Plane harmonic measures live on sets of σ-finite length

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The purpose of this paper is to prove Theorem 1 below. Let $E \subset \mathbf{C}$ be compact and $\Omega = \mathbf{C}^* \setminus E$ where $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$ is the Riemann sphere. We always assume that Ω is regular for the Dirichlet problem and denote by $\omega(\Omega, Y, z)$ the harmonic measure relative to Ω of the set $Y \subset \mathbf{C}^*$ (equivalently, of $Y \cap \partial\Omega$), evaluated at $z \in \Omega$. We also use $\omega(\Omega, Y)$ when discussing properties invariant when z is changed.

Theorem 1. *If Ω is as above there is a set $F \subset \partial\Omega$ satisfying $\omega(\Omega, F) = 1$ and with σ-finite one-dimensional Hausdorff measure.*

Remarks. (1) The assumption that Ω be regular for the Dirichlet problem is made only for convenience and in fact is no loss of generality in view of A. Ancona's result [2] which implies that an arbitrary domain Ω whose complement has positive capacity may be expressed as $\bigcap_n \Omega_n \setminus P$ where each Ω_n is regular for the Dirichlet problem and P has zero capacity. A set with full harmonic measure for each Ω_n will have full harmonic measure for Ω, and since a countable union of sets with σ-finite length clearly has σ-finite length Theorem 1 for the Ω_n's implies the corresponding statement for Ω.

(2) Theorem 1 sharpens the result of [5] which says the same with "σ-finite one-dimensional Hausdorff measure" replaced by "Hausdorff dimension one". For simply connected domains, Theorem 1 is proved in [6,7].

(3) In a sense we obtain a semiexplicit set F namely

$$F = \left\{ \zeta : \limsup_{r \to 0} \frac{\omega(\Omega, D(\zeta, r))}{r} > 0 \right\},$$

where $D(\zeta, r)$ is the (euclidean) disc with center ζ and radius r. However, it is easy to see that if any set F will work then so will this one, so we do not stress this point.

Theorem 1 will be a corollary of the following somewhat more precise result.

Theorem 2. *With notation as above suppose that* $\operatorname{diam} E \le 1$. *Then for any* $0<\delta<1$, $0<\varrho<1$ *and sufficiently large* M *(how large depends on* δ *only) there is a set* $F\subset E$ *such that* $\omega(\Omega, F, \infty)\ge C^{-1}\delta$ *and with a covering* $F\subset\bigcup_i D(z_i, r_i)$ *where* (i) $\sum_i r_i\le CM^\delta$ *and* (ii) $\sum_{i:r_i>\varrho} r_i\le CM^{-1}$.

Here C is an absolute constant. Theorem 1 follows from Theorem 2 by a formal argument as we will explain shortly. It first seems necessary to make a few nonmathematical remarks.

I first proved Theorem 1 (and 2) in 1986. I circulated a handwritten manuscript at the time but did not have it typed up. The proof was related to the proof of the corresponding dimension one statement given in the preliminary version [4] of a joint paper with P. Jones. Subsequently a more elegant approach to the dimension one result was given by L. Carleson and the published version [5] of the paper by Jones and myself uses Carleson's argument. It is not immediately clear how to adapt the latter argument to give the σ-finite length statement but it is natural to expect that this can be done. However, I have now decided to publish the original argument, with a few technical simplifications, which is what will be found below. This argument is quite precise if also quite long-winded and conceivably it could be of interest for other problems.

Since paper [4] was written jointly with Jones and is not published it is also appropriate to mention that the Lemma 2.1 below was in [4] and that the general scheme of the argument in the subsequent sections is also similar to [4], although in the case of the dimension one result proved there the latter argument is more straightforward and simpler.

Now let us return to mathematics and explain why Theorem 2 implies Theorem 1.

Proof of Theorem 1. We require only the case $\delta=\frac{1}{2}$ of Theorem 2.

Let ϕ be any rate function strictly weaker than one dimensional, i.e., ϕ: $[0,\infty)\to[0,\infty)$ is continuous and increasing with $\lim_{t\to 0}\phi(t)/t=0$. Let

$$h_\phi(E)=\inf\left\{\sum_i \phi(r_i): E\subset\bigcup_i D(z_i, r_i)\right\}$$

be the associated Hausdorff content.

Assume first that $\Omega=\mathbf{C}^*\setminus E$ with $\operatorname{diam} E\le 1$. If $\varepsilon>0$, then an appropriate choice of ϱ and M in Theorem 2 (namely: $M>\varepsilon^{-1}$ and ϱ small enough that $\phi(\varrho)<\varepsilon M^{-\delta}\varrho$) yields a set F_ε with $\omega(\Omega, F_\varepsilon, \infty)\ge C^{-1}$ and $h_\phi(F_\varepsilon)<C\varepsilon$. Taking $\limsup_{j\to\infty} F_{1/j^2}$ we obtain a set F_∞ such that $\omega(\Omega, F_\infty, \infty)\ge C^{-1}$ and $h_\phi(F_\infty)=0$.

Next let Ω be arbitrary and let $\{z_k\}$ be a countable dense subset of Ω. Moving the result in the preceding paragraph around using linear fractional transformations we obtain sets $\{F_k\}$ with $\omega(\Omega, F_k, z_k)\geq C^{-1}$ and $h_\phi(F_k)=0$. It is well known that then $F_\phi=\bigcup_k F_k$ will satisfy $\omega(\Omega, F_\phi)=1$, and clearly $h_\phi(F_\phi)=0$.

Finally let

$$F=\left\{\zeta:\limsup_{r\to 0}\frac{\omega(\Omega, D(\zeta,r))}{r}>0\right\}.$$

Suppose to get a contradiction that $\omega(\Omega, F)\neq 1$ or in other words that

$$\omega\left(\Omega,\left\{\zeta:\lim_{r\to 0}\frac{\omega(\Omega, D(\zeta,r),\infty)}{r}=0\right\}\right)>0. \tag{0.1}$$

By Egorov's theorem there is a set Y with nonzero harmonic measure on which the limit in (0.1) may be taken uniformly, i.e., a rate function ϕ with the property $\lim_{t\to 0}\phi(t)/t=0$ and a set Y with $\omega(\Omega, Y)>0$ such that $\zeta\in Y$ implies $\omega(\Omega, D(\zeta,r),\infty)\leq\phi(r)$ for all r. With F_ϕ as above we have $\omega(\Omega, Y\cap F_\phi,\infty)=\omega(\Omega, Y,\infty)>0$ and $h_\phi(Y\cap F_\phi)\leq h_\phi(F_\phi)=0$. Choose a covering of $Y\cap F_\phi$ by discs $D(\zeta_i,r_i)$, $\zeta_i\in Y\cap F_\phi$, with $\sum\phi(r_i)<\omega(\Omega, Y\cap F_\phi,\infty)$. Then

$$\begin{aligned}\omega(\Omega, Y\cap F_\phi,\infty)&\leq\sum_i\omega(\Omega, D(\zeta_i,r_i),\infty)\\&\leq\sum_i\phi(r_i)\\&<\omega(\Omega, Y\cap F_\phi,\infty)\end{aligned}$$

and we have our contradiction. So $\omega(\Omega, F)=1$.

On the other hand it is well-known that F will have σ-finite one-dimensional Hausdorff measure. This is proved as follows. It suffices to show that

$$F^\delta=\left\{\zeta:\limsup_{r\to 0}\frac{\omega(\Omega, D(\zeta,r),\infty)}{r}>\delta\right\}$$

has finite one-dimensional Hausdorff measure. Clearly each point $\zeta\in F^\delta$ has arbitrarily small neighborhoods $D(\zeta,r)$ such that $\omega(\Omega, D(\zeta,r),\infty)\geq\delta r$. By the Besicovitch lemma there are discs $D_i=D(\zeta_i,r_i)$ such that r_i is less than any preassigned ε, $r_i\leq\delta^{-1}\omega(\Omega, D(\zeta_i,r_i),\infty)$ and no point belongs to more than a fixed finite number C of the D_i's. Then

$$\sum_i r_i\leq\delta^{-1}\sum_i\omega(\Omega, D(\zeta_i,r_i),\infty)\leq C\delta^{-1}$$

and if we now let $\varepsilon\to 0$ we are done. $\square$

We note that the more natural looking way to quantify Theorem 1, namely, that if Ω is normalized as in Theorem 2 then there is a set F with $\omega(\Omega, F, \infty) \geq \frac{1}{2}$ (say) and with one dimensional Hausdorff measure bounded by a universal constant, is readily seen to be false using conformal mapping. Start with a simply connected domain Ω whose Riemann mapping function $f: \{z: |z|>1\} \to \Omega$ satisfies $\limsup_{r\to 1} |f'(re^{i\theta})| = \infty$ a.e. Delete a sequence of discs of uniform hyperbolic radius centered at points $f(z_j)$ where $\{z_j\}$ is a nontangentially dense set with $|f'(z_j)| \geq N$, and N is a given large constant. Harmonic measure for the resulting domain will be supported on the discs, and its density relative to length measure will be $\lesssim N^{-1}$.

The rest of the paper is concerned with the proof of Theorem 2. As in [4], [5] the proof is based on a recursive domain modification construction. In Section 1 we give some preliminary lemmas and then explain the construction in a special case. In Section 2 we prove two key lemmas, 2.1 which gives one of the building blocks for the construction, and 2.6. Sections 3 and 4 contain the recursive construction and Section 5 the main estimate on the resulting domain. In Section 6 we finish up the proof.

We conclude this introduction by giving a list of notation.

$\bar{D}$:	closure of the set D
$D(a,r)$:	euclidean disc with center a and radius r
$l(Q)$:	side length of the square Q
λD:	dilation of the disc or square D by factor $\lambda > 0$ around its center, e.g. $\lambda D(a,r) = D(a, \lambda r)$
$d\sigma$:	arc length measure on a given smooth curve
$h_1(E)$:	one-dimensional Hausdorff content of E, i.e., $h_1(E) = \inf(\sum_i r_i : E \subset \bigcup_i D(a_i, r_i))$
$\operatorname{cap} E$:	capacity of E, as defined in [1]
$\omega(\Omega, Y, z)$:	harmonic measure for Ω of $Y \cap \partial\Omega$, evaluated at z
g_Ω:	Green's function of Ω with pole at ∞, normalized so that (i) $g_\Omega > 0$ and (ii) if $\partial\Omega$ is smooth $d\omega(\Omega, \cdot, \infty) = |\nabla g_\Omega| d\sigma$.

We denote fixed constants by C, and use the notation $x \lesssim y$ to mean $x \leq Cy$ and $x \approx y$ for "$x \lesssim y$ and $y \lesssim x$". However, in many cases we have given numerical values for constants instead of calling them C. With one exception (the $\frac{5}{4}$ and $\frac{3}{2}$ at the beginning of Section 3) the values of these constants are irrelevant and are given only for the convenience of the author for whom it is easier to write 3, 4, and 12 instead of C_1, C_2 and $C_1 C_2$.

1. Auxiliary lemmas and thick case

Fix $\varepsilon>0$. Let $\Omega=\mathbf{C}^*\setminus E$ be a domain containing ∞, $D=D(z,r)$ a disc and assume that

$$\operatorname{cap}(E\cap D)\geq e^{-1/\varepsilon}r. \tag{1.1}$$

Condition (1.1) (called capacity density condition) and its negation will play a significant role in this paper as in [4]. If (1.1) is satisfied we can "smooth" the domain Ω by deleting $\overline{D}$ from it, without distorting harmonic measure by very much, as described in the following lemma.

Lemma 1.1. *Suppose Ω and D satisfy* (1.1). *Then for $\lambda>1$,*

$$\omega(\Omega\setminus\overline{D},\overline{D},\infty)\leq C_{\varepsilon,\lambda}\omega(\Omega,\lambda D,\infty).$$

Proof. We have $\omega(\lambda D\setminus E, E\cap\lambda D, z)\geq C_{\varepsilon,\lambda}^{-1}$ for all $z\in\partial D$. This is a corollary of Lemma 2.1 below so we skip the proof at present. By the maximum principle $\omega(\Omega, E\cap\lambda D, z)\geq C_{\varepsilon,\lambda}^{-1}$ for $z\in\partial D$. So by the maximum principle on $\Omega\setminus\overline{D}$, $\omega(\Omega\setminus\overline{D},\overline{D},z)\leq C_{\varepsilon,\lambda}\omega(\Omega,\lambda D,z)$ for all $z\in\Omega\setminus\overline{D}$ and in particular when $z=\infty$. $\square$

We will also want to control the density of harmonic measure. In the following lemma, σ is surface measure on ∂D.

Lemma 1.2. *Suppose $\Omega=\mathbf{C}^*\setminus E$ is a domain containing ∞, $\omega=\omega(\Omega,\cdot,\infty)$ and that $D=D(a,r)$ is a disc contained in E and $\lambda>1$. Then $\omega|_{\partial D}$ is absolutely continuous with respect to σ and for $z\in\partial D$,*

$$\left|\frac{d\omega}{d\sigma}(z)\right|\leq C_\lambda r^{-1}\omega(\Omega\setminus\lambda\bar{D},\lambda\bar{D},\infty).$$

Proof. Equivalently

$$\omega(\Omega,Y,\infty)\leq C_\lambda\frac{\sigma(Y)}{r}\omega(\Omega\setminus\lambda\bar{D},\lambda\overline{D},\infty) \tag{1.2}$$

for all $Y\subset\partial D$. To prove (1.2) define $N=\max_{\partial(\lambda D)}\omega(\Omega,Y,\cdot\,)$ and choose $z_0\in\partial(\lambda D)$ with $\omega(\Omega,Y,z_0)=N$. Let A be the annulus $\lambda^2D\setminus\overline{D}$. Then

$$\begin{aligned}N-\omega(\Omega\cap A,Y,z_0)&=\omega(\Omega,Y,z_0)-\omega(\Omega\cap A,Y,z_0)\\&=\int_{\partial(\lambda^2D)}\omega(\Omega,Y,\cdot\,)\,d\omega(\Omega\cap A,\cdot\,,z_0)\\&\leq N\omega(\Omega\cap A,\partial(\lambda^2D),z_0),\end{aligned}$$

where the last step follows by the maximum principle on $\Omega\setminus\lambda\bar{D}$. Also

$$\omega(\Omega\cap A,\partial(\lambda^2 D),z_0)\le\omega(A,\partial(\lambda^2 D),z_0)=\tfrac{1}{2}.$$

Combining this with the preceeding shows that $\omega(\Omega\cap A,Y,z_0)\ge\frac{1}{2}N=\frac{1}{2}\omega(\Omega,Y,z_0)$. By the maximum principle again, $\omega(A,Y,z_0)\ge\frac{1}{2}\omega(\Omega,Y,z_0)$. Since $\omega(A,Y,z_0)$ is comparable to $\sigma(Y)/r$ we obtain $\max_{\partial(\lambda D)}\omega(\Omega,Y,\cdot\,)\le C_\lambda\sigma(Y)/r$ and then (1.2) follows by the maximum principle on $\Omega\setminus\lambda\bar{D}$. □

Remarks. (1) We can combine Lemmas 1.1 and 1.2: with the assumptions of Lemma 1.1, let $\omega(\Omega\setminus\bar{D},\cdot\,,\infty)$. Then $|d\omega/d\sigma|\le C_{\varepsilon,\lambda}r^{-1}\omega(\Omega,\lambda D,\infty)$ on the set $\partial(\Omega\setminus\bar{D})\cap\partial D$.

(2) Lemma 1.1 and the result in Remark (1) remain valid if $\Omega\setminus\bar{D}$ is replaced by any domain contained in $\Omega\setminus\bar{D}$, because of the maximum principle. We will make this type of extension routinely and sometimes without explicitly saying so.

By way of motivation for the rest of the paper, we will now present a proof of Theorem 2 (hence Theorem 1) under the assumption that every disc of radius ≤ 1 centered at a point of E satisfies (1.1) This case of Theorem 1 does not originate with us, however. It is implicit in [6] given certain known facts about the covering map onto such a domain, as was observed independently by a large number of people in or around 1985. The proof below is implicit in [4] and the reason we make it explicit here is that it provides a simple model for the proof in the general case.

Let $\omega=\omega(\Omega,\cdot\,,\infty)$. For each $z\in E$ choose a disc $D(z,r)$ which is maximal subject to the following condition

$$\text{either } r=\varrho \quad\text{or}\quad \omega(D(z,r))\ge Mr. \tag{1.3}$$

Choose a subcover $D_j=D(z_j,r_j)$ with the Besicovitch property, i.e. $E\subset\bigcup_j D_j$ and no point belongs to more than C D_j's where C is a universal constant. Let $\widetilde{\Omega}=\Omega\setminus\bigcup_j\bar{D}_j$, $\widetilde{\omega}=\omega(\widetilde{\Omega},\cdot\,,\infty)$, $\tilde{g}$ the Green's function of $\widetilde{\Omega}$ with pole at ∞. Note that $\partial\widetilde{\Omega}$ is smooth except at finitely many points so that $d\widetilde{\omega}/d\sigma$ can be identified with $|\nabla\tilde{g}|$. Also

$$\widetilde{\omega}(\bar{D}_j)\lesssim\omega(2D_j) \tag{1.4}$$

by Lemma 1.1 (and the maximum principle), and in fact

$$|\nabla\tilde{g}|\lesssim\frac{\omega(2D_j)}{r_j}\quad\text{on } \partial\widetilde{\Omega}\cap\partial D_j \tag{1.5}$$

by Lemmas 1.1 and 1.2. In particular (1.5) implies

$$|\nabla\tilde{g}|\lesssim M \tag{1.6}$$

on $\partial\widetilde{\Omega}$, because of the stopping rule (1.3). On the other hand

$$\int_{\partial\widetilde{\Omega}} |\nabla\tilde{g}|\,d\sigma = 1, \tag{1.7}$$

$$\int_{\partial\widetilde{\Omega}} |\nabla\tilde{g}| \log|\nabla\tilde{g}|\,d\sigma \geq \text{const} \tag{1.8}$$

(see [3,5]), and therefore for large M it follows as in [4,5] that

$$\int_{|\nabla\tilde{g}|\geq M^{-\delta}} |\nabla\tilde{g}|\,d\sigma \geq C^{-1}\delta. \tag{1.9}$$

The simple calculation showing that (1.6)–(1.8) imply (1.9) is also done in Section 6 of the present paper. Now let $\{D_{j_k}\}$ be all D_j's such that either $r_j>\varrho$, or else $r_j=\varrho$ but ∂D_j intersects the set $\{\zeta\in\partial\widetilde{\Omega}: |\nabla\tilde{g}(\zeta)|\geq M^{-\delta}\}$. By (1.9) we have $\sum_k \widetilde{\omega}(\overline{D}_{j_k})\geq C^{-1}\delta$, and then (1.4) implies $\sum_k \omega(2D_{j_k})\geq C^{-1}\delta$. If $r_{j_k}>\varrho$ then the stopping rule (1.3) implies $\omega(2D_{j_k})\lesssim\omega(D_{j_k})$ so if we let

$$A_{j_k}=\begin{cases} D_{j_k}, & r_{j_k}>\varrho,\\ 2D_{j_k}, & r_{j_k}=\varrho,\end{cases}$$

then $\sum_k \omega(A_{j_k})\geq C^{-1}\delta$. Let $F=\bigcup_k A_{j_k}$. The A_{j_k} clearly have the Besicovitch property so $\omega(F)\geq C^{-1}\delta$. On the other hand, if $r_{j_k}>\varrho$ then $r_{j_k}\leq M^{-1}\omega(A_{j_k})$ by (1.3). Thus

$$\sum_{r_{j_k}>\varrho} r_{j_k}\lesssim M^{-1}.$$

If $r_{j_k}=\varrho$ then $r_{j_k}\lesssim M^{\delta}\omega(A_{j_k})$ by (1.5) since $|\nabla\tilde{g}|\geq M^{-\delta}$ somewhere on ∂D_{j_k}. Therefore

$$\sum_{r_{j_k}=\varrho} r_{j_k}\lesssim M^{\delta},$$

so the discs A_{j_k} give a covering of F of the type in Theorem 2. $\square$

The difficulty when (1.1) fails is of course that we cannot conclude (1.4) and (1.5). We therefore cannot simply delete the $\overline{D}_j$'s from Ω and instead will use a construction described in Lemma 2.1 below.

2. Domain modification procedure near thin parts of the boundary

Lemma 2.1. *There are absolute constants $R<\infty$, $\varepsilon_0>0$, $A<\infty$ making the following true.*

Suppose $\Omega=\mathbf{C}^*\setminus E$ is a domain containing ∞, $\omega=\omega(\Omega,\cdot,\infty)$, Q is a square with diameter r, and $\operatorname{cap}(E\cap Q)\le e^{-1/\varepsilon_0}r$. Then there are r_1 and r_2, $\frac{5}{6}<r_1<r_2<1$, such that if we let B be any closed disc contained in $\frac{1}{4}Q$ with radius $r(r^{-1}\operatorname{cap}(E\cap r_2Q))^{1/R}$ and define $\widetilde{E}=(E\setminus r_1Q)\cup B$, $\widetilde{\Omega}=\mathbf{C}^*\setminus\widetilde{E}$, $\widetilde{\omega}=\omega(\widetilde{\Omega},\cdot,\infty)$, then

(i) $\widetilde{\omega}(Y)\le\omega(Y)$ *for all Y with $Y\cap B=\emptyset$,*
(ii) $\widetilde{\omega}(B)\le A\omega(E\cap r_2Q)$,
(iii) $h_1(B)\ge A^{-1}h_1(E\cap r_2Q)$.

Remarks. (1) What the lemma says is that we delete the part of E which is contained in a certain square r_1Q and replace it with a disc whose capacity is roughly that of the part of E contained in a slightly larger square r_2Q. This makes harmonic measure decrease except of course on the disc, where it increases in a controlled way, and also makes h_1 content increase.

(2) Part (iii) of the lemma, while important for us, is trivial provided $R\ge1$ since a disc has essentially the largest h_1 content among all sets with the same capacity. It will be clear later on that we can take $R\ge1$, so we regard (iii) as proved and will say no more about it.

(3) Part (i) on the other hand may appear a bit strange, since if $E\cap r_1Q$ is deleted from E one would normally expect harmonic measure to increase drastically on the part of E which is just outside r_1Q. However, the small capacity assumption will allow us to choose r_1 so that E is very thin near $\partial(r_1Q)$ and then we will be able to show (i). This is the main point in the proof.

(4) By shrinking Q slightly we can guarantee (provided ε_0 is small enough) that $\partial Q\cap E=\emptyset$. This well-known fact (i.e. capacity dominates projected linear measure) is also part of Lemma 2.4 below. We may (and will) therefore assume $\partial Q\cap E=\emptyset$ in the proof.

(5) It is not hard to see (particularly if one thinks probabilistically) that it suffices to prove the following local statements on Q instead of (i) and (ii).

(i′) There is $r_{3/2}\in(r_1,r_2)$ such that $\omega(r_2Q\setminus\widetilde{E},Y,z)\le\omega(r_2Q\setminus E,Y,z)$ for all $z\in\partial(r_{3/2}Q)$ and all Y with $Y\cap B=\emptyset$.

(ii′) There is $r_3\in(r_2,1)$ such that $\omega(Q\setminus\widetilde{E},B,z)\le A\omega(Q\setminus E,E\cap r_2Q,z)$ for all $z\in\partial(r_3Q)$.

We will concentrate at first on proving (i′), (ii′) and after this has been done, will give the details of the reduction of (i), (ii) to (i′), (ii′).

We will need some estimates of harmonic measure which are special to the situation at hand, i.e., a domain obtained from a nice domain by deleting a set with small capacity. The following four lemmas are of this type. The sets G appearing in these lemmas are assumed closed relative to Q and Wiener regular for the sake of simplicity, although the latter assumption is not really needed. The first lemma is wellknown.

Lemma 2.2. *Let Q be a square with diameter* 1. *Fix $\lambda<1$ and $\varrho>0$. Suppose $G\subset\lambda Q$. Then $\omega(Q\setminus G, G, a)\gtrsim(\log(1/\operatorname{cap} G))^{-1}$ for all $a\in\lambda Q$. The inequality is reversible if* $\operatorname{dist}(a,G)\geq\varrho$. *(Constants depend on λ and ϱ.)*

Remarks. (1) Similar statements when $\operatorname{diam} Q\neq 1$ are obtained by scaling.

(2) Q could equally well be a disc instead of a square. This justifies the first sentence in the proof of Lemma 1.1.

Proof. Let μ be the capacitary measure of G, i.e., $\|\mu\|=(\log(1/\operatorname{cap} G))^{-1}$ and $V_\mu(x)\stackrel{\text{def}}{=}\int\log(1/|x-y|)d\mu(y)=1$ when $x\in G$. Let $\Gamma(x,y)$ be the Green's function of Q and consider the function $\chi(x)=\int\Gamma(x,y)d\mu(y)$. Then χ vanishes on ∂Q and is bounded above and below on G since $\Gamma(x,y)\approx\log(1/|x-y|)$ when $x,y\in\lambda Q$. Hence $\chi(x)\approx\omega(Q\setminus G,G,x)$ for all $x\in Q$. If $x\in\lambda Q$, then $\Gamma(x,y)$ is bounded below by a constant for all $y\in G$ and therefore $\chi(x)\gtrsim\|\mu\|$. If $\operatorname{dist}(x,G)>\varrho$ then $\Gamma(x,y)$ is bounded above by a constant so that $\chi(x)\lesssim\|\mu\|$. The lemma follows. $\square$

Lemma 2.3. *Let ε_0 be a sufficiently small constant. Fix $\lambda<1$ and $\varrho>0$. Suppose Q is a square of diameter* 1, *$G\subset Q$ and $Y\subset\partial(Q\setminus G)$. Suppose $a,b\in\lambda Q\setminus G$ satisfy $\omega(Q\setminus G,G,a)\leq\varepsilon_0$, $\omega(Q\setminus G,G,b)\leq\varepsilon_0$, $\operatorname{dist}(a,Y)\geq\varrho$ and $\operatorname{dist}(b,Y)\geq\varrho$. Then $\omega(Q\setminus G,Y,a)\approx\omega(Q\setminus G,Y,b)$ (constants depend on λ and ϱ).*

Proof. This will be based on the following "fine" version of Harnack's inequality. Suppose Δ is the unit disc, $E\subset\Delta$ is relatively closed, $a\in\frac{1}{2}\Delta$, $\omega(\Delta\setminus E,E,a)<\varepsilon_0$ where ε_0 is sufficiently small. Suppose u is continuous on Δ, positive harmonic on $\Delta\setminus E$ and zero on E. Then $\sup_{(1/2)\Delta}u(x)\leq Cu(a)$.

The proof we give for the Harnack inequality was suggested by P. Jones. Let $M=\sup_{(1/2)\Delta}u(x)$ and $F=\{x\in\Delta:u(x)\geq M\}$. Then F intersects every circle centered at zero with radius between $\frac{1}{2}$ and 1 so by the Beurling projection theorem, $\omega(\Delta\setminus F,F,a)\geq C^{-1}$. Then also

$$\omega(\Delta\setminus(E\cup F),F,a)\geq\omega(\Delta\setminus F,F,a)-\omega(\Delta\setminus E,E,a)\geq C^{-1}-\varepsilon_0\geq(2C)^{-1}$$

provided ε_0 is small. So by harmonic estimation

$$u(a)\geq M\omega(\Delta\setminus(E\cup F),F,a)\geq(2C)^{-1}M$$

and we are done with the Harnack inequality.

In proving the lemma we may assume $\operatorname{diam} Y \leq \varrho$ (else subdivide Y in sets with this property). Note that by Lemma 2.2, $\operatorname{cap}(G \cap \mu Q)$ will be small, where $\mu = \frac{1}{2}(1+\lambda)$. Choose discs Δ_j, $j=1, \dots, N$ so that the following hold:

(i) Δ_1 is centered at a, Δ_N is centered at b;

(ii) $\Delta_j \subset \mu Q \setminus Y$;

(iii) the diameters of the Δ_j are bounded below;

(iv) N is bounded above;

(v) $\frac{1}{4}\Delta_j \subset \frac{1}{2}\Delta_{j+1}$.

We note that (iii) and (iv) are possible because $\operatorname{diam} Y$ is small which implies that $Q \setminus Y$ is for all intents and purposes an annulus. Because of (iii) the capacity of $G \cap \Delta_j$ will be small compared with the radius of Δ_j. Thus, by Lemma 2.2, for any j there will be a point $a_j \in \frac{1}{4}\Delta_j$ such that $\omega(\Delta_j \setminus G, G, a_j)$ is small. If $j=1$, we can take $a_j = a$. In view of (v), the Harnack type inequality of the first part of the proof implies $\max_{(1/2)\Delta_{j+1}} u(z) \lesssim u(a_j)$, so by iterating and using (iv) we obtain $u(b) \lesssim u(a)$. $\square$

We mentioned in Remark 3 at the beginning of the section, that Lemma 2.1 requires choosing r_1 so that E is appropriately "thin near $\partial(r_1 Q)$". The next two lemmas give a meaning to this.

Lemma 2.4. *Let Q be a square with diameter 1, $G \subset Q$ and define $G_\lambda = \{z \in Q : \omega(Q \setminus G, G, z) \geq \lambda\}$ and $\gamma_\lambda = \{r : G_\lambda \cap \partial(rQ) \neq \emptyset\}$. Then the linear measure of γ_λ is $\leq C(\operatorname{cap} G)^\lambda$.*

Proof. By standard projection theorems for capacity it suffices to prove $\operatorname{cap} G_\lambda \leq (\operatorname{cap} G)^\lambda$. Let μ and μ_λ be the capacitary measures of G and G_λ respectively. Let $V_\mu(x) = \int \log(1/|x-y|)\, d\mu(y)$. Then $\omega(Q \setminus G, G, \cdot\,) \leq V_\mu$ by the maximum principle. So

$$
\begin{aligned}
\left(\log \frac{1}{\operatorname{cap} G_\lambda}\right)^{-1} = \|\mu_\lambda\| &\leq \lambda^{-1} \int \omega(Q \setminus G, G, \cdot\,)\, d\mu_\lambda(\cdot\,) \\
&\leq \lambda^{-1} \int V_\mu \, d\mu_\lambda \\
&= \lambda^{-1} \int V_{\mu_\lambda} \, d\mu \\
&\leq \lambda^{-1} \|\mu\| = \lambda^{-1} \left(\log \frac{1}{\operatorname{cap} G}\right)^{-1}
\end{aligned}
$$

as claimed. $\square$

Lemma 2.5. *Suppose Q is a square of diameter 1, $G\subset Q$, and $a\in Q$. Then the set*

$$\{t\in\mathbf{R}^+ : \exists\varepsilon>0 : \omega\big(Q\backslash G, G\cap((t+\varepsilon)Q\backslash(t-\varepsilon)Q), a\big)\geq M\varepsilon\omega(Q\backslash G, G, a)\}$$

has linear measure $\leq CM^{-1}$.

Remark. Needless to say we are setting $sQ=\emptyset$ if $s<0$.

Proof. This is simply the one-dimensional Hardy–Littlewood maximal theorem applied to the projected measure $\sigma([r_1,r_2))=\omega(Q\backslash G, G\cap(r_2Q\backslash r_1Q), a)$. $\square$

We won't define precisely what we mean by "E is thin near $\partial(r_1Q)$." However, the idea is as follows: fix a point a located well inside Q such that $\omega(Q\backslash E, E, a)\leq(\log(1/\operatorname{cap}(E\cap Q)))^{-1}$. Such points exist because of Lemma 2.4. Then choose r_1 by Lemma 2.5 so that $\omega(Q\backslash E, E\cap((r_1+\varepsilon)Q\backslash(r_1-\varepsilon)Q), a)\leq\varepsilon(\log(1/\operatorname{cap}(E\cap Q)))^{-1}$ for all $\varepsilon>0$. This makes E about as thin as it can be near $\partial(r_1Q)$, at least as far as harmonic measures are concerned. Some variants on this idea will also be used (later on, that is, when we give the precise definition of r_1, r_2, r_3 and $r_{3/2}$).

Now we start the proof of Lemma 2.1. We can assume that Q has diameter 1 and (see Remark 4 above) ∂Q does not intersect E. The main work in the proof is contained in the following inequality (2.2). Fix a large number $M<\infty$. Suppose $a, b\in\lambda Q$ with $\lambda=\frac{59}{60}$, $E\subset Q$ is relatively closed and Wiener regular. Let $r\in(\frac{1}{60},\frac{59}{60})$ be such that $E\cap\partial(rQ)=\emptyset$ and

$$\left\{\begin{array}{l}\omega(Q\backslash E, E\cap((r+\varepsilon)Q\backslash(r-\varepsilon)Q), a)\leq m(a)\varepsilon\\ \omega(Q\backslash E, E\cap((r+\varepsilon)Q\backslash(r-\varepsilon)Q), b)\leq m(b)\varepsilon\end{array}\right.\tag{2.1}$$

for all $\varepsilon>0$, where $m(a)$ and $m(b)$ are sufficiently small. Let F be $E\cap(Q\backslash rQ)$. Then we claim that

$$\omega(Q\backslash F, Y, a)\leq\omega(Q\backslash E, Y, a)+Cm(a)\omega(Q\backslash E, Y, b)\tag{2.2}$$

for all $Y\subset\partial(Q\backslash F)$.

Remark. In other words, deleting $E\cap rQ$ does not drive harmonic measures up by very much.

Proof of (2.2). Let σ_a (respectively σ_b) be the projection of $\omega(Q\backslash E,\cdot\,,a)$ (respectively $\omega(Q\backslash E,\cdot\,,b)$) i.e. $\sigma_a([s,t))=\omega(Q\backslash E, E\cap(tQ\backslash sQ), a)$ etc. Then

$$\begin{aligned}\omega(Q\backslash F, Y, a)-\omega(Q\backslash E, Y, a)&=\int_{E\backslash F}\omega(Q\backslash F, Y,\cdot\,)\,d\omega(Q\backslash E,\cdot\,,a)\\ &\leq\int_{t<r}\max_{z\in\partial(tQ)}\omega(Q\backslash F, Y, z)\,d\sigma_a(t).\end{aligned}$$

We now show the following inequality on the integrand

$$\omega(Q\backslash F, Y, z) \le C\left(\log\frac{1}{r-t}\right)\omega(Q\backslash F, Y, b), \quad z\in\partial(tQ),\ t<r. \tag{2.3}$$

To prove (2.3) let D be a closed disc centered at z with radius half its distance to $\partial(rQ)$ (approximately $r-t$). Then using the maximum principle

$$\begin{aligned}\omega\big(Q\backslash(D\cup F), D, b\big) &\ge \omega\big(Q\backslash(D\cup E), D, b\big)\\ &= \omega(Q\backslash D, D, b) - \int_{E\backslash D}\omega(Q\backslash D, D, \cdot\,)\,d\omega(Q\backslash(D\cup E), \cdot\,, b)\\ &\ge \omega(Q\backslash D, D, b) - \int_{E\backslash D}\omega(Q\backslash D, D, \cdot\,)\,d\omega(Q\backslash E, \cdot\,, b).\end{aligned}$$

$Q\backslash D$ is essentially an annulus so that

$$\begin{aligned}\omega(Q\backslash D, D, b) &\gtrsim \left(\log\frac{1}{r-t}\right)^{-1};\\ \omega(Q\backslash D, D, \zeta) &\lesssim \left(\log\frac{1}{r-t}\right)^{-1}\log\frac{1}{|\zeta-z|}\\ &\lesssim \left(\log\frac{1}{r-t}\right)^{-1}\log\frac{1}{|r-s|} \quad \text{if } \zeta\in\partial(sQ).\end{aligned}$$

Hence

$$\omega\big(Q\backslash(D\cup F), D, b\big) \gtrsim \left(\log\frac{1}{r-t}\right)^{-1} - \int\left(\log\frac{1}{r-t}\right)^{-1}\log\left|\frac{1}{r-s}\right|\,d\sigma_b(s). \tag{2.4}$$

By the assumption on $\sigma_b([r-\varepsilon, r+\varepsilon))$ and monotonicity of the logarithm we may replace $d\sigma_b$ by $m(b)ds$ and may therefore conclude that

$$\omega\big(Q\backslash(D\cup F), D, b\big) \gtrsim \left(\log\frac{1}{r-t}\right)^{-1}\big(1-Cm(b)\big) \gtrsim \left(\log\frac{1}{r-t}\right)^{-1}$$

provided $m(b)$ is small enough. Next

$$\begin{aligned}\omega(Q\backslash F, Y, z) &\lesssim \min_{\zeta\in D}\omega(Q\backslash F, Y, \zeta)\\ &\lesssim \omega(Q\backslash F, Y, b)/\omega\big(Q\backslash(D\cup F), D, b\big),\end{aligned}$$

where the first inequality follows from Harnack's inequality and the second from harmonic estimation of the function $\omega(Q\backslash F, Y, \cdot\,)$ on the domain $Q\backslash(D\cup F)$. Since we have already shown that $\omega(Q\setminus(D\cup F), D, b) \gtrsim (\log 1/(r-t))^{-1}$ we may conclude (2.3).

To prove (2.2) substitute (2.3) into the inequality preceding it obtaining

$$\omega(Q\setminus F, Y, a) - \omega(Q\setminus E, Y, a) \lesssim \omega(Q\setminus F, Y, b) \int_{t<r} \log \frac{1}{r-t}\, d\sigma_a(t).$$

$$\omega(Q\setminus F, Y, a) - \omega(Q\setminus E, Y, a) \lesssim m(a)\omega(Q\setminus F, Y, b), \tag{2.5}$$

where we used the assumption on $\sigma_a([r-\varepsilon, r+\varepsilon))$ and monotonicity of the logarithm. If we apply (2.5) with $a=b$ we obtain

$$\omega(Q\setminus F, Y, b) - \omega(Q\setminus E, Y, b) \lesssim m(b)\omega(Q\setminus F, Y, b)$$

and therefore $\omega(Q\setminus F, Y, b) \lesssim \omega(Q\setminus E, Y, b)$ provided $m(b)$ is sufficiently small. Substituting this last inequality back into the right side of (2.5) gives (2.2).

We will also use the following slight variant of (2.2) where one deletes an annulus instead of a disc. Let Q, E, a, b as in (2.1) and suppose $r, \varrho \in (\frac{1}{60}, \frac{59}{60})$, $\varrho<r$, are such that ϱ as well as r satisfies (2.1), i.e.

$$\begin{cases} \omega(Q\setminus E, E\cap((r+\varepsilon)Q\setminus(r-\varepsilon)Q), a) \le m(a)\varepsilon \\ \omega(Q\setminus E, E\cap((\varrho+\varepsilon)Q\setminus(\varrho-\varepsilon)Q), a) \le m(a)\varepsilon \\ \omega(Q\setminus E, E\cap((r+\varepsilon)Q\setminus(r-\varepsilon)Q), b) \le m(b)\varepsilon \\ \omega(Q\setminus E, E\cap((\varrho+\varepsilon)Q\setminus(\varrho-\varepsilon)Q), b) \le m(b)\varepsilon \end{cases} \tag{2.1'}$$

for all $\varepsilon>0$, where the numbers $m(a)$ and $m(b)$ are sufficiently small. Let us define $F=E\cap(\varrho Q\cup(Q\setminus rQ))$. Then the statement analogous to (2.2) holds, i.e.

$$\omega(Q\setminus F, Y, a) \le \omega(Q\setminus E, Y, a) + C\, m(a)\omega(Q\setminus E, Y, b). \tag{2.2'}$$

We indicate the necessary changes in the proof. The inequality preceding (2.3) is now replaced by

$$\omega(Q\setminus F, Y, a) - \omega(Q\setminus E, Y, a) \le \int_{t\in(\varrho,r)} \max_{z\in\partial(tQ)} \omega(Q\setminus F, Y, z)\, d\sigma_a(t)$$

and (2.3) is replaced by

$$\omega(Q\setminus F, Y, z) \le C \max\left(\log \frac{1}{r-t}, \log \frac{1}{t-\varrho}\right)\omega(Q\setminus F, Y, b), \tag{2.3'}$$

$z\in\partial(tQ)$, $\varrho<t<r$. The proof of (2.3′) is essentially the same as of (2.3). Take D to be a disc of radius half the distance from z to $\partial(\varrho Q)\cup\partial(rQ)$. Considering separately the cases $r-t\le t-\varrho$ and $t-\varrho\le r-t$ we obtain

(2.4′)
$$\begin{cases} \omega(Q\setminus D\cup F, D, b)\gtrsim\left(\log\frac{1}{r-t}\right)^{-1}-\int\left(\log\frac{1}{r-t}\right)^{-1}\log\frac{1}{|r-s|}d\sigma_b(s), & r-t\le t-\varrho \\ \omega(Q\setminus D\cup F, D, b)\gtrsim\left(\log\frac{1}{t-\varrho}\right)^{-1}-\int\left(\log\frac{1}{t-\varrho}\right)^{-1}\log\frac{1}{|\varrho-s|}d\sigma_b(s), & t-\varrho\le r-t. \end{cases}$$

We conclude that $\omega(Q\setminus D\cup F, D, b)\gtrsim(\log(1/(r-t)))^{-1}$ in the first case and $\omega(Q\setminus D\cup F, D, b)\gtrsim(\log 1/(t-\varrho))^{-1}$ in the second case, and then (2.3′) follows similarly to (2.3). From (2.3′) and the inequality preceding it we conclude that

$$\begin{aligned} &\omega(Q\setminus F, Y, a)-\omega(Q\setminus E, Y, a) \\ &\qquad\lesssim\omega(Q\setminus F, Y, b)\left[\int_{t<r}\log\frac{1}{r-t}\,d\sigma_a(t)+\int_{t>\varrho}\log\frac{1}{t-\varrho}\,d\sigma_a(t)\right] \end{aligned}$$

and then (2.2′) follows like (2.2).

Now we define r_1, r_2, r_3, and $r_{3/2}$.

Choice of r_3. We require $r_3\in(\frac{56}{60},\frac{57}{60})$, $E\cap\partial(r_3Q)=\emptyset$ and

$$\omega(Q\setminus E, E, z)\le C_3\left(\log\frac{1}{\operatorname{cap}(E\cap Q)}\right)^{-1}$$

for all $z\in\partial(r_3Q)$. This is possible by Lemma 2.4 provided C_3 is a sufficiently large universal constant and ε_0 is small enough.

Choice of r_2. Let $a\in\partial(r_3Q)$ be such that $\omega(Q\setminus E, E, a)$ is as small as possible and choose $r_2\in(\frac{54}{60},\frac{55}{60})$ such that $E\cap\partial(r_2Q)=\emptyset$ and (for a sufficiently large universal constant C_2)

$$\omega\big(Q\setminus E, E\cap((r_2+\varepsilon)Q\setminus(r_2-\varepsilon)Q), a\big)\le C_2\varepsilon\,\omega(Q\setminus E, E, a) \tag{2.6}$$

for all $\varepsilon>0$. This is possible by Lemma 2.5.

The remaining choices—$r_{3/2}$ and r_1—are analogous to the preceding but with r_2Q replacing Q.

Choice of $r_{3/2}$. We require $r_{3/2}\in(\frac{52}{60},\frac{53}{60})$, $E\cap\partial(r_{3/2}Q)=\emptyset$ and

$$\omega(r_2Q\setminus E, E, z)\le C_{3/2}\left(\log\frac{1}{\operatorname{cap}(E\cap r_2Q)}\right)^{-1}$$

for all $z\in\partial(r_{3/2}Q)$, which is possible by Lemma 2.4.

Choice of r_1. Let $\alpha\in\partial(r_{3/2}Q)$ be such that $\omega(Q\setminus E, E, \alpha)$ is as small as possible and choose $r_1\in\left(\frac{50}{60}, \frac{51}{60}\right)$ such that $E\cap\partial(r_1Q)=\emptyset$ and

$$(2.7)\qquad \omega\big(r_2Q\setminus E, E\cap((r_1+\varepsilon)Q\setminus(r_1-\varepsilon)Q), \alpha\big)\leq C_1\varepsilon\omega(r_2Q\setminus E, E, \alpha)$$

for all $\varepsilon>0$, which is possible by Lemma 2.5.

Next we want to prove (i′) and (ii′). Let us note first that (2.6) remains valid (with a different constant C_2) if the fixed point a is replaced by any other point $b\in\overline{r_3Q}$ satisfying the following conditions: $b\notin\left(\frac{56}{60}Q\setminus\frac{53}{60}Q\right)$ and $\omega(Q\setminus E, E, b)$ is sufficiently small. This is a tautology for $\varepsilon>\frac{1}{120}$ and follows from Lemma 2.3 for $\varepsilon<\frac{1}{120}$ (so that the distance from $(r_2+\varepsilon)Q\setminus(r_2-\varepsilon)Q$ to b is bounded below), since $\omega(Q\setminus E, E, b)\geq\omega(Q\setminus E, E, a)$ by the maximum principle.

In the same way (2.7) remains valid if α is replaced by any point $\beta\in\overline{r_{3/2}Q}$ such that $\beta\notin\frac{52}{60}Q\setminus\frac{49}{60}Q$ and $\omega(r_2Q\setminus E, E, \beta)$ is small enough.

In particular inequality (2.6) holds for all $a\in\partial(r_3Q)$ and for some $a\in\frac{1}{4}Q$ (the latter by Lemma 2.4) while inequality (2.7) holds for all $\alpha\in\partial(r_{3/2}Q)$ and for some $\alpha\in\frac{1}{4}Q$.

We are now set up to prove (i′), (ii′).

Proof of (ii′). Fix $z_0\in\partial(r_3Q)$. Choose $r\in\left(\frac{58}{60}, \frac{59}{60}\right)$ by Lemma 2.5 so that $\omega(Q\setminus E, E\cap((r+\varepsilon)Q\setminus(r-\varepsilon)Q), z_0)\leq C\varepsilon\omega(Q\setminus E, E, z_0)$ for all $\varepsilon>0$ and apply (2.2′) with this r and with $\varrho=r_2$, $b=a=z_0$, taking $Y=E\cap r_2Q$. The hypothesis (2.1′) is satisfied with $m(a)=m(b)=C\omega(Q\setminus E, E, z_0)$. We conclude that

$$\omega(Q\setminus F, E\cap r_2Q, z_0)\leq\omega(Q\setminus E, E\cap r_2Q, z_0)+C\omega(Q\setminus E, E, z_0)\omega(Q\setminus E, E\cap r_2Q, z_0).$$

The second term on the right side can be absorbed leading to

$$\begin{aligned}\omega(Q\setminus E, E\cap r_2Q, z_0)&\gtrsim\omega(Q\setminus F, E\cap r_2Q, z_0)\\&\geq\omega(rQ\setminus E\cap r_2Q, E\cap r_2Q, z_0)\\&\gtrsim\left(\log\frac{1}{\operatorname{cap}(E\cap r_2Q)}\right)^{-1}\end{aligned}$$

by the maximum principle and Lemma 2.2. On the other hand

$$\begin{aligned}\omega(Q\setminus\widetilde{E}, B, z_0)&\leq\omega(Q\setminus B, B, z_0)\\&\approx\left(\log\frac{1}{\operatorname{cap}B}\right)^{-1}\\&\approx R\left(\log\frac{1}{\operatorname{cap}(E\cap r_2Q)}\right)^{-1}\end{aligned}$$

using the definition of B. This gives (ii′) with $A=\text{const}\cdot R$.

Proof of (i′). We will apply (2.2) with Q replaced by r_2Q, $F=E\cap(r_2Q\setminus r_1Q)$, and taking a to be an arbitrary point of $\partial(r_{3/2}Q)$ and b a point of $\frac{1}{4}Q$ such that $\omega(r_2Q\setminus E,E,b)$ is small. Then (2.7) holds with $\alpha=a$ or b, so (2.1) holds with $m(a)=C\omega(r_2Q\setminus E,E,a)$, $m(b)=C\omega(r_2Q\setminus E,E,b)$. Thus for $Y\subset\partial(r_2Q\setminus F)$

$$\omega(r_2Q\setminus F,Y,a)-\omega(r_2Q\setminus E,Y,a)\lesssim\omega(r_2Q\setminus E,E,a)\omega(r_2Q\setminus E,Y,b). \tag{2.8}$$

On the other hand

$$\begin{aligned}\omega(r_2Q\setminus F,Y,a)-\omega(r_2Q\setminus\widetilde{E},Y,a)&=\int_B\omega(r_2Q\setminus F,Y,\cdot\,)\,d\omega(r_2Q\setminus\widetilde{E},\cdot\,,a)\\&\geq\min_{\zeta\in B}\omega(r_2Q\setminus F,Y,\zeta)\omega(r_2Q\setminus\widetilde{E},B,a).\end{aligned}$$

The values of $\omega(r_2Q\setminus F,Y,\cdot\,)$ at any two points of $\frac{1}{4}Q(\supset B)$ are comparable by Harnack's inequality. Taking $\zeta=b$ we get

$$\begin{aligned}\omega(r_2Q\setminus F,Y,a)-\omega(r_2Q\setminus\widetilde{E},Y,a)&\gtrsim\omega(r_2Q\setminus F,Y,b)\omega(r_2Q\setminus\widetilde{E},B,a)\\&\geq\omega(r_2Q\setminus E,Y,b)\omega(r_2Q\setminus\widetilde{E},B,a).\end{aligned}\tag{2.9}$$

Also by the maximum principle

$$\begin{aligned}\omega(r_2Q\setminus\widetilde{E},B,a)&\geq\omega(r_2Q\setminus B,B,a)-\omega(r_2Q\setminus F,F,a)\\&\geq\omega(r_2Q\setminus B,B,a)-\omega(r_2Q\setminus E,E,a)\\&\geq C^{-1}R\left(\log\frac{1}{\operatorname{cap}(r_2Q\cap E)}\right)^{-1}-C\left(\log\frac{1}{\operatorname{cap}(r_2Q\cap E)}\right)^{-1}\end{aligned}$$

using that $a\in\partial(r_{3/2}Q)$. So for large R,

$$\begin{aligned}\omega(r_2Q\setminus\widetilde{E},B,a)&\gtrsim R\left(\log\frac{1}{\operatorname{cap}(r_2Q\cap E)}\right)^{-1}\\&\approx R\omega(r_2Q\setminus E,E,a)\end{aligned}$$

using $a\in\partial(r_{3/2}Q)$ again. If we substitute this into (2.9) we obtain

$$\omega(r_2Q\setminus F,Y,a)-\omega(r_2Q\setminus\widetilde{E},Y,a)\gtrsim R\omega(r_2Q\setminus E,E,a)\omega(r_2Q\setminus E,Y,b).$$

Comparing with (2.8) gives

$$\omega(r_2Q\setminus F,Y,a)-\omega(r_2Q\setminus\widetilde{E},Y,a)\gtrsim R\big(\omega(r_2Q\setminus F,Y,a)-\omega(r_2Q\setminus E,Y,a)\big)$$

and therefore (i′) provided R is sufficiently large.

In order to complete the proof of Lemma 2.1 we must now carry out the localization argument showing that (i′) and (ii′) imply (i) and (ii).

Suppose first that Ω is any domain (not necessarily containing ∞) with $(\mathbf{C}^*\backslash\Omega)\cap Q=E\cap Q$, and let $\widetilde{\Omega}=(\Omega\cup E)\backslash\widetilde{E}$. We claim that if $Y\cap B=\emptyset$ then $\omega(\widetilde{\Omega},Y,z)\le\omega(\Omega,Y,z)$ for all $z\in\Omega\backslash r_2Q$.

It suffices by the maximum principle to prove this when $z\in\partial(r_2Q)$. Let $P=\max_{\partial(r_2Q)}\omega(\widetilde{\omega},Y,\cdot\,)/\omega(\Omega,Y,\cdot\,)$ which is well-defined because $\partial(r_2Q)\cap E=\emptyset$. For $z\in\partial(r_{3/2}Q)$

$$\begin{aligned}\omega(\widetilde{\Omega},Y,z)-\omega(r_2Q\backslash\widetilde{E},Y,z)&=\int_{\partial(r_2Q)}\omega(\widetilde{\Omega},Y,\cdot\,)\,d\omega(r_2Q\backslash\widetilde{E},\cdot\,,z)\\&\le\int_{\partial(r_2Q)}\omega(\widetilde{\Omega},Y,\cdot\,)\,d\omega(r_2Q\backslash E,\cdot\,,z)\\&\le P\int_{\partial(r_2Q)}\omega(\Omega,Y,\cdot\,)\,d\omega(r_2Q\backslash E,\cdot\,,z)\\&=P\big(\omega(\Omega,Y,z)-\omega(r_2Q\backslash E,Y,z)\big).\end{aligned}$$

The first inequality followed from (i′) applied to subsets of $\partial(r_2Q)$. Using (i′) again, this time with the given Y,

$$\omega(\widetilde{\Omega},Y,z)\le P\omega(\Omega,Y,z)+(1-P)\omega(r_2Q\backslash E,Y,z).$$

If $P>1$ this is a contradiction since $\omega(\widetilde{\Omega},Y,\cdot\,)-P\omega(\Omega,Y,\cdot\,)$ would be a harmonic function on $\Omega\backslash\overline{r_{3/2}Q}$ nonpositive and not identically zero on the boundary but vanishing at some point of $\partial(r_2Q)$. This proves the claim.

Lemma 2.1 follows immediately by taking Ω to be the given domain. For (ii) we define $\Pi=\max_{\partial Q}\omega(\widetilde{\omega},B,\cdot\,)/\omega(\Omega,r_2Q\cap E,\cdot\,)$ where now Ω is the given domain. For $z\in\partial(r_3Q)$ we have

$$\omega(\widetilde{\Omega},B,z)-\omega(Q\backslash\widetilde{E},B,z)=\int_{\partial Q}\omega(\widetilde{\Omega},B,\cdot\,)\,d\omega(Q\backslash\widetilde{E},\cdot\,,z).$$

If we apply the claim with $\Omega=Q\backslash E$ we bound this by

$$\int_{\partial Q}\omega(\widetilde{\Omega},B,\cdot\,)\,d\omega(Q\backslash E,\cdot\,,z)$$

which is then

$$\begin{aligned}&\le\Pi\int_{\partial Q}\omega(\Omega,r_2Q\cap E,\cdot\,)\,d\omega(Q\backslash E,\cdot\,,z)\\&=\Pi\big(\omega(\Omega,r_2Q\cap E,z)-\omega(Q\backslash E,r_2Q\cap E,z)\big).\end{aligned}$$

So by (ii′)

$$\omega(\widetilde{\Omega}, B, z) \le \Pi\omega(\Omega, r_2 Q \cap E, z) + (A-\Pi)\omega(Q\setminus E, r_2 Q\cap E, z)$$

with A as in (ii′). If $\Pi > A$ this means that $\omega(\widetilde{\Omega}, B, \cdot) - \Pi\omega(\Omega, r_2 Q\cap E, \cdot)$ would be a nonpositive harmonic function on $\Omega\setminus\overline{r_3 Q}$, not identically zero but zero at an interior part. A contradiction which finishes the proof of Lemma 2.1. □

As has already been mentioned Lemma 2.1 is intended as a building block in a domain modification construction. However, let us first use it to prove the following estimate which will be needed in Section 5. A somewhat cruder result than Lemma 2.1 would also suffice for this.

Lemma 2.6. *Suppose $\Omega = \mathbf{C}^* \setminus E$ is a domain containing ∞, $\omega = \omega(\Omega, \cdot, \infty)$, Q a square, and assume that $\omega(Q) \le M h_1(E\cap\frac{1}{2}Q)$. Then*

$$\left|\int_{\mathbf{C}\setminus Q} \frac{d\omega(\zeta)}{\zeta - z}\right| \le CM, \quad z\in \tfrac{1}{8}Q.$$

Proof. We show first that if α is a small fixed constant and B^* a disc of radius $\alpha h_1(E\cap\frac{1}{2}Q)$ centered at z, then $\omega(\Omega\setminus B^*, B^*, \infty) \lesssim M\cdot$ radius of B^*.

For this we let ε_0 be as in Lemma 2.1, let $\widetilde{Q} = \frac{3}{4}Q$ and consider two cases

(i) $\operatorname{cap}(E\cap\widetilde{Q}) \ge e^{-1/\varepsilon_0}\operatorname{diam}\widetilde{Q}$

(ii) $\operatorname{cap}(E\cap\widetilde{Q}) < e^{-1/\varepsilon_0}\operatorname{diam}\widetilde{Q}$.

In case (i) we use Lemma 1.1 (more precisely, the analogous statement with squares instead of discs, which is proved exactly the same) to conclude that $\omega(\Omega\setminus\overline{\widetilde{Q}}, \overline{\widetilde{Q}}, \infty) \lesssim \omega(Q)$. Hence by the maximum principle $\omega(\Omega\setminus B^*, B^*, \infty) \lesssim \omega(Q)$, and since $\omega(Q) \le M h_1(E\cap\frac{1}{2}Q) \approx M\cdot$ radius of B^* we are done.

For case (ii) we note that the radius of B^* is smaller than the radius of the concentric disc B obtained by applying Lemma 2.1 to the square $\widetilde{Q}$. This follows from Lemma 2.1 (iii) provided α is small, since $\frac{1}{2}Q \subset \frac{5}{6}\widetilde{Q} \subset r_2\widetilde{Q}$. Using the maximum principle, then Lemma 2.1 (ii) therefore gives that

$$\omega(\Omega\setminus B^*, B^*, \infty) \le \omega(\widetilde{\Omega}, B, \infty) \lesssim \omega(Q)$$

where $\widetilde{\Omega}$ is the domain resulting from Lemma 2.1. So it follows as in the previous case that $\omega(\Omega\setminus B^*, B^*, \infty) \lesssim M\cdot$ radius of B^*.

Now let $\omega^* = \omega(\Omega\setminus B^*, \cdot, \infty)$. Then

$$\int_{\mathbf{C}^*\setminus Q} \frac{d\omega^*(\zeta)}{\zeta - z} = -\int_Q \frac{d\omega^*(\zeta)}{\zeta - z}.$$

This is because $\int_{\mathbf{C}^*} d\omega^*(\zeta)/(\zeta-z)$ is the z-derivative of the Green's function of $\Omega\setminus B^*$ (with pole at ∞) and therefore vanishes at interior points of $\mathbf{C}^*\setminus(\Omega\setminus B^*)$.

Now $\omega^*(Q)\le\omega^*(B^*)+\omega(Q)\lesssim M\cdot$ radius of B^* and on the other hand the distance from z to $\operatorname{supp}\omega^*$ is the radius of B^*. We conclude that

$$\left|\int_Q \frac{d\omega^*(\zeta)}{(\zeta-z)}\right| \lesssim M$$

and therefore that

$$\left|\int_{\mathbf{C}^*\setminus Q} \frac{d\omega^*(\zeta)}{\zeta-z}\right| \lesssim M. \tag{2.10}$$

Now consider

$$\left|\int_{\mathbf{C}^*\setminus Q} \frac{d(\omega-\omega^*)(\zeta)}{\zeta-z}\right|. \tag{2.11}$$

$\omega-\omega^*$ is a positive measure on $\mathbf{C}^*\setminus Q$ by the maximum principle, and $\omega(\mathbf{C}^*\setminus Q)-\omega^*(\mathbf{C}^*\setminus Q)=\omega^*(Q)-\omega(Q)\le\omega^*(B^*)\le Ml(Q)$. And $\operatorname{dist}(z,\mathbf{C}^*\setminus Q)\gtrsim l(Q)$, so (2.11) is $\lesssim M$ and if we combine this with (2.10) we are done. $\square$

3. The domain Ω_r

For $n>0$ let $\mathcal{G}_n$ be the nth 8-adic grid on the unit square

$$\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\times\left[-\tfrac{1}{2},\tfrac{1}{2}\right],$$

i.e. $\mathcal{G}_n$ is the set of all squares

$$\left[\frac{j}{8^n},\frac{j+1}{8^n}\right]\times\left[\frac{k}{8^n},\frac{k+1}{8^n}\right],\quad -\tfrac{1}{2}8^n\le j,k\le\tfrac{1}{2}8^n-1.$$

We also define $\mathcal{G}_n$ for $n\le 0$ to be the singleton $\{Q\}$ where Q is the square of side 8^{-n} centered at zero.

For $Q\in\mathcal{G}_n$ define

$$Q^*=\tfrac{5}{4}Q,\quad Q^{**}=\tfrac{3}{2}Q,\quad \mathring{Q}= \text{disc concentric with } Q \text{ of radius } \tfrac{1}{100}l(Q).$$

Thus for $n\ge 0$, Q^* is obtained from Q by adjoining all $\mathcal{G}_{n+1}$ squares which touch Q, and Q^{**} is obtained from Q^* by adjoining all $\mathcal{G}_{n+1}$ squares which touch Q^*. $\mathring{Q}$ is a disc which is small enough for our purposes, in particular, small enough that (I) below holds. We record the following two properties.

(I) If $Q\in\mathcal{G}_n$, $R\in\mathcal{G}_m$, $m\ge n$, and $R^*\not\subset Q^*$ then $R^{**}\cap 10\mathring{Q}=\emptyset$. We also have $\operatorname{int} Q\cap\operatorname{int} R=\emptyset$.

(II) If $\{Q_j\}$ is any family of squares in $\bigcap_n \mathcal{G}_n$, such that $j\ne k\Rightarrow Q_j^*\not\subset Q_k^*$, then no point belongs to more than C Q_j^*'s, where we may take $C=148$.

Statement I is trivial but we will sketch the proof of II. Each square Q_j^* is the union of 37 zones, where the first zone is Q_j and the others are the 36 $\mathcal{G}_{n+1}$ squares adjoined to Q_j to form Q_j^*, numbered clockwise from the lower left, say. It is easy to see (for fixed $i\in\{1,\dots,37\}$) that if the interiors of the ith zones of Q_j^* and Q_k^* intersect then one of Q_j^*, Q_k^* is contained in the other. So no point is contained in more than 37 sets of the form $Q_j^*\setminus Z_j$ where Z_j is the union of the boundaries of the various zones of Q_j^*. If a point belongs to a given Q_j^* then a nearby point located to one of the "southwest", "northwest", "northeast" or "southeast" directions will belong to $Q_j^*\setminus Z_j$. Consequently, if a point belongs to some collection of Q_j^*'s then a suitable nearby point will belong to at least one quarter as many $Q_j^*\setminus Z_j$ so we are done.

We now choose $M<\infty$ and $\varrho>0$ as in Theorem 2 and let $\mathcal{G}=\bigcup_{8^{-n}\geq\varrho}\mathcal{G}_n$. We assume ϱ is a power of 8 and order $\mathcal{G}$ as $\{Q_j\}_{j=-\infty}^{r}$ in such a way that $j<k$ implies $l(Q_j)\geq l(Q_k)$. We can take $l(Q_j)=8^{-j}$ for $j\leq 0$.

Let $\Omega=\mathbf{C}^*\setminus E$ be our given domain, $\omega=\omega(\Omega,\cdot,\infty)$. We may assume that $E\subset[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}]$, and also that $Mh_1(E)>1$, since if the latter assumption fails we can take $F=E$. Then for $j\leq 0$, $\omega(Q_j^*)<Mh_1(E)=Mh_1(E\cap Q_j^*)$. This will guarantee that the definition of Ω_j given below is consistent for nonpositive j.

We now perform the following recursive construction. Set $\Omega_{-\infty}=\Omega$, and for $j\leq 0$, set $\Omega_j=\Omega$ also.

If Ω_j has been defined let $E_j=\mathbf{C}^*\setminus\Omega_j$, $\omega_j=\omega(\Omega_j,\cdot,\infty)$. We then say that Q_{j+1}^* is *chosen* if (i) it is not contained in any previously chosen square and (ii) either $l(Q_j)=\varrho$, or

$$\omega_j(Q_{j+1}^*)\geq M\max_{R\geq 1}R^{-\beta}h_1(E_j\cap RQ_{j+1}^*). \tag{3.1}$$

Here β is a fixed constant which should be >1. For definiteness we will take $\beta=2$.

If Q_{j+1}^* is chosen it is *thin* if

$$\operatorname{cap}(E_j\cap Q_{j+1}^{**})<e^{-1/\varepsilon_0}l(Q_{j+1}^{**})$$

with ε_0 as in Lemma 2.1 and *thick* if $\operatorname{cap}(E_j\cap Q_{j+1}^{**})\geq e^{-1/\varepsilon_0}l(Q_{j+1}^{**})$. If Q_{j+1}^* is chosen and thin then we define Ω_{j+1} by applying Lemma 2.1 to Ω_j with the squares $Q=Q_{j+1}^{**}$ and $\frac{5}{6}Q=Q_{j+1}^*$. We denote the quantities r_1, r_2 and B appearing in Lemma 2.1 by $r_1^{(j)}, r_2^{(j)}$ and B_j. We take B_j concentric with Q_j, and may assume (by shrinking ε_0 if necessary) that $B_j\subset\frac{1}{8}\mathring{Q}_j$. In all other cases (i.e. Q_{j+1}^* is chosen and thick or is not chosen) we let $\Omega_{j+1}=\Omega_j$.

In this way we eventually obtain a domain Ω_r which we will analyze in this section. In Section 4 we will make a further domain modification involving the thick chosen squares.

Remarks. (1) We note again that the definition of Ω_j is consistent for nonpositive j since (3.1) must fail. The nonpositive values of j are of course just a technicality and the reader can safely ignore them. For clarity, we also note that $\Omega_{j+1}=\Omega_j$ unless Q^*_{j+1} is chosen and thin, in which case Ω_{j+1} differs from Ω_j only inside Q^{**}_{j+1}.

(2) It seems worth comparing this construction with the constructions in [4,5] and in the special case treated in Section 1 of the present paper. It differs from all of them in various technical respects. However, the most significant difference is the role of the h_1 content in the stopping rule, i.e. on the right hand side of (3.1). The constructions in [4,5] and in Section 1 all use instead a comparison between $\omega(Q)$ and $l(Q)$. Such a stopping rule does not permit the estimate in Lemma 5.2 below. Instead there is a ϱ dependent estimate (see [5]), although the dependence on ϱ was weak enough to permit a proof of the Øksendal conjecture.

(3) The lemmas in this section will involve the behavior of h_1 measures, harmonic measures and capacity in passing from Ω to Ω_r. However, let us first note some general properties of the construction.

(a) If Q^*_j is any square in $\mathcal{G}$, then by (I) above, Q^{**}_j does not intersect B_i for any $i<j$. Therefore $Q^{**}_j\cap E_{j-1}$ is contained in E_i for all $i<j$ and in particular in E.

(b) E_r consists of discs B_i where Q^*_i is a thin chosen square, together with a certain subset of $\bigcup\{Q^*_j : Q^*_j \text{ chosen and thick}\}$.

(c) No point can belong to more than 148 chosen squares Q^*_j. This follows from (II) above. Furthermore by (I) above, if Q^*_j and Q^*_k are chosen then $\operatorname{int} Q_j$ and $\operatorname{int} Q_k$ are disjoint.

These properties, (a) in particular, will be used frequently below.

Now suppose Q^*_j is a thin chosen square and define a set $A_j\subset\bigcap_{i<j} E_i$ as follows.

$$A_j=\begin{cases} Q^*_j\cap E_{j-1} & \text{if } l(Q_j)>\varrho,\\ r_2^{(j)}Q^{**}_j\cap E_{j-1} & \text{if } l(Q_j)=\varrho.\end{cases}$$

The sets A_j with $l(Q_j)>\varrho$ are disjoint, since A_j is deleted from E_{j-1} in forming E_j. The A_j with $l(Q_j)=\varrho$ are not necessarily disjoint, but since they are contained in the Q^{**}_j no point can belong to more than four of them. So no point belongs to more than five A_j in all.

Lemma 3.1 (h_1 contents increase).

(i) *For any thin chosen square Q^*_j, $h_1(B_j)\gtrsim h_1(A_j)$,*

(ii) *For any j, any $k>j$ and any square Q with $l(Q)\ge l(Q_j)$, $h_1(4Q\cap E_k)\gtrsim h_1(Q\cap E_j)$ (Q need not be 8-adic here).*

Proof. Part (i) is immediate from part (iii) of Lemma 2.1. To prove (ii) it suffices to show that $h_1(Q\cap(E_j\setminus E_k))\lesssim h_1(4Q\cap E_k)$. We have

$$Q\cap(E_j\setminus E_k)\subset \bigcup_{\substack{j<i\le k\\ Q_i^* \text{ chosen and thin}\\ Q_i^{**}\cap Q\neq\emptyset}} C_i,$$

where C_i denotes $r_2^{(i)}Q_i^{**}\cap E_{i-1}$. Since $l(Q)\ge l(Q_i)$ it follows that $Q_i^{**}\subset 4Q$ and therefore that $B_i\subset 4Q$, for all such i. Accordingly, we will be done if we show the following:

Claim. If $Q_{i_1}^*, ..., Q_{i_m}^*$ is any collection of thin chosen squares, then

$$h_1\Big(\bigcup_j r_2^{(i_j)}Q_{i_j}^{**}\cap E_{i_j-1}\Big)\le Ch_1\Big(\bigcup_j B_{i_j}\Big).$$

To prove the claim choose a covering of $\bigcup_j B_{i_j}$ by discs $D_k=D(a_k,r_k)$, with $\sum r_k\approx h_1(\bigcup_j B_{i_j})$.

For each j, consider two possibilities.

(i) Every D_k with $D_k\cap B_{i_j}\neq\emptyset$ is contained in Q_{i_j},

(ii) Not (i).

Then

$$h_1\Big(\bigcup_{j \text{ type (i)}} C_{i_j}\Big)\lesssim \sum_{j \text{ type (i)}}\ \sum_{D_k\subset Q_{i_j}} r_k\le \sum_k r_k\approx h_1\Big(\bigcup_j B_{i_j}\Big)$$

where the second inequality follows because the (interiors of the) Q_{i_j} are disjoint.

On the other hand, if j is type (ii) then one of the D_k's which intersects B_{i_j} has radius $\gtrsim l(Q_{i_j})$, so a suitable fixed multiple of D_k will contain C_{i_j}. Hence

$$h_1\Big(\bigcup_{j \text{ type (ii)}} C_{i_j}\Big)\le C\sum r_k$$

and we are done. □

Next we consider harmonic measures which is somewhat more involved. The information we need is contained in the following four lemmas.

Lemma 3.2. *$\omega_k(B_j)\le C\omega_l(A_j)$ for any thin chosen square Q_j^* and any $l<j\le k$ (and in particular, if $l=-\infty, k=r$).*

Lemma 3.3. (harmonic measures decrease) *If Q is any square which is not strictly contained in a chosen square and if $k>l$ then $\omega_k(Q)\lesssim\omega_l(5Q)$.*

Lemma 3.4. (doubling property) *If Q_j^* is a chosen square with $l(Q_j)>\varrho$ and if $k\geq j-1\geq l$, $1\leq T<\infty$ then $\omega_k(TQ_j^*)\lesssim T^\beta \omega_l(Q_j^*)$.*

Lemma 3.5. *If Q_j^* is a chosen square then there is $T_j\geq 1$ such that*

$$\omega_k(TQ_j^*)\lesssim M\left(\frac{T}{T_j}\right)^\beta h_1(E_r\cap T_jQ_j^*)$$

for all $T\geq 1$ and all $k\geq m-1$, where $m=m(T)$ denotes an index such that $5TQ_j^*\subset Q_m^*$ and $l(Q_m)\leq 100Tl(Q_j)$. In particular,

$$\omega_k(TQ_j^*)\lesssim M\left(\frac{T}{T_j}\right)^{\beta-1} l(TQ_j^*).$$

Proofs. These lemmas are closely related and we start with some observations which are relevant to the proofs of all of them. First,

$$\text{(3.2)}\qquad \text{if } k>l \text{ and if } Y\cap B_i=\emptyset \text{ for all } i\in\{l+1,...,k\} \text{ then } \omega_k(Y)\leq\omega_l(Y).$$

This follows by induction on Lemma 2.1 (i). Next, the stopping rule (3.1) will be used in the following way. Let Q_j^* be a chosen square and let Q_m be a $\mathcal{G}$ square containing Q_j, with length $l(Q_m^*)=Tl(Q_j^*)$. Then Q_m^* was not chosen so

$$\omega_{m-1}(Q_m^*)\leq M\max_{R\geq 1} h_1(E_{m-1}\cap RQ_m^*)R^{-\beta}.$$

By Lemma 3.1 (ii)

$$\begin{aligned}\omega_{m-1}(Q_m^*)&\lesssim M\max_{R\geq 1} h_1(E_{j-1}\cap 4RQ_m^*)R^{-\beta}\\ &\lesssim M\max_{R\geq 1} h_1(E_{j-1}\cap RQ_m^*)R^{-\beta}.\end{aligned}$$

Q_m^* contains and is comparable to $T'Q_j^*$ where $T'=\max(1,T/100)$. So

$$\omega_{m-1}(Q_m^*)\lesssim M\max_{R\geq 1} h_1(E_{j-1}\cap RT'Q_j^*)R^{-\beta},$$

$$\text{(3.3)}\qquad \omega_{m-1}(Q_m^*)\lesssim M\max_{R\geq 1} h_1(E_{j-1}\cap RQ_j^*)\left(\frac{T}{R}\right)^\beta.$$

If $l(Q_j)>\varrho$ then we can further conclude by (3.1) that

$$\text{(3.4)}\qquad \omega_{m-1}(Q_m^*)\lesssim T^\beta\omega_{j-1}(Q_j^*).$$

If we apply (3.4) with Q_m the immediate predecessor of Q_j (i.e. $T=8$) and note that $Q_j^{**}\subset Q_m^*$ we obtain $\omega_{m-1}(Q_j^{**})\lesssim\omega_{j-1}(Q_j^*)$. However Q_j^{**} is disjoint from B_i for $i<j$ (Remark 3a above) so, by (3.2), $\omega_{j-1}(Q_j^{**})\le\omega_{m-1}(Q_j^{**})$. We conclude the following "preliminary version of the doubling property"

$$\omega_{j-1}(Q_j^{**})\lesssim\omega_{j-1}(Q_j^*),\quad Q_j^* \text{ chosen, } l(Q_j)>\varrho. \tag{3.5}$$

Proof of Lemma 3.2. Since the B_i are disjoint we have $\omega_k(B_j)\le\omega_j(B_j)$ by (3.2). By Lemma 2.1 (ii), $\omega_j(B_j)\lesssim\omega_{j-1}(r_2^{(j)}Q_j^{**})$. By (3.5) we can replace $r_2^{(j)}Q_j^{**}$ by A_j here, so $\omega_k(B_j)\lesssim\omega_{j-1}(A_j)$. But $A_j\cap B_i=\emptyset$ for $i<j$ so $\omega_{j-1}(A_j)\le\omega_l(A_j)$ and we are done.

Proof of Lemma 3.3. Write

$$Q\cap E_k=Y\cup\Big(\bigcup\{B_i: l<i\le k \text{ and } B_i\cap Q\neq\emptyset\}\Big) \tag{3.6}$$

where $Y\subset Q$ is a set which is disjoint from all B_i's with $l<i\le k$ and therefore satisfies $\omega_k(Y)\le\omega_l(Y)$. The chosen squares Q_i^* yielding B_i's in (3.6) all have side length $l(Q_i^*)\le 4l(Q)$, since Q intersects B_i which is situated near the middle of Q_i^*, but is not contained in Q_i^*. Therefore each A_i is contained in $5Q$. Using Lemma 3.2 we obtain

$$\begin{aligned}\omega_k(Q)&\le\omega_k(Y)+\sum_{l<i\le k}\omega_k(B_i)\\&\lesssim\omega_l(Y)+\sum_{l<i\le k}\omega_l(A_i)\end{aligned}$$

with Y and the A_i being contained in $5Q$. At most five A_i's contain any given point so the sum is $\le 5\omega_l(5Q)$ and we are done.

Proof of Lemma 3.4. We may assume $k=j-1=l$, since $\omega_k(TQ_j^*)\lesssim\omega_{j-1}(5TQ_j^*)$ by Lemma 3.3 and $\omega_{j-1}(Q_j^*)\le\omega_l(Q_j^*)$ by Remark 3a and (3.2).

Choose Q_m^* containing $5TQ_j^*$ and with comparable side length. Then

$$\omega_{m-1}(5TQ_j^*)\lesssim T^\beta\omega_{j-1}(Q_j^*)$$

by (3.4). On the other hand $5TQ_j^*$ is not contained in any chosen square (else Q_j^* could not have been chosen) so by Lemma 3.3

$$\omega_{j-1}(TQ_j^*)\lesssim\omega_{m-1}(5TQ_j^*)$$

and we are done. $\square$

Proof of Lemma 3.5. Choose T_j to be such that $T_j^{-\beta}h_1(E_r\cap T_jQ_j^*)$ is as large as possible. If T is given and Q_m is as in the lemma, then by (3.3) there is R such that

$$\omega_{m-1}(5TQ_j^*)\lesssim Mh_1(E_{j-1}\cap RQ_j^*)\left(\frac{T}{R}\right)^{\beta}.$$

By Lemma 3.1,

$$\begin{aligned}\omega_{m-1}(5TQ_j^*)&\lesssim Mh_1(E_r\cap 4RQ_j^*)\left(\frac{T}{4R}\right)^{\beta}\\ &\le Mh_1(E_r\cap T_jQ_j^*)\left(\frac{T}{T_j}\right)^{\beta}.\end{aligned}$$

By Lemma 3.3,

$$\omega_k(TQ_j^*)\lesssim Mh_1(E_r\cap T_jQ_j^*)\left(\frac{T}{T_j}\right)^{\beta}$$

for all $k\ge m-1$, and we are done with the first part of the lemma. The "in particular" part follows by estimating $h_1(E_r\cap T_jQ_j^*)\le T_jl(Q_j^*)$. $\square$

Finally we need to keep track of capacity.

Lemma 3.6. (capacities increase) *If Q_j^* is a thick chosen square then*

$$\operatorname{cap}(E_r\cap 3Q_j^*)\ge e^{-1/\varepsilon_1}l(3Q_j^*)$$

where ε_1 depends on ε_0.

Proof. If Q_i^* is a thin chosen cube with $i>j$ then we denote $r_1^{(i)}Q_i^{**}\cap E_{i-1}$ (the set deleted from E_{i-1} in forming E_i) by C_i. If $C_i\cap Q_j^{**}\neq\emptyset$ then $B_i\subset 3Q_j^*$. The lemma will follow from this and the fact that $\operatorname{cap}B_i\ge\operatorname{cap}C_i$ (for this, see the statement of Lemma 2.1). Namely, the squares Q_i with Q_i^* chosen are disjoint except for edges. So for any x, there is at most one i such that $\operatorname{dist}(x,B_i)\le\frac{1}{3}\max_{y\in C_i}|x-y|$, since Q_i would have to contain x. Hence it suffices to prove the following general fact.

Suppose E and F are subsets of a square of diameter r, and have the following structure: $E=E_0\cup(\bigcup_i C_i)$ and $F=E_0\cup(\bigcup_i B_i)$ where B_i is a disc, $\operatorname{cap}B_i\ge\operatorname{cap}C_i$, and for any given x we have $\operatorname{dist}(x,B_i)\ge\frac{1}{3}\max_{y\in C_i}|x-y|$ for all i with one possible exception. Then $\operatorname{cap}F/r\ge\frac{1}{3}(\operatorname{cap}E/r)^2$.

This statement is scale invariant and we may assume $r=1$. We may also assume the C_i are disjoint from each other and from E_0 since otherwise we replace C_i

with $C_i\setminus(E_0\cup(\bigcup_{j<i}C_j))$. Let μ be the capacitary measure of E (i.e. $V_\mu(x)\overset{\text{def}}{=}\int\log(1/|x-y|)\,d\mu(y)=1$ on E except for a set of zero capacity). Define

$$\nu=\mu|_{E_0}+\sum\mu(C_i)\,d\sigma_i,$$

where σ_i is a uniform mass distribution on the boundary of B_i with total mass 1. Given x, let i_0 be the exceptional index with $\operatorname{dist}(x,B_{i_0})<\frac{1}{3}\max_{y\in C_{i_0}}|x-y|$. (We assume for notational purposes that such an exceptional i_0 exists.) Then

$$\begin{aligned}V_\nu(x)&\le\int_{E_0}\log\frac{1}{|x-y|}\,d\mu(y)+\sum_{i\ne i_0}\mu(C_i)\int\log\frac{1}{|x-y|}\,d\sigma_i(y)\\&\qquad+\mu(C_{i_0})\int\log\frac{1}{|x-y|}\,d\sigma_{i_0}(y)\\&\le\int_{E_0}\log\frac{1}{|x-y|}\,d\mu(y)+\sum_{i\ne i_0}\int_{C_i}\log\frac{3}{|x-y|}\,d\mu(y)\\&\qquad+\mu(C_{i_0})\int\log\frac{1}{|x-y|}\,d\sigma_{i_0}(y).\end{aligned}$$

Since $\operatorname{cap}C_{i_0}\le\operatorname{cap}B_{i_0}$ the last term here must be bounded by 1. So for $x\in F$,

$$\begin{aligned}V_\nu(x)&\le\int_{E_0\cup(\bigcup_{i\ne i_0}C_i)}\log\frac{1}{|x-y|}\,d\mu(y)+(\log 3)\|\mu\|+1\\&\le V_\mu(x)+(\log 3)\|\mu\|+1,\end{aligned}$$

where we used $\operatorname{diam}E\le1$ in the first term. Hence $\log(1/\operatorname{cap}F)\le2\log(1/\operatorname{cap}E)+\log 3$ and we are done. □

4. The domain $\widetilde{\Omega}$

This domain is defined as follows. For each thick chosen square Q_j^* let R_j be the circumscribed disc. Let

$$\widetilde{R}_j=R_j\cup\Big(\bigcup\{\mathring{Q}_i:Q_i^*\text{ is chosen and thin and }R_j\cap\tfrac{1}{2}\mathring{Q}_i\ne\emptyset\}\Big).$$

We regard R_j and $\mathring{Q}_i$ as being closed discs here. Let

$$\widetilde{\Omega}=\Omega_r\setminus\bigcup\{\widetilde{R}_j:Q_j^*\text{ chosen and thick}\}$$

and let $\widetilde{E}=\mathbf{C}^*\setminus\widetilde{\Omega}$, $\widetilde{\omega}=\omega(\widetilde{\Omega},\cdot,\infty)$.

Note that each $\frac{1}{2}\mathring{Q}_i$ (Q_i^* chosen and thin) is either contained in $\widetilde{E}$ or disjoint from it. Thus the same is true for each B_i. Also it is clear that each $\mathring{Q}_i$ adjoined to R_j to form $\widetilde{R}_j$ must satisfy $l(Q_i)\le l(Q_j)$. Hence $\widetilde{R}_j\subset 3R_j$, say.

For each chosen square Q_j^* we now define two sets $A_j\subset E$ and $\Delta_j\subset\widetilde{E}$, which we regard as the parts of E and of $\widetilde{E}$ associated to Q_j. Namely

$$A_j=\begin{cases} Q_j^*\cap E_{j-1} & \text{if } l(Q_j)>\varrho,\\ r_2^{(j)}Q_j^{**}\cap E_{j-1} & \text{if } l(Q_j)=\varrho \text{ and } Q_j^* \text{ is thin},\\ 100Q_j^*\cap E & \text{if } l(Q_j)=\varrho \text{ and } Q_j^* \text{ is thick}.\end{cases}$$

If Q_j^* is thin this definition agrees with the one in the previous section. Note that no point belongs to more than C A_j's with C a fixed constant. This follows easily using that no point belongs to more than 148 Q_j^*'s. Also

$$\Delta_j=\begin{cases} B_j & \text{if } Q_j^* \text{ is thin},\\ \widetilde{R}_j & \text{if } Q_j^* \text{ is thick}.\end{cases}$$

Lemma 4.1. *If Q_j^* is a thick chosen square then $\widetilde{\omega}(\widetilde{R}_j)\lesssim\omega_r(10R_j)\lesssim\omega_l(C_jQ_j^*)$ for all $l\le j-1$, where $C_j=1$ if $l(Q_j)>\varrho$ and $C_j=100$ if $l(Q_j)=\varrho$.*

Proof. First of all

$$\widetilde{\omega}(\widetilde{R}_j)\le\omega(\Omega_r\setminus 3R_j,3R_j,\infty)\lesssim\omega_r(10R_j),$$

where the first inequality follows by the maximum principle and the second by Lemmas 3.6 and 1.1.

Now consider cases. If $l(Q_j)=\varrho$ then

$$\omega_r(10R_j)\le\omega_r(20Q_j^*)\lesssim\omega_l(100Q_j^*)$$

where the second inequality follows from Lemma 3.3. If $l(Q_j)>\varrho$ then

$$\omega_r(10R_j)\le\omega_r(20Q_j^*)\lesssim\omega_{j-1}(100Q_j^*)\lesssim\omega_{j-1}(Q_j^*)\le\omega_l(Q_j^*),$$

where the last three inequalities use respectively Lemma 3.3, Lemma 3.4 and (3.2). □

Lemma 4.2. *For any chosen square Q_j^**

(i) $h_1(\Delta_j)\gtrsim h_1(A_j)$,

(ii) $\widetilde{\omega}(\Delta_j)\lesssim\omega(A_j)$,

(iii) *If $l(Q_j)>\varrho$ then $\omega(A_j)\gtrsim Mh_1(A_j)$.*

Proof. Part (i) is identical to Lemma 3.1 (i) if Q_j^* is thin and is trivial if Q_j^* is thick. Part (ii) follows from Lemma 3.2 if Q_j^* is thin, since the maximum principle implies $\tilde{\omega}(B_j)\le\omega_r(B_j)$. If Q_j^* is thick then part (ii) is identical with the $l=-\infty$ case of Lemma 4.1. It remains to prove (iii). But $\omega_{j-1}(Q_j^*)\lesssim\omega(Q_j^*\cap E_{j-1})=\omega(A_j)$ by (3.2), and on the other hand $\omega_{j-1}(Q_j^*)\ge Mh_1(E_{j-1}\cap Q_j^*)=Mh_1(A_j)$ by (3.1), so we are done. □

Let us also prove the following fact.

Lemma 4.3. *Each set $\widetilde{R}_j$ (Q_j^* chosen and thick) satisfies $\sigma(Q\cap\partial\widetilde{R}_j)\lesssim l(Q)$ for all squares Q.*

In other words $\widetilde{R}_j$ is what is called Ahlfors–David regular, with uniform bounds.

Proof. This is certainly true if $\widetilde{R}_j$ is replaced by R_j. Hence it suffices to prove it with $\widetilde{R}_j$ replaced by $\bigcup\{\mathring{Q}_i: R_j\cap\frac{1}{2}\mathring{Q}_i\neq\emptyset\}$. Fix a square Q. If $Q\subset Q_{i_0}$ for some thin chosen square $Q_{i_0}^*$ then Q cannot intersect $\mathring{Q}_i$ for $i\neq i_0$ and the lemma follows. If Q is not contained in any such Q_{i_0} and if $Q\cap\partial\mathring{Q}_i\neq\emptyset$ then $\mathring{Q}_i\subset 3Q$, and clearly if $R_j\cap\frac{1}{2}\mathring{Q}_i\neq\emptyset$ then $\sigma(\partial\mathring{Q}_i)\lesssim\sigma(\mathring{Q}_i\cap\partial R_j)$. Hence

$$\sigma\Big(\bigcup\{\partial\mathring{Q}_i\cap Q: R_j\cap\tfrac{1}{2}\mathring{Q}_i\neq\emptyset\}\Big)\le\sum_{\substack{\mathring{Q}_i\subset 3Q\\ R_j\cap\frac{1}{2}\mathring{Q}_i\neq\emptyset}}\sigma(\partial\mathring{Q}_i)$$
$$\lesssim\sum_{\mathring{Q}_i\subset 3Q}\sigma(\mathring{Q}_i\cap\partial R_j).$$

The subsets of ∂R_j appearing here are disjoint so the sum is bounded by $\sigma(\partial R_j\cap 3Q)\lesssim l(Q)$ and we are done. □

5. The gradient of the Green's function

Let $\tilde{g}$ be the Green's function of $\widetilde{\Omega}$ with pole at ∞. $\widetilde{E}$ consists of B_i's and $\widetilde{R}_j$'s and $\partial\widetilde{\Omega}$ is therefore smooth except at finitely many points. So we may identify $|\nabla\tilde{g}|$ with $d\tilde{\omega}/d\sigma$. We will prove the following estimates.

Lemma 5.1. (thick case) *If Q_j^* is a thick chosen square and $x\in\partial\widetilde{R}_j\cap\partial\widetilde{\Omega}$ then $|\nabla\tilde{g}(x)|\lesssim\min(M,\omega(A_j)/h_1(A_j))$.*

Lemma 5.2. (thin case) *If Q_j^* is a thin chosen square and $\mathring{Q}_j$ is not contained in any $\widetilde{R}_i$ then*

(a) *if $x\in\partial B_j$ then $|\nabla\tilde{g}(x)|\lesssim\omega(A_j)/h_1(A_j)$,*

(b) *if $x\in\frac{1}{4}\mathring{Q}_j\cap\Omega$ then $|\nabla\tilde{g}(x)|\lesssim\tilde{\omega}(B_j)/|z-z_j|+M$.*

Here z_j is the center of B_j. We also let r_j be the radius of B_j. Note that the estimates are independent of ϱ.

Proof of Lemma 5.1. First let $\Gamma_0=\Omega_r\setminus R_j$, g_{Γ_0} its Green's function with pole at ∞.

Claim 1. $|\nabla g_{\Gamma_0}|\lesssim\min(M,\omega(A_j)/h_1(A_j))$ on $\partial R_j\cap\partial\Gamma_0$.

Proof. By Lemma 1.2 it suffices to show that

$$\omega(\Omega_r\setminus 2R_j, 2R_j, \infty)\lesssim l(Q_j)\min\left(M, \frac{\omega(A_j)}{h_1(A_j)}\right).$$

So by Lemmas 3.6 and 1.1 it suffices to show that

$$\omega_r(3R_j)\lesssim l(Q_j)\min\left(M, \frac{\omega(A_j)}{h_1(A_J)}\right).$$

However $\omega_r(3R_j)\lesssim Ml(Q_j)$ by Lemma 3.5, and $\omega_r(3R_j)\lesssim\omega(A_j)$ by Lemma 4.1, so we are done.

Now let $\mathring{Q}_i$ be one of the discs adjoined to R_j to form $\widetilde{R}_j$. Let $\Gamma_1=\Omega_r\setminus(R_j\cup\mathring{Q}_i)$ and g_{Γ_1} its Green's function with pole at ∞.

Claim 2. $|\nabla g_{\Gamma_1}|\lesssim\min(M,\omega(A_j)/h_1(A_j))$ on $\partial(R_j\cup\mathring{Q}_i)\cap\partial\Gamma_1$.

Proof. On ∂R_j such an estimate follows from Claim 1 by the maximum principle, so it suffices to prove the estimate on $\partial\mathring{Q}_i$. For this it suffices by Lemma 1.2 to prove

$$\omega(\Gamma_0\setminus 2\mathring{Q}_i, 2\mathring{Q}_i, \infty)\lesssim l(Q_i)\min\left(M, \frac{\omega(A_j)}{h_1(A_j)}\right).$$

Consider the domain $\Omega_{\mathrm{loc}}=Q_i\cap\Gamma_0$ whose complement relative to Q_i consists of $R_j\cap Q_i$ together with the disc B_i (which may or may not be contained in R_j). Since ∂R_j is a continuum which intersects $\frac{1}{2}\mathring{Q}_i$ and is not contained in Q_i, and since B_i is contained in $\frac{1}{4}\mathring{Q}_i$, it is clear that $\omega(\Omega_{\mathrm{loc}},\partial R_j,\infty)\geq$ const on $\Omega_{\mathrm{loc}}\cap\partial(2\mathring{Q}_i)$ (see Figure 1).

It follows by the maximum principle that

$$\omega(\Gamma_0, Q_i\cap\partial R_j,\cdot\,)\geq\text{const}$$

on $\partial(2\mathring{Q}_i)$, and therefore, using the maximum principle as in the proof of Lemma 1.1, that

$$\omega(\Gamma_0\setminus 2\mathring{Q}_i, 2\mathring{Q}_i, \infty)\lesssim\omega(\Gamma_0, Q_i\cap\partial R_j, \infty).$$

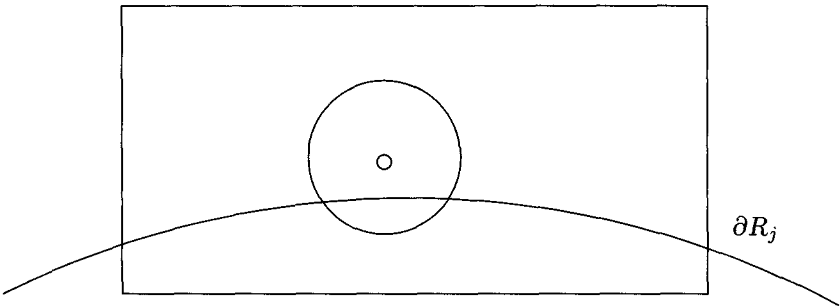

Figure 1. The square is Q_i. The large disc is $2\mathring{Q}_i$ and the small disc is B_i. Ω_{loc} is the part of the square lying above ∂R_j, with B_i deleted.

The right side here is $\lesssim l(Q_i)\min(M, \omega(A_j)/h_1(A_j))$ by Claim 1 so we are done with Claim 2.

Lemma 5.1 follows immediately from Claim 2 using the maximum principle.

Proof of Lemma 5.2. Part (a) is basically trivial from Lemma 4.2. Namely, since $B_j \subset \frac{1}{4}\mathring{Q}_j$ and $\widetilde{E} \cap \frac{1}{2}\mathring{Q}_j = \emptyset$, $\tilde{g}$ may be extended by the reflection principle to a harmonic function on $2B_j \setminus \frac{1}{2}B_j$. Using polar coordinates based at z_j we have for $1 \le s \le 2$ (actually, for any s such that $sB_j \setminus B_j \subset \widetilde{\Omega}$)

$$\int_{\partial(sB_j)} \tilde{g}\frac{d\theta}{2\pi} = \int_{r_j}^{sr_j}\int \frac{d\tilde{g}}{dr} r \frac{d\theta}{2\pi}\frac{dr}{r} = \int_{r_j}^{sr_j} \frac{\widetilde{\omega}(B_j)}{2\pi}\frac{dr}{r} = \frac{\widetilde{\omega}(B_j)}{2\pi}\log s.$$

It follows by Harnack's inequality that $|\tilde{g}| \lesssim \widetilde{\omega}(B_j)$ on $\partial(\frac{3}{2}B_j)$ and therefore on $\frac{3}{2}B_j \setminus \frac{2}{3}B_j$. By standard derivative bounds for harmonic functions $|\nabla \tilde{g}| \lesssim \widetilde{\omega}(B_j)/r_j$ on ∂B_j and (a) now follows from (i) and (ii) of Lemma 4.2.

Proof of (*b*). The preceding argument may be applied not just on ∂B_j, but on $\frac{4}{3}B_j \setminus B_j$, say, so we know $|\nabla \tilde{g}| \lesssim \widetilde{\omega}(B_j)/r_j$ there and in proving (b), may restrict attention to points of $\frac{1}{4}\mathring{Q}_j \setminus \frac{4}{3}B_j$.

We claim there is a scale $T_j' \ge 1$ such that

(i) $\widetilde{\omega}(T_j'Q_j^*) \lesssim M h_1(\widetilde{E} \cap \frac{1}{2}T_j'Q_j^*)$,

(ii) $\widetilde{\omega}(TQ_j^*) \lesssim M(T/T_j')^\beta l(T_j'Q_j^*)$ when $1 \le T \le T_j'$.

T_j' is defined as follows. Let T_j be the scale from Lemma 3.5. Let

$$\mathcal{F} = \{\widetilde{R}_i : Q_i^* \text{ is a thick chosen square and } \operatorname{diam} \widetilde{R}_i > \operatorname{dist}(z_j, \widetilde{R}_i)\}.$$

If no $\widetilde{R}_i \in \mathcal{F}$ intersects $2T_jQ_j^*$, then we let $T_j' = 2T_j$. Otherwise we let $T_j' = 1000\min\{T : TQ_j^* \cap \widetilde{R}_i \ne \emptyset \text{ for some } \widetilde{R}_i \in \mathcal{F}\}$.

Thus $T_j' \le 1000 T_j$, and also $T_j' \ge 1$ since no $\widetilde{R}_i$ intersects $\frac{1}{2}\mathring{Q}_j$. We will now prove (i) and (ii).

Claim. Suppose $T \ge 1$ is such that no $\widetilde{R}_i \in \mathcal{F}$ intersects TQ_j^*. Then $\widetilde{\omega}(TQ_j^*) \lesssim M(T/T_j)^\beta h_1(\widetilde{E} \cap T_j Q_j^*)$.

Proof. By the maximum principle

$$\widetilde{\omega}(TQ_j^*) \le \omega_r(TQ_j^*) + \sum_{\widetilde{R}_i \cap TQ_j^* \ne \emptyset} \widetilde{\omega}(\partial \widetilde{R}_i). \tag{5.1}$$

Let C be a suitable constant. Choose m so that $5CTQ_j^* \subset Q_m^*$ and $l(Q_m) \le 500CTl(Q_j)$. Consider one of the $\widetilde{R}_i$ appearing in the sum in (5.1) and the corresponding thick chosen square Q_i^*. $\widetilde{R}_i$ cannot belong to $\mathcal{F}$ and must therefore be contained in $5TQ_j^*$, and consequently if C is large we will have $i > m$. So by Lemma 4.1, $\widetilde{\omega}(\widetilde{R}_i) \lesssim \omega_{m-1}(C_i Q_i^*)$ where $C_i = 1$ if $l(Q_i) > \varrho$ and $C_i = 100$ if $l(Q_i) = \varrho$. By Lemma 3.3, $\omega_r(TQ_j^*) \lesssim \omega_{m-1}(5TQ_j^*)$. Thus

$$\widetilde{\omega}(TQ_j^*) \lesssim \omega_{m-1}(5TQ_j^*) + \sum_{\widetilde{R}_i \cap TQ_j^* \ne \emptyset} \omega_{m-1}(C_i Q_i^*).$$

For large C all the $C_i Q_i^*$ will be contained in CTQ_j^*, and no point belongs to more than a fixed finite number of them. So

$$\widetilde{\omega}(TQ_j^*) \lesssim \omega_{m-1}(CTQ_j^*)$$

and therefore by Lemma 3.5,

$$\begin{aligned}\widetilde{\omega}(TQ_j^*) &\lesssim M\left(\frac{T}{T_j}\right)^\beta h_1(E_r \cap T_j Q_j^*)\\ &\le M\left(\frac{T}{T_j}\right)^\beta h_1(\widetilde{E} \cap T_j Q_j^*)\end{aligned}$$

as claimed.

Now we consider cases. Suppose first that no $\widetilde{R}_i \in \mathcal{F}$ intersects $2T_j Q_j^*$. Then $T_j = \frac{1}{2} T_j'$ and both (i) and (ii) follow immediately from the claim. Now suppose that some $\widetilde{R}_i \in \mathcal{F}$ intersects $2T_j Q_j^*$. For (ii) there are two subcases $1 \le T < \frac{1}{1000} T_j'$ and $T_J' \ge T \ge \frac{1}{1000} T_j'$. If $1 \le T < \frac{1}{1000} T_j'$ then by definition of T_j' no $\widetilde{R}_i \in \mathcal{F}$ intersects TQ_j^*, and the claim implies

$$\begin{aligned}\widetilde{\omega}(TQ_j^*) &\lesssim M\left(\frac{T}{T_j}\right)^{\beta} h_1(\widetilde{E}\cap T_jQ_j^*)\\ &\lesssim M\left(\frac{T}{T_j}\right)^{\beta} l(T_jQ_j^*)\\ &\lesssim M\left(\frac{T}{T_j'}\right)^{\beta} l(T_j'Q_j^*)\end{aligned}$$

since $T_j'\lesssim T_j$ and $\beta>1$. This gives (ii) for $T<\frac{1}{1000}T_j'$. It remains only to prove (i) since (ii) for $T\geq\frac{1}{1000}T_j'$ is clearly a corollary of (i). Moreover, to prove (i) it suffices to prove

$$\widetilde{\omega}(T_j'Q_j^*)\lesssim Ml(T_j'Q_j^*) \tag{5.2}$$

since $h_1(\frac{1}{2}T_j'Q_j^*\cap\widetilde{E})$ is comparable to $l(T_j'Q_j^*)$ due to the fact that some $\widetilde{R}_i\in\mathcal{F}$ intersects $\frac{1}{1000}T_j'Q_j^*$. To prove (5.2), denote T_j' by T and fix i with $\widetilde{R}_i\cap\frac{1}{1000}TQ_j^*\neq\emptyset$. By the maximum principle

$$\widetilde{\omega}(TQ_j^*)\leq\omega(\Omega_r\setminus(\widetilde{R}_i\cup TQ_j^*), TQ_j^*, \infty)$$

and since $\operatorname{cap}(TQ_j^*\cap\widetilde{R}_i)\gtrsim l(TQ_j^*)$ (this is because $\widetilde{R}_i\in\mathcal{F}$) Lemma 1.1 with squares instead of discs now implies that

$$\widetilde{\omega}(TQ_j^*)\lesssim\omega(\Omega_r\setminus\widetilde{R}_i, 2TQ_j^*, \infty),$$

which by the maximum principle is

$$\lesssim\omega_r(2TQ_j^*)+\omega(\Omega_r\setminus\widetilde{R}_i, \partial\widetilde{R}_i\cap 2TQ_j^*, \infty).$$

Here $\omega_r(2TQ_j^*)\lesssim Ml(TQ_j^*)$ by Lemma 3.5. On the other hand by Lemma 5.1 and then by Lemma 4.3,

$$\begin{aligned}\omega(\Omega_r\setminus\widetilde{R}_i, \partial\widetilde{R}_i\cap 2TQ_j^*, \infty) &\lesssim M\sigma(\partial\widetilde{R}_i\cap 2TQ_j^*)\\ &\lesssim Ml(TQ_j^*).\end{aligned}$$

We conclude that (5.2) holds and have therefore proved (i) and (ii).

To finish the proof of Lemma 5.2 write $\nabla\tilde{g}(z)$ for $z\in\frac{1}{4}\mathring{Q}_j\setminus\frac{4}{3}B_j$ as a Cauchy integral

$$\frac{d\tilde{g}}{dz}=\int_{\widetilde{E}}\frac{1}{z-\zeta}\,d\widetilde{\omega}(\zeta)$$

and split the integral as

$$\int_{\mathbf{C}\setminus T_j'Q_j^*}+\int_{T_j'Q_j^*\setminus\frac{1}{2}\mathring{Q}_j}+\int_{B_j}.$$

The punch line is that by (i) above and Lemma 2.6, the first term is $\lesssim M$.

The other terms may be estimated in a straightforward manner. The last term is clearly $\leq\widetilde{\omega}(B_j)/\operatorname{dist}(z,B_j)\approx\widetilde{\omega}(B_j)/|z-z_j|$. The bound for the second term uses the decay property (ii). Namely

$$\begin{aligned}\left|\int_{T_j'Q_j^*\setminus\frac{1}{2}\mathring{Q}_j}\frac{1}{z-\zeta}\,d\widetilde{\omega}(\zeta)\right|&\leq\int_{T_j'Q_j^*\setminus\frac{1}{2}\mathring{Q}_j}\frac{1}{|z-\zeta|}\,d\widetilde{\omega}(\zeta)\\&\approx\int_{1/200}^{T_j'}\widetilde{\omega}(TQ_j^*)(Tl(Q_j^*))^{-2}d(Tl(Q_j^*))\end{aligned}\tag{5.3}$$

where we used that $\operatorname{dist}(z,\partial(TQ_j^*))\approx Tl(Q_j^*)$ and that $\operatorname{dist}(z,\mathbf{C}\setminus\frac{1}{2}\mathring{Q}_j)\approx l(Q_j^*)$. Using (ii) we bound (5.3) by

$$M\int_{1/200}^{T_j'}\left(\frac{T}{T_j'}\right)^{\beta}l(T_j'Q_j^*)(Tl(Q_j^*))^{-2}d(Tl(Q_j^*))\lesssim M$$

where the last inequality follows since $\beta>1$. This finishes Lemma 5.2. $\square$

6. Completion of the proof

The technical part of the proof is now over and we will finish up as in [4,5].

Step 1. For each thin chosen cube Q_j^* such that $\mathring{Q}_j$ is not contained in any $\widetilde{R}_i$ we will define a certain level set component $\mathcal{L}_j$ for $\tilde{g}$, contained in $\frac{1}{4}\mathring{Q}_j$. Namely let C_1 and C_2 be appropriate large constants. If $\widetilde{\omega}(B_j)\leq C_1C_2Mr_j$ then let $\mathcal{L}_j=\partial B_j$. If $\widetilde{\omega}(B_j)>C_1C_2Mr_j$ define $s_1=\widetilde{\omega}(B_j)/(C_1C_2M)$, $s_2=\widetilde{\omega}(B_j)/(C_1M)$. Then $r_j\leq s_1<s_2<$ radius of $\frac{1}{4}\mathring{Q}_j$, the last inequality following (for large C_1) because $\widetilde{\omega}(B_j)\lesssim\omega_{j-1}(A_j)$ (Lemma 3.2 and the maximum principle) and $\omega_{j-1}(A_j)\lesssim Ml(Q_j)$ (Lemma 3.5). For $s=s_1,s_2$ we have

$$\int\tilde{g}(z_j+se^{i\theta})\,d\theta=\widetilde{\omega}(B_j)\log\frac{s}{r_j}$$

as in the proof of Lemma 5.2(a). Therefore by Lemma 5.2(b)

$$2\pi\tilde{g}(z_j+s_1e^{i\theta})\leq\widetilde{\omega}(B_j)\log\frac{s_1}{r_j}+C\left(M+\frac{\widetilde{\omega}(B_j)}{s_1}\right)s_1,$$

$$2\pi\tilde{g}(z_j+s_2e^{i\theta})\geq\widetilde{\omega}(B_j)\log\frac{s_2}{r_j}-C\left(M+\frac{\widetilde{\omega}(B_j)}{s_2}\right)s_2,$$

i.e.

$$2\pi\tilde{g}(z_j+s_1e^{i\theta})\leq\widetilde{\omega}(B_j)\left[\log\frac{\widetilde{\omega}(B_j)}{C_1C_2Mr_j}+C+\frac{C}{C_1C_2}\right],$$
$$2\pi\tilde{g}(z_j+s_2e^{i\theta})\geq\widetilde{\omega}(B_j)\left[\log\frac{\widetilde{\omega}(B_j)}{C_1Mr_j}-C-\frac{C}{C_1}\right].$$

So, if $\log C_2\geq 2C+C/(C_1C_2)+C/C_1$ (as we may arrange by taking C_2 large) then

$$\min_{|z-z_j|=s_2}\tilde{g}\geq\max_{|z-z_j|=s_1}\tilde{g}$$

and it follows that there is a level set component $\mathcal{L}_j$ contained in $s_1\leq|z-z_j|\leq s_2$. On $\mathcal{L}_j$, we have $|\nabla\tilde{g}|\leq M$ by Lemma 5.2(b). We also have this when $\mathcal{L}_j=\partial B_j$ by the definition and Lemma 5.2(a).

Step 2. Let $\mathcal{L}=\bigcup_j\mathcal{L}_j\cup(\bigcup_i\partial\widetilde{R}_i)$ where i runs over thick chosen squares and j over thin chosen squares not contained in any $\widetilde{R}_i$. Since $\mathcal{L}$ is a union of level set components it follows (see [5]) that

$$\int_{\mathcal{L}}|\nabla\tilde{g}|\log|\nabla\tilde{g}|\,d\sigma\geq\text{const}.$$

Also $\int_{\mathcal{L}}|\nabla\tilde{g}|d\sigma=1$. On the other hand $|\nabla\tilde{g}|\lesssim M$ on $\mathcal{L}$ by Lemma 5.1 and the last sentences in Step 1. So as in [5]

$$\int_{\mathcal{L}}|\nabla\tilde{g}|\log^+|\nabla\tilde{g}|\,d\sigma\leq(\log M+C)\int_{\mathcal{L}\cap\{|\nabla\tilde{g}|\geq 1\}}|\nabla\tilde{g}|\,d\sigma$$

and then also

$$\int_{\mathcal{L}}|\nabla\tilde{g}|\log^-|\nabla\tilde{g}|\,d\sigma\leq(\log M)\int_{\mathcal{L}\cap\{|\nabla\tilde{g}|\geq 1\}}|\nabla\tilde{g}|\,d\sigma+C.$$

Therefore

$$\begin{aligned}\delta\log M\int_{\mathcal{L}\cap\{|\nabla\tilde{g}|\leq M^{-\delta}\}}|\nabla\tilde{g}|\,d\sigma&\leq(\log M)\int_{\mathcal{L}\cap\{|\nabla\tilde{g}|\geq 1\}}|\nabla\tilde{g}|\,d\sigma+C\\&\leq(\log M)\int_{\mathcal{L}\cap\{|\nabla\tilde{g}|\geq M^{-\delta}\}}|\nabla\tilde{g}|\,d\sigma+C,\end{aligned}$$

$$\int_{\mathcal{L}\cap\{|\nabla\tilde{g}|\geq M^{-\delta}\}}|\nabla\tilde{g}|\,d\sigma\geq\frac{\delta}{1+\delta}-\frac{C}{(1+\delta)\log M}\geq C^{-1}\delta$$

for large M.

Now we define our set F. Namely

$$F=\bigcup_j A_j,$$

where the union is over all j such that Q_j^* is chosen, Δ_j intersects $\partial\widetilde{\Omega}$ and one of the following holds.

(a) $l(Q_j)>\varrho$,

(b) $l(Q_j)=\varrho$, Q_j^* is thin and $\mathcal{L}_j\neq\partial B_j$,

(c) $l(Q_j)=\varrho$, Q_j^* is thin, $\mathcal{L}_j=\partial B_j$ and $|\nabla\tilde{g}|\geq M^{-\delta}$ somewhere on $\mathcal{L}_j$,

(d) $l(Q_j)=\varrho$, Q_j^* is thick and $|\nabla\tilde{g}|\geq M^{-\delta}$ somewhere on $\partial\widetilde{R}_j$.

We denote the set of all j satisfying (a), (b), (c) or (d) by $\mathcal{J}$.

We show first that $\omega(F)\geq C^{-1}\delta$. Since the A_j have finite overlap it suffices to prove

$$\sum_{j\in\mathcal{J}}\omega(A_j)\geq C^{-1}\delta.$$

By Lemma 4.2, it even suffices if

$$\sum_{j\in\mathcal{J}}\tilde{\omega}(\Delta_j)\geq C^{-1}\delta.$$

However, since $\int_{\partial B_j}|\nabla\tilde{g}|d\sigma=\int_{\mathcal{L}_j}|\nabla\tilde{g}|d\sigma$ for thin squares Q_j^*,

$$\begin{aligned}\sum_{j\in\mathcal{J}}\tilde{\omega}(\Delta_j)&=\sum_{\substack{j\in\mathcal{J}\\ Q_j^*\text{ thick}}}\int_{\partial\widetilde{R}_j}|\nabla\tilde{g}|\,d\sigma+\sum_{\substack{j\in\mathcal{J}\\ Q_j^*\text{ thin}}}\int_{\mathcal{L}_j}|\nabla\tilde{g}|\,d\sigma\\ &\geq\int_{\mathcal{L}\cap\{|\nabla\tilde{g}|\geq M^{-\delta}\}}|\nabla\tilde{g}|\,d\sigma\end{aligned}$$

by choice of $\mathcal{J}$. This we know is $\geq\delta$ so we have proved that $\omega(F)\gtrsim\delta$.

What remains is to show that F has a covering as described in Theorem 2. First we consider the set $\bigcup_j A_j$, Q_j^* satisfying (a) or (b). If Q_j^* satisfies (a), then by Lemma 4.2 (iii),

$$h_1(A_j)\lesssim M^{-1}\omega(A_j).$$

If Q_j^* satisfies (b), then by definition of $\mathcal{L}_j$, $r_j\lesssim M^{-1}\tilde{\omega}(A_j)$. Using this and Lemma 4.2 (i) (ii),

$$h_1(A_j)\lesssim M^{-1}\omega(A_j).$$

So we conclude that

$$\sum_{j \text{ satisfies (a) or (b)}} h_1(A_j) \lesssim M^{-1} \sum \omega(A_j) \lesssim M^{-1},$$

i.e. there is a covering where radii sum to $\lesssim M^{-1}$. Next consider j satisfying (c) or (d). By Lemma 5.1 or 5.2(a)we know that

$$M^{-\delta} \lesssim \frac{\omega(A_j)}{h_1(A_j)}.$$

Therefore

$$\sum_{j \text{ satisfies (c) or (d)}} h_1(A_j) \lesssim M^{\delta} \sum \omega(A_j) \lesssim M^{\delta}.$$

Since A_j has diameter $\lesssim \varrho$ it is clear that only discs of radius $\leq \varrho$ need be used in an economical covering of A_j. Consequently the union of A_j satisfying (c) or (d) has a covering by discs of radius $\leq \varrho$, whose radii sum to $\lesssim M^{\delta}$, and the theorem is proved. □

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