varkappa r_g^{1+\sigma}))$ together imply that, for θ less than $c(g)$,

$$\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})) \gg \mu(\mathcal{H}_{g(p_0)}(r_g(\theta))) + \sum_{q\in P} \sum_{\substack{h \in Q_q \\ r_h \geq \theta}} \mu(\mathcal{H}_{h(q)}(r_h(\theta))). \tag{1}$$

On the other hand, it is easily checked, using the fact that r_g^σ is less than $(4\varkappa)^{-1}$, that we have, for θ less than $c(g)$,

$$r_g(\theta) < r_g^{1+\sigma} - 2\varkappa r_g^{1+2\sigma},$$

and hence that

$$\mathcal{H}_{g(p_0)}(r_g(\theta)) \subset \mathcal{H}_{g(p_0)}(r_g^{1+\sigma} - 2\varkappa r_g^{1+2\sigma}).$$

Using once again the Dirichlet-type theorem and also the fact that

$$\tfrac{1}{2} r_g^{1+\sigma} \leq r_g^{1+\sigma} - 2\varkappa r_g^{1+2\sigma} \leq r_g^{1+\sigma},$$

which implies that $\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma}))$ is comparable to $\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma} - 2\varkappa r_g^{1+2\sigma}))$, we derive, for θ less than $c(g)$,

$$\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})) \ll \mu(\mathcal{H}_{g(p_0)}(r_g(\theta))) + \sum_{q\in P} \sum_{\substack{h \in Q_q \\ r_h \geq \theta}} \mu(\mathcal{H}_{h(q)}(r_h(\theta))). \tag{2}$$

Combining (1) and (2), and using the global measure formula (Lemma 3), we deduce that, for positive constants c_1 and c_2, and for θ less than $c(g)$,

(3)

$$c_1\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma}))\theta^{-\delta} \le \left(\frac{r_g}{\theta}\right)^{k_{\max}/2} + \sum_{q\in P}\sum_{\substack{h\in Q_q\\ r_h\ge\theta}}\left(\frac{r_h}{\theta}\right)^{k(q)/2} \le c_2\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma}))\theta^{-\delta}.$$

Now, we let $\theta =: \lambda^m$, for some m in $\mathbf{N}$ and positive λ which will be specified in a moment. From the above we have that

$$\left(\frac{r_g}{\lambda^{m+1}}\right)^{k_{\max}/2} + \sum_{q\in P}\sum_{\substack{h\in Q_q\\ r_h\ge\lambda^m}}\left(\frac{r_h}{\lambda^{m+1}}\right)^{k(q)/2} \le c_2\lambda^{-m\delta}\lambda^{-k_{\max}/2}\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma}))$$

$$= c_2\lambda^{-(m+1)\delta}\lambda^{\delta-k_{\max}/2}\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma}));$$

and hence

$$\begin{aligned}\sum_{q\in P}\sum_{\substack{h\in Q_q\\ \lambda^{m+1}\le r_h<\lambda^m}}\left(\frac{r_h}{\lambda^{m+1}}\right)^{k(q)/2} &\ge \sum_{q\in P}\sum_{\substack{h\in Q_q\\ r_h\ge\lambda^{m+1}}}\left(\frac{r_h}{\lambda^{m+1}}\right)^{k(q)/2} \\ &\quad - \sum_{q\in P}\sum_{\substack{h\in Q_q\\ r_h\ge\lambda^m}}\left(\frac{r_h}{\lambda^{m+1}}\right)^{k(q)/2} \\ &\ge \lambda^{-(m+1)\delta}(c_1 - c_2\lambda^{\delta-k_{\max}/2})\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})).\end{aligned} \tag{4}$$

In an analogous way we derive

$$\sum_{q\in P}\sum_{\substack{h\in Q_q\\ \lambda^{m+1}\le r_h<\lambda^m}}\left(\frac{r_h}{\lambda^{m+1}}\right)^{k(q)/2} \le \lambda^{-(m+1)\delta}(c_2 - c_1\lambda^{\delta-k_{\min}/2})\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})). \tag{5}$$

Now, for λ less than $(2c_2c_1^{-1})^{-2/(2\delta-k_{\max})}$, (4) and (5) together imply

$$\tfrac{1}{2}c_1\lambda^{-(m+1)\delta}\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})) \le \sum_{q\in P}\sum_{\substack{h\in Q_q\\ \lambda^{m+1}\le r_h<\lambda^m}}\left(\frac{r_h}{\lambda^{m+1}}\right)^{k(q)/2}$$

$$\le c_2\lambda^{-(m+1)\delta}\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})),$$

and hence, if $c_3 := (2\lambda)^{-1}c_1$,

$$c_3\lambda^{-(m+1)\delta},\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})) \le \sum_{q\in P}\sum_{\substack{h\in Q_q\\ \lambda^{m+1}\le r_h<\lambda^m}} 1 \le c_2\lambda^{-(m+1)\delta}\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})).$$

The decoupling lemma (Lemma 1) then gives the existence of a positive number ϱ such that

(6) $$\sum_{\substack{h\in Q_{p_0}\\ \varrho^{m+1}\le r_h<\varrho^m}} 1 \asymp \varrho^{-(m+1)\delta}\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})).$$

Recalling the definition of $Q_m(g,\sigma,\varrho)$, we have now shown that there exist positive constants c_4 and c_5 such that

$$c_4\varrho^{-(m+1)\delta}\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma}))\le \mathrm{card}(Q_m(g,\sigma,\varrho))\le c_5\varrho^{-(m+1)\delta}\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})).$$

Let $n\in\mathbf{N}$ be such that $g\in A_n(\varrho)$. Using the global measure formula, we derive the existence of positive constants c_6 and c_7 such that

(7) $$c_6\varrho^{(n+1)\delta+n\sigma(2\delta-k_{\max})}\le\mu(\mathcal{H}_{g(p_0)}(r_g^{1+\sigma}))\le c_7\varrho^{(n+1)\delta+n\sigma(2\delta-k_{\max})}.$$

Now, (6) and (7) together imply that

$$c_4c_6\varrho^{(n-m)\delta+n\sigma(2\delta-k_{\max})}\le \mathrm{card}(Q_m(g,\sigma,\varrho))\le c_5c_7\varrho^{(n-m)\delta+n\sigma(2\delta-k_{\max})}.$$

If we define $\iota_0(n):=\delta^{-1}(n(\delta+\sigma(2\delta-k_{\max}))+(\log c_4c_6-\log 2)/\log\varrho)$, then it is easy to see that we have in particular, for m greater than $\iota_0(n)$,

$$Q_m(g,\sigma,\varrho)\ge 2.$$

Finally, if we convert the imposed conditions on r_g and m into conditions on n and m, an elementary calculation shows that the above holds for

1. n greater than $k_0:=(\log\varrho)^{-1}\min\{-\sigma^{-1}\log 4\varkappa,\log k_D\}$,
2. m greater than $\iota(n):=\max\{n(1+2\sigma)-(2\log 2\varkappa/\log\varrho),\iota_0(n)\}$. □

II. The construction of the probability space $(\mathcal{I}_\sigma,\nu_\sigma)$

For the remaining part of the paper let ϱ denote the positive number which we derived in Proposition 1. For ease of notation, we put, for n in $\mathbf{N}$ and g in $\mathcal{T}_{\max}$,

$$A_n:=A_n(\varrho)\quad\text{and}\quad Q_n(g):=Q_n(g,\sigma,\varrho).$$

Now, we define a rapidly increasing sequence $\{n_k\}_{k\in\mathbf{N}}$ of positive integers which satisfies the following three properties:

1. n_0 is greater than $\max\{k_0,2\sigma^{-1}\}$;
2. if n_{k-1} is given for k in $\mathbf{N}$, then n_k is greater than $\iota(n_{k-1})$;
3. $\lim_{m\to\infty}((1/n_m)\sum_{j=0}^{m-1}n_j)=0$.

The sequence $\{n_k\}_{k\in\mathbf{N}}$ will be central in the following construction.

For k in $\mathbf{N}$, we define

$$N_k := \min_{g\in A_{n_{k-1}}} \operatorname{card} Q_{n_k}(g).$$

Further, for each k in $\mathbf{N}$ and g in $A_{n_{k-1}}$, we let $\widetilde{Q}_{n_k}(g)$ be an arbitrary subset of $Q_{n_k}(g)$ such that

$$N_k = \operatorname{card} \widetilde{Q}_{n_k}(g).$$

Also, for a given $g_0 \in A_{n_0}$, we define sets I_k^σ by induction as follows:

1. $I_0^\sigma := \{\mathcal{H}_{g_0(p_0)}(r_{g_0}^{1+\sigma})\}$;
2. if I_{k-1}^σ is defined for k in $\mathbf{N}$, then let

$$I_k^\sigma := \{\mathcal{H}_{h(p_0)}(r_h^{1+\sigma}) : h \in \widetilde{Q}_{n_k}(g) \text{ for some } g \in A_{n_{k-1}} \text{ such that } \mathcal{H}_{g(p_0)}(r_g^{1+\sigma}) \in I_{k-1}^\sigma\}.$$

Clearly, we have that each element in I_{k-1}^σ contains exactly N_k elements of I_k^σ.

The 'level sets' I_k^σ form the basis for the Cantor-like set $\mathcal{I}_\sigma$, which we now define by

$$\mathcal{I}_\sigma := \bigcap_{k\geq 0} \bigcup_{I\in I_k^\sigma} I.$$

We would like to warn the reader that the set $\mathcal{I}_\sigma$ is not a 'spherical Cantor set' in the sense of Beardon ([1, Definition 1]).

Distributing the mass $(N_1 \dots N_k)^{-1}$ uniformly on each of the $N_1 \dots N_k$ horoball shadows in the level set I_k^σ, we derive a probability measure on $\mathcal{I}_\sigma$. To be precise, for each k in $\mathbf{N}$, we let $\nu_\sigma^{(k)}$ denote the probability measure on I_k^σ, which is defined for Borel sets E in S^N by

$$\nu_\sigma^{(k)}(E) = \sum_{I\in I_k^\sigma} (N_1 \dots N_k)^{-1} \mu(E\cap I)/\mu(I).$$

Using *Helly's theorem*, we then derive a mass distribution ν_σ on $\mathcal{I}_\sigma$; i.e. we obtain the probability measure ν_σ on $\mathcal{I}_\sigma$ as the weak limit of the sequence of measures $\{\nu_\sigma^{(k)}\}_{k\in\mathbf{N}}$.

It is easily checked that, for each k in $\mathbf{N}$, I in I_k^σ and m greater than k, one has

$$\nu_\sigma^{(k)}(I) = \nu_\sigma^{(m)}(I).$$

This observation then implies that

$$\nu_\sigma^{(k)}(I) = \nu_\sigma(I),$$

for all k in $\mathbf{N}$ and I in I_k^σ.

III. An analysis of the system $(\mathcal{I}_\sigma, \nu_\sigma)$

We shall give an estimate for the ν_σ-measure of sufficiently small balls centred around elements of $\mathcal{I}_\sigma$. In order to obtain this estimate, we require the following lemma.

Lemma 6. *There exists a positive constant k_3 such that if $B(\xi, r)$ is a spherical N-ball with centre $\xi \in \mathcal{I}_\sigma$ and radius r satisfying $\varrho^{n_k+1} \le r < \varrho^{n_{k-1}+1}$ for some k in* $\mathbf{N}$, *then*

$$\operatorname{card}\{I \in I_k^\sigma : I \cap B(\xi, r) \neq \emptyset\} \le \min\{N_k, k_3 \varrho^{-\delta n_k} \mu(B(\xi, r))\}.$$

Proof. Let $B(\xi, r)$ be given as stated in the lemma. We shall first see that $B(\xi, r)$ intersects at most one element in I_{k-1}^σ. For this, we observe that, since ξ is an element of $\mathcal{I}_\sigma$, ξ is contained in exactly one element $\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})$ of I_{k-1}^σ. Let $\mathcal{H}_{h(p_0)}(r_h^{1+\sigma})$ be an arbitrary second element in I_{k-1}^σ. We then have, for sufficiently small ϱ and for $n_0 \ge 2\sigma^{-1}$,

$$\begin{aligned} r_g^{1+\sigma} + r_h^{1+\sigma} + \max_{\varrho^{n_k+1} \le r < \varrho^{n_{k-1}+1}} r &< 2\varrho^{n_{k-1}(1+\sigma)} + \varrho^{n_{k-1}+1} \\ &\le \varrho^{n_{k-1}+1}(2\varrho^{\sigma n_{k-1}-1} + 1) \le 2\varrho^{n_{k-1}+1}. \end{aligned}$$

On the other hand, the pairwise disjointness of the standard horoballs implies that

$$|g(p_0) - h(p_0)| \ge 2\sqrt{r_g r_h} \ge 2\varrho^{n_{k-1}+1}.$$

It hence follows that

$$r_g^{1+\sigma} + r_h^{1+\sigma} + \max_{\varrho^{n_k+1} \le r < \varrho^{n_{k-1}+1}} r < |g(p_0) - h(p_0)|.$$

Since $\mathcal{H}_{h(p_0)}(r_h^{1+\sigma})$ was an arbitrary element in I_{k-1}^σ, the latter estimate gives that $\mathcal{H}_{g(p_0)}(r_g^{1+\sigma})$ is the only element of $I \in I_{k-1}^\sigma$ which has non-trivial intersection with $B(\xi, r)$. It now follows that

$$\operatorname{card}\{I \in I_k^\sigma : I \cap B(\xi, r) \neq \emptyset\} \le N_k.$$

In order to derive the second upper bound which is stated in the lemma, we observe that for each $\mathcal{H}_{f(p_0)}(r_f^{1+\sigma}) \in I_k^\sigma$ intersecting $B(\xi, r)$ non-trivially,

$$\mathcal{H}_{f(p_0)}(\varrho r_f) \subset B\Big(\xi, r + 2 \max_{\mathcal{H}_{h(p_0)}(r_h^{1+\sigma}) \in I_k^\sigma} r_h^{1+\sigma} + \varrho \max_{\mathcal{H}_{h(p_0)}(r_h^{1+\sigma}) \in I_k^\sigma} r_h\Big).$$

Since, for sufficiently small ϱ, we have that $\mathcal{H}_{e(p_0)}(\varrho r_e)\cap\mathcal{H}_{f(p_0)}(\varrho r_f)=\emptyset$ for distinct elements e, f in A_{n_k}, it follows that

$$\begin{aligned}\operatorname{card}\{I\in I_k^\sigma : I\cap B(\xi,r)\neq\emptyset\} & \min_{\mathcal{H}_{h(p_0)}(r_h^{1+\sigma})\in I_k^\sigma} \mu(\mathcal{H}_{h(p_0)}(\varrho r_h))\\ &\leq \mu\Big(B\Big(\xi, r+2\max_{\mathcal{H}_{h(p_0)}(r_h^{1+\sigma})\in I_k^\sigma} r_h^{1+\sigma}+\varrho\max_{\mathcal{H}_{h(p_0)}(r_h^{1+\sigma})\in I_k^\sigma} r_h\Big)\Big)\\ &\leq \mu(B(\xi, r+2\varrho^{n_k(1+\sigma)}+\varrho^{n_k+1}))\leq\mu(B(\xi,4r)).\end{aligned}$$

Using the global measure formula for the Patterson measure (Lemma 3), it follows, for some positive constant k_3,

$$\operatorname{card}\{I\in I_k^\sigma : I\cap B(\xi,r)\neq\emptyset\}\leq k_3\,\varrho^{-\delta n_k}\mu(B(\xi,r)). \quad \square$$

Proposition 2. *There exists a positive constant k_4 such that, for each positive ε, there exists a positive number $r_0=r_0(\varepsilon)$ such that, for all $\xi\in\mathcal{I}_\sigma$ and for all r less than r_0, we have that*

$$\nu_\sigma(B(\xi,r))\leq k_4\,\mu(B(\xi,r))^{(\delta-\varepsilon)/(\delta+\sigma(2\delta-k_{\max}))}.$$

Proof. Let ξ be in $\mathcal{I}_\sigma$ and let r be positive and sufficiently small. Without loss of generality we may assume that $\varrho^{n_k+1}\leq r<\varrho^{n_{k-1}+1}$, for some k in $\mathbf{N}$. If θ denotes an arbitrary element of the closed unit interval, i.e. if $0\leq\theta\leq1$, then we derive from the construction of the measure ν and from Lemma 6, that

$$\begin{aligned}\nu_\sigma(B(\xi,r)) &\leq \prod_{j=0}^{k} N_j^{-1}\operatorname{card}\{I\in I_k^\sigma : I\cap B(\xi,r)\neq\emptyset\}\\ &\leq \prod_{j=0}^{k} N_j^{-1}\min\{N_k, k_3\varrho^{-\delta n_k}\mu(B(\xi,r))\}\\ &\leq k_3^\theta\varrho^{-n_k\delta\theta}\mu(B(\xi,r))^\theta N_k^{1-\theta}\prod_{j=0}^{k}N_j^{-1}.\end{aligned}$$

Using Proposition 1, it follows that

$$\begin{aligned}&\nu_\sigma(B(\xi,r))\\ &\quad\leq k_3^\theta k_1^{-\theta}\varrho^{-n_k\delta\theta}\mu(B(\xi,r))^\theta k_1^{-(k-1)}\varrho^{\theta\delta(n_k-n_{k-1})}\varrho^{-\theta\sigma n_{k-1}(2\delta-k_{\max})}\\ &\qquad\times\varrho^{\delta(n_{k-1}-n_0)}\varrho^{-\sigma(2\delta-k_{\max})\sum_{j=0}^{k-2}n_j}\\ &\quad=k_3^\theta k_1^{-\theta}\mu(B(\xi,r))^\theta\\ &\qquad\times\varrho^{n_{k-1}(\delta-\delta\theta-\theta\sigma(2\delta-k_{\max})-n_{k-1}^{-1}(n_0+\sigma(2\delta-k_{\max})\sum_{j=0}^{k-2}n_j+(k-1)(\log k_1)(\log\varrho)^{-1}))}.\end{aligned}$$

The growth condition which we imposed on the sequence $\{n_k\}_{k\in\mathbf{N}}$ now yields, for each positive ε, the existence of a positive number $c_0=c_0(\varepsilon)$ such that, for all k greater than c_0,

$$\frac{n_0+\sigma(2\delta-k_{\max})\sum_{j=0}^{k-2} n_j+(k-1)(\log k_1)(\log \varrho)^{-1}}{n_{k-1}}<\varepsilon.$$

For k greater than c_0, it hence follows that

$$\nu_\sigma(B(\xi,r))\leq k_3^\theta k_1^{-\theta}\mu(B(\xi,r))^\theta \varrho^{n_{k-1}(\delta-\delta\theta-\theta\sigma(2\delta-k_{\max})-\varepsilon)}.$$

If we specify θ by

$$\theta:=\frac{\delta-\varepsilon}{\delta+\sigma(2\delta-k_{\max})},$$

then the proposition follows. □

Now, if we define

$$\mathcal{I}_\sigma^+:=\mathcal{I}_\sigma\cap\mathcal{J}_\sigma^+(G),$$

Proposition 2 immediately implies the following result.

Corollary 1.

$$\mathcal{I}_\sigma^+\subset\left\{\xi\in L(G):\liminf_{r\to 0}\frac{\log\nu_\sigma(B(\xi,r))}{\log\mu(B(\xi,r))}\geq\frac{\delta}{\delta+\sigma(2\delta-k_{\max})}\right\}.$$

The following Proposition is required in the proof of Theorem A.

Proposition 3. *Where $\mathcal{I}_\sigma^+$ and ν_σ are as above,*

$$\nu_\sigma(\mathcal{I}_\sigma^+)=1.$$

Proof. Let ε, τ be positive such that $\varepsilon<\tau\delta(2\delta-k_{\max})/(\delta+(\sigma+\tau)(2\delta-k_{\max}))$. We first make the following observation. If g denotes an element in $\mathcal{T}_p$ such that $r_g\leq r_0(\varepsilon)$ and such that

$$\mathcal{H}_{g(p)}(r_g^{1+\sigma+\tau})\cap\mathcal{I}_\sigma\neq\emptyset,$$

then there exists $\xi\in\mathcal{H}_{g(p)}(r_g^{1+\sigma+\tau})\cap\mathcal{I}_\sigma$ such that (using Proposition 2)

$$\begin{aligned}\nu_\sigma(\mathcal{H}_{g(p)}(r_g^{1+\sigma+\tau})\cap\mathcal{I}_\sigma)&\leq\nu_\sigma(B(\xi,2r_g^{1+\sigma+\tau}))\\&\ll\mu(B(\xi,2r_g^{1+\sigma+\tau}))^{(\delta-\varepsilon)/(\delta+\sigma(2\delta-k_{\max}))}\\&\leq\mu(\mathcal{H}_{g(p)}(4r_g^{1+\sigma+\tau}))^{(\delta-\varepsilon)/(\delta+\sigma(2\delta-k_{\max}))}\\&\ll\mu(\mathcal{H}_{g(p)}(r_g^{1+\sigma+\tau}))^{(\delta-\varepsilon)/(\delta+\sigma(2\delta-k_{\max}))}.\end{aligned}$$

Now, the latter observation, the global measure formula for μ and the fact that δ is the exponent of convergence of G imply, by the above choice of ε and τ,

$$\sum_{\substack{g\in\mathcal{T}\\ r_g\le r_0(\varepsilon)}} \nu_\sigma(\mathcal{H}_{g(p)}(r_g^{1+\sigma+\tau})) \ll \sum_{g\in\mathcal{T}} \mu(\mathcal{H}_{g(p)}(r_g^{1+\sigma+\tau}))^{(\delta-\varepsilon)/(\delta+\sigma(2\delta-k_{\max}))}$$
$$\ll \sum_{g\in\mathcal{T}} r_g^{\delta-\varepsilon+\tau(\delta-\varepsilon)(2\delta-k_{\max})/(\delta+\sigma(2\delta-k_{\max}))} < \infty.$$

If, for $g\in\mathcal{T}_p$, we define

$$E(g):=\mathcal{H}_{g(p)}(r_g^{1+\sigma+\tau}),$$

and interpret $E(g)$ as an event in the probability space $(\mathcal{I}_\sigma,\nu_\sigma)$, then the previous calculation implies

$$\sum_{g\in\mathcal{T}} \nu_\sigma(E(g))<\infty.$$

Hence, we can apply the first part of the *Borel–Cantelli lemma*, which then yields that

$$\nu_\sigma\Big(\limsup_{g\in\mathcal{T}} E(g)\Big)=0.$$

Since we have that

$$\limsup_{g\in\mathcal{T}} E(g)=\mathcal{W}_{\sigma+\tau},$$

the proposition follows. □

IV. Fractal dimensions of $\mathcal{J}_\sigma(G)$ and $\mathcal{J}_\sigma^+(G)$

Proof of Theorem A. Using Proposition 3, we derive, for positive τ,

$$\inf_{\{U_i\}\in\mathcal{U}_\tau^{\nu_\sigma}(\mathcal{I}_\sigma^+)} \sum_i \nu_\sigma(U_i) \ge \inf_{\{U_i\}\in\mathcal{U}_\tau^{\nu_\sigma}(\mathcal{I}_\sigma^+)} \nu_\sigma\Big(\bigcup_i U_i\Big)=1.$$

It hence follows that

$$\dim_{\nu_\sigma}(\mathcal{I}_\sigma^+)\ge 1.$$

From Corollary 1 and from Billingsley's lemma (Lemma 4) we now have that

$$\dim_\mu \mathcal{I}_\sigma^+ \ge \frac{\delta}{\delta+\sigma(2\delta-k_{\max})}\dim_{\nu_\sigma}\mathcal{I}_\sigma^+ \ge \frac{\delta}{\delta+\sigma(2\delta-k_{\max})}.$$

It hence follows that

$$\dim_\mu \mathcal{J}_\sigma(G)\ge \dim_\mu \mathcal{J}_\sigma^+(G)\ge \frac{\delta}{\delta+\sigma(2\delta-k_{\max})}.$$

In order to derive the upper bound, we consider $\phi\in\mathcal{F}$ and define

$$\mathcal{W}_\sigma^\phi := \bigcap_{n\in\mathbf{N}} \bigcup_{p\in P} \bigcup_{\substack{g\in\mathcal{T}_p\\ r_g\le 1/n}} \mathcal{H}_{g(p)}(\phi(r_g)r_g^{1+\sigma}).$$

For $\tau>0$ and $\varepsilon>0$, the global measure formula yields that

$$\begin{aligned}\inf_{\{U_i\}\in\mathcal{U}_\tau^\mu(\mathcal{W}_\sigma^\phi)} \sum_i \mu(U_i)^{\delta/(\delta+\sigma(2\delta-k_{\max}))+\varepsilon} &\le \sum_{p\in P}\sum_{g\in\mathcal{T}_p} \mu(\mathcal{H}_{g(p)}(\phi(r_g)r_g^{1+\sigma}))^{\delta/(\delta+\sigma(2\delta-k_{\max}))+\varepsilon} \\ &\ll \sum_{p\in P}\sum_{g\in\mathcal{T}_p} (r_g^{\delta+\sigma(2\delta-k(p))}\phi(r_g)^{2\delta-k_{\max}})^{\delta/(\delta+\sigma(2\delta-k_{\max}))+\varepsilon} < \infty.\end{aligned}$$

This implies that

$$\dim_\mu \mathcal{W}_\sigma^\phi \le \frac{\delta}{\delta+\sigma(2\delta-k_{\max})}.$$

Since $\mathcal{J}_\sigma(G)=\bigcup_{\phi\in\mathcal{F}}\mathcal{W}_\sigma^\phi$ and since $\mathcal{J}_\sigma^+(G)\subset\mathcal{J}_\sigma(G)$, it follows that

$$\dim_\mu \mathcal{J}_\sigma^+(G) \le \dim_\mu \mathcal{J}_\sigma(G) \le \frac{\delta}{\delta+\sigma(2\delta-k_{\max})}. \qquad \square$$

For the proof of Theorem B and Proposition C we require the following fact, which is immediately implied by the global measure formula and the definition of $\mathcal{J}_\sigma^+(G)$.

Lemma 7.

(i) *If* $\delta\le k_{\max}$, *then*

$$\mathcal{J}_\sigma^+(G) \subset \left\{\xi\in L(G) : \liminf_{r\to 0} \frac{\log\mu(B(\xi,r))}{\log r} = \frac{\delta+\sigma(2\delta-k_{\max})}{1+\sigma}\right\}.$$

(ii) *If* $\delta>k_{\max}$, *then*

$$\mathcal{J}_\sigma^+(G) \subset \left\{\xi\in L(G) : \liminf_{r\to 0} \frac{\log\mu(B(\xi,r))}{\log r} = \delta\right\}.$$

Proof of Theorem B. The upper bound $\delta/(1+\sigma)$ for the Hausdorff dimension of $\mathcal{J}_\sigma(G)$ is trivial. The calculation is analogous to the derivation of the upper bound in the proof of Theorem A.

In order to derive the lower bound for the Hausdorff dimension, we use Billingsley's lemma (Lemma 4), Lemma 7(i) and Theorem A, which imply that

$$\dim_H(\mathcal{J}_\sigma^+(G)) \geq \frac{\delta+\sigma(2\delta-k_{\max})}{1+\sigma}\dim_\mu(\mathcal{J}_\sigma^+(G)) \geq \frac{\delta}{1+\sigma}.$$

Since $\mathcal{J}_\sigma^+(G) \subset \mathcal{J}_\sigma(G)$, Theorem B follows. □

Proof of Proposition C. As in the proof of Theorem B, the upper bound $\delta/(1+\sigma)$ for the Hausdorff dimension of $\mathcal{J}_\sigma(G)$ is trivial.

In order to derive the lower bound $\delta/(1+\sigma(2\delta-k_{\max})/\delta)$ for the Hausdorff dimension, we use Billingsley's lemma, Lemma 7(ii) and Theorem A, which imply that

$$\dim_H(\mathcal{J}_\sigma(G)) \geq \dim_H(\mathcal{J}_\sigma^+(G)) \geq \delta \dim_\mu(\mathcal{J}_\sigma^+(G)) \geq \frac{\delta^2}{\delta+\sigma(2\delta-k_{\max})}.$$

For the lower bound $\delta/(1+\sigma)$ of the packing dimension, we observe that by Proposition 2 and by the the global measure formula (Lemma 3) we have, for arbitrary positive ε and for each $\xi \in \mathcal{I}_\sigma^+$,

$$\liminf_{r\to 0} \frac{\nu_\sigma(B(\xi,r))}{r^{\delta/(1+\sigma)-\varepsilon}} \ll 1.$$

Hence, using Lemma 5(iv), it follows that

$$\dim_p(\mathcal{J}_\sigma(G)) \geq \dim_p(\mathcal{I}_\sigma^+) \geq \frac{\delta}{1+\sigma}. \quad \square$$

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