The Milnor fiber of a generic arrangement

Peter Orlik and Richard Randell

In memoriam Deane Montgomery

1.Introduction

Let $f=f(z_1,...,z_l)$ be a homogeneous polynomial of degree $n\geq 2$ in l complex variables. The Milnor fibration [6] of f is usually defined in a neighborhood of the origin. Since f is homogeneous there is a global fibration

$$f\colon \mathbf{C}^l\setminus f^{-1}(0)\to \mathbf{C}\setminus\{0\}$$

and $F=f^{-1}(1)$ is the *Milnor fiber* of the map f. Let $\xi=\exp(2\pi i/n)$. Let $h^*\colon H^*(F)\to H^*(F)$ be the monodromy induced by $h(z_1,...,z_l)=(\xi z_1,...,\xi z_l)$. Consider all homology and cohomology with complex coefficients and let $b_k=\dim H^k(F)$ be the k-th Betti number of F. Since F is a Stein space of dimension $(l-1)$ we have $H^k(F)=0$ for $k\geq l$.

If f has an isolated singularity it is known from Milnor's work that $b_k(F)=0$ for $1\leq k\leq l-2$ and that $b_{l-1}(F)=(n-1)^l$. The characteristic polynomial of the automorphism induced by the monodromy on $H^{l-1}(F)$ was computed in [7]. In [9] we gave an explicit basis of differential forms for the nonvanishing group $H^{l-1}(F)$. The classes are all represented as restrictions to F of differential forms $q\omega$ where q is a homogeneous polynomial and

$$\omega=\sum_{k=1}^{l}(-1)^{k-1}z_k dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_l.$$

If the singularity of f is not isolated very little is known about the cohomology of F. Special cases have been studied by Dimca [3], Esnault [4], Randell [10], Siersma [11], and others. In this note we consider the case where f is the product of distinct linear forms which define an arrangement. Let V be a vector space of dimension l over $\mathbf{C}$. An *arrangement* in V is a finite set $\mathcal{A}$ of hyperplanes. It is central if all hyperplanes contain the origin and affine otherwise. If $H \in \mathcal{A}$ is a hyperplane, let α_H be a polynomial of degree 1 with kernel H. Call

$$Q = Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

a defining polynomial of $\mathcal{A}$.

If $\mathcal{A}$ is central then $Q(\mathcal{A})$ is homogeneous of degree $n = |\mathcal{A}|$ and for $l \geq 3$ the singularity is not isolated. The hyperplane complement $M = M(\mathcal{A}) = V \setminus Q^{-1}(0)$ is the total space of the Milnor fibration with fiber $F = Q^{-1}(1)$. From Brieskorn's work [2] we have a complete description of the cohomology of M in terms of differential forms. This allows us to detect some cohomology in F as follows. Recall the Hopf bundle $p: \mathbf{C}^l \setminus \{0\} \to \mathbf{C}P^{l-1}$ with fiber $\mathbf{C}^*$. Let $B = p(M)$. It is easy to see that the restriction $p_M: M \to B$ is trivial. Moreover B is the complement of an affine arrangement in $\mathbf{C}^{l-1}$ and therefore a Stein space of dimension $(l-1)$. Thus $H^k(B) = 0$ for $k \geq l$. The Betti numbers of B may be computed in terms of the intersection lattice of $\mathcal{A}$, see [8]. In fact there is a complete description of $H^*(B)$ in terms of differential forms. The monodromy map h generates a cyclic group G of order $n = |\mathcal{A}|$. The restriction of the Hopf bundle $p_F: F \to B$ is the orbit map of the free action of G. These fibrations fit into a commutative diagram.

$$\begin{array}{ccccc} & & F & \rightarrow & F/G \\ & & \downarrow & & \downarrow \\ \mathbf{C}^* & \rightarrow & M & \rightarrow & B \\ \downarrow & & \downarrow & & \\ \mathbf{C}^*/G & = & \mathbf{C}^* & & \end{array}$$

Since we use cohomology with complex coefficients we get $[H^k(F)]^G = H^k(B)$. This describes the 1-eigenspace of the monodromy. The eigenspaces of the other n-th roots of unity are harder to detect in general, but we get lower bounds:

$$b_k(F) \geq b_k(B). \tag{1}$$

Definition 1.1. A central l-arrangement with n hyperplanes is called *generic* and denoted $\mathcal{G}_n^l$, if $n > l$ and the intersection of every subset of l distinct hyperplanes is the origin.

In this note we compute the cohomology groups of the Milnor fiber and the characteristic polynomial of the monodromy for a generic arrangement. We also find a basis of Kähler differentials for $H^k(F)$ provided $0 \leq k \leq l-2$. We close with the description of a space of Kähler $(l-1)$-forms which we conjecture to be isomorphic to $H^{l-1}(F)$.

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2. The cohomology of F

Definition 2.1. An affine l-arrangement with n hyperplanes is called a *general position* arrangement and denoted $\mathcal{B}_n^l$, if $n>l$, the intersection of every subset of l distinct hyperplanes is a point, and the intersection of every subset of $l+1$ distinct hyperplanes is empty.

Proposition 2.2. *Let $\mathcal{G}_n^l$ be a generic arrangement. Let $M=M(\mathcal{G}_n^l)$ and let $p_M: M \to B$ be the restriction of the Hopf bundle. Then B is the complement of a general position arrangement, $B=M(\mathcal{B}_{n-1}^{l-1})$.*

Proof. Fix $H_0 \in \mathcal{G}_n^l$ and choose coordinates so that $H_0=\ker(z_l)$. Let $Q=Q(\mathcal{G}_n^l)$. A defining polynomial for $\mathcal{B}_{n-1}^{l-1}$ is obtained by setting $z_l=1$ in Q.

Hattori [5] obtained a complete description of the homotopy type of B. Let $J=\{1,\dots,n-1\}$. If $I \subset J$ let $|I|$ denote its cardinality. Define the subtorus T_I of T^{n-1} by

$$T_I = \{z_1,\dots,z_{n-1} \in T^{n-1} \mid z_j = 1 \text{ for } j \notin I\}.$$

Theorem 2.3. *Let $\mathcal{B}_{n-1}^{l-1}$ be a general position arrangement with $l \geq 3$ and let $B=M(\mathcal{B}_{n-1}^{l-1})$. Then B has the homotopy type of*

$$B_0 = \bigcup_{|I|=l-1} T_I.$$

Corollary 2.4. *Let $\mathcal{B}_{n-1}^{l-1}$ be a general position arrangement with $l \geq 3$ and let $B=M(\mathcal{B}_{n-1}^{l-1})$. Then*

(i) *$\pi_1(B)$ is free abelian of rank $n-1$,*

(ii) *$\pi_k(B)=0$ for $2 \leq k \leq l-2$,*

(iii) *for $0 \leq k \leq l-1$*

$$b_k(B) = \binom{n-1}{k},$$

(iv) *the Euler characteristic of B is*

$$\chi(B) = (-1)^{l-1}\binom{n-2}{l-1}.$$

Proof. We think of T^{n-1} as the $(n-1)$-dimensional hypercube with opposite faces identified. Then B_0 is obtained from T^{n-1} by removing cells in dimensions $n-1, n-2, \ldots, l$ corresponding to the interior of the cube, and to pairs of faces of the cube. Thus B_0 has the same $(l-1)$-skeleton as T^{n-1}. The boundaries of the removed l-cells give rise to nonvanishing homotopy classes but they are nullhomologous. This proves parts (i), (ii), and (iii). Part (iv) follows from Lemma 2.5 below.

Lemma 2.5. *For $m>k$ we have*

$$\binom{m-1}{k} = \binom{m}{k} - \binom{m}{k-1} + \ldots + (-1)^k \binom{m}{0}.$$

Proof. We use induction on k. The formula holds for $k=1$, and if we assume it for $k-1$ then it follows for k from the formula

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}.$$

Theorem 2.6. *Assume that $l \geq 3$. Let $\mathcal{G}_n^l$ be a generic arrangement with total space $M = M(\mathcal{G}_n^l)$. Let $p_M: M \to B$ be the restriction of the Hopf bundle. Let $Q: M \to \mathbf{C}^*$ be the Milnor fibration and let F be the associated Milnor fiber. Let $\xi = \exp(2\pi i/n)$. Let $h^*: H^*(F) \to H^*(F)$ be the monodromy induced by $h(z_1, \ldots, z_l) = (\xi z_1, \ldots, \xi z_l)$. Let $u = \binom{n-2}{l-2}$ and let $v = \binom{n-2}{l-1}$. Then*

(i) *$\pi_1(F)$ is a free abelian group of rank $(n-1)$,*

(ii) *$b_k(F) = b_k(B)$ for $0 \leq k \leq l-2$ and hence the monodromy is trivial in this range,*

(iii) *$b_{l-1}(F) = u + nv$,*

(iv) *the characteristic polynomial of the monodromy on $H^{l-1}(F)$ is*

$$\Delta_{l-1}(t) = (1-t)^u (1-t^n)^v.$$

Proof. If we think of the universal cover of T^{n-1} as $\mathbf{R}^{n-1}$ subdivided into hypercubes by the integer lattice, then the universal cover of B is a giant "Swiss cheese" since in each hypercube the same cells are removed as the cells removed to get B_0. Since the restriction of the Hopf map $p_F: F \to B$ is an n-fold covering, F has the homotopy type of the union of n such hypercubes with the appropriate identifications. This proves (i) and (ii). Part (iii) follows from (ii) together with the formula for the Euler characteristic of a covering $\chi(F) = n\chi(B)$, and the calculation of $\chi(B)$ in 2.4(iv).

To prove (iv) we use Milnor's work [6, pp. 76–77]. The Weil ζ function of the mapping h can be expressed as a product

$$\zeta(t)=\prod_{d|n}(1-t^d)^{-r_d}$$

where the exponents $-r_d$ can be computed from the formula

$$\chi_j=\sum_{d|j} dr_d.$$

Here χ_j is the Lefschetz number of the mapping h^j, the j-fold iterate of h. Milnor showed that χ_j is the Euler characteristic of the fixed point manifold of h^j. Since h^j has no fixed points for $1\leq j<n$ and $\chi(F)=n\chi(B)$ we conclude that

$$\zeta(t)=(1-t^n)^{-\chi(B)}. \tag{2}$$

The zeta function can be expressed as an alternating product of polynomials

$$\zeta(t)=\Delta_0(t)^{-1}\Delta_1(t)\Delta_2(t)^{-1}\dots\Delta_{l-1}(t)^{\pm 1} \tag{3}$$

where $\Delta_k(t)$ is the characteristic polynomial of the monodromy on $H^k(F)$. Part (iv) now follows from (2), (3), and the fact that $\Delta_k(t)=(1-t)^{b_k(F)}$ for $0\leq k\leq l-2$, which is a consequence of (ii).

Remark 2.7. If $l=2$ then $\pi_1(F)$ is free of rank $(n-1)^2$. Conclusions (ii)–(iv) of the theorem are valid.

A central 2-arrangement is always generic. In this case Q has an isolated singularity at the origin. Thus $b_0(F)=1$ and $b_1(F)=(n-1)^2$. In fact F has the homotopy type of a wedge of $(n-1)^2$ circles. In this case B is the complex line with $(n-1)$ points removed. Thus $b_0(B)=1$ and $b_1(B)=n-1$. This agrees with assertions (ii) and (iii). The characteristic polynomial of the monodromy on $H^1(F)$ may be computed using the divisor formula in [7]:

$$\delta(h)=(nE_n-1)^2=n(n-2)E_n+1=(n-2)\Lambda_n+1.$$

Thus $\Delta_1(t)=(1-t)(1-t^n)^{n-2}$, which agrees with (iv).

Remark 2.8. It is shown in [1] that the complexification of the D_3 arrangement has $b_1(F)=7$, while $b_1(B)=5$. Thus Theorem 2.6 does not hold in general.

It follows from Milnor's fibration theorem that F is the interior of a closed manifold with boundary. Let F^c denote this closed manifold and let ∂F^c be its boundary, a smooth closed orientable $(2l-3)$-manifold. Sard's theorem implies that there exists a closed ball B^{2l} centered at the origin whose boundary S^{2l-1} intersects F transversely. Then $F^c = F \cap B^{2l}$ and $\partial F^c = F \cap S^{2l-1}$. Since the singularity of $Q^{-1}(0)$ is not isolated, the compact set $K = Q^{-1}(0) \cap S^{2l-1}$ is not a manifold. The degeneration map $\partial F^c \to K$ is a resolution of the singularities of K. Since the monodromy h leaves S^{2l-1} invariant, there is an induced monodromy $h: \partial F^c \to \partial F^c$. Norbert A'Campo has informed us that he can prove the following.

Theorem 2.9. *The induced monodromy $h: \partial F^c \to \partial F^c$ acts trivially on $H^*(\partial F^c)$.*

Kähler differentials

We define Kähler differentials as in [9]. Let $S = K[z_1, \dots, z_l]$ be the polynomial ring over the field K with its usual grading, so $\deg z_j = 1$ for all j. Let S_r be the homogeneous component of degree r. Write $(\Omega, d) = (\Omega_S, d)$ for the cochain complex of Kähler differential forms on S.

Definition 3.1. For $0 \le p \le l$ and $J = (j_1, \dots, j_p)$ let $\sigma_J = dz_{j_1} \dots dz_{j_p}$ and let

$$\omega_J = \sum_{k=1}^{p} (-1)^{k-1} z_{j_k} dz_{j_1} \wedge \dots \wedge \widehat{dz_{j_k}} \wedge \dots \wedge dz_{j_p}.$$

The symbol ω_J is skew symmetric in its indices. Since Ω^p is a free S-module with basis consisting of the elements σ_J with $|J| = p$ and $j_1 < \dots < j_p$, and the symbol σ_J is also skew symmetric in its indices, we may define an S-linear map $\delta: \Omega^p \to \Omega^{p-1}$ for $p \ge 1$ by $\delta(\sigma_J) = \omega_J$. For $p = 0$ let $\delta = 0$. Computation shows that $\delta^2 = 0$. In fact the complex (Ω, δ) is the Koszul complex based on $z_1, \dots, z_l$. The next result is the Poincaré lemma in our setting. For a proof see [9, Lemma 4.5].

Lemma 3.2. *Let $a \in S_r$ and let $J = (j_1, \dots, j_p)$. Then*

$$\delta d(a\sigma_J) = ra\sigma_J - da \wedge \omega_J \tag{1}$$

$$d\delta(a\sigma_J) = pa\sigma_J + da \wedge \omega_J \tag{2}$$

$$(d\delta + \delta d) a\sigma_J = (p+r) a\sigma_J. \tag{3}$$

Now assume that $\mathcal{A}$ is any arrangement over $K = \mathbf{C}$. According to Brieskorn [2] the cohomology of M is represented by rational forms.

Theorem 3.3. *Let $\mathcal{A}$ be an arrangement. For $H\in\mathcal{A}$ let $\eta_H=d\alpha_H/\alpha_H$. Then $H^*(M)$ is isomorphic to the* **C**-*algebra $R(\mathcal{A})$ generated by* 1 *and the* 1-*forms η_H for $H\in\mathcal{A}$.*

If $\mathcal{A}$ is a central arrangement then the α_H are linear forms. Define an operator $\partial: R\to R$ by $\partial 1=0$, $\partial\eta_H=1$ and for $p\geq 2$

$$\partial\eta_1 \dots \eta_p = \sum_{i=1}^{p}(-1)^{i-1}\eta_1 \dots \eta_{i-1}\widehat{\eta_i}\eta_{i+1} \dots \eta_p.$$

It is clear that $\partial\partial=0$ and it is known that (R,∂) is an acyclic complex, see [8]. Let $R_0=\ker\partial$. Since ∂ is a derivation, it follows that R_0 is a subalgebra. Given $H_0\in\mathcal{A}$ write $\eta_0=\eta_{H_0}$. We have $R=R_0\oplus\eta_0 R_0$. Denote by A_0 the subalgebra generated by 1 and η_0. Then $R=R_0\otimes A_0$. We obtain from (2.2):

Proposition 3.4. *Let $\mathcal{A}$ be a central arrangement and let $p_M: M\to B$ be the restriction of the Hopf bundle. Then $p_M^*: H^*(B)\to H^*(M)$ is injective and we may identify $p_M^*(H^*(B))$ with R_0.*

Proposition 3.5. *Let $\mathcal{A}$ be a central arrangement and let $p_F: F\to B$ be the restriction of the Hopf bundle. We may identify $p_F^*(H^*(B))$ with $QR_0=\{Q\varrho\,|\,\varrho\in R_0\}$.*

If $\mathcal{G}_n^l$ is a generic arrangement then Proposition 3.5 and Theorem 2.6 provide Kähler differential form representatives for all cohomology except $H^{l-1}(F)$. In the rest of this section we discuss the problem of finding Kähler form representatives for $H^{l-1}(F)$. Let $T=S/(Q-1)S$ be the coordinate ring of F. Let $\pi^0: S\to T$ be the natural projection, and let $\pi: \Omega_S\to\Omega_T$ be the induced map. In the top dimension we use special notation.

Definition 3.6. Let

$$\tau = \sigma_{1,\dots,l} = dz_1\wedge\dots\wedge dz_l,$$
$$\tau_j = (-1)^{j-1}\sigma_{1,\dots,\hat{j},\dots,l} = (-1)^{j-1}dz_1\wedge\dots\wedge\widehat{dz_j}\wedge\dots\wedge dz_l,$$
$$\omega = \delta(\tau) = \omega_{1,\dots,l} = \sum_{k=1}^{l}(-1)^{k-1}z_k dz_1\wedge\dots\wedge\widehat{dz_k}\wedge\dots\wedge dz_l,$$

and $\omega_j=\delta(\tau_j)$. Note that $dz_j\wedge\tau_j=\tau$. Also $d\omega=l\tau$ and $d\omega_j=(l-1)\tau_j$.

Definition 3.7. If $\alpha,\alpha'\in\Omega$ write $\alpha\equiv\alpha'$ if $\pi\alpha=\pi\alpha'$. If $\beta\in\Omega_T$ is a cocycle, let $[\beta]$ denote its cohomology class in $H^*(\Omega_T)$. If $\alpha,\alpha'\in\Omega$ are cocycles, write $\alpha\sim\alpha'$ if $[\pi(\alpha)]=[\pi(\alpha')]$.

In the next result we establish certain inhomogeneous relations in Ω_T.

Lemma 3.8. *Let $a \in S_r$. Then*

$$n\tau_k \equiv \frac{\partial Q}{\partial z_k}\omega, \tag{4}$$

$$nd(\delta a\tau_k) \equiv \left[(n+r-1)a\frac{\partial Q}{\partial z_k} - n\frac{\partial a}{\partial z_k}\right]\omega, \tag{5}$$

$$(n+r-1)a\frac{\partial Q}{\partial z_k}\omega \sim n\frac{\partial a}{\partial z_k}\omega. \tag{6}$$

Proof. For $J=(j_1,\dots,j_p)$, define $(j,J)=(j,j_1,\dots,j_p)$. From (3.2.1) we get for an arbitrary index set J

$$ra\sigma_J - da\wedge\omega_J = \sum_{j=1}^{l}\frac{\partial a}{\partial z_j}\omega_{(j,J)}.$$

To prove (4) let $a=Q$ so $r=n$, $J=(1,\dots,\hat{k},\dots,l)$ and note that $Q\equiv 1$ and $dQ\equiv 0$. From (3.2) and the equation above we get for an arbitrary index set J with $|J|=p$

$$d\delta(a\sigma_J) = (p+r)a\sigma_J - \sum_{j=1}^{l}\frac{\partial a}{\partial z_j}\,\omega_{(j,J)}.$$

Now choose J as above, multiply the last equation by n and substitute (4) to obtain (5). The inhomogeneous relation (6) follows from (5).

Proposition 3.9. *Every cohomology class of $H^{l-1}(\Omega_T)$ has the form $[\pi(p\omega)]$ where $p\in S$.*

Proof. Since $\tau_1,\dots,\tau_l$ generate Ω^{l-1} as S-module, it follows that their images under π generate Ω_T^{l-1} as T-module. The assertion follows from (3.8).

If f has an isolated singularity then the Jacobi ideal I generated by the partials of f has finite codimension in S. In [9] we showed that there is a homogeneous subspace H with $S=H\oplus I$ such that every cohomology class of $H^{l-1}(F)$ has the form $[\pi(h\omega)]$ with $h\in H$.

If the singularity of f is not isolated then the Jacobi ideal has infinite codimension. In the case of a generic arrangement we have an explicit conjecture for a finite dimensional subspace which carries the cohomology. First we need some notation. Let $\mathcal{G}_n^l$ be a generic arrangement. Let $\mathcal{M}\subset\mathcal{G}_n^l$ be a subarrangement with $|\mathcal{M}|=l-1$. Write $\mathcal{M}=\{H_1,\dots,H_{l-1}\}$. Then $Q(\mathcal{M})=\alpha_{H_1}\dots\alpha_{H_{l-1}}$. Define $Q^{\mathcal{M}}=Q(\mathcal{G}_n^l)/Q(\mathcal{M})$. Given $s\in S$ let $J_{\mathcal{M}}(s)$ be the determinant of the Jacobian matrix of $(s,\alpha_1,\dots,\alpha_{l-1})$. The next result proves the existence of certain homogeneous relations in Ω_T.

Lemma 3.10. *For every* $a \in S_r$ *and* $\mathcal{M} \subset \mathcal{G}_n^l$ *with* $|\mathcal{M}| = l-1$ *we have*

$$raJ_{\mathcal{M}}(Q^{\mathcal{M}})\omega \sim nQ^{\mathcal{M}}J_{\mathcal{M}}(a)\omega.$$

Proof. We may choose coordinates so that $Q(\mathcal{M}) = z_2 \dots z_l$. Then $J_{\mathcal{M}}(s) = \partial s/\partial z_1$. In the notation of (3.6):

$$d(aQ^{\mathcal{M}}\omega_1) = Q^{\mathcal{M}}da\wedge\omega_1 + adQ^{\mathcal{M}}\wedge\omega_1 + aQ^{\mathcal{M}}d\omega_1.$$

Direct calculation gives

$$\begin{aligned} Q^{\mathcal{M}}da\wedge\omega_1 &= Q^{\mathcal{M}}\left[ra\tau_1 - \frac{\partial a}{\partial z_1}\omega\right] \\ adQ^{\mathcal{M}}\wedge\omega_1 &= a\left[(n-l+1)Q^{\mathcal{M}}\tau_1 - \frac{\partial Q^{\mathcal{M}}}{\partial z_1}\omega\right] \\ aQ^{\mathcal{M}}d\omega_1 &= (l-1)aQ^{\mathcal{M}}\tau_1. \end{aligned}$$

Recall from (3.8) that $n\tau_1 \equiv (\partial Q/\partial z_1)\omega$. Since $\partial Q/\partial z_1 = Q(\mathcal{M})(\partial Q^{\mathcal{M}}/\partial z_1)$ we have

$$nQ^{\mathcal{M}}\tau_1 \equiv Q^{\mathcal{M}}Q(\mathcal{M})\frac{\partial Q^{\mathcal{M}}}{\partial z_1}\omega \equiv \frac{\partial Q^{\mathcal{M}}}{\partial z_1}\omega.$$

It follows that

$$\begin{aligned} d(naQ^{\mathcal{M}}\omega_1) &= n(n+r)aQ^{\mathcal{M}}\tau_1 - na\frac{\partial Q^{\mathcal{M}}}{\partial z_1}\omega - nQ^{\mathcal{M}}\frac{\partial a}{\partial z_1}\omega \\ &\equiv \left[ra\frac{\partial Q^{\mathcal{M}}}{\partial z_1} - nQ^{\mathcal{M}}\frac{\partial a}{\partial z_1}\right]\omega. \end{aligned}$$

Lemma 3.11. *Define* $\phi: \Omega^{l-2} \to S$ *by* $dQ\wedge d\varrho = \phi(\varrho)\tau$ *for* $\varrho \in \Omega^{l-2}$. *Define* $E = \{e \in S \mid eQ \in \text{im } \phi\}$. *For every* $a \in S_r$ *and* $\mathcal{M} \subset \mathcal{G}_n^l$ *with* $|\mathcal{M}| = l-1$ *we have*

$$raJ_{\mathcal{M}}(Q^{\mathcal{M}}) - nQ^{\mathcal{M}}J_{\mathcal{M}}(a) \in E.$$

Proof. As in (3.10) we may choose coordinates so that $Q(\mathcal{M}) = z_2 \dots z_l$. Since $dQ\wedge\omega = nQ\tau$ we get as in the proof of (3.10)

$$\begin{aligned} dQ\wedge d(aQ^{\mathcal{M}}\omega_1) &= dQ\wedge\left[(n+r)aQ^{\mathcal{M}}\tau_1 - a\frac{\partial Q^{\mathcal{M}}}{\partial z_1}\omega - Q^{\mathcal{M}}\frac{\partial a}{\partial z_1}\right]\omega \\ &= \left[ra\frac{\partial Q^{\mathcal{M}}}{\partial z_1} - nQ^{\mathcal{M}}\frac{\partial a}{\partial z_1}\right]Q\tau. \end{aligned}$$

Conjecture 3.12. *Let $\mathcal{G}_n^l$ be a generic arrangement defined by Q.*

(i) *There exists a finite dimensional homogeneous subspace $U \subset S$ such that*

$$S \approx E \oplus \mathbf{C}[Q] \otimes U.$$

(ii) $\Omega_T^{l-1} = \pi(U\omega) \oplus d_T \Omega_T^{l-2}$, *and the map $U \to H^{l-1}(F)$ defined by $u \to [\pi(u\omega)]$ is an isomorphism.*

(iii) *Let $U_r = U \cap S_r$, let $u_r = \dim U_r$, and let $P(U,t) = \sum_r u_r t^r$ be the Poincaré polynomial of U. Then*

$$u_r = \begin{cases} \binom{r+l-1}{l-1} & \text{for } 0 \le r \le n-l, \\ \binom{n-2}{l-1} & \text{for } n-l+1 \le r \le n-1, \\ \binom{n-2}{l-1} - \binom{r-n+l-1}{l-1} & \text{for } n \le r \le 2n-l-2. \end{cases}$$

Example 3.13. Consider $\mathcal{G}_5^3$ and use coordinates x, y, z. Let $Q = xyz(x+y+z) \times (x+2y+3z)$. The cohomology of its Milnor fiber is described as follows.

Label the linear forms $\alpha_1, \ldots, \alpha_5$. For $i<j$ define 1-forms in Ω^1 by

$$\zeta_{i,j} = Q\left(\frac{d\alpha_i}{\alpha_i} - \frac{d\alpha_j}{\alpha_j}\right).$$

It follows from (3.5) that a $\mathbf{C}$-basis for $H^1(F)$ consists of $\pi^1(\zeta_{i,5})$ for $i = 1,2,3,4$.

In the description of $H^2(F)$ note that

$$\omega = x dy \wedge dz - y dx \wedge dz + z dx \wedge dy.$$

Direct calculation shows that in our case $U_r = S_r$ for $r = 0,1,2$. For $r = 3,4$ only homogeneous relations occur, but for $r = 5$ the inhomogeneous relation $Q\omega \sim \omega$ is also required. We get

$$P(U,t) = 1 + 3t + 6t^2 + 3t^3 + 3t^4 + 2t^5.$$

This agrees with (3.12). The monodromy h has order $n = 5$. Let $\xi = \exp(2\pi/5)$. Recall that the action of h is contragradient in S. Thus $h\omega = \xi^{-3}\omega$ and U_r is an eigenspace with eigenvalue ξ^{-r-3}. It follows that the eigenvalues computed from this Poincaré polynomial agree with the characteristic polynomial of the monodromy from (2.6):

$$\Delta_2(t) = (1-t)^3(1-t^5)^3.$$

See also the preprint "On Milnor fibrations of arrangements" by D. C. Cohen and A. I. Suciu.

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Peter Orlik
Department of Mathematics
University of Wisconsin
Madison, WI 53706
U.S.A.
email: orlik@math.wisc.edu

Richard Randell
Department of Mathematics
University of Iowa
Iowa City, IA 52242
U.S.A.
email: randell@math.uiowa.edu