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Analytic discs attached to a generating CR-manifold

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Abstract. Perturbations by analytic discs along generating CR-submanifolds of $\mathbf{C}^n$ are considered. In the case all partial indices of a closed path p in a generating CR-fibration $\{M(\xi)\}_{\xi\in\partial D}$ are greater or equal to -1 we can completely parametrize all small holomorphic perturbations of the path p along the fibration $\{M(\xi)\}_{\xi\in\partial D}$. In this case we also study the geometry of perturbations by analytic discs and their relation to the conormal bundle of the fibration.

1. Introduction

Given an analytic disc in $\mathbf{C}^n$ with boundary in a generating CR-submanifold $M\subseteq\mathbf{C}^n$, one would like to describe the family of all nearby analytic discs in $\mathbf{C}^n$ attached to M. This problem is the object of a considerable research in the recent years. The following list of authors and their papers related to the problem is not at all meant to be complete: Alexander [A], Baouendi, Rothschild and Trepreau [BRT], Bedford [B1], [B2], Bedford and Gaveau [BG], Eliashberg [E], Forstnerič [Fo1], [Fo2], Globevnik [Gl1], [Gl2], Gromov [Gr], Y.-G. Oh [O1], [O2], Tumanov [T].

The technique of perturbing an analytic disc with boundary in a given manifold has found several applications in the problems of the analysis of several complex variables. Two, probably the most known problems, where this technique can be used, are the problem of describing the polynomial hull of a given set in $\mathbf{C}^n$ and the problem of extending CR-functions from a given CR-submanifold of $\mathbf{C}^n$ into some open subset of $\mathbf{C}^n$. Recently has J. Globevnik in his paper [Gl1], which was inspired by the work [Fo1] by F. Forstnerič, found very elegant sufficient conditions on a given analytic disc p with boundary in a maximal real submanifold M of $\mathbf{C}^n$ which imply finite dimensional parametrization of all nearby holomorphic discs attached to M.

To each, not necessary holomorphic, disc p with boundary in a maximal real submanifold $M\subseteq\mathbf{C}^n$ one associates n integers $k_1,\dots,k_n$ called the partial indices of the disc p. Their sum $k:=k_1+...+k_n$ is called the total index of the disc p. A part of Globevnik's work [Gl1] is the theorem in which he proves that if the pull-back bundle $p^*(TM)$ of the tangent bundle TM is trivial and if all partial indices of the disc p are greater or equal to 0, then there exists an $n+k$ dimensional parametrization of all nearby discs of the form

$$p+\text{analytic disc}$$

with boundary in M. Later we proved, [Če], that Globevnik's theorem extends in the same form to the case where the pull-back bundle $p^*(TM)$ is non-trivial. The final version of the result was given by Y.-G. Oh in [O1], where Globevnik's result is generalized, but using a different approach, to the case where all partial indices are greater or equal to -1 and arbitrary pull-back bundle $p^*(TM)$.

The present work is organized as follows. Section 2 introduces notation and terminology which we use throughout the paper. In Section 3 we give a short overview of some results by Vekua [V1], [V2] and Globevnik [Gl1] related to our problem and introduce the partial indices of a maximal real bundle over the unit circle $\partial D\subseteq\mathbf{C}$. In Section 4 we first show that a generating CR-bundle L over ∂D of class C^1 can be split into the direct sum

$$L:=L^{\mathbf{C}}\oplus\mathcal{L},$$

where $L^{\mathbf{C}}$ denotes the maximal complex subbundle of L and $\mathcal{L}$ is some maximal real bundle over ∂D. The main result of this section is Theorem 1 in which we generalize Globevnik's result to the case of generating CR-fibrations over the unit circle $\partial D\subseteq\mathbf{C}$. Also some related remarks and examples are given. The last section was inspired by the work [BRT] by Baouendi, Rothschild and Trepreau on the geometry of small analytic discs attached to a CR-submanifold of $\mathbf{C}^n$, where some of their notions, such as defect, and results related to the evaluation maps are generalized to large analytic discs with boundaries in a generating CR-submanifold of $\mathbf{C}^n$, Theorem 2 and Theorem 3. See also the paper [T] by Tumanov. The section ends with some examples.

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2. Notation and terminology

Let $D=\{z\in\mathbf{C};|z|<1\}$ and let ∂D denote the unit circle in $\mathbf{C}$, the boundary of D. If K is either $\overline{D}$ or ∂D, and $0<\alpha<1$, we denote by $C^{0,\alpha}(K)$ the Banach algebra of Hölder continuous complex-valued functions on K with finite Lipschitz norm of exponent α

$$\|f\|_\alpha=\sup_{x\in K}|f(x)|+\sup_{\substack{x,y\in K\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\alpha}<\infty.$$

For every $m\in\mathbf{N}\cup\{0\}$ we also define the algebra

$$C^{m,\alpha}(K)=\left\{f\in C^m;\|f\|_{m,\alpha}=\sum_{|j|\leq m}\|D^jf\|_\alpha<\infty\right\}.$$

The subalgebra of the real-valued functions from $C^{m,\alpha}(K)$ will be denoted by $C^{m,\alpha}_{\mathbf{R}}(K)$.

Let $A(D)$ denote the disk algebra and let $A(\partial D)=\{f|_{\partial D};f\in A(D)\}$. We define

$$A^{m,\alpha}(D)=C^{m,\alpha}(\overline{D})\cap A(D)$$

and

$$A^{m,\alpha}(\partial D)=C^{m,\alpha}(\partial D)\cap A(\partial D).$$

Note that if $f\in A(D)$, then $f\in A^{m,\alpha}(D)$ if and only if $f|_{\partial D}\in A^{m,\alpha}(\partial D)$, [Go].

We will also need some other not so standard function spaces. Let $r(\xi)$, $\xi\in\partial D\backslash\{1\}$, denote the principal branch of the square root, i.e., the complex plane is cut along the positive real line and $r(-1)=i$. Let $\mathcal{E}^{m,\alpha}_{\mathbf{R}}$ be the space consisting of the real continuous functions on $\partial D\backslash\{1\}$ with the property that a continuous function g on $\partial D\backslash\{1\}$ is in $\mathcal{E}^{m,\alpha}_{\mathbf{R}}$ if and only if there exists an odd function g_0 in $C^{m,\alpha}_{\mathbf{R}}(\partial D)$, i.e., $g_0(-\xi)=-g_0(\xi)$, $\xi\in\partial D$, such that

$$g(\xi)=g_0(r(\xi))\quad(\xi\in\partial D\backslash\{1\}).$$

In other words, this is the space of continuous functions g on $\partial D\backslash\{1\}$ such that

(a) there exist the limits

$$\lim_{\theta\to0^+}g(e^{i\theta})\quad\text{and}\quad\lim_{\theta\to2\pi^-}g(e^{i\theta})\tag{1}$$

which we denote by $g(1^+)$ and $g(1^-)$, respectively, and are related by the equation

$$g(1^+)+g(1^-)=0,$$

(b) the function

$$(Hg)(\xi):=\begin{cases} g(\xi^2); & \operatorname{Im}\xi>0, \\ -g(\xi^2); & \operatorname{Im}\xi<0 \end{cases} \tag{2}$$

is in $C^{m,\alpha}_{\mathbf{R}}(\partial D)$.
Obviously $\mathcal{E}^{m,\alpha}_{\mathbf{R}}$ is an $\mathbf{R}$-linear space and for the norm on it we take

$$\|g\|_{m,\alpha}:=\|Hg\|_{m,\alpha} \quad (g\in\mathcal{E}^{m,\alpha}_{\mathbf{R}}).$$

So $\mathcal{E}^{m,\alpha}_{\mathbf{R}}$ is a Banach space that is via H isometrically isomorphic to the closed subspace of odd functions in $C^{m,\alpha}_{\mathbf{R}}(\partial D)$.

Remark. Another equivalent description of the space $\mathcal{E}^{m,\alpha}_{\mathbf{R}}$ can be given in terms of Fourier series. Namely, each element $g\in\mathcal{E}^{m,\alpha}_{\mathbf{R}}$ has a unique expansion of the form

$$\sum_{k=-\infty}^{\infty} c_k e^{i(2k+1)\theta/2},$$

where the sum

$$\sum_{k=-\infty}^{\infty} c_k e^{i(2k+1)\theta}$$

represents the Fourier series of some odd function $g_0\in C^{m,\alpha}_{\mathbf{R}}(\partial D)$. We will refer to c_k, $k\in\mathbf{Z}$, as the Fourier coefficients of the function g.

One can define a Hilbert transform T on $\mathcal{E}^{m,\alpha}_{\mathbf{R}}$. Let T_0 be the standard harmonic conjugate function operator on $C^{m,\alpha}_{\mathbf{R}}(\partial D)$. Then

$$T:\mathcal{E}^{m,\alpha}_{\mathbf{R}}\to\mathcal{E}^{m,\alpha}_{\mathbf{R}}$$

is defined by

$$Tg=H^{-1}T_0Hg \quad (g\in\mathcal{E}^{m,\alpha}_{\mathbf{R}}).$$

Note that T_0 takes the subspace of odd functions in $C^{m,\alpha}_{\mathbf{R}}(\partial D)$ into itself. Thus for every $g\in\mathcal{E}^{m,\alpha}_{\mathbf{R}}$ the function

$$H(g+iTg):=Hg+iHTg=Hg+iT_0(Hg)$$

is an odd function on ∂D from the space $A^{m,\alpha}(\partial D)$. We denote the space of functions of the form

$$g+iTg \quad (g\in\mathcal{E}^{m,\alpha}_{\mathbf{R}})$$

by $\mathcal{A}^{m,\alpha}$. Observe that all functions from the space $\mathcal{A}^{m,\alpha}$ are of the form $r(\xi)f(\xi)$ for some $f\in A^{m,\alpha}(\partial D)$. Observe also that since for any two functions g and h from $\mathcal{E}_{\mathbf{R}}^{m,\alpha}$ the following identity holds

$$(Hg)(\xi)(Hh)(\xi)=g(\xi^2)h(\xi^2) \quad (\xi\in\partial D),$$

the product of two functions from $\mathcal{E}_{\mathbf{R}}^{m,\alpha}$ gives a function in $C_{\mathbf{R}}^{m,\alpha}(\partial D)$, and the product of two functions of the form $g+iHg$, $g\in\mathcal{E}_{\mathbf{R}}^{m,\alpha}$, gives a function in $A^{m,\alpha}(\partial D)$.

The spaces we will most often consider are the finite products of the spaces $C_{\mathbf{R}}^{0,\alpha}(\partial D)$ and $\mathcal{E}_{\mathbf{R}}^{0,\alpha}$. A product with n factors will be denoted by $\mathcal{E}_\sigma$, where σ is an n-vector with 0's and 1's as its entries. The entry 0 in the jth place represents the space $C_{\mathbf{R}}^{0,\alpha}(\partial D)$ as the jth factor and the entry 1 in the jth place means that the jth factor is the space $\mathcal{E}_{\mathbf{R}}^{0,\alpha}$. By analogy we also define the spaces $\mathcal{A}_\sigma$ which are the products of finitely many copies of $A^{0,\alpha}(\partial D)$ and $\mathcal{A}^{0,\alpha}$.

We extend the definition of the Hilbert transform in a natural way (componentwise) to the space $\mathcal{E}_\sigma$. We denote the extension by T_σ. It is a bounded linear map from $\mathcal{E}_\sigma$ into itself and it has the property that the vector function $v+iT_\sigma v$ belongs to the space $\mathcal{A}_\sigma$ for every $v\in\mathcal{E}_\sigma$. We also define the map

$$H_\sigma\colon \mathcal{E}_\sigma\to(C_{\mathbf{R}}^{0,\alpha}(\partial D))^n$$

which is defined as the identity map on each factor $C_{\mathbf{R}}^{0,\alpha}(\partial D)$, and is defined as the map (2) on each factor $\mathcal{E}_{\mathbf{R}}^{0,\alpha}$.

3. Maximal real bundles over the circle

Let L be a maximal real subspace of $\mathbf{C}^n$, i.e., its real dimension is n and $L\oplus iL=\mathbf{C}^n$. To any such maximal real subspace L one can associate an $\mathbf{R}$-linear map R_L on $\mathbf{C}^n$, called the reflection about L, given by

$$z=x+i\tilde{x}\mapsto x-i\tilde{x} \quad (x,\tilde{x}\in L),$$

where $z=x+i\tilde{x}$ is the unique decomposition of z into the sum of vectors from L and iL. The mapping

$$R_L\colon \mathbf{C}^n\to\mathbf{C}^n$$

is an $\mathbf{R}$-linear automorphism of $\mathbf{C}^n$ which is also $\mathbf{C}$-antilinear, i.e., $R_L(iv)=-iR_L(v)$ for every $v\in\mathbf{C}^n$. The reflection about the maximal real subspace $\mathbf{R}^n\subseteq\mathbf{C}^n$ will be denoted by R_0. Note that in the standard notation R_0 is just the ordinary conjugation on $\mathbf{C}^n$ and that for any $n\times n$ complex matrix A the following identity holds

$$\bar{A}=R_0AR_0.$$

Lemma 1. *Let L be a maximal real subspace of $\mathbf{C}^n$ and let $x_1, \dots, x_n$ be any set of vectors spanning L. Let $A := [x_1, \dots, x_n]$ be the matrix whose columns are the given vectors x_j, $j=1, \dots, n$, and let $B := A\overline{A^{-1}}$. Then*

$$B = R_L R_0.$$

Moreover, the matrix B does not depend on the basis of L, i.e., B remains the same even if a different basis for L is selected, and

$$\overline{B} = B^{-1}, \quad |\det B| = 1.$$

Remark. In the above lemma $n \times n$ matrices A and B are identified with $\mathbf{C}$-linear automorphisms of $\mathbf{C}^n$ in the standard basis.

Proof. Observe that A is a $\mathbf{C}$-linear automorphism of $\mathbf{C}^n$ which maps $\mathbf{R}^n$ onto L. Consider the following composition of automorphisms of $\mathbf{C}^n$

$$S := R_0 A^{-1} R_L A.$$

Then S is a $\mathbf{C}$-linear automorphism of $\mathbf{C}^n$ which equals the identity on $\mathbf{R}^n$. Since $\mathbf{R}^n$ is a maximal real subspace of $\mathbf{C}^n$, S is the identity on $\mathbf{C}^n$, and hence

$$R_0 A^{-1} = A^{-1} R_L.$$

Finally, since

$$B = A\overline{A^{-1}} = A R_0 A^{-1} R_0,$$

we get

$$B = A A^{-1} R_L R_0 = R_L R_0.$$

The rest is obvious. $\square$

The following definition is taken from [Gl1], see also [Fo1].

Definition 1. Let $L = \{L_\xi ; \xi \in \partial D\}$ be a real rank n subbundle of the product bundle $\partial D \times \mathbf{C}^n$ of class $C^{0,\alpha}$. If for each $\xi \in \partial D$ the fiber L_ξ is a maximal real subspace of $\mathbf{C}^n$, the bundle L is called maximal real.

Example. A very important example of a maximal real bundle over ∂D one gets in the following case. Let M be a C^2 maximal real submanifold of $\mathbf{C}^n$ and let $p: \partial D \to M$ be a C^2 closed curve in M. Then the pull-back bundle $p^*(TM)$, where TM is the tangent bundle of the submanifold M, is a maximal real bundle over ∂D of rank n.

It is known, see [V1], that for every closed path B in $GL(n,\mathbf{C})$ of class $C^{0,\alpha}$ one can find holomorphic matrix functions

$$F^+\colon \overline{D}\to GL(n,\mathbf{C}),\quad F^-\colon \overline{\mathbf{C}}\backslash D\to GL(n,\mathbf{C})$$

of class $C^{0,\alpha}$ and n integers $k_1\geq k_2\geq\ldots\geq k_n$ such that

$$B=F^+(\xi)\Lambda(\xi)F^-(\xi)\quad (\xi\in\partial D),$$

where

$$\Lambda(\xi):=\begin{pmatrix} \xi^{k_1} & 0 & \ldots & \ldots & 0\\ 0 & \xi^{k_2} & 0 & \ldots & 0\\ \vdots & 0 & \xi^{k_3} & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \ldots & \ldots & 0 & \xi^{k_n}\end{pmatrix}.$$

The matrix Λ will be called the *characteristic matrix* of the path B. One can prove that under the condition $k_1\geq\ldots\geq k_n$, the characteristic matrix Λ does not depend on the factorization of the matrix function B of the above form, see [V1], [Gl1], [CG] for more details. The integers $k_1,\ldots,k_n$ are called the *partial indices* of the path B, and their sum

$$k:=k_1+\ldots+k_n$$

is called the *total index* of the matrix function B.

Definition 2. Let L be a maximal real bundle over the unit circle ∂D. The partial indices of the $GL(n,\mathbf{C})$ closed path

$$B_L\colon\xi\mapsto R_{L_\xi}R_0 \tag{3}$$

of class $C^{0,\alpha}$, are called the partial indices of the bundle L and their sum is called the total index of L.

Remark. The total index of a closed path p on a maximal real submanifold $M\subseteq\mathbf{C}^n$ is also called the Maslov index of p.

Although Globevnik in [Gl1] works only with the trivial bundles over the circle ∂D, Lemma 5.1 in [Gl1] still applies and one can conclude:

Lemma 2. *The $C^{0,\alpha}$ closed path in $GL(n,\mathbf{C})$*

$$B_L\colon\xi\mapsto R_{L_\xi}R_0\quad(\xi\in\partial D)$$

can be decomposed in the form

$$B_L(\xi)=\Theta(\xi)\Lambda(\xi)\overline{\Theta(\xi)^{-1}} \quad (\xi\in\partial D),$$

where the map $\Theta:\overline{D}\to GL(n,\mathbf{C})$ *is of class* $C^{0,\alpha}$ *and holomorphic on* D, *i.e., the* $n\times n$ *matrix* Θ *is in* $A^{0,\alpha}(D)^{n\times n}$.

The characteristic matrix Λ can be decomposed further as

$$\Lambda=\Lambda_0^2=\Lambda_0\overline{\Lambda_0^{-1}},$$

where

$$\Lambda_0(\xi):=\begin{pmatrix} \xi^{k_1/2} & 0 & \dots & \dots & 0 \\ 0 & \xi^{k_2/2} & 0 & \dots & 0 \\ \vdots & 0 & \xi^{k_3/2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \xi^{k_n/2} \end{pmatrix}.$$

Here $\xi^{k/2}$ stands for ξ^m if $k=2m$ and for $\xi^m r(\xi)$ if $k=2m+1$. We will refer to Λ_0 as the *square root* of the characteristic matrix Λ and we say that the matrix function

$$\xi\mapsto\Theta(\xi)\Lambda_0(\xi) \quad (\xi\in\partial D)$$

represents the *normal form* of the bundle L. To the $C^{0,\alpha}$ closed path B_L in $GL(n,\mathbf{C})$ we also associate the corresponding Banach space $\mathcal{E}_\sigma$ we will work with, see the previous section for the definition. The n-vector σ is defined as

$$\sigma:=(k_1 \bmod 2,\dots,k_n \bmod 2).$$

Corollary 1. *If all partial indices of a maximal real bundle* L *are nonnegative, then there exists an* $n\times n$ *matrix function* $A(\xi)$, $\xi\in\partial D$, *with the rows from the space* $\mathcal{A}_\sigma$ *and such that its columns* $X_1(\xi),\dots,X_n(\xi)$ *span the fiber* L_ξ *for every* $\xi\in\partial D$.

Remark. For $\xi=1$ the above statement still makes sense in terms of the limits (1) when $\xi\neq1$ approaches 1.

4. CR-vector bundles and CR-fibrations

We begin with a definition.

Definition 3. Let $L=\{L_\xi\subseteq\mathbf{C}^{m+n};\xi\in\partial D\}$ be a real vector bundle over ∂D of class $C^{0,\alpha}$. If for each $\xi\in\partial D$ the fiber L_ξ is a real vector subspace of CR-dimension m, the bundle L is called a CR-bundle of CR-dimension m over the unit circle ∂D. If, in addition, for each $\xi\in\partial D$ the fiber L_ξ is a generating subspace of $\mathbf{C}^{m+n}$, i.e., $L_\xi+iL_\xi=\mathbf{C}^{n+m}$, $\xi\in\partial D$, then the bundle L is called a generating CR-bundle over ∂D.

Remarks. 1. Every maximal real bundle over ∂D is a generating CR-bundle with CR-dimension 0.

2. To each CR-bundle L over the unit circle one can associate a complex m-dimensional vector subbundle $L^{\mathbf{C}}\subseteq L$ which is just the bundle of the maximal complex subspaces of the bundle L, i.e., for each $\xi\in\partial D$ the fiber $L_\xi^{\mathbf{C}}$ equals to $L_\xi\cap iL_\xi$.

Lemma 3. *Let V be a C^1 complex vector bundle over ∂D such that for each $\xi\in\partial D$ the fiber V_ξ is an m-dimensional complex subspace of $\mathbf{C}^{m+n}$. Then there exists a linear change of coordinates in $\mathbf{C}^{m+n}$ such that in the new coordinates each fiber V_ξ projects isomorphically onto $\mathbf{C}^m\times\{0\}\subseteq\mathbf{C}^{m+n}$. Moreover, the set of invertible $(m+n)\times(m+n)$ matrices satisfying this property is open and dense in $GL(m+n,\mathbf{C})$.*

Proof. We denote by $\mathcal{G}$ the set of invertible $(m+n)\times(m+n)$ complex matrices having the above property. Clearly $\mathcal{G}$ is open in $GL(m+n,\mathbf{C})$. So, to prove the lemma, we have to show that the complement of $\mathcal{G}$ in $GL(m+n,\mathbf{C})$ has no interior.

Fix $\xi_0\in\partial D$. Let A_{ξ_0} be any $(m+n)\times m$ matrix such that its columns form a basis of the fiber V_{ξ_0}. We define the mapping

$$\Phi_{\xi_0}: GL(m+n,\mathbf{C})\to\mathbf{C}$$

by

$$\Phi_{\xi_0}(U)=\det([I_m,0]UA_{\xi_0})$$

where $[I_m,0]$ is an $m\times(m+n)$ matrix which has the identity matrix in its first m columns and the 0 matrix in its last n columns. The mapping Φ_{ξ_0} depends on the matrix A_{ξ_0}, i.e., on the basis of the fiber V_{ξ_0}, but the set

$$\mathcal{U}_{\xi_0}:=\Phi_{\xi_0}^{-1}(0)$$

does not. The equation

$$\Phi_{\xi_0}(U)=0$$

is algebraic and so $\mathcal{U}_{\xi_0}$ is an algebraic subset of $GL(m+n,\mathbf{C})$. Hence, locally the set $\mathcal{U}_{\xi_0}$ has finite $2((m+n)^2-1)$-dimensional Hausdorff measure.

Let

$$\mathcal{U} := \bigcup_{\xi\in\partial D} \{\xi\}\times\mathcal{U}_\xi \subseteq \partial D\times GL(m+n,\mathbf{C}).$$

Then, because of the smoothness assumptions on the bundle V, the $2(m+n)^2-1$ dimensional Hausdorff measure of the set $\mathcal{U}$ is locally finite and so for every compact set $K\subseteq GL(m+n,\mathbf{C})$ we have

$$\mathcal{H}_{2(m+n)^2-1}(\pi(\mathcal{U})\cap K) < \infty$$

where π is the projection

$$\pi: \partial D\times GL(m+n,\mathbf{C}) \to GL(m+n,\mathbf{C}).$$

Since $\pi(\mathcal{U})$ is exactly the complement of the set $\mathcal{G}$, the lemma is proved. $\square$

Let $\Sigma\subseteq\mathbf{C}^{m+n}$ be a generating CR-subspace of CR-dimension m such that its maximal complex subspace $\Sigma^{\mathbf{C}}$ projects isomorphically onto $\mathbf{C}^m\times\{0\}$.

Lemma 4. *The subspace*

$$S := \Sigma\cap(\{0\}\times\mathbf{C}^n)$$

is a maximal real subspace of $\{0\}\times\mathbf{C}^n$ *and the only subspace of* $\{0\}\times\mathbf{C}^n$ *for which*

$$\Sigma = \Sigma^{\mathbf{C}}\oplus S.$$

Proof. We denote by $\pi: \mathbf{C}^{m+n}\to\mathbf{C}^m\times\{0\}$ the orthogonal projection onto $\mathbf{C}^m\times\{0\}$. Since π projects $\Sigma^{\mathbf{C}}$ isomorphically onto $\mathbf{C}^m\times\{0\}$, we conclude that for every $x\in\Sigma$ there exists exactly one vector $v\in\Sigma^{\mathbf{C}}$ such that $\pi(x)=\pi(v)$. Hence the vector $x-v\in\Sigma$ is in the kernel of the projection π, i.e., $x-v$ is in $\{0\}\times\mathbf{C}^n$. Therefore $x-v$ is in S. The assumption on the projection π also implies

$$\Sigma^{\mathbf{C}}\cap(\{0\}\times\mathbf{C}^n) = \{0\}, \tag{4}$$

and so S is a totally real subspace of $\{0\}\times\mathbf{C}^n$ for which

$$\Sigma = \Sigma^{\mathbf{C}}\oplus S.$$

Finally, the subspace Σ is a generating CR-subspace of $\mathbf{C}^{m+n}$ and thus S is a maximal real subspace of $\{0\}\times\mathbf{C}^n$. The uniqueness follows from (4). $\square$

Let $L\subseteq\partial D\times\mathbf{C}^{m+n}$ be a generating CR-bundle of the class $C^{0,\alpha}$ over the unit circle and of CR-dimension m. We assume that each fiber $L_\xi^{\mathbf{C}}$, $\xi\in\partial D$, projects isomorphically onto $\mathbf{C}^m\times\{0\}\subseteq\mathbf{C}^{m+n}$. By Lemma 3 this assumption can always be realized in the case where the bundle L is of class C^1. The above Lemma 4 implies that there is a unique maximal real bundle $\mathcal{L}\subseteq\partial D\times\mathbf{C}^n$ such that for each $\xi\in\partial D$ we have

$$L_\xi = L_\xi^{\mathbf{C}}\oplus\mathcal{L}_\xi.$$

Definition 4. Let $L\subseteq\partial D\times\mathbf{C}^{m+n}$ be a generating CR-bundle over ∂D of CR-dimension m whose fibers project isomorphically onto $\mathbf{C}^m\times\{0\}$. We define the partial indices and the total index of the bundle L as the partial indices and the total index of the maximal real bundle $\mathcal{L}\subseteq\partial D\times\mathbf{C}^n$.

We fix $\xi_0\in\partial D$. Let $N(\xi_0)$ be any $(m+n)\times n$ matrix whose columns span the real orthogonal complement $L_{\xi_0}^{\perp}$. Then the equations of the fibers L_{ξ_0} and $L_{\xi_0}^{\mathbf{C}}$ are

$$\operatorname{Re}\left(N^*(\xi_0)\begin{bmatrix} z \\ w \end{bmatrix}\right)=0 \quad\text{and}\quad N^*(\xi_0)\begin{bmatrix} z \\ w \end{bmatrix}=0,$$

respectively. Here $z\in\mathbf{C}^m$ and $w\in\mathbf{C}^n$. Since we are assuming that each fiber of the bundle $L^{\mathbf{C}}$ projects isomorphically onto $\mathbf{C}^m\times\{0\}\subseteq\mathbf{C}^{m+n}$, the matrix $N(\xi_0)$ can be written in the block form

$$N(\xi_0)=\begin{bmatrix} G_0(\xi_0) \\ N_0(\xi_0) \end{bmatrix},$$

where $N_0(\xi_0)$ is an invertible $n\times n$ complex matrix. The definition of the bundle $\mathcal{L}$ immediately implies that $\mathcal{L}_{\xi_0}$ is given by the equations

$$\operatorname{Re}(N_0^*(\xi_0)w)=0.$$

Therefore the columns of the matrix $N_0(\xi_0)$ span the real orthogonal space $\mathcal{L}_{\xi_0}^{\perp}\subseteq\mathbf{C}^n$.

Let $k_1,k_2,\dots,k_n$ be the partial indices of the bundle $\mathcal{L}$ and let $\Lambda(\xi)$ be its characteristic matrix. Then there exists an $n\times n$ invertible holomorphic matrix function Θ_0 on $\bar{D}$ such that the columns of the matrix function

$$A_0(\xi):=\Theta_0(\xi)\Lambda_0(\xi) \quad (\xi\in\partial D)$$

span the fibers of the maximal real bundle $\mathcal{L}$. Here Λ_0 denotes the square root of the characteristic matrix Λ. Once A_0 is fixed, there is a naturally given basis of the bundle $L^{\perp}$, namely, there is an $(m+n)\times n$ matrix function $N(\xi)$, $\xi\in\partial D$, whose rows are from the space $\mathcal{E}_\sigma$, whose columns for each $\xi\in\partial D$ span $L_\xi^{\perp}$, and such that

$$N_0^*=iA_0^{-1}.$$

We consider now the nonlinear case. Let $\{M(\xi)\}_{\xi\in\partial D}$ be a family of generating CR-submanifolds of CR-dimension m in $\mathbf{C}^{m+n}$ and let

$$p\colon\partial D\to\mathbf{C}^{m+n}$$

be a map of class $C^{0,\alpha}$ such that

$$p(\xi)\in M(\xi) \quad (\xi\in\partial D).$$

We say, see also [Gl1], that the family $\{M(\xi)\}_{\xi\in\partial D}$ is a $C^{0,\alpha}$ generating CR-fibration over the unit circle ∂D with C^2 fibers if for each $\xi_0\in\partial D$ there are a neighbourhood $U_{\xi_0}\subseteq\partial D$ of ξ_0, an open ball $B_{\xi_0}\subseteq\mathbf{C}^{m+n}$ centered at the origin and maps $\varrho_1^{\xi_0},\dots,\varrho_n^{\xi_0}$ from the space $C^{0,\alpha}(U_{\xi_0},C^2(B_{\xi_0}))$ such that for every $\xi\in U_{\xi_0}$

1. the CR-submanifold $M(\xi)\cap(p(\xi)+B_{\xi_0})$ equals

$$\{(z,w)\in p(\xi)+B_{\xi_0}\,;\varrho_j^{\xi_0}(\xi,(z,w)-p(\xi))=0,\ j=1,\dots,n\},$$

2. $\varrho_j^{\xi_0}(\xi,0,0)=0$, $j=1,\dots,n$, and
3. $\bar\partial_{z,w}\varrho_1^{\xi_0}(\xi,z,w)\wedge\ldots\wedge\bar\partial_{z,w}\varrho_n^{\xi_0}(\xi,z,w)\neq0$ on B_{ξ_0}.

The following theorem is well known in the case the CR-dimension of the fibration $\{M(\xi)\}_{\xi\in\partial D}$ is 0, i.e., in the case of a maximal real fibration over ∂D, see [Fo1], [Gl1], [Gl2], [O1]. Our proof follows the one given by Globevnik in [Gl1] and thus not every detail will be given, see also [Fo1]. On the other hand this theorem represents a generalization of Globevnik's results in [Gl1] even in the case the fibration $\{M(\xi)\}_{\xi\in\partial D}$ is maximal real.

Theorem 1. *Let $M(\xi)\subseteq\mathbf{C}^{m+n}$, $\xi\in\partial D$, be a $C^{0,\alpha}$ generating CR-fibration over the unit circle ∂D with C^2 fibers and CR-dimension m. Let*

$$p\colon\partial D\to\mathbf{C}^{m+n}$$

be a $C^{0,\alpha}$ closed path in $\mathbf{C}^{m+n}$ such that

$$p(\xi)\in M(\xi)\quad(\xi\in\partial D).$$

Assume that there exists a linear change of coordinates in $\mathbf{C}^{m+n}$ such that in the new coordinate system all maximal complex subspaces of the generating CR-bundle

$$L:=\bigcup_{\xi\in\partial D}\{\xi\}\times T_{p(\xi)}M(\xi)$$

project isomorphically onto the subspace $\mathbf{C}^m\times\{0\}$. Assume also that all partial indices of the corresponding maximal real bundle $\mathcal{L}\subseteq\partial D\times\mathbf{C}^n$ are greater or equal to -1 and that the total index is k. Then there are an open neighbourhood U of $0\in\mathbf{R}^{n+k}$, an open neighbourhood V of the function 0 in $(A^{0,\alpha}(\partial D))^m$, an open neighbourhood W of p in $(C^{0,\alpha}(\partial D))^{m+n}$ and a map

$$\psi\colon U\times V\to W$$

of class C^1 such that

(1) $\psi(0,0)=p$,

(2) *for each* $(t,f)\in U\times V$ *the map* $\tilde p:=\psi(t,f)-p$ *extends holomorphically to* D *and* $(p+\tilde p)(\xi)\in M(\xi)$ *for all* $\xi\in\partial D$,

(3) $\psi(t_1,f)\neq\psi(t_2,f)$ *for* $t_1\neq t_2$ *from the neighbourhood* U *and any* $f\in V$,

(4) *if* $\tilde p\in W$ *satisfies the condition* $\tilde p(\xi)\in M(\xi)$, $\xi\in\partial D$, *and it is such that* $\tilde p-p$ *extends holomorphically to* D, *then there are* $t\in U$ *and* $f\in V$ *such that* $\psi(t,f)=\tilde p$.

Proof. Since we are assuming that all maximal complex subspaces project isomorphically onto the subspace $\mathbf{C}^m\times\{0\}$, one can find a set

$$\varrho(\xi,z,w)=(\varrho_1(\xi,z,w),\dots,\varrho_n(\xi,z,w))$$

of "global" defining functions for the fibration $\{M(\xi)\}_{\xi\in\partial D}$, i.e., there exist an $r_0>0$ and functions

$$\varrho_j^0\in C^{0,\alpha}_{\mathbf{R}}(\partial D,C^2(B_{r_0}))\quad(1\le j\le n)$$

such that for every odd partial index k_j the function ϱ_j^0 has the property

$$\varrho_j^0(-\xi,z,w)=-\varrho_j^0(\xi,z,w)\quad((\xi,z,w)\in\partial D\times B_{r_0}),$$

and such that for the functions

$$\varrho_j(\xi,z):=\begin{cases}\varrho_j^0(r(\xi),z,w); & k_j \text{ is odd,}\\ \varrho_j^0(\xi,z,w); & k_j \text{ is even}\end{cases}$$

the following holds

(a) $M(\xi)\cap(p(\xi)+B_{r_0})=\{(z,w)\in p(\xi)+B_{r_0};\varrho_j(\xi,(z,w)-p(\xi))=0;j=1,\dots,n\}$,

(b) $\bar\partial_w\varrho_1\wedge\ldots\wedge\bar\partial_w\varrho_n\neq0$ on $\partial D\times B_{r_0}$.

One may also assume that for each $\xi\in\partial D$ one has

$$(\bar\partial_w\varrho)^*(\xi,0,0)=(\partial_w\varrho)^t(\xi,0,0)=N_0^*(\xi),$$

where $\bar\partial_w\varrho$ denotes the $n\times n$ matrix whose columns are the coefficients of the $(0,1)$ forms $\bar\partial_w\varrho_1,\dots,\bar\partial_w\varrho_n$.

We define

$$\Psi\colon(A^{0,\alpha}(\partial D))^m\times\mathcal{E}_\sigma\times\mathcal{E}_\sigma\to\mathcal{E}_\sigma$$

by

$$\Psi(f,u,v)(\xi):=\varrho(\xi,f(\xi),A_0(u+i(v+iT_\sigma v))(\xi))\quad(\xi\in\partial D).$$

Then for every $v\in\mathcal{E}_\sigma$ and $\xi\in\partial D$ we have (see [Gl1] for the argument that Ψ is of class C^1)

$$(D_v\Psi(0,0,0)v)(\xi)=2\,\mathrm{Re}((\partial_w\varrho)^t(\xi,0,0)A_0(\xi)i(v(\xi)+i(T_\sigma v)(\xi)))=-2v(\xi)$$

and thus the partial derivative of the mapping Ψ with respect to the variable v is an invertible linear map from the space $\mathcal{E}_\sigma$ into itself. By the implicit mapping theorem one can find a neighbourhood V of the zero function in $(A^{0,\alpha}(\partial D))^m$, neighbourhoods $\widetilde{W}$ and $\widetilde{U}$ of the zero function in $\mathcal{E}_\sigma$ and a unique mapping

$$\tilde{\psi}: \widetilde{U} \times V \to \widetilde{W}$$

such that a triple $(f,u,v) \in V \times \widetilde{U} \times \widetilde{W}$ solves the equation

$$\Psi(f,u,v)=0 \tag{5}$$

if and only if $v=\tilde{\psi}(u,f)$. Finally one would like to select from the above family of all possible $C^{0,\alpha}$ closed curves in the CR-fibration $\{M(\xi)\}_{\xi\in\partial D}$ near p those which bound a sum

$$p+\text{analytic disc.}$$

At this point one should assume that all partial indices of the maximal real bundle $\mathcal{L}$ are greater or equal to -1. In this case the vector function

$$A_0(v+iT_\sigma v)$$

extends holomorphically to D. This follows from the fact that for any odd partial index k_j the function v_j+iTv_j is of the form $r(\xi)g_j^0(\xi)$ for some function $g_j^0 \in A^{0,\alpha}(\partial D)$. We recall that $r(\xi)$ represents the principal branch of the square root $\sqrt{\xi}$. So the condition on the vector function

$$\xi \mapsto A_0(u+i(v+iT_\sigma v))(\xi) \quad (\xi \in \partial D)$$

to extend holomorphically into D is in the case where $k_j \geq -1$, $j=1,\dots,n$, equivalent to the condition that the vector function

$$\xi \mapsto A_0(\xi)u(\xi) \quad (\xi \in \partial D)$$

extends holomorphically to D. To find all such functions $u\in\mathcal{E}_\sigma$ one has to find all vector functions $a \in (A^{0,\alpha}(\partial D))^n$ such that on ∂D

$$\Lambda\bar{a}=a,$$

i.e., for all $j=1,\dots,n$

$$\xi^{k_j}\overline{a_j(\xi)}=a_j(\xi) \quad (\xi \in \partial D).$$

For any partial index $k_j=-1$ the only solution of the above equation is $a_j=0$ and for $k_j\geq 0$ one has a k_j+1 dimensional parameter family of solutions. A parametrization ϕ of all functions $u\in\mathcal{E}_\sigma$ such that the vector function A_0u extends holomorphically to D is for each component function u_j given by

$$\phi_j(t_0,\dots,t_{k_j})(\xi):=t_0+\mathrm{Re}\left(\sum_{s=1}^{k_j/2}(t_{2s-1}+it_{2s})\xi^s\right) \tag{6}$$

in the case the partial index k_j is an even integer and by

$$\phi_j(t_0,\dots,t_{k_j})(\xi):=\mathrm{Re}\left(\sum_{s=0}^{(k_j-1)/2}(t_{2s}+it_{2s+1})r(\xi)^{2s+1}\right) \tag{7}$$

in the case k_j is an odd integer. See [Gl1] for more details. Hence, altogether one gets a $k+n$ parameter family of solutions. □

Remarks. 1. Since our choice of a change of coordinates in $\mathbf{C}^{m+n}$ involves quite a lot of freedom, it can happen that one could get, using a different change of coordinates, also a different set of attached discs to the fibration $\{M(\xi)\}_{\xi\in\partial D}$. It is quite easy to construct an example, e.g., the real normal bundle $L^\perp\subseteq\partial D\times\mathbf{C}^2$ is given by the matrix $N^t(\xi):=[\xi,\xi^2]$, where different linear changes of coordinates result in different sets of partial indices of the associated bundle $\mathcal{L}$. Moreover, in the example of the CR-bundle $L\subseteq\partial D\times\mathbf{C}^2$, where its real normal bundle $L^\perp$ is given by the matrix $N^t(\xi)=[\bar{\xi}^2,\xi^2]$, one can see that it can also happen that a certain linear change of coordinates can produce only positive partial indices but some other only negative partial indices. But, as already the above argument shows, as soon as all partial indices of the bundle $\mathcal{L}$ are greater or equal to -1, we know how to parametrize all small holomorphic perturbations of the path p with boundaries in the fibration $\{M(\xi)\}_{\xi\in\partial D}$. Observe also that in the case of positive CR-dimension the condition that all partial indices are negative does not necessary imply, as in the case of maximal real bundles [Fo1], [Gl1], [O1], that there is no nearby analytic discs attached to $\{M(\xi)\}_{\xi\in\partial D}$.

2. The set of discs attached to a generating CR-bundle of positive CR-dimension is always, even in the case where all partial indices of the associated bundle $\mathcal{L}$ are negative, parametrized by an infinite dimensional Banach space.

3. The following example shows that in the nonlinear case some assumptions on the partial indices are really needed. We already know that this can happen in the maximal real case, but when the CR-dimension is positive the difference can be even more striking. Namely, although in the linear model the set of solutions is always parametrized by an infinite dimensional vector space, it can happen that the set of local nearby solutions on a CR-manifold is only finite dimensional.

Example. Let

$$M_\xi := \{(z,w)\in \mathbf{C}^2\,; \operatorname{Im}(\xi w)=|z|^2\} \quad (\xi\in\partial D).$$

Then the disc

$$\xi \mapsto (0,0) \quad (\xi\in\partial D)$$

is the only analytic disc with boundary in the fibration $\{M_\xi\}_{\xi\in\partial D}$.

Proof. Let (f,g) be an analytic disc with boundary in the fibration $\{M_\xi\}_{\xi\in\partial D}$. Then

$$\operatorname{Im}(\xi g(\xi)) = |f(\xi)|^2 \quad (\xi\in\partial D).$$

But

$$0=\int_0^{2\pi} \operatorname{Im}(\xi g(\xi))\,d\theta = \int_0^{2\pi} |f(\xi)|^2\,d\theta$$

and so $f=0$. Then

$$\operatorname{Im}(\xi g(\xi))=0$$

on ∂D and so also $g=0$. □

Observe that in the above example the matrix $N^t(\xi)$ equals to $(0, i\bar{\xi})$ and thus the only partial index is -2.

One can also define a 4-dimensional submanifold of $\mathbf{C}^3$ of CR-dimension 1 with a similar property. Let

$$M := \bigcup_{\xi\in\partial D} \{\xi\}\times M_\xi.$$

Then any holomorphic disc with boundary in M and close to the disc

$$\xi\mapsto(\xi,0,0) \quad (\xi\in\partial D)$$

is of the form

$$\xi\mapsto(a(\xi),0,0) \quad (\xi\in\partial D),$$

where a is an automorphism of the unit disc close to the identity. Thus the family of such discs is 3-dimensional. □

Remark. After this paper was finished, the author received a preprint of the paper *On the global Bishop equation* by J.-M. Trepreau, where similar questions as here, i.e., when the family of small holomorphic perturbations along a CR-manifold is a manifold, were considered. Trepreau's sufficient condition is given in terms of the conormal bundle of a CR-manifold and thus more geometric than ours. Also, it is obviously invariant under a holomorphic change of coordinates. However, it can be shown (modulo some smoothness assumptions) that our condition on the existence of a coordinate system in which all partial indices are greater than or equal to -1 and Trepreau's condition on the conormal bundle are equivalent under a suitable fiberwise change of coordinates, i.e., a change of coordinates which is a nonsingular linear transformation $Q(\xi)$ of $\mathbf{C}^{m+n}$ for each fixed $\xi\in\overline{D}$ and the map $\xi\mapsto Q(\xi)$ is holomorphic on D.

5. Geometry of perturbations

The results of this section were inspired by the work [BRT] by Baouendi, Rothschild and Trepreau. See also the paper [T] by Tumanov for some related results and definitions.

We recall the definition of the conormal bundle of a CR-submanifold $M\subseteq\mathbf{C}^N$ as given in [BRT]. We identify the complex bundle $\Lambda^{1,0}\mathbf{C}^N$ of $(1,0)$ forms on $\mathbf{C}^N$ with the real cotangent bundle $T^*\mathbf{C}^N$ as follows. To a real 1-form $\Gamma=\sum c_j\,dz_j+\bar{c}_j\,d\bar{z}_j$ on $\mathbf{C}^N$ we associate the complex $(1,0)$ form $\gamma=2i\sum c_j\,dz_j$ so that the pairings between the vectors and covectors are related by the identity

$$\langle\Gamma,X\rangle=\operatorname{Im}\langle\gamma,X\rangle$$

for all $X\in T_z\mathbf{C}^N$. Under this identification, the fiber of the conormal bundle $\Sigma(M)$ on a CR-submanifold M at the point $p\in M$ is given by

$$\Sigma_p(M)=\{\gamma\in\Lambda^{1,0}\mathbf{C}^N\,;\operatorname{Im}\langle\gamma,X\rangle=0,\ X\in T_pM\}.$$

If the manifold M is generating, then the conormal bundle can be naturally identified with the characteristic bundle $(T^{\mathbf{C}}M)^{\perp}$ of the CR-structure on M. Namely, if locally, near some point $p\in M$, the submanifold M is generating and is given by the set of equations $\varrho=(\varrho_1,\dots,\varrho_n)=0$, then the fiber of the conormal bundle over the point p is given by

$$\Sigma_p(M)=\left\{is^t\partial\varrho(p)=i\sum_j s_j\frac{\partial\varrho_j}{\partial z}(p)\,;s_j\in\mathbf{R},\ 1\le j\le n\right\}.$$

From now on let $M(\xi)\subseteq\mathbf{C}^{m+n}$, $\xi\in\partial D$, be a generating CR-fibration over the unit circle ∂D of class $C^{0,\alpha}$ with C^2 fibers and with CR-dimension m. Let

$$p\colon\partial D\to\mathbf{C}^{m+n}$$

be a $C^{0,\alpha}$ curve such that

$$p(\xi)\in M(\xi)\quad(\xi\in\partial D).$$

Let V_p be the set of all holomorphic discs $c(\xi)=(c_1(\xi),\dots,c_{m+n}(\xi))$ of class $C^{0,\alpha}$ such that for each $\xi\in\partial D$ the $(1,0)$ form

$$\sum_{j=1}^{m+n} c_j(\xi)\,dz_j \tag{8}$$

belongs to the space $\Sigma_{p(\xi)}M(\xi)$. For each $\xi\in\partial D$ we denote by $V_p(\xi)\subseteq\Sigma_{p(\xi)}M(\xi)$ the subset consisting of all such forms (8),

$$V_p(\xi)=\Big\{\gamma\in\Sigma_{p(\xi)}M(\xi)\,;\gamma=\sum c_j(\xi)\,dz_j,\ c\in V_p\Big\}.$$

Clearly $V_p(\xi)$ is a real linear subspace of $\Sigma_{p(\xi)}M(\xi)$.

Henceforth we will assume that the coordinates in $\mathbf{C}^{m+n}$ can be chosen so that each maximal complex subspace of the tangent space $T_{p(\xi)}M(\xi)$, $\xi\in\partial D$, projects isomorphically onto $\mathbf{C}^m\times\{0\}$. We recall that this is always possible in the case the fibration $M(\xi)\subseteq\mathbf{C}^{m+n}$, $\xi\in\partial D$, is of at least class C^1, i.e., the defining functions of the fibration belong to the space $C^1(\partial D, C^2(B_{r_0}))$ for some $r_0>0$, and the closed path p is of class C^1. We also recall that for each $\xi\in\partial D$ the columns of the matrix function

$$\bar\partial\varrho(\xi,0,0)=N(\xi)=\begin{bmatrix} G_0(\xi)\\ N_0(\xi)\end{bmatrix}\quad(\xi\in\partial D),$$

span the fiber of the normal bundle of the submanifold $M(\xi)$ at the point $p(\xi)$.

The following characterization of elements of $V_p(\xi)$, $\xi\in\partial D$, is immediate, see also [BRT, Proposition 3.6].

Proposition 1. *Let $\xi_0\in\partial D$. A covector $\gamma=\sum c_j\,dz_j\in T^*_{p(\xi_0)}M(\xi_0)$ belongs to the subspace $V_p(\xi_0)$ if and only if there is a real function $s=(s_1,\dots,s_n)\in\mathcal{E}_\sigma$ such that*

1. $c^t=[c_1,\dots,c_{m+n}]=is^t[G_0^*,N_0^*](\xi_0)$ *and*
2. *the covector function $s^t[G_0^*,N_0^*]$ extends holomorphically to D.*

Remark. For any real vector function $s\in\mathcal{E}_\sigma$ for which the second property holds we will say that it *generates* an element from V_p.

Corollary 2. *If all partial indices of the associated maximal real fibration $\mathcal{L}$ are greater or equal to 1, then $V_p=\{0\}$ and so each of the subspaces $V_p(\xi)$, $\xi\in\partial D$, is trivial.*

Proof of the corollary. Let A_0 denote the matrix function whose columns for every $\xi\in\partial D$ span the fibers of $\mathcal{L}$. Then $A_0=\Theta\Lambda_0$, where Θ is an invertible holomorphic matrix on $\overline{D}$ and Λ_0 is the square root of the characteristic matrix Λ of the maximal real vector bundle $\mathcal{L}$. Then

$$N_0^*=i\bar\Lambda_0\Theta^{-1}$$

and a necessary condition to get an element from V_p is that there exists a real function $s \in \mathcal{E}_\sigma$ such that

$$s^t \bar{\Lambda}_0$$

extends holomorphically to D. But since all partial indices of Λ are greater or equal to 1, one concludes that s has to be 0. □

Since our method gives all nearby analytic discs of class $C^{0,\alpha}$ attached to the CR-fibration $M(\xi) \subseteq \mathbf{C}^{m+n}$, $\xi \in \partial D$, only in the case when all partial indices of the associated maximal real bundle are greater or equal to -1, this will be the case we will consider from now on. In this case we have already proved, Theorem 1, that the family of all nearby holomorphic discs, i.e., all holomorphic discs $F \in (A^{0,\alpha}(\partial D))^{m+n}$ with the property that the disc $p+F$ is attached to the fibration $M(\xi)$, $\xi \in \partial D$, forms a Banach submanifold $\mathcal{A}$ of the Banach space $(A^{0,\alpha}(\partial D))^{m+n}$. In the case where the CR-dimension of the fibration is 0, this submanifold is of finite real dimension $n+k$, where k is the total index of the fibration, but in the case of a positive CR-dimension we get an infinite dimensional submanifold. Also, the differentiation of the equation (5) with respect to u and f at the point $(0,0)$ yields

$$(D_u \tilde{\psi})u = 0 \quad \text{and} \quad (D_f \tilde{\psi})f = \operatorname{Re}(G_0^* f)$$

for all $(u, f) \in \mathcal{E}_\sigma \times (A^{0,\alpha}(\partial D))^m$. Thus all vectors of the tangent space $T_0\mathcal{A}$ to the submanifold $\mathcal{A}$ at the point 0 are of the form

$$(f, A_0(u + i(v + iT_\sigma v))),$$

where $f \in (A^{0,\alpha}(\partial D))^m$, and the functions $u, v \in \mathcal{E}_\sigma$ are such that $v = \operatorname{Re}(G_0^* f)$ and $\Lambda_0 u$ extends holomorphically to D.

Remark. In the case considered by Baouendi, Rothschild and Trepreau in [BRT] one works only in a neighbourhood of a point on a given CR-submanifold and so all partial indices of any nearby holomorphic disc attached to the manifold are 0. It is easy to see that all partial indices of a constant map are 0. On the other hand, this condition is stable under small perturbations of the disc, see [V2].

Proposition 2. *The dimension of the subspace $V_p(\xi) \subseteq \Sigma_{p(\xi)}(M(\xi))$ does not depend on $\xi \in \partial D$, i.e., it is the same for every $\xi \in \partial D$.*

Remark. This proposition extends the Proposition 3.6 from [BRT].

Proof. We split the space $\mathbf{R}^n$ into three subspaces

$$\mathbf{R}^n = \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_0} \oplus \mathbf{R}^{n_{-1}},$$

where n_1 is the number of positive partial indices, n_0 is the number of partial indices which are equal to 0, and n_{-1} is the number of partial indices which are equal to -1. With respect to this splitting we denote the coordinates on $\mathbf{R}^n$ by (q, y, t).

We recall that every element of the space V_p is of the form

$$is^t[G_0^*, N_0^*]$$

for some real function $s \in \mathcal{E}_\sigma$. We also recall that $N_0^* = iA_0^{-1}$. Since the first n_1 partial indices are positive, any real vector function s from the space $\mathcal{E}_\sigma$ which generates an element from V_p, must have, by the same argument as in the proof of Corollary 2, the first n_1 coordinate functions being identically equal to 0. Because of that we may assume without loss of generality that $n_1 = 0$.

Each element of V_p is now generated by a real function of the form

$$(y, \operatorname{Re}(\omega r(\xi)))$$

where $y \in \mathbf{R}^{n_0}$, $\omega \in \mathbf{C}^{n_{-1}}$ and $r(\xi)$ is the principal branch of the square root. Let k_0 be the dimension of the space $V_p(1)$. We shall prove that for each $\xi \in \partial D$ the dimension of the space $V_p(\xi)$ is also k_0. Since for each $\xi_0 \in \partial D$ there exists an automorphism of the unit disc D which takes 1 to 1 and ξ_0 to -1, it is enough to prove the above claim for $\xi_0 = -1$.

Let $(y_j, \operatorname{Re}(\omega_j r(\xi)))$, $j = 1, \dots, k_0$, be a set of real functions on ∂D which for each j generate an element from V_p, and such that the real vectors

$$(y_j, \operatorname{Re}(\omega_j)) \quad (j = 1, \dots, k_0)$$

are linearly independent. If also the set of vectors

$$(y_j, \operatorname{Re}(i\omega_j)) \quad (j = 1, \dots, k_0)$$

is linearly independent, the claim is already proved and we are done. Let us assume now that this is not the case and that these vectors are not linearly independent. Then there are real numbers $\lambda_1, \dots, \lambda_{k_0}$, not all equal to 0, such that

$$\sum_{j=1}^{k_0} \lambda_j y_j = 0$$

and

$$\sum_{j=1}^{k_0} \lambda_j \operatorname{Re}(i\omega_j) = 0.$$

The second equation is equivalent to

$$\sum_{j=1}^{k_0} \lambda_j \omega_j = t_1$$

for some real vector t_1 from $\mathbf{R}^{n-1}$. The way how t_1 is defined immediately implies that $t_1 \neq 0$ and that the real vector function

$$(0, \mathrm{Re}(t_1 r(\xi)))$$

generates an element from V_p. Since t_1 is a real vector, both functions

$$(0, \mathrm{Re}(t_1 r(\xi))) \quad \text{and} \quad (0, \mathrm{Re}(it_1 r(\xi)))$$

generate an element from the space V_p. This follows from the following claim.

Claim. Let $f = u + iv$, $u, v \in (\mathcal{E}_{\mathbf{R}}^{0,\alpha})^n$, be a vector function such that the function

$$\xi \mapsto \mathrm{Re}(r(\xi)) f(\xi) \quad (\xi \in \partial D)$$

extends holomorphically into D. Then $f \in (\mathcal{A}^{0,\alpha})^n$. In particular, also the function

$$\xi \mapsto \mathrm{Re}(ir(\xi)) f(\xi) \quad (\xi \in \partial D)$$

extends holomorphically into D.

Proof of the claim. Since the function

$$\xi \mapsto \mathrm{Re}(r(\xi)) f(\xi) \quad (\xi \in \partial D)$$

extends holomorphically into D, all its negative Fourier coefficients have to vanish. This implies that for every $j \in \mathbf{N}$ we have

$$\hat{f}(-j) + \hat{f}(-j-1) = 0.$$

Since we also have

$$\lim_{j \to \infty} \hat{f}(-j) = 0,$$

we conclude that all negative Fourier coefficients of the function f are 0 and the claim is proved. $\square$

Also, since not all real numbers λ_j, $j = 1, \dots, k_0$, are 0, we may assume, without loss of generality, that $\lambda_1 \neq 0$. We repeat the above argument on the set of real functions $(y_j, \mathrm{Re}(\omega_j r(\xi)))$, $j=2,\ldots,k_0$, and the vector function $(0, \mathrm{Re}(it_1 r(\xi)))$. If at $\xi=-1$ these vectors are still linearly dependent, one can find real numbers $\tilde{\lambda}_1,\ldots,\tilde{\lambda}_{k_0}$, not all equal to 0, such that

$$\sum_{j=2}^{k_0} \tilde{\lambda}_j y_j = 0 \quad \text{and} \quad \tilde{\lambda}_1 it_1 + \sum_{j=2}^{k_0} \tilde{\lambda}_j \omega_j = t_2$$

for some nonzero real vector $t_2 \in \mathbf{R}^{n-1}$. We observe that it can not happen that $\tilde{\lambda}_2 = \ldots = \tilde{\lambda}_{k_0} = 0$ since the vectors t_1 and t_2 are real. We also observe that t_1 and t_2 are linearly independent vectors. Repeating the above argument we either stop on the jth step, $j<k_0$, or we produce k_0 linearly independent vectors which span $V_p(-1)$. $\square$

So we can define the defect of a closed curve p in a generating CR-fibration over ∂D with partial indices greater or equal to -1 in the same way as Baouendi, Rothschild and Trepreau do in [BRT]. See Definition 3.5 and Proposition 3.6 in [BRT]. See also [T].

Definition 5. The defect $\mathrm{def}(p)$ of the curve p is defined as the dimension of the real vector spaces $V_p(\xi)$, $\xi \in \partial D$.

From now on we will restrict our discussion to the set $\mathcal{A}_*$ of holomorphic perturbations of p which leave one of the points, say $p(1)$, on the curve p fixed. But to prove that the set $\mathcal{A}_*$ is in fact a manifold, we have to assume that all partial indices of the path p are nonnegative. See the examples at the end of this section. Let $\mathcal{E}_{\sigma,*} \subseteq \mathcal{E}_\sigma$ and $(A_*^{0,\alpha}(\partial D))^m \subseteq (A^{0,\alpha}(\partial D))^m$ denote the subspaces of the functions which are 0 at $\xi=1$. Also, let

$$T_*: C_{\mathbf{R}}^{0,\alpha}(\partial D) \to C_{\mathbf{R}}^{0,\alpha}(\partial D)$$

denote the Hilbert transform which assigns to a function $v \in C_{\mathbf{R}}^{0,\alpha}(\partial D)$ the harmonic conjugate function $\tilde{v}$ for which $\tilde{v}(1)=0$. Observe that since T_* does not preserve the subspace of odd functions in $C_{\mathbf{R}}^{0,\alpha}(\partial D)$, there is no natural way of defining an appropriate Hilbert transform on $\mathcal{E}_{\sigma,*}$.

Let k be the total index of the associated maximal real bundle $\mathcal{L} \subseteq \partial D \times \mathbf{C}^n$ and assume that all its partial indices are nonnegative. Then the following lemma holds.

Lemma 5. *The set $\mathcal{A}_*$ is a Banach submanifold of the manifold $\mathcal{A}$ of infinite dimension in the case the CR-dimension of the fibration $\{M(\xi)\}_{\xi \in \partial D}$ is positive, and of real dimension k in the case of maximal real fibration over ∂D.*

Proof. We define the map

$$F: (A_*^{0,\alpha}(\partial D))^m \times \mathcal{E}_{\sigma,*} \times \mathcal{E}_{\sigma,*} \to \mathcal{E}_{\sigma,*}$$

by

$$F(f,\tilde{u},v)(\xi):=\varrho(\xi,f(\xi),A_0((\tilde{u}+T_\sigma v)+i(v+iT_\sigma v))(\xi))\quad(\xi\in\partial D).$$

Here $\varrho=(\varrho_1,\dots,\varrho_n)$ is the set of defining functions of the fibration $\{M(\xi)\}_{\xi\in\partial D}$ along the path p. Using the implicit mapping theorem as in the proof of Theorem 1 one gets a neighbourhood $\mathcal{N}$ of the zero function in $(A_*^{0,\alpha}(\partial D))^m$, neighbourhoods $\widetilde{U}$ and V of 0 in $\mathcal{E}_{\sigma,*}$, and a unique mapping $\psi:\mathcal{N}\times\widetilde{U}\to V$ such that the triple $(f,\tilde{u},v)\in\mathcal{N}\times\widetilde{U}\times V$ solves the equation $F(f,\tilde{u},v)=0$ if and only if $v=\psi(f,\tilde{u})$.

As we already know, a necessary and sufficient condition for any disc from the above family to be the boundary value of a holomorphic disc is that the mapping

$$A_0(\tilde{u}+T_\sigma v)$$

extends holomorphically to D. Let $\phi(t)$, $t\in\mathbf{R}^{n+k}$, denote the linear parametrization (6), (7) of all real functions $u\in\mathcal{E}_\sigma$ such that A_0u extends holomorphically to D. Thus to extract from the above family of discs $A_0(\tilde{u}+i\psi(f,\tilde{u}))$ all holomorphic discs which are 0 at $\xi=1$, we have to find all functions $\tilde{u}\in\mathcal{E}_{\sigma,*}$ and values $t\in\mathbf{R}^{m+n}$ which solve the equations

$$\tilde{u}+T_\sigma\psi(f,\tilde{u})=\phi(t)\quad\text{and}\quad T_\sigma\psi(f,\tilde{u})(1)=\phi(t)(1).$$

Since all partial indices are nonnegative, the $n\times(n+k)$ matrix $D_t(\phi(t)(1))|_{t=0}$ has maximal rank. Since we also have

$$(D_{\tilde{u}}\psi)(f,\tilde{u})|_{f=0,\tilde{u}=0}=0,$$

one gets, using the implicit mapping theorem again, a unique mapping μ from a neighbourhood of the point $(0,0)\in(A_*^{0,\alpha}(\mathcal{D}))^m\times\mathbf{R}^k$ into $\mathcal{E}_{\sigma,*}$ such that all small holomorphic discs (f,g) from $(A_*^{0,\alpha}(\partial D))^{m+n}$ which solve the equation

$$\varrho(\xi,f(\xi),g(\xi))=0\quad(\xi\in\partial D)$$

are of the form

$$(f,A_0(\mu(f,s)+i\psi(f,\mu(f,s))))$$

for a unique pair (f,s) from a neighbourhood of the point $(0,0)$ in $(A_*^{0,\alpha}(\mathcal{D}))^m\times\mathbf{R}^k$. $\square$

Note that every element of the tangent space $T_0\mathcal{A}_*$ is of the form

$$(f,A_0(u+i(v+iT_\sigma v)))$$

for some $f\in(A_*^{0,\alpha}(\partial D))^m$, $v\in\mathcal{E}_{\sigma,*}$ such that $v=\mathrm{Re}(G_0^*f)$, and $u\in\mathcal{E}_\sigma$ such that A_0u extends holomorphically to D and for which one also has $u-T_\sigma v\in\mathcal{E}_{\sigma,*}$.

Henceforth our goal will be to reprove and to generalize Theorem 1 from [BRT]. In fact we can prove the same statement for an arbitrary closed path p in a generating CR-fibration $M(\xi)\subseteq\mathbf{C}^{m+n}$, $\xi\in\partial D$, for which there exists a linear change of coordinates in $\mathbf{C}^{m+n}$ such that the partial indices of the corresponding maximal real bundle are all greater or equal to 0.

We recall the definition of the evaluation maps $\mathcal{F}_\xi$ defined on the manifold $\mathcal{A}_*$, see [BRT] for more details. See also [T]. For every $\xi\in\partial D$ and $F\in\mathcal{A}_*$ we define

$$\mathcal{F}_\xi(F):=(p+F)(\xi).$$

Then for every $\xi\in\partial D$ the derivative $\mathcal{F}'_\xi(0)$ maps the tangent space $T_0\mathcal{A}_*$ into $T_{p(\xi)}M(\xi)$.

Theorem 2. *Let p and $M(\xi)\subseteq\mathbf{C}^{m+n}$, $\xi\in\partial D$, be as above. Then for each $\xi\in\partial D$, $\xi\neq1$, one has*

$$\mathcal{F}'_\xi(0)(T_0\mathcal{A}_*)=V_p(\xi)^\perp. \tag{9}$$

Proof. We first prove the following partial statement, namely,

$$\mathcal{F}'_\xi(0)(T_0\mathcal{A}_*)\subseteq V_p(\xi)^\perp$$

for each $\xi\in\partial D$. To prove this claim let

$$(f,A_0(u+i(v+iT_\sigma v))) \tag{10}$$

be an arbitrary element of $T_0\mathcal{A}_*$. We recall that $f\in(A_*^{0,\alpha}(\partial D))^m$, that $u\in\mathcal{E}_\sigma$, that $v,u-T_\sigma v\in\mathcal{E}_{\sigma,*}$, and that also

$$v=\mathrm{Re}(G_0^*f).$$

On the other hand let

$$u_0^t[G_0^*,N_0^*] \tag{11}$$

be an element of V_p. Here $u_0\in\mathcal{E}_\sigma$. Since both vector functions (10) and (11) extend holomorphically to D, their product also has to extend holomorphically to D. But on the other hand the multiplication of (10) and (11) yields a purely imaginary vector function

$$iu_0^t(u-T_\sigma v+\mathrm{Im}(G_0^*f)).$$

Since the vector function (10) is 0 at $\xi=1$, the claim is proved.

To prove that in the above inclusion in fact the equality holds it is enough to prove that the dimension of the space $\mathcal{F}'_\xi(0)(T_0\mathcal{A}_*)$ is $2m+n-\operatorname{def}(p)$. Since for each $\xi\in\partial D$, $\xi\neq 1$, one can find an automorphism of the unit disc D which takes 1 to 1 and ξ to -1, it is enough to prove the claim for $\xi=-1$. For this it is enough, since the set of function values $\{f(-1); f\in(A_*^{0,\alpha}(\partial D))^m\}$ already spans a $2m$-dimensional subspace, to prove that the subspace

$$\{(u-T_\sigma v)(-1)\,; v=\operatorname{Re}(G_0^* f),\ f(-1)=0,\ A_0 u\in(A^{1,\alpha}(\partial D))^n,\ u-T_\sigma v\in\mathcal{E}_{\sigma,*}\}$$

of $\mathbf{R}^n$ has dimension $n-\operatorname{def}(p)$.

We denote by n_1 the number of positive indices and by n_0 the number of indices which are equal to 0, and split the space $\mathbf{R}^n$ correspondingly. For each positive partial index k_j the set of real functions u_j such that the function $\xi\mapsto\xi^{k_j/2}u_j(\xi)$ extends holomorphically to D and $(u_j-Tv_j)(1)=0$ is at least 1-dimensional. Thus the proof of the claim will be finished once we prove that the following subspace of $\mathbf{R}^{n_0}$

$$\{(u-T_\sigma v)(-1)\,; v=\operatorname{Re}(G_0^* f),\ f(-1)=0,\ u-T_\sigma v\in\mathcal{E}_{\sigma,*}\}\cap\mathbf{R}^{n_0}$$

has dimension $n-n_1-\operatorname{def}(p)=n_0-\operatorname{def}(p)$.

To prove the last claim we will show that for a vector $u_0\in\mathbf{R}^{n_0}$ the condition

$$u_0^t(u-T_\sigma v)(-1)=0$$

for every $v\in(C_{\mathbf{R}}^{0,\alpha}(\partial D))^{n_0}$ such that v is given as the last n_0 component functions of $\operatorname{Re}(G_0^* f)$, $f(-1)=0$, and every constant vector $u\in\mathbf{R}^{n_0}$ such that $(u-T_\sigma v)(1)=0$, implies

$$(0,u_0^t)[G_0^*,N_0^*](-1)\in V_p(-1).$$

This will complete the proof of (9). But since every real vector function $\tilde{u}_0\in\mathcal{E}_\sigma$ which generates an element from V_p has the first n_1 coordinate functions identically equal to 0 (see the proof of Corollary 2), it is enough to prove the statement for the case where all partial indices are 0 and $n_1=0$.

From here on the argument goes very much the same as the one given by Baouendi, Rothschild and Trepreau in [BRT]. We include it for the sake of completeness.

We recall that T_* denotes the Hilbert transform on $(C_{\mathbf{R}}^{0,\alpha}(\partial D))^n$ such that for every $v\in(C_{\mathbf{R}}^{0,\alpha}(\partial D))^n$ we have

$$(T_*v)(1)=0.$$

Also, since all partial indices are 0, the vector function u is in fact a constant such that $(T_\sigma v-u)(1)=0$ and hence

$$T_\sigma v-u=T_*v.$$

Let $u_0\in\mathbf{R}^{n_0}$ be a vector with the property that

$$u_0^t(T_*v)(-1)=0$$

for every $v\in(C_{\mathbf{R}}^{0,\alpha}(\partial D))^n$ such that $v=\mathrm{Re}(G_0^* f)$, $f(-1)=0$. We recall ([BRT]) that for $v\in C_{\mathbf{R}}^{0,\alpha}(\partial D)$ and $\xi_0\in\partial D$ one has

$$(T_*v)(\xi_0)=\mathrm{p.v.}\,\frac{i}{\pi}\int_0^{2\pi}\frac{v(\xi)(1-\xi_0)}{(\xi-1)(\xi-\xi_0)}\xi\,d\theta,$$

where ξ stands for $e^{i\theta}$. We denote the vector function $u_0^t G_0^*$ by a_0^t. Then for every nonnegative integer q, every vector $z_0\in\mathbf{C}^m$, and a function f of the form

$$f(\xi)=(\xi^2-1)\xi^q z_0 \quad (\xi\in\partial D)$$

one has

$$(u_0^t(T_*v))(-1)=\mathrm{p.v.}\,\frac{i}{\pi}\int_0^{2\pi}2\frac{\mathrm{Re}(a_0^t(\xi)f(\xi))}{(\xi-1)(\xi+1)}\xi\,d\theta \tag{12}$$

$$=\frac{2i}{2\pi}\int_0^{2\pi}[\xi^{q+1}a_0^t(\xi)z_0-\overline{\xi^{q+1}a_0^t(\xi)z_0}\,]\,d\theta. \tag{13}$$

By our assumption the integrals (12) and (13) are equal 0 for every nonnegative integer q and every vector $z_0\in\mathbf{C}^m$. Since one can also take iz_0 instead of z_0, one gets that

$$\frac{2i}{2\pi}\int_0^{2\pi}\xi^{q+1}a_0^t(\xi)\,d\theta=0$$

for every nonnegative integer q. The above identity can be written in terms of Fourier coefficient as

$$\hat{a}_0(-q-1)=0 \quad (q=0,1,2,\ldots)$$

which immediately implies that the real vector u_0^t generates an element of V_p. The identity (9) is proved. $\square$

For the next theorem we have to assume more regularity on the fibration $\{M(\xi)\}_{\xi\in\partial D}$ and the closed path p. We assume now that we have a $C^{1,\alpha}$ fibration with C^3 fibers, i.e., the fibration is given by a set of real functions from the space $C^{1,\alpha}(\partial D, C^3(B_{r_0}))$, and the closed path p shall be of class $C^{1,\alpha}$. Under these conditions one can repeat the proofs of Theorem 1 and Theorem 2 in the $C^{1,\alpha}$ category. We recall the definition of the mapping $\mathcal{G}$ from [BRT]. Let

$$\mathcal{G}: T_0\mathcal{A}_* \to \mathbf{C}^{m+n}$$

be defined by

$$\mathcal{G}(F) := \frac{\partial}{\partial\theta} F(e^{i\theta}) \Big|_{\theta=0}.$$

Theorem 3. *Let p and $M(\xi) \subseteq \mathbf{C}^{m+n}$, $\xi \in \partial D$, be as above. Then $\mathcal{G}$ maps $T_0\mathcal{A}_*$ into $T_{p(1)}M(1)$ and*

$$\mathcal{G}(T_0\mathcal{A}_*) = V_p(1)^{\perp}. \tag{14}$$

Proof. We first observe that for every $F \in T_0\mathcal{A}_*$ one has

$$\operatorname{Re}([G_0^*, N_0^*]F) = 0$$

on ∂D. Differentiation with respect to θ and setting $\theta=0$ implies that $\mathcal{G}$ maps $T_0\mathcal{A}_*$ into $T_{p(1)}M(1)$.

The proof of (14) is quite similar to the proof of (9). The inclusion

$$\mathcal{G}(T_0\mathcal{A}_*) \subseteq V_p(1)^{\perp}$$

follows as above since the product of any two functions $G \in V_p$ and $F \in T_0\mathcal{A}_*$ equals 0,

$$G^t F = 0. \tag{15}$$

Namely, the differentiation of (15) with respect to θ and setting $\theta=0$ yields

$$G^t(1) \frac{\partial}{\partial\theta} F(e^{i\theta}) \Big|_{\theta=0} = 0.$$

To prove the opposite inclusion in (14) we proceed similarly as in the proof of (9) and reduce the problem to the case where all partial indices of the associated maximal real bundle are 0.

Claim. The vector space

$$\left\{ \frac{\partial}{\partial\theta}(T_* v)(e^{i\theta}) \Big|_{\theta=0} ; v = \operatorname{Re}(G_0^* f),\ f(\xi) = (\xi-1)^2 \xi^q z_0,\ q \in \mathbf{N} \cup \{0\},\ z_0 \in \mathbf{C}^m \right\}$$

has dimension $n-\operatorname{def}(p)$.

Proof of the claim. We are using a similar notation as in the proof of (9). Let $u_0 \in \mathbf{R}^n$ be a real n-vector which annihilates the above vector space. Also, let a_0^t be the vector $u_0^t G_0^*$. We recall ([BRT]) that

$$u_0^t \frac{\partial}{\partial\theta}(T_* v)(e^{i\theta})\bigg|_{\theta=0} = \frac{1}{\pi}\int_0^{2\pi} \frac{\operatorname{Re}(a_0^t(\xi) f(\xi))}{(\xi-1)^2}\xi \, d\theta$$
$$= \frac{1}{\pi}\int_0^{2\pi} [a_0^t(\xi)\xi^{q+1} z_0 + \overline{a_0^t(\xi)\xi^{q+1} z_0}\,]\, d\theta.$$

Replacing z_0 by iz_0 and adding the identities one gets

$$\int_0^{2\pi} a_0^t(\xi)\xi^{q+1}\, d\theta = 0$$

for every nonnegative integer q. In terms of Fourier coefficients we have

$$\hat{a}_0(-q-1) = 0$$

for every $q \in \mathbf{N} \cup \{0\}$. Thus the real vector u_0 generates an element from V_p. This finishes the proof of the claim and so also the theorem. □

Remarks and examples. In the case when the partial indices of the path p in the generating CR-fibration $\{M(\xi)\}_{\xi\in\partial D}$ are not all nonnegative, the conclusions in Theorems 2 and 3 are not true even if we consider only the case where all indices are greater or equal to -1. One problem, of course, occurs if the total index k happens to be negative. Then the number of free parameters is strictly less than the number of additional equations we have to satisfy. Here we give two examples in $\mathbf{C}^2$ for which $k \geq 0$ but the conclusions of Theorems 2 and 3 still do not hold.

Example 1. In this example we find a maximal real fibration in $\mathbf{C}^2$ for which the set of attached discs passing through the point $(0,0)$ does not form a manifold.

Let the maximal real fibration $\{M(\xi)\}_{\xi\in\partial D}$ be given by the set of equations

$$\operatorname{Im}(z\bar{\xi}) = 0$$

and

$$\operatorname{Re}(wr(\xi)) = \operatorname{Re}(r(\xi)) \operatorname{Re}((z\bar{\xi})^2).$$

It is easy to check that the partial indices of the path $p(\xi)=0$, $\xi \in \partial D$, are 2 and -1, and so the total index k equals 1 and the defect of the path p is 1. Hence the dimension of the spaces

$$V_p(\xi)^{\perp} \quad (\xi \in \partial D)$$

is also 1.

Claim. The family of holomorphic discs with boundaries in the maximal real fibration $\{M(\xi)\}_{\xi\in\partial D}$ which all pass through the point $(0,0)$ at $\xi=1$ is not a manifold.

Proof of the claim. Let (z,w) be a holomorphic disc with boundary in the maximal real fibration $\{M(\xi)\}_{\xi\in\partial D}$ and such that $(z(1),w(1))=(0,0)$. Then from the first equation

$$\operatorname{Im}(z(\xi)\bar{\xi})=0$$

we get

$$z(\xi)\bar{\xi}=\omega\xi-2\operatorname{Re}(\omega)+\bar{\omega}\bar{\xi}$$

for some complex number ω. The second equation

$$\operatorname{Re}(w(\xi)r(\xi))=\operatorname{Re}(r(\xi))\operatorname{Re}((z(\xi)\bar{\xi})^2)\quad(\xi\in\partial D)$$

implies

$$\operatorname{Re}(w(\xi^2)\xi)=\operatorname{Re}(\xi)\operatorname{Re}((z(\xi^2)\bar{\xi}^2)^2)\quad(\xi\in\partial D).$$

A short calculation shows that the right hand side of the last equation equals to

$$\operatorname{Re}(\omega^2\xi^5+(\omega^2-4\omega\operatorname{Re}(\omega))\xi^3+(2|\omega|^2+4(\operatorname{Re}(\omega))^2-4\omega\operatorname{Re}(\omega))\xi).$$

Thus

$$w(\xi)=\omega^2\xi^2+(\omega^2-4\omega\operatorname{Re}(\omega))\xi+(2|\omega|^2+4(\operatorname{Re}(\omega))^2-4\omega\operatorname{Re}(\omega))$$

and so at $\xi=1$ one must have

$$\omega^2+(\omega^2-4\omega\operatorname{Re}(\omega))+(2|\omega|^2+4(\operatorname{Re}(\omega))^2-4\omega\operatorname{Re}(\omega))=0$$

or after division by 2

$$\omega^2-4\omega\operatorname{Re}(\omega)+|\omega|^2+2(\operatorname{Re}(\omega))^2=0.$$

If we write $\omega=x+iy$, then the imaginary part of the last equation yields

$$-2xy=0.$$

Thus the constant ω has to be either real or purely imaginary. So the set of solutions of the above equations is the union of two intersecting curves in $(A^{0,\alpha}(\partial D))^2$ and therefore not a manifold. $\square$

Example 2. Let the maximal real fibration in $\mathbf{C}^2$ be given by

$$\mathrm{Im}(z\overline{r(\xi)}\,) = 0$$

and

$$\mathrm{Im}(wr(\xi)) = \mathrm{Re}((z\overline{r(\xi)}\,)^3).$$

Then the partial indices of the closed path $p(\xi)=0$, $\xi\in\partial D$, are 1 and -1 and the total index is 0. Also, the defect $\mathrm{def}(p)$ is 1 and the dimension of the spaces $V_p^{\perp}(\xi)$, $\xi\in\partial D$, is 1. Let (z,w) be a holomorphic disc with boundary in the maximal real fibration $\{M(\xi)\}_{\xi\in\partial D}$ and such that $(z(1),w(1))=(0,0)$. Then from the first equation we get

$$z(\xi) = ia(\xi-1)$$

for some real number a. The second equation now implies

$$\mathrm{Im}(w(\xi^2)\xi) = \mathrm{Re}((ia(\xi-\bar{\xi}))^3)$$

or

$$\mathrm{Im}(w(\xi^2)\xi) = 2a^3\,\mathrm{Im}(\xi^3-3\xi).$$

Hence we get

$$w(\xi) = 2a^3(\xi-3)$$

which can be 0 at $\xi=1$ if only if $a=0$. Thus we showed that the only holomorphic disc attached to the fibration $\{M(\xi)\}_{\xi\in\partial D}$ and which is passing through the point $(0,0)$, is the zero disc $p(\xi)=0$, $\xi\in\partial D$.

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