Ark. Mat., **32** (1994), 195–211

Almost everywhere convergence of the inverse spherical transform on $SL(2, \mathbf{R})$

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Abstract. We prove the almost everywhere convergence of the inverse spherical transform of L_p bi-K-invariant functions on the group $SL(2, \mathbf{R})$, $\frac{4}{3} < p \leq 2$. The result appears to be sharp.

1. Preliminary material about $SL(2, \mathbf{R})$

Before stating the theorem proved below, we present some background material on spherical functions on $SL(2, \mathbf{R})$.

The group $SL(2, \mathbf{R})$, consisting of 2×2 matrices with real entries and with determinant equal to 1, is often the testing ground for questions in analysis on noncompact semisimple real Lie groups. In all that follows G will denote $SL(2, \mathbf{R})$. Inside G there is the compact subgroup $K = SO(2)$, consisting of all orthogonal matrices in G, so that elements of K are matrices of the form

$$k(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

In addition to the subgroup K, there is the subgroup of diagonal elements of G with positive diagonal entries,

$$A = \left\{ a(s) = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix} : s \in \mathbf{R} \right\}.$$

This normalizes the subgroup

$$N = \left\{ n(\xi) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} : \xi \in \mathbf{R} \right\}.$$

([1]) Partially supported by the MPI

([2]) Partially supported by the CMA

In fact $a(s)n(\xi)a(-s)=n(e^s\xi)$ for all s and ξ in $\mathbf{R}$.

As basic references for calculations on G, there is the survey article of T. Koornwinder [Ko].

Lemma 1 (Cartan decomposition). *Each element $x\in G$ can be decomposed into a product of the form $x=k_1ak_2$, with k_1, $k_2\in K$ and $a\in A$. If $x\neq 1$, then there are exactly two elements a, $a'\in A$ such that $x\in KaK$ and $x\in Ka'K$. The elements a, a' are inverses of each other, i.e. $a'=a^{-1}$.*

The decomposition $G=KAK$ is also called the polar decomposition and is analogous to using polar coordinates in Euclidean space. We can equip K with normalized Lebesgue measure, so that

$$\int_K f(k)\,dk=\frac{1}{2\pi}\int_0^{2\pi} f(k(\theta))\,d\theta$$

for all continuous functions f on K. Similarly, since A is isomorphic with the real line, it can also be equipped with Lebesgue measure. Let μ denote Haar measure on G, normalized according to the following integral formula.

Lemma 2 (Integration formula). *For every compactly supported continuous function f on G,*

$$\int_G f(x)\,d\mu(x)=\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\int_0^{2\pi} f(k(\theta_1)a(s)k(\theta_2))\sinh(s)\,d\theta_1 d\theta_2\,ds.$$

Consider the action of G on the upper half plane $\mathcal{H}=\{z\in\mathbf{C}:\operatorname{Im}(z)>0\}$. If $g\in G$ is of the form

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix},$$

and if $z\in\mathcal{H}$, then the action of g on z is $g\cdot z=(az+b)/(cz+d)\in\mathcal{H}$. In particular, N acts by translation parallel to the real axis $(n(\xi)\cdot z=z+\xi)$ and A acts by dilations $(a(s)\cdot z=e^s z)$. This shows that the action is transitive. K fixes the point $i\in\mathcal{H}$, which we will treat as the origin. Hence $\mathcal{H}$ can be identified with the homogeneous space G/K, so that $g\cdot i$ is identified with the coset gK in G/K.

If f is a right-K-invariant function on G, then it can be identified with a function $f^\sharp$ on $\mathcal{H}$ by assigning $f^\sharp(g\cdot i)=f(g)$, for all $g\in G$. Similarly, every function F on $\mathcal{H}$ is equivalent to a right-K-invariant function $F^\flat$ on G, with $F^\flat(g)=F(g\cdot i)$, for all $g\in G$. The set $\mathcal{H}$ can be equipped with the Poincaré metric $ds^2=dx^2+dy^2/y^2$, which is invariant under fractional linear transformations $z\mapsto g\cdot z$. The corresponding G-invariant measure on $\mathcal{H}$ is

$$\int_{\mathcal{H}} f(x+iy)\,\frac{dxdy}{y^2}.$$

When $\mathcal{H}$ is equipped with the Poincaré metric, it carries the Laplace–Beltrami operator

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

and this is G-invariant. Clearly Δ acts on smooth right-K-invariant functions on G by forming $\Delta f(g)=(\Delta f^\sharp)^\flat(g)$.

We now concentrate on analysis of bi-K-invariant functions on G. From the Cartan decomposition, we see that if f satisfies $f(k_1gk_2)=f(g)$ for all k_1, $k_2\in K$ and $g\in G$, then f is completely determined by its restriction to $\{a(s):s\geq 0\}$. In particular,

$$\int_G f(g)\,d\mu(g) = 2\pi\int_0^\infty f(a(s))\sinh(s)\,ds. \tag{1}$$

In addition, if f is bi-K-invariant then $s\mapsto f(a(s))$ is an even function on the real line. Another interpretation of bi-K-invariant functions comes from viewing them as functions on $\mathcal{H}$ with the property that $F(k\cdot z)=F(z)$ for all $k\in K$ and $z\in\mathcal{H}$. In this setting, they are "radial functions" on $\mathcal{H}$, depending only on the distance from i with respect to the Poincaré metric. When thinking in these terms, it is natural that there should be an integral transform analogous to the Hankel transform of radial functions and coming from the eigenfunctions of the Laplace–Beltrami operator. The transform which plays this role is called the *spherical transform*. For an introduction to this theory, see the notes of Godement [Gd]. The elementary spherical functions are radial eigenfunctions of Δ.

Definition. A continuous function φ on G is said to be an elementary spherical function if it satisfies the following three conditions:

- $\varphi(1)=1$,
- φ is bi-K-invariant,
- there is a complex number α such that $\Delta\varphi=\alpha\varphi$.

These are given by Legendre functions on $[0,\infty)$. That is,

$$\varphi_\lambda(a(s)) = \frac{1}{2\pi}\int_0^{2\pi}(\cosh(s)+\sinh(s)\cos(\theta))^{-(1/2)+i\lambda}\,d\theta$$

for all nonnegative numbers s and complex λ, with $\Delta\varphi_\lambda=-(\lambda^2+\frac{1}{4})\varphi_\lambda$.

The spherical transform of an integrable bi-K-invariant function f on G is

$$\mathcal{F}f(\lambda)=\int_G f(g)\varphi_\lambda(g)\,d\mu(g) = 2\pi\int_0^\infty f(a(s))\varphi_\lambda(a(s))\sinh(s)\,ds.$$

In particular, there is the Plancherel formula for square-integrable bi-K-invariant functions on G. There is a measure ν on $[0,\infty)$ such that $f\mapsto\mathcal{F}f|_{[0,\infty)}$ extends from ${}^K L_1^K(G)\cap L_2(G)$ to be an isometry

$$\mathcal{F}:{}^K L_2^K(G)\longrightarrow L_2([0,\infty),\nu).$$

The density for this measure is $d\nu(\lambda)=\lambda\tanh(\pi\lambda)\,d\lambda/\pi$, and the inversion formula is

$$f(a(s))=\frac{1}{\pi}\int_0^\infty \mathcal{F}f(\lambda)\varphi_\lambda(a(s))\lambda\tanh(\pi\lambda)\,d\lambda. \tag{2}$$

When f is a smooth, compactly supported bi-K-invariant function, this inversion formula converges absolutely for all $s\geq 0$. Hence, in order to prove almost everywhere convergence for the inverse spherical transform on some space ${}^K L_p^K(G)$, it suffices to prove a boundedness result for the maximal function

$$S^*f(a(t))=\sup_{R>1}\left|\int_1^R \mathcal{F}f(\lambda)\varphi_\lambda(a(t))\,d\nu(\lambda)\right|.$$

2. Statement of results

For simplicity we write t instead of $a(t)$ and we set

$$\psi(t,\lambda)=|\lambda\tanh\,(\pi\lambda)|^{1/2}(\sinh(t))^{1/2}\varphi_\lambda(t).$$

We now state our main theorem, its consequences and make comments on the sharpness of the results.

Theorem. *Let* $S_Rf(t)=\int_0^\infty f(r)H_R(t,r)dr$ *where*

$$H_R(t,r)=\sqrt{\frac{\sinh(r)}{\sinh(t)}}\int_0^R \psi(t,\lambda)\psi(r,\lambda)\,d\lambda.$$

Then $S^*f(t)=\sup_{R>1}|S_Rf(t)|$ *is a bounded operator from* ${}^K L_p^K(G)$ *into* $L_2(G)+L_p(G)$ *for* $\frac{4}{3}<p\leq 2$.

Corollary 1. *With the same notation,* S^* *maps* ${}^K L_p^K(G)$ *boundedly into itself if and only if* $p=2$.

Proof. Because of the theorem, we need only show that S^* is unbounded on ${}^K L_p^K(G)$ when $1\leq p<2$. However, it is true that if $f\neq 0$ belongs to ${}^K L_p^K(G)$ and

$1\le p<2$ then $S_1 f$ is not in $L_p(G)$. This follows from Theorem 4.4 [ST] where it is shown that if g is in ${}^K L_p^K(G)$ then its spherical transform $\hat{g}$ may be extended to be an analytic function in the strip

$$\left\{ z\in\mathbf{C} : |\operatorname{Im}(z)| < \frac{1}{2}\left(\frac{2}{p}-1\right)\right\}.$$

Notice that $\widehat{(S_1 f)}$ cannot be analytic on such a set unless f is identically zero. The case $p>2$ is handled by an argument which goes back to Rubio de Francia [CS]. The point being that $S_1 f$ is not even a tempered distribution on G. If it were the case that $S_1: {}^K L_p^K(G)\to C_c^\infty(G)'$ with $p>2$, then by duality we would have $S_1: C_c^\infty\to L^{p'}(G)^p$, with $p'<2$. The analyticity argument used above excludes this possibility.

Corollary 2. *If $f\in {}^K L_p^K(G)$, $\frac{4}{3}<p\le 2$, then $S_R f(t)\to f(t)$ a.e. as $R\to\infty$.*

Now let us comment on the sharpness of the main result. By Rubio's argument our theorem cannot hold for $p>2$. In addition the lower bound $\frac{4}{3}$ appears to be sharp. As we shall see below, if $f(r)$ is supported near the origin then $S_R f(t)$ for t close to zero is essentially the spherical partial sums operator of the Euclidean space $\mathbf{R}^2$, that we also denote by S_R, applied to radial functions. Now Kanjin in [Ka] proved that there exists h in $L_{4/3}(\mathbf{R}^2)$ compactly supported and radial such that $S_R h(x)$ diverges a.e. as $\mathbf{R}\to\infty$.

Our proof is based on Schindler's asymptotic estimates for $\psi(t,\lambda)$ of Section 3, on an analogous result in the Euclidean space $\mathbf{R}^2$ [P] and on the $L_p\to L_p+L_2$ estimate for the Carleson operator with exponential weights of Section 4.

3. Schindler's estimates and preparatory lemmas

The following formulas (S1)–(S4) give the asymptotic properties of the function $\psi(t,\lambda)$. They can be found in [Sc], since $\psi(t,\lambda)$ is a special case of $K^m(x,y)$ in [Sc] with $m=0, x=\lambda, y=t$.

In the region $t\ge 1, \lambda\le 1$

$$\text{(S1)}\qquad \begin{aligned}\psi(t,\lambda) = \pm\left(\frac{2}{\pi}\right)^{1/2}&\sin(\lambda t)+\cos(\lambda t)h_1(\lambda)+\sin(\lambda t)h_2(\lambda)\\ &+\cos(\lambda t)s_1(\lambda,t)+\sin(\lambda t)s_2(\lambda,t)\end{aligned}$$

where

$$s_j, \frac{\partial s_j}{\partial\lambda} = \mathbf{O}(e^{-2t}),$$
$$h_j(\lambda)=\mathbf{O}(\lambda),$$
$$h_j'(\lambda), h_j''(\lambda) = \mathbf{O}(1),\ \ j=1,2.$$

In the region $t \geq 1, \lambda \geq 1$

$$\begin{aligned}\psi(t,\lambda) = {} & \left(\frac{2}{\pi}\right)^{1/2} \cos\left(\lambda t - \frac{\pi}{4}\right) + \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{8\lambda} \sin\left(\lambda t - \frac{\pi}{4}\right) \\ & + \cos\left(\lambda t - \frac{\pi}{4}\right)\left[S_1(\lambda) + \frac{T_1(\lambda)}{e^{2t}-1}\right] + \sin\left(\lambda t - \frac{\pi}{4}\right)\left[S_2(\lambda) + \frac{T_2(\lambda)}{e^{2t}-1}\right] \\ & + \cos\left(\lambda t - \frac{\pi}{4}\right) D_1(\lambda,t) + \sin\left(\lambda t - \frac{\pi}{4}\right) D_2(\lambda,t)\end{aligned} \tag{S2}$$

where

$$\begin{aligned} S_j, T_j, S_j'' T_j'' &= \mathbf{O}(\lambda^{-2}), \\ S_j', T_j' &= \mathbf{O}(\lambda^{-3}), \\ D_j, \frac{\partial D_j}{\partial \lambda} &= \mathbf{O}(\lambda^{-2} e^{-4t}). \end{aligned}$$

In the region $t \leq 1, \lambda \leq 1$

$$\psi(t,\lambda) = \sqrt{\pi}\lambda \sinh^{1/2}(t) + \sqrt{\pi}\lambda t_1(t,\lambda) + \sinh^{1/2}(t) s(\lambda) + s(\lambda) t_1(t,\lambda) \tag{S3}$$

where

$$\begin{aligned} t_1, \frac{\partial t_1}{\partial t} &= \mathbf{O}(t^{5/2}), \\ s(\lambda) &= \mathbf{O}(\lambda^3), \\ s'(\lambda) &= \mathbf{O}(\lambda^2). \end{aligned}$$

In the region $t \leq 1, \lambda \geq 1$

$$\psi(t,\lambda) = (t\lambda)^{1/2} J_0(t\lambda) + \varphi_1(t) t^{3/2} \lambda^{-1/2} J_1(t\lambda) + F_0(t,\lambda) \tag{$S4$}$$

where $\varphi_1(t) = \mathbf{O}(1)$,

$$F_0(t,\lambda) = (t\lambda)^{1/2} \widetilde{K}_0(\lambda) J_0(t\lambda) + \begin{cases} \mathbf{O}(\lambda^{-3/2} t^{5/2}), & \lambda t \leq 1, \\ \mathbf{O}(\lambda^{-2} t^2), & \lambda t \geq 1, \end{cases}$$

and $\widetilde{K}_0(\lambda) = \mathbf{O}(\lambda^{-2})$, $\widetilde{K}_0'(\lambda) = \mathbf{O}(\lambda^{-3})$,

$$\frac{\partial F_0}{\partial \lambda} = \frac{\partial}{\partial \lambda}\{\widetilde{K}_0(\lambda)(t\lambda)^{1/2} J_0(t\lambda)\} + \begin{cases} \mathbf{O}\left(\lambda^{-5/2} t^{5/2}\right), & \lambda t \leq 1, \\ \mathbf{O}\left(\lambda^{-2} t^3\right), & \lambda t \geq 1. \end{cases}$$

The following lemma computes the Fourier transform of a smooth function, zero at the origin and equal to $1/\lambda$ at infinity.

Lemma 1. (a) *There exists* $\Psi\in C^\infty$*, supported on* $\left[\frac{1}{2},2\right]$ *such that*

$$\sum_{k=0}^{\infty}\Psi(2^{-k}\lambda)=1 \quad \text{on } [1,+\infty).$$

(b) *Set* $\lambda\Phi(\lambda)=\Psi(\lambda)$*, then* $\sum_{k=0}^{\infty}2^{-k}\Phi\left(2^{-k}\lambda\right)=1/\lambda$ *on* $[1,+\infty)$*, and*

$$\left|\left(\sum_{k=0}^{\infty}2^{-k}\Phi(2^{-k}\lambda)\right)^{\wedge}(x)\right|\le\begin{cases} c\lg\dfrac{1}{|x|}, & |x|<1,\\ \dfrac{C_M}{|x|^M}, & |x|\ge 1,\end{cases}$$

for every $M\ge 1$.

The proof is easy and it is left to the reader. The next lemma carries well known estimates on Bessel functions $J_k(t)$ that we shall need only for $k=0,1$.

Lemma 2.

$$|J_k(t)|\le c_k,\quad t\ge 0, \tag{1}$$

$$J_k(t)=\sqrt{\frac{2}{\pi t}}\cos\left(t-\frac{\pi}{2}k-\frac{\pi}{4}\right)+E_k(t), \tag{2}$$

where $|E_k(t)|\le\bar{c}_k/t^{3/2}$ *for* $t>1$.

4. A simpler case

Recall that

$$S_Rf(t)=\int_0^\infty f(r)H_R(t,r)\,dr,$$

$$H_R(t,r)=\sqrt{\frac{\sinh(r)}{\sinh(t)}}\int_0^R\psi(t,\lambda)\psi(r,\lambda)\,d\lambda,$$

$$S^*f(t)=\sup_{R>1}|S_Rf(t)|.$$

Using Schindler's estimates one sees that the kernel $H_R(t,r)$ shows a singularity along the line $t=r$, so we are going to focus on the region $t\sim r$. We write

$$H_R(t,r)=\sqrt{\frac{\sinh(r)}{\sinh(t)}}\left(\int_0^1\ldots d\lambda+\int_1^R\ldots d\lambda\right)=H^1+H_R^2$$

and consequently we have $S^*f(t)\leq S_1^*f(t)+S_2^*f(t)$.

First we consider the case $r,t\leq 1$. By (S3) we have $|H^1(t,r)|\leq r$ and so the corresponding operator $S_1^*f(t)\leq c\|f\|_{L_p(G)}$ by Hölder's inequality. Therefore

$$\|S_1^*f\|_{L_p(G)}\leq c_p\|f\|_{L_p(G)},\quad 1<p<\infty.$$

By (S4) we see that $H_R^2(t,r)$ contains a typical term like

$$\sqrt{\frac{\sinh(r)}{\sinh(t)}}t^{1/2}r^{1/2}\int_1^R J_0(t,\lambda)J_0(r,\lambda)\lambda\,d\lambda.$$

Adding a term that can be controlled as we did with H^1, we can say that in first approximation $H_R^2(t,r)\simeq r\int_0^R J_0(t\lambda)J_0(r\lambda)\lambda\,d\lambda$. We still denote by S_2^* the corresponding operator and we handle it as follows.

In [P] it has been shown that

$$(P1)\qquad \sup_{R>1}\left|\int_0^\infty\int_0^R J_0(t,\lambda)J_0(r,\lambda)\lambda\,d\lambda\,r f(r)\,dr\right|\leq ct^{-1/2}M(f(r)r^{1/2})(t),$$

where M denotes the sum of the maximal function, of the maximal Hilbert transform and of the maximal Carleson operator. Therefore the same estimate holds for $S_2^*f(t)$.

Then by a weighted estimate [GR] we have

$$\begin{aligned}\|S_2^*f(t)\|_{L_p(G)}&\leq c_p\left(\int_0^1|M(f(r)r^{1/2})(t)|^p t^{1-p/2}dt\right)^{1/p}\\&\leq c_p\left(\int_0^1|f(t)|^p t\,dt\right)^{1/p}\leq c_p\|f\|_{L_p(G)},\end{aligned}$$

if $\frac{4}{3}<p<4$.

Now we consider the case $r,t\geq 1$. Again one finds that H^1 is much simpler than H_R^2, so let us disregard it for the time being.

By (S2) we can see that in first approximation

$$(P2)\qquad S_2^*f(t)\simeq\frac{1}{\sqrt{\sinh(t)}}\sup_{R>1}\left|\int_1^\infty\frac{e^{iR(t-r)}}{t-r}f(r)\sqrt{\sinh(r)}dr\right|.$$

We write the above operator as

$$\frac{1}{\sqrt{\sinh(t)}}\sum_{k=1}^\infty\left(\int_k^{k+1}\frac{e^{iR(t-r)}}{t-r}f(r)\sqrt{\sinh(r)}dr\right)(\varphi_k(t)+\psi_k(t))=\sum_{k=1}^\infty A_{k,R}+B_{k,R},$$

where $\varphi_k(t)+\psi_k(t)=1$ for every $t\geq 1$ and $\varphi_k=\chi_{[k-1,\,k+2)}(t)$. Correspondingly

$$S_2^*f(t)\leq \sum_{k=1}^{\infty} A_k^*f(t)+\sup_{R>1}\left|\sum_{k=1}^{\infty} B_{k,R}f(t)\right|.$$

We shall prove

$$\text{(i)}\qquad \left\|\sum_{k=1}^{\infty} A_k^*f(t)\right\|_{L_p(G)}\leq c_p\|f\|_{L_p(G)},\quad 1<p<\infty,$$

$$\text{(ii)}\qquad \left\|\sup_{R>1}\left|\sum_{k=1}^{\infty} B_{k,R}f(t)\right|\right\|_{L_2(G)}\leq c_p\|f\|_{L_p(G)},\quad 1<p\leq 2.$$

From this the desired estimate for S_2^* follows. To prove (i) observe that

$$A_k^*f(t)=\frac{1}{\sqrt{\sinh(t)}}\varphi_k(t)(Cg_k)(t),$$

where $g_k(r)=f(r)\sqrt{\sinh(r)}\chi_{[k,k+1]}(r)$ and C denotes the Carleson operator. Moreover two different $A_k^*f(t)$'s have essentially disjoint supports. So for $1<p<\infty$ we have

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} A_k^*f(t)\right\|_{L_p(G)}^p&\leq c_p\sum_{k=1}^{\infty}\|A_k^*f(t)\|_{L_p(G)}^p\\&\leq c_p\sum_{k=1}^{\infty} e^{k(1-p/2)}\int_{k-1}^{k+2}|Cg_k(t)|^p dt\\&\leq c_p\sum_{k=1}^{\infty} e^{k(1-p/2)}\int_{0}^{\infty}|g_k(r)|^p dr\leq c_p\|f\|_{L_p(G)}^p.\end{aligned}$$

Now we turn to (ii). Observe that (ii) holds for $p=2$ by the L_2 boundedness of S_2^* that can be easily proved by the L_2 boundedness of the Carleson operator and by (i) with $p=2$.

We shall now prove a restricted (L_p,L_2) estimate for $\sum_{k=1}^{\infty} B_k^*f(t)$, $1\leq p<2$. By interpolation (ii) follows. Let $E\subset[1,+\infty]$ be any measurable set. If $\|\chi_E\|_{L_2(G)}\geq 1$ then $\|\sum_k B_k^*\chi_E\|_{L_2(G)}\leq c\|\chi_E\|_{L_2(G)}\leq c\|\chi_E\|_{L_p(G)}$ for $p\leq 2$. If $\|\chi_E\|_{L_2(G)}<1$ then observe that

$$|B_k^*\chi_E(t)|\leq e^{-t/2}\left(\int_k^{k+1}\frac{1}{|t-r|}\chi_E(r)\sqrt{\sinh(r)}\,dr\right)\psi_k(t)\simeq\frac{e^{-t/2}}{|t-k|}e^{k/2}|E_k|\psi_k(t),$$

where $E_k = E\cap[k,k+1]$ and $|E_k|$ denotes its Euclidean measure. Now

$$\Big\|\sum_k B_k^*\chi_E\Big\|_{L_2(G)} \le \sum_k \|B_k^*\chi_E(t)\|_{L_2(G)} \le c\sum_k e^{k/2}|E_k| \le c\sum_k e^k|E_k|$$
$$\le c\int_1^\infty \chi_E(r)\sinh(r)\,dr \le c\|\chi_E\|_{L_p(G)}$$

for $1\le p\le\infty$. We suggest the interested reader to see [CV] for a closely related argument.

5. The complete proof

In this section we shall see that

(1) the main term of our operator is slightly more complicated than the one considered in Section 4, namely it will be the one of the previous section applied not directly to $f(r)\sqrt{\sinh(r)}$ but to $\sigma*(f(r)\sqrt{\sinh(r)})$ where $\sigma\in L^1(\mathbf{R})$,

(2) we will take into account the error terms,

(3) we will take into consideration the region $\{t\le 1, r\ge 1\}$ and $\{t\ge 1, r\le 1\}$.

Let us start from the last point and subdivide the region $\{(t,r): t\ge 0, r\ge 0\}$ into four pieces

1^{st} region $= \{t\ge\frac12, r\ge\frac12\}$,
2^{nd} region $= \{0\le t\le\frac12, 0\le r\le 1\}\cup\{0\le t\le 1, 0\le r\le\frac12\}$,
3^{rd} region $= \{0\le t\le\frac12, r\ge 1\}$,
4^{th} region $= \{t\ge 1, 0\le r\le\frac12\}$.

Observe that the 3^{rd} and 4^{th} regions are far away from the line $r=t$ where $H(t,r)$ is "singular".

The first region

For technical reasons we are going to break up smoothly the domain of integration in λ. Let $g_1(\lambda)$ be a C^∞ function supported on $|\lambda|\le 1$ and such that $g_1(\lambda)=1$ for $|\lambda|\le\frac12$. Let $g_2(\lambda)$ be defined by $g_1(\lambda)+g_2(\lambda)=1$ for every $\lambda\ge 0$.

Then $H_R = H^1 + H_R^2$ where

$$H^1(t,r) = \frac12\sqrt{\frac{\sinh(r)}{\sinh(t)}}\int_{-1}^1 \psi(t,\lambda)\psi(r,\lambda)g_1(\lambda)\,d\lambda$$

is independent of R. From (S1) it follows that

$$\sqrt{\frac{\sinh(t)}{\sinh(r)}}H^1(t,r)$$

is the sum of terms like

$$\int_{-1}^{1} e^{i\lambda(t-r)} g_1(\lambda)\,d\lambda = \hat{g}_1(t-r),$$

$$\int_{-1}^{1} e^{i\lambda(t-r)} h_1(\lambda) g_1(\lambda)\,d\lambda = (h_1 g_1)^{\wedge}(t-r),$$

$$\int_{-1}^{1} e^{i\lambda(t-r)} h_1(\lambda) s_1(\lambda,t) g_1(\lambda)\,d\lambda,$$

$$\int_{-1}^{1} e^{i\lambda(t-r)} h_1(\lambda) s_1(\lambda,r) g_1(\lambda)\,d\lambda,$$

$$\int_{-1}^{1} e^{i\lambda(t-r)} s_1(\lambda,t) s_1(\lambda,r) g_1(\lambda)\,d\lambda$$

and similar terms containing $t+r$ in the exponent. Simpler operators are associated to the latter and we leave their study to the interested reader.

The first two terms are $L^1(\mathbf{R})$ convolution kernels, since $g_1, h_1 g_1 \in C^{(2)}(\mathbf{R})$. Hence the corresponding operator, being simpler than the operator (P2) handled in the previous section, maps boundedly ${}^K L_p^K(G)$ into $L_p(G)+L_2(G)$, $1<p\le 2$. Let us consider the third term; its corresponding kernel, that by an abuse of notation we still denote by $H^1(t,r)$, is dominated by $e^{-3/2t} e^{r/2}/r$. For, if $t>r/2$ taking absolute values inside the integral we obtain a bound of

$$\sqrt{\frac{\sinh(r)}{\sinh(t)}}\, e^{-2t},$$

which implies the claimed estimate; if $t<r/2$ one integration by parts gives a bound of

$$\sqrt{\frac{\sinh(r)}{\sinh(t)}}\, \frac{e^{-2t}}{r}$$

as claimed.

Then, if p and q are dual exponents and $1<p\le 2$ we have

$$\begin{aligned}\int_{1/2}^{\infty} H^1(t,r) f(r)\,dr &\le e^{-3/2t} \|f\|_{L_p(G)} \left(\int_{1/2}^{\infty} e^{r(1/2-1/p)q} r^{-q} dr \right)^{1/q} \\ &\le c_p e^{-3/2t} \|f\|_{L_p(G)}\end{aligned}$$

and so

$$\left(\int_{1/2}^{\infty} \left| \int_{1/2}^{\infty} H^1(t,r) f(r)\,dr \right|^p \sinh(t)\,dt \right)^{1/p} \le c_p \|f\|_{L_p(G)}.$$

Exchanging the roles of t and r we handle the fourth term; its corresponding kernel, that we still denote by $H^1(t,r)$, is dominated by $(e^{-r/2}e^{-t/2})/t$. Hence for $1<p\leq 2$,

$$\int_{1/2}^{\infty} H^1(t,r)f(r)\,dr \leq \frac{e^{-t/2}}{t}\|f\|_{L_p(G)}\left(\int_{1/2}^{\infty} e^{-r(1/2+q/p)}dr\right)^{1/q}$$
$$\leq c_p\frac{e^{-t/2}}{t}\|f\|_{L_p(G)}$$

and so

$$\left(\int_{1/2}^{\infty}\left|\int_{1/2}^{\infty} H^1(t,r)f(r)\,dr\right|^2 \sinh(t)\,dt\right)^{1/2} \leq c_p\|f\|_{L_p(G)}.$$

The fifth term is even easier. So we proved that $S_1^*: {}^K L_p^K(G)\to L_2(G)+L_p(G), 1<p\leq 2$.

Let us turn to H_R^2. From (S2) it follows that

$$\sqrt{\frac{\sinh(t)}{\sinh(r)}}H_R^2$$

is the sum of terms like

$$\int_0^R e^{i\lambda(t-r)}g_2(\lambda)\,d\lambda,$$
$$\int_0^R \frac{e^{i\lambda(t-r)}}{\lambda}g_2(\lambda)\,d\lambda,$$
$$\int_0^R e^{i\lambda(t-r)}S_j(\lambda)g_2(\lambda)\,d\lambda,$$
$$\int_0^R e^{i\lambda(t-r)}D_j(\lambda,t)g_2(\lambda)\,d\lambda.$$

Let us examine the first term which is equal to

$$\int_0^1 e^{i\lambda(t-r)}g_2(\lambda)\,d\lambda+\int_1^R e^{i\lambda(t-r)}d\lambda = G(t-r)+\frac{e^{iR(t-r)}}{i(t-r)}-\frac{e^{i(t-r)}}{i(t-r)},$$

where $G(y)$ is bounded at the origin and for $|y|>1$ is equal to e^{iy}/y plus an error. The error is dominated by c/y^2, as one can see integrating by parts twice. So the work done to control the operator in $(P2)$ applies here. The second term, by inspection, should not be any worse than the first one. We shall back up this claim by a precise argument. We write the second term as a sum of three pieces as we did above. Let us follow the most complicated one, namely the one that carries the dependency upon R.

Setting $y=t-r$

$$\int_1^R \frac{e^{i\lambda y}}{\lambda} g_2(\lambda)\, d\lambda = \left(\chi_{[0,R]}(\lambda)\cdot\frac{g_2(\lambda)}{\lambda}\right)^\wedge(y)$$
$$= \left[\frac{e^{iRx}}{x} * \left(\frac{g_2(\lambda)}{\lambda}\right)^\wedge\right](y).$$

Now $\sigma(y)=((g_2(\lambda))/\lambda)^\wedge(y)$ is described in Lemma 1. So we have to handle

$$(P3) \qquad \frac{1}{\sqrt{\sinh(t)}}\sup_{R>1}\left|\frac{e^{iRx}}{x} * \sigma * \left(f(r)\sqrt{\sinh(r)}\,\right)(t)\right|.$$

By the method used to control the operator in $(P2)$ we can prove that

$$Tf(y) = \frac{1}{\sqrt{\sinh(y)}}\int_{1/2}^\infty \sigma(y-r) f(r)\sqrt{\sinh(r)}\, dr$$

maps boundedly ${}^K L_p^K(G)$ into L_2+L_p.

Now we have to apply to $Tf(y)$ the operator in (P2) namely

$$\sup_R\left|\int_0^\infty \frac{e^{iR(t-y)}}{t-y}\, Tf(y)\sqrt{\sinh(y)}\, dy\right|, \quad t\ge\tfrac12,$$

that we know maps L_p into L_2+L_p, $1<p\le 2$.

This proves that the operator in (P3) does the same.

The third term is a convolution kernel uniformly bounded at the origin and equal to $(e^{iRy}/y)S_j(R)$ plus negligible terms at infinity, as one can see integrating by parts twice and using (S2).

The fourth term can be handled in a way similar to that used for the third term of the preceding $H^1(t,r)$ kernel. So we proved that

$$S_2^* : {}^K L_p^K(G) \to L_2(G)+L_p(G), \quad 1<p\le 2.$$

The second region

As before we break up the domain of integration in λ in a smooth way by means of $g_1(\lambda)$ and $g_2(\lambda)$ and we have $H_R=H^1+H_R^2$. By $(S3)$ we obtain

$$H^1(t,r)\le c\sqrt{\frac{\sinh(r)}{\sinh(t)}}\left(\int_0^2 \lambda^2 \sinh^{1/2}(t)\sinh^{1/2}(r) g_1(\lambda)\, d\lambda + \text{similar terms}\right)\le cr$$

and so the corresponding operator S_1^* satisfies

$$\text{(P4)} \qquad |S_1^* f(t)| \le \int_0^2 r f(r) dr \le c_p \|f\|_{L_p(G)}, \quad 1<p<\infty.$$

Therefore $\|S_1^* f\|_{L_p(G)} \le c_p \|f\|_{L_p(G)}$, for $1<p<\infty$.

By $(S4)$ it follows that

$$\sqrt{\frac{\sinh(t)}{\sinh(r)}} H_R^2$$

breaks up in several terms like

$$t^{1/2} r^{1/2} \int_0^R J_0(t\lambda) J_0(r\lambda) \lambda g_2(\lambda)\, d\lambda,$$

$$t^{1/2} r^{3/2} \varphi_1(r) \int_0^R J_0(t\lambda) J_1(r\lambda) g_2(\lambda)\, d\lambda,$$

$$t^{3/2} r^{1/2} \varphi_1(t) \int_0^R J_1(t\lambda) J_0(r\lambda) g_2(\lambda)\, d\lambda,$$

$$t^{3/2} r^{3/2} \varphi_1(r) \varphi_1(t) \int_0^R J_1(t\lambda) J_1(r\lambda) \frac{g_2(\lambda)}{\lambda}\, d\lambda,$$

$$r^{1/2} \int_0^R \lambda^{1/2} J_0(r\lambda) F_0(t,\lambda) g_2(\lambda)\, d\lambda,$$

$$t^{1/2} \int_0^R \lambda^{1/2} J_0(t\lambda) F_0(r,\lambda) g_2(\lambda)\, d\lambda$$

and other negligible terms.

We consider the first term that we write as a main term plus an error as follows

$$t^{1/2} r^{1/2} \left[\int_0^R J_0(t\lambda) J_0(r\lambda) \lambda\, d\lambda - \int_0^2 J_0(t\lambda) J_0(r\lambda)(1-g_2(\lambda)) \lambda\, d\lambda \right].$$

For the operator that corresponds to the main term see (P1).

The error term, independent of R, is dominated by $ct^{1/2} r^{1/2}$ and so the corresponding operator satisfies the estimate in (P4). Let us move to the second term that, by inspection, should not be any worse than the first one. Suppose e.g. that $r<t$ and break up the domain of integration $[0,R]=[0,1/r]\cup[1/r,R]$ in a smooth way, obtaining an error and a main term. Since $|\int_0^{1/r} J_0(t\lambda) J_1(r\lambda) g_2(\lambda) d\lambda| \le c/r$, the error term is harmless; the corresponding operator satisfies the estimate in (P4). Consider the main term. By Lemma 2

$$\int_{1/r}^R J_0(t\lambda) J_1(r\lambda)\, d\lambda = c t^{-1/2} r^{-1/2} \int_{1/r}^R \frac{e^{i\lambda(t-r)}}{\lambda}\, d\lambda + \text{errors}.$$

We leave to the interested reader the study of the errors. To decode what remains recall that we broke up the domain of integration in λ in a smooth way, so the corresponding operator is dominated by

$$t^{-1/2}\sup_{R>1}\left|\frac{e^{iRx}}{x}*\varrho_r*\big(f(r)r^{3/2}\big)(t)\right|,$$

where $\varrho_r\in L^1(\mathbf{R})$ satisfies the estimates of Lemma 1 independently of r. (By $K*g(t)$ we denote $\int K(t-r)g(r)dr$.) This is slightly more complicated than the operator in (P1). We already encountered a similar situation and we know that the L_p estimate remains the same. Similarly if $r>t$. The third and fourth term are not any worse than the second one and they can be handled similarly. The remaining terms can be controlled by taking absolute values inside the integration in λ. Let us consider for instance $r^{1/2}\int_{1/2}^{R}\lambda^{1/2}J_0(r\lambda)F_0(t,\lambda)g_2(\lambda)d\lambda$. Suppose $r<t$ (and proceed similarly if $r>t$). Then break up the domain of integration $\left[\frac12,R\right]=\left[\frac12,1/r\right]\cup[1/r,R]$. Now observe that $|r^{1/2}\int_{1/2}^{1/r}\lambda^{1/2}J_0(r\lambda)F_0(t,\lambda)d\lambda|\le r^{1/2}\lg(r)t^{1/2}$.

Therefore the corresponding operator, let us call it again S_1^*, satisfies the estimate $|S_1^*f(t)|\le c_p\|f\|_{L_p(G)}\cdot(\int_0^1 r\lg^q(1/r)dr)^{1/q}\le c_p\|f\|_{L_p(G)}$ and so S_1^* is bounded from ${}^KL_p^K(G)$ to itself, $1<p<\infty$. We are left with

$$\left|r^{1/2}\int_{1/r}^{R}\lambda^{1/2}J_0(r\lambda)F_0(t,\lambda)\,d\lambda\right|\le 1.$$

Therefore the corresponding operator, that we call again S_2^*, satisfies the estimate

$$\left|S_2^*f(t)\right|\le t^{-1/2}\int_0^1|f(r)|r^{-1/2}dr\le c_pt^{-1/2}\left\|f\right\|_{L_p(G)},\quad \text{if } p>\tfrac43.$$

Then

$$\left\|S_2^*f\right\|_{L_p(G)}\le c_p\left\|f\right\|_{L_p(G)},\quad \text{if } p<4.$$

The third region

We shall prove that the corresponding operator maps boundedly ${}^KL_p^K(G)$ into itself, $1<p\le 2$. This easily follows from the estimate

$$|H_R(t,r)|\le\sqrt{\frac{\sinh(r)}{\sinh(t)}}\frac{1}{r},$$

that we are going to prove after breaking up $H_R=H^1+H_R^2$ as usual. Using (S1) and (S3) we write H^1 as a sum of several terms. Let us consider one of them, for instance the first one $\sqrt{\sinh(r)}\int_0^1\lambda e^{i\lambda r}d\lambda$.

One integration by parts gives a bound of $\sqrt{\sinh(r)}/r$, stronger then what claimed. All the other terms can be handled similarly since we have good estimate on the derivatives with respect to λ of all functions involved.

Using (S2) and (S4), H_R^2 breaks up in several terms. Let us consider the first one

$$\sqrt{\frac{\sinh(r)}{\sinh(t)}}\,t^{1/2}\int_1^R \lambda^{1/2}J_0(\lambda t)e^{i\lambda r}\,d\lambda=\sqrt{\frac{\sinh(r)}{\sinh(t)}}\,t^{1/2}\left[\int_1^{1/t}+\int_{1/t}^R\right].$$

Recall that $J_0'(t)=-J_1(t)$. Then one integration by parts proves that

$$\sqrt{\frac{\sinh(r)}{\sinh(t)}}\,t^{1/2}\left|\int_1^{1/t}\lambda^{1/2}J_0(\lambda t)e^{i\lambda r}\,d\lambda\right|\le\frac{c}{r}\sqrt{\frac{\sinh(r)}{\sinh(t)}}.$$

Since, up to negligible terms, we have

$$t^{1/2}\int_{1/t}^R \lambda^{1/2}J_0(\lambda t)e^{i\lambda r}\,d\lambda=t^{1/2}\int_{1/t}^R \lambda^{1/2}\frac{e^{\pm i\lambda t}}{(\lambda t)^{1/2}}e^{i\lambda r}\,d\lambda=\frac{e^{iR(\pm t+r)}}{i(\pm t+r)}-\frac{1}{i(\pm t+r)},$$

the claimed estimate is proved for the first term of H_R^2. Proceed similarly for the others.

The fourth region

The corresponding operator maps boundedly ${}^K L_p^K(G)\to L_2(G)$, $p>\frac{4}{3}$. To prove this, observe that

$$|H_R(t,r)|\le\sqrt{\frac{\sinh(r)}{\sinh(t)}}\cdot\frac{1}{t}.$$

This follows from the estimates for H_R in the third region, exchanging the roles of t and r. Now

$$\left|\int_0^1 H_R(t,r)f(r)\,dr\right|\le\frac{c}{t\sqrt{\sinh(t)}}\|f\|_{L_p(G)}$$

if $p>\frac{4}{3}$. From this, it follows easily that the L_2-norm of the corresponding operator S^* satisfies

$$\|S^*f(t)\|_{L_2(G)}\le C\|f\|_{L_p(G)},\quad p>\tfrac{4}{3}.$$

This concludes the proof.

In [MP] we extend the present result to all rank one, non compact, connected Riemannian symmetric spaces G/K using a different approach that produces a shorter proof, but works only for square integrable functions.

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Received March 22, 1993

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