Representation of quasianalytic ultradistributions

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Abstract. We give the following representation theorem for a class containing quasianalytic ultradistributions and all the non-quasianalytic ultradistributions: Every ultradistribution in this class can be written as

$$u = P(\Delta)g(x) + h(x)$$

where $g(x)$ is a bounded continuous function, $h(x)$ is a bounded real analytic function and $P(d/dt)$ is an ultradifferential operator. Also, we show that the boundary value of every heat function with some exponential growth condition determines an ultradistribution in this class. These results generalize the theorem of Matsuzawa [M] for the above class of quasianalytic ultradistributions and partially solve a question of A. Kaneko [Ka]. Our interest lies in the quasianalytic case, although the theorems do not exclude non-quasianalytic classes.

1. A class of quasianalytic ultradistributions

We use the multi-index notations such as $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n}$, $\partial_j = \partial/\partial x_j$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{N}_0^n$ where $\mathbf{N}_0$ is the set of non-negative integers.

Let M_p, $p = 0, 1, 2, \ldots$, be a sequence of positive numbers, and let Ω be an open subset of $\mathbf{R}^n$. An infinitely differentiable function ϕ on Ω is called an ultradifferentiable function of class (M_p) (of class $\{M_p\}$ respectively) if for any compact set K of Ω and for each $h > 0$ (there exist constants $h > 0$ such that)

$$|\phi|_{M_p,K,h} = \sup_{x \in K, \alpha \in \mathbf{N}_0^n} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|} M_{|\alpha|}} \tag{1.1}$$

is finite. We impose the following conditions on M_p:

(M.0) For any $A > 0$ there exists a constant $C > 0$ such that

$$p! \le C A^p M_p, \quad p = 0, 1, 2, \ldots .$$

(M.1) $M_p^2 \le M_{p-1} M_{p+1}$, $p = 1, 2, \ldots$.

(M.2) There are constants C and H such that

$$M_{p+q} \le CH^{p+q} M_p M_q, \quad p, q = 0, 1, 2, \dots .$$

We call the above sequence M_p the defining sequence and denote by $\mathcal{E}_{(M_p)}(\Omega)$ ($\mathcal{E}_{\{M_p\}}(\Omega)$ respectively) the space of all ultradifferentiable functions of class (M_p) (of class $\{M_p\}$ respectively) on Ω.

If $M_p = p!$ then we obtain the class of analytic functions by Pringsheim's theorem. The condition (M.1) can be naturally fulfilled by Gorny's theorem ([Ma], p. 226). Thus the condition (M.2) is the only significant condition.

The topology of such spaces is defined as follows : A sequence $\phi_j \to 0$ in $\mathcal{E}_{(M_p)}(\Omega)$ ($\mathcal{E}_{\{M_p\}}(\Omega)$ respectively) if for any compact set K of Ω and for any $h>0$ (for some $h>0$ respectively) we have

$$\sup_{x\in K, \alpha \in \mathbf{N}_0^n} \frac{|\partial^\alpha \phi_j(x)|}{h^{|\alpha|} M_{|\alpha|}} \to 0, \quad \text{as } j \to \infty.$$

As usual, we denote by $\mathcal{E}'_{(M_p)}(\Omega)$ ($\mathcal{E}'_{\{M_p\}}(\Omega)$ respectively) the strong dual space of $\mathcal{E}_{(M_p)}(\Omega)$ (of $\mathcal{E}_{\{M_p\}}(\Omega)$ respectively) and we call its elements ultradistributions of Beurling type (of Roumieu type respectively) with compact support in Ω. Let $K \subset \mathbf{R}^n$ be a compact set. We denote by $\mathcal{E}'_{(M_p)}(K)$ ($\mathcal{E}'_{\{M_p\}}(K)$ respectively) the set of ultradistributions of class (M_p) (of class $\{M_p\}$ respectively) with support in K. In fact $u \in \mathcal{E}'_{(M_p)}(K)$ if and only if for any neighborhood Ω of K there exist constants $h>0$ and $C>0$ such that

$$|u(\phi)| \le C \sup_{x\in\Omega, \alpha\in\mathbf{N}_0^n} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|} M_{|\alpha|}}, \quad \phi \in \mathcal{E}_{(M_p)}(\mathbf{R}^n). \tag{1.2}$$

For each defining sequence M_p we define for $t>0$

$$\begin{aligned} M(t) &= \sup_p \log \frac{t^p M_0}{M_p}, \\ M^*(t) &= \sup_p \log \frac{p! t^p M_0}{M_p}, \\ \overline{M}(t) &= \sup_p \log \frac{p! t^p M_0^2}{M_p^2}. \end{aligned} \tag{1.3}$$

An operator of the form

$$P(\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha \partial^\alpha, \quad a_\alpha \in \mathbf{C} \tag{1.4}$$

is called an ultradifferential operator of class (M_p) (of class $\{M_p\}$ respectively) if there are constants L and C (for every $L>0$ there is a constant $C>0$ respectively) such that

$$|a_\alpha| \le CL^{|\alpha|}/M_{|\alpha|}, \quad \alpha \in \mathbf{N}_0^n. \tag{1.5}$$

It is well known that if $P(\partial)$ is an ultradifferential operator of class $*$ then

$$P(\partial): \mathcal{E}_*(\Omega) \to \mathcal{E}_*(\Omega) \tag{1.6}$$

and

$$P(\partial): \mathcal{E}'_*(\Omega) \to \mathcal{E}'_*(\Omega) \tag{1.7}$$

are continuous where $*=(M_p)$ or $\{M_p\}$. The condition (1.5) is equivalent to the condition that

$$|P(\zeta)| \le C \exp M(L|\zeta|), \quad \zeta \in \mathbf{C}^n. \tag{1.8}$$

2. Structure theorems

In this section it will be shown that every $u \in \mathcal{E}'_{(M_p)}(K)$ can be written as an infinite sum of derivatives of a continuous function modulo a bounded real analytic function and that every $u \in \mathcal{E}'_{(M_p)}(K)$ can be represented by the boundary value of a heat function satisfying some exponential growth condition. The structure theorems for $u \in \mathcal{E}'_{\{M_p\}}(K)$ can be shown by similar arguments.

We denote by $E(x,t)$ the n-dimensional heat kernel:

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t>0, \\ 0, & t \le 0. \end{cases}$$

Lemma 2.1 ([M]). *$E(\cdot,t)$ is an entire function of order* 2 *for every $t>0$. It has the following properties:*

(i) $\int_{\mathbf{R}^n} E(x,t)dx = 1, \ t>0.$

(ii) *There are positive constants C and a such that*

$$|\partial_x^\alpha E(x,t)| \le C^{|\alpha|} t^{-(n+|\alpha|)/2} \alpha!^{1/2} \exp[-a|x|^2/4t], \quad t>0,$$

where a can be chosen as close as desired to 1 *and* $0<a<1$.

The following lemma can be shown by similar arguments as in Proposition 2.1 of Matsuzawa [M], or as in Theorem 4 of Neymark [N].

Lemma 2.2. *Let K be a compact subset of $\mathbf{R}^n$ and Ω be a bounded open set containing K. For every $\phi \in \mathcal{E}_{(M_p)}$ let*

$$\phi_t(x) = \int_\Omega E(x-y,t)\phi(y)\,dy.$$

Then ϕ_t converges to ϕ in $\mathcal{E}_{(M_p)}(\Omega)$ as $t \to 0+$.

For each defining sequence M_p we impose the following condition:

(C) There exists a positive integer k such that

$$\liminf_{p\to\infty} \left(\frac{m_{kp}}{m_p}\right)^2 > k,$$

where $m_p = M_p/M_{p-1}$, $p=1,2,\dots$.

Remark 2.3. (i) Let $m_p = p(\log p)^\alpha$, $\alpha > 0$. Then $M_p = m_2 \dots m_p$ satisfies (C). Thus the defining sequence for this standard quasianalytic class satisfies (C).

(ii) The Gevrey sequence $M_p = p!^s$, $s > 1$, satisfies (C).

(iii) Furthermore, if M_p satisfies the strong non-quasianalytic condition (M.3) in Komatsu [K1] then it satisfies (C). In fact, (M.3) is equivalent to the fact that for some integer $k > 0$

$$(M.3'') \qquad \liminf_{p\to\infty} \frac{m_{kp}}{m_p} > k.$$

Thus the condition (C) is equivalent to the fact that $N_p = M_p^2$ satisfies (M.3) (see [P], p. 300).

The following lemma will be very useful later on. For the details of the proof we refer to Komatsu [K1], Lemma 11.4 and Matsuzawa [M], Lemma 4.1.

Lemma 2.4. *Let L be an arbitrary positive number and let*

$$P(\zeta) = (1+\zeta)^2 \prod_{p=1}^\infty \left(1 + \frac{L\zeta}{m_p}\right), \quad \zeta \in \mathbf{C}^n. \tag{2.1}$$

(i) *If M_p satisfies* (M.1), (M.2) *and* (M.3) *then $P(\partial)$ is an ultradifferential operator of class (M_p).*

(ii) *If M_p satisfies* (M.1) *and $\sum_{p=1}^\infty M_{p-1}/M_p < \infty$ then for any $\varepsilon > 0$ there exist functions $v, w \in C_0^\infty(\mathbf{R})$ such that*

$$\operatorname{supp} v \subset [0,\varepsilon], \quad \operatorname{supp} w \subset [\varepsilon/2, \varepsilon], \tag{2.2}$$

$$|v(t)| \le C \exp[-M^*(L/t)], \quad t > 0, \tag{2.3}$$

$$P(d/dt)v(t) = \delta(t) + w(t), \tag{2.4}$$

where δ is a Dirac measure.

Now we are in a position to state and prove the main theorem of this paper.

Theorem 2.5. *Let M_p be a defining sequence satisfying* (C) *and* $u\in\mathcal{E}'_{(M_p)}(K)$. *Then there exists an ultradifferential operator $P(d/dt)$ such that for some $C>0$ and $L>0$*

$$P(d/dt)=\sum_{k=0}^{\infty}a_k(d/dt)^k,\quad |a_k|\le CL^k/M_k^2 \tag{2.5}$$

and there exist a bounded continuous function $g(x)$ and a bounded real analytic function $h(x)$ such that

$$u=P(\Delta)g(x)+h(x) \tag{2.6}$$

where $g(x)\in C^\infty(\mathbf{R}^n\backslash K)$, $P(\Delta)g(x)+h(x)=0$ in $\mathbf{R}^n\backslash K$, and Δ is the Laplacian.

Proof. Let $U(x,t)=u_y(E(x-y,t))$. Since $E(x,t)$ is an entire function of x for each $t>0$, $U(x,t)$ is well defined and entire analytic for each $t>0$. Furthermore $U(x,t)$ satisfies

$$(\partial_t-\Delta)U(x,t)=0\quad\text{in }\mathbf{R}^{n+1}_+ \tag{2.7}$$

where $\mathbf{R}^{n+1}_+=\{(x,t)\in\mathbf{R}^{n+1}: x\in\mathbf{R}^n,\ t>0\}$. Let $K_\delta=\{x\in\mathbf{R}^n: d(x,K)\le\delta\}$. Then $u\in\mathcal{E}'_{(M_p)}(K)$ means that for any $\delta>0$ there exist $h>0$ and $C>0$ such that

$$|U(x,t)|\le C\sup_{y\in K_\delta,\alpha\in\mathbf{N}_0^n}\frac{|\partial_y^\alpha E(x-y,t)|}{h^{|\alpha|}M_{|\alpha|}},\quad t>0.$$

By Lemma 2.1 (ii) we have for $t>0$

$$\begin{aligned}|U(x,t)|&\le C_1\sup_\alpha\left[\frac{\alpha!t^{-|\alpha|}C^{2|\alpha|}/h^{2|\alpha|}}{M_{|\alpha|}^2}\right]^{1/2}\sup_{y\in K_\delta}\exp\left[-\frac{|x-y|^2}{8t}\right]\\&\le C_1\exp\left[\overline{M}(\varepsilon/t)-d(x,K_\delta)^2/8t\right]\end{aligned} \tag{2.8}$$

for some $\varepsilon>0$. Let Ω be a bounded open neighborhood of K and

$$G(y,t)=\int_\Omega E(x-y,t)\phi(x)\,dx,\quad \phi\in\mathcal{E}_{(M_p)}.$$

Then by Lemma 2.2 we can easily see that

$$G(\cdot,t)\to\phi\quad\text{in }\mathcal{E}_{(M_p)}(\Omega),\quad\text{as }t\to0+ \tag{2.9}$$

and

$$\int_\Omega U(x,t)\phi(x)\,dx = u_y(G(y,t)) \tag{2.10}$$

by taking the limit of the Riemann sum of the left hand side. Thus it follows from (2.9) and (2.10) that

$$U(x,t) \to u \quad \text{as } t \to 0+$$

in the following sense:

$$\lim_{t\to 0+}\int_\Omega U(x,t)\phi(x)\,dx = u(\phi), \quad \phi \in \mathcal{E}_{(M_p)}. \tag{2.11}$$

Now let $N_p = M_p^2$. Then N_p is also a defining sequence satisfying (M.3) and $\sum_{p=1}^{\infty} N_{p-1}/N_p < \infty$ by (C) and (M.0). Then applying Lemma 2.4 to the sequence $N_p = M_p^2$ we can choose an ultradifferential operator $P(d/dt)$ such that for some $C>0$ and for some L_0

$$P(d/dt) = \sum a_k (d/dt)^k, \quad |a_k| \le C L_0^k / M_k^2 \tag{2.12}$$

and choose $v, w \in C_0^\infty(\mathbf{R})$ such that

$$\operatorname{supp} v \subset [0,2], \quad \operatorname{supp} w \subset [1,2], \tag{2.13}$$

$$|v(t)| \le C\exp[-N^*(L/t)], \quad t>0, \tag{2.14}$$

$$P(d/dt)v(t) = \delta(t) + w(t). \tag{2.15}$$

Here we note that $N^*(t) = \sup_p \log \dfrac{p!t^p M_0^2}{M_p^2} = \overline{M}(t)$.

Let

$$\widetilde{U}(x,t) = \int_0^\infty U(x,t+s)v(s)\,ds. \tag{2.16}$$

Then it follows from (2.8) that

$$|\widetilde{U}(x,t)| \le C_1 \int_0^\infty \exp\left[\overline{M}\left(\frac{\varepsilon}{t+s}\right) - \overline{M}\left(\frac{L}{s}\right)\right] ds. \tag{2.17}$$

Choosing $L>\varepsilon$, we can easily see that $\widetilde{U}(x,t)$ is uniformly bounded on $\overline{\mathbf{R}_+^{n+1}} = \{(x,t) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n,\ t \ge 0\}$. Thus $\widetilde{U}(x,t)$ is continuous on $\overline{\mathbf{R}_+^{n+1}}$. It follows that $g(x) = \widetilde{U}(x,0)$ is a bounded continuous function. Let

$$H(x,t) = -\int_0^\infty U(x,t+s)w(s)\,ds. \tag{2.18}$$

Since $\operatorname{supp} w \subset [1,2]$, $H(x,t)$ can be analytically continued to

$$\{(x,t)\in \mathbf{R}^{n+1} : x\in \mathbf{R}^n,\ t> -1\}.$$

Thus $h(x)=H(x,0)$ is a bounded real analytic function. On the other hand, (2.15) implies that

$$P(-\Delta)\widetilde{U}(x,t)=U(x,t)-H(x,t). \tag{2.19}$$

Hence we can easily see that in the sense of (2.11),

$$u=\lim_{t\to 0+} U(x,t)=P(-\Delta)g(x)+h(x).$$

The condition (M.0) implies that for any $L>0$

$$\overline{M}(t)\le Lt+C.$$

Therefore, it follows from (2.8) that for $x\notin K$

$$u=P(-\Delta)g(x)+h(x)=\lim_{t\to 0+} U(x,t)=0$$

which completes the proof.

Every distribution and hyperfunction with compact support can be represented as the boundary value of a holomorphic function. Here we will give a similar result for $\mathcal{E}'_{(M_p)}(K)$. In fact, this follows from (2.11) in the proof of Theorem 2.5. Thus we will prove that every heat function with growth condition (2.8) defines an element in $\mathcal{E}'_{(M_p)}(K)$.

Theorem 2.6. *Let M_p be a defining sequence satisfying* (C) *and $U(x,t)$ be an infinitely differentiable function in $\mathbf{R}^{n+1}_+$ satisfying the following conditions:*

(i) $(\partial_t-\Delta)U(x,t)=0$ *in* $\mathbf{R}^{n+1}_+$,

(ii) *for any $\delta>0$ there exist $C>0$ and $\varepsilon>0$ such that*

$$|U(x,t)|\le C\exp\left[\overline{M}(\varepsilon/t)-d(x,K_\delta)^2/8t\right] \quad \text{in } \mathbf{R}^{n+1}_+. \tag{2.20}$$

Then there exists a unique element $u\in\mathcal{E}'_{(M_p)}(K)$ such that

$$U(x,t)=u_y(E(x-y,t)),\quad t>0 \tag{2.21}$$

and

$$\lim_{t\to 0+} U(x,t)=u \tag{2.22}$$

in the following sense:

$$u(\phi)=\lim_{t\to 0+}\int_{\Omega} U(x,t)\phi(x)\,dx, \quad \phi\in\mathcal{E}_{(M_p)}(\mathbf{R}^n) \tag{2.23}$$

where Ω is an arbitrary bounded neighborhood of K.

Proof. Consider the function, as in (2.16)

$$\widetilde{U}(x,t)=\int_0^{\infty} U(x,t+s)v(s)\,ds.$$

Then it follows from (2.17)–(2.19) that

$$U(x,t)=P(-\Delta)\widetilde{U}(x,t)+H(x,t). \tag{2.24}$$

Furthermore, $g(x)=\widetilde{U}(x,0)$ and $h(x)=H(x,0)$ are bounded continuous functions on $\mathbf{R}^n_x$. Define u as

$$u=P(-\Delta)g(x)+h(x). \tag{2.25}$$

Since $P(-\Delta)$ is an ultradifferential operator of class (M_p) by (2.12) u belongs to $\mathcal{E}'_{(M_p)}(K)$ and $U(x,t)\to u$ as $t\to 0+$. Thus the existence of u is proved.

Now define heat functions for $t>0$ as

$$A(x,t)=g(x)*E(x,t)=\int_{\mathbf{R}^n} g(y)E(x-y,t)\,dy$$

and

$$B(x,t)=h(x)*E(x,t)=\int_{\mathbf{R}^n} h(y)E(x-y,t)\,dy.$$

Then it is easy to show that $A(x,t)$ and $B(x,t)$ converge locally uniformly to $g(x)$ and $h(x)$ respectively so that they are continuous on $\overline{\mathbf{R}^{n+1}_+}$ and $A(x,0)=g(x)$, $B(x,0)=h(x)$. Then $\widetilde{U}(x,0)=A(x,0)$ and $H(x,0)=B(x,0)$. Since they are bounded on $\overline{\mathbf{R}^{n+1}_+}$ we have by the uniqueness theorem of heat equations in Friedman [F]

$$\widetilde{U}(x,t)=g(x)*E(x,t)$$

and

$$H(x,t)=h(x)*E(x,t).$$

Then it follows from these facts and (2.24) that

$$\begin{aligned} u*E &= [P(-\Delta)g(x)+h(x)]*E \\ &= P(-\Delta)\widetilde{U}(x,t)+H(x,t) \\ &= U(x,t), \end{aligned}$$

which gives the relation (2.21). Also the uniqueness is easily obtained by (2.21).

For a compact set K of $\mathbf{R}^n$ we denote by $\mathcal{M}_K^{\text{tame}}$ the totality of C^∞ solutions $U(x,t)$ of the heat equation $(\partial_t-\Delta)U(x,t)=0$ in $\mathbf{R}_+^{n+1}$ which satisfy the following condition:

For any $\delta>0$ there exist C and $\varepsilon>0$ such that

$$|U(x,t)| \le C\exp\left[\overline{M}(\varepsilon/t)-d(x,K_\delta)^2/8t\right] \quad \text{in } \mathbf{R}_+^{n+1}. \tag{2.26}$$

Note that $\mathcal{M}_K^{\text{tame}}$ is a DF-space with the best constants C as semi-norms. Then we have the following theorem in view of Theorem 2.5 and 2.6:

Theorem 2.7. *Let M_p be a defining sequence satisfying* (C). *Then there exists an isomorphism:*

$$\mathcal{M}_K^{\text{tame}} \cong \mathcal{E}'_{(M_p)}(K).$$

Matsuzawa [M] has proved similar theorems for the case of hyperfunctions and ultradistributions of Gevrey class. Thus the above theorem is an extension of Matsuzawa's result for a class of quasianalytic ultradistributions.

Remark 2.8. We note that in Theorems 2.5 and 2.6 the conditions (2.20) and (2.26) can be replaced by the following:

$$|U(x,t)| \le C\exp\overline{M}(\varepsilon/t), \quad t>0$$

and $U(x,t)$ converges uniformly to zero in $\mathbf{R}^n\backslash K_\delta$ as $t\to 0+$.

Remark 2.9. In this section we have proved theorems only for $u\in\mathcal{E}'_{(M_p)}(K)$. For $u\in\mathcal{E}'_{\{M_p\}}(K)$ we can prove similar theorems under the same conditions. But, in fact (M.0) can be replaced by the less restrictive condition (M.0′) as follows:

(M.0′) There exist constants $A>0$ and $C>0$ such that

$$p! \le CA^pM_p, \quad p=0,1,2,\dots .$$

Finally, we conjecture that our assertions should also remain valid without the condition (C), i.e., for the general ultradistributions, both quasianalytic and non-quasianalytic.

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