Commutators of Littlewood–Paley sums

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Introduction

For every interval $I\subset\mathbf{R}$ we denote by S_I the partial sum operator:

$$(S_I f)^\wedge = \hat{f}\mathcal{X}_I.$$

Given a sequence $\{I_j\}$ of disjoint intervals and a function b, we form the square function

$$[\Delta, b]f(x) = \left(\sum_j \left|b(x)S_{I_j}f(x) - S_{I_j}(bf)(x)\right|^2\right)^{1/2}.$$

We aim to prove inequalities of the type

$$\left\|[\Delta, b]f\right\|_{L^p(\beta)} \le C_p \|f\|_{L^p(\alpha)},$$

for some classes of weights α, β, depending on the family $\{I_j\}$ and on the function b. See Theorem (3.2) and (3.5).

Inequalities of the aforementioned type are new, even in the unweighted case, for general families of intervals $\{I_j\}$. In the case of the family of dyadic intervals some results are known see [ST2], for the smooth operators $\widetilde{S}_{I_j}$, see Definition (3.11).

We shall need a vector-valued commutator theorem (see Theorem (2.2)) for a kind of vector-valued L^r-Dini singular integrals. The use of these vector-valued L^r-Dini singular integrals in the Littlewood–Paley theory was introduced in the beautiful paper of J.L. Rubio de Francia [RF2].

To prove the commutator theorem, we shall need an extrapolation theorem (see Section 1) for pairs of weights α and β that satisfy the relation $\alpha=\nu^p\beta$, where ν is a given positive function and α and β belong to A_p. For notation and the general theory of A_p weights, we indicate [GCRF] for instance.

Throughout this paper we shall work in $\mathbf{R}$, endowed with the Lebesgue measure. Given a Banach space E we shall denote by $L^p_E(\mathbf{R})$ or L^p_E the Bochner–Lebesgue space of E-valued strongly measurable functions such that

$$\int_{\mathbf{R}} \|f(x)\|_E^p \, dx < +\infty.$$

Given a positive measurable function $\alpha(x)$, we shall denote by $L^p_E(\alpha)$ the space of E-valued strongly measurable functions such that

$$\int_{\mathbf{R}} \|f(x)\|_E^p \, \alpha(x) \, dx < +\infty.$$

Given two Banach spaces E and F, we shall denote by $\mathcal{L}(E,F)$ the Banach space of all continuous linear operators from E into F.

1. An extrapolation theorem

Let $1 \le p < \infty$ and $1 \le \lambda \le p$. Let ν be a measurable function. We shall say that a weight (positive measurable function) ω belongs to $A^{(\nu)}_{p,\lambda}$ if

$$\omega \in A_{p/\lambda} \quad \text{and} \quad \nu^p \omega \in A_{p/\lambda}.$$

If $\lambda = 1$ we shall write $A^{(\nu)}_p$. Observe that $A^{(\nu)}_{p,\lambda} = A^{(\nu^\lambda)}_{p/\lambda}$.

Now we list the basic properties of the classes $A^{(\nu)}_{p,\lambda}$. See [ST1].

(1.1) The class $A^{(\nu)}_{p,\lambda}$ is not empty if and only if $\nu^\lambda \in A_2$.

(1.2) Given $\omega \in A^{(\nu)}_{p,\lambda}$, there exists $\varepsilon > 0$ such that if $p < q < p + \varepsilon$ then $\omega \in A^{(\nu)}_{q,\lambda}$.

(1.3) Given $\omega \in A^{(\nu)}_{p,\lambda}$, $\lambda < p$, there exists $\varepsilon > 0$ such that ω belongs to $A^{(\nu)}_{q,\lambda}$ for $p - \varepsilon < q < p$.

(1.4) (Factorization). The weight ω belongs to $A^{(\nu)}_{p,\lambda}$ if and only if $\omega = \omega_0 \omega_1^{1-p/\lambda}$, where $\omega_0 \in A_1^{(\nu^\lambda)}$ and $\omega_1 \in A_1^{(\nu^{-\lambda})}$.

The classes $A^{(\nu)}_{p,\lambda}$ also satisfy extrapolation results. We are interested in the following theorems (see [ST1]).

(1.5) Theorem. *Let S be a sublinear operator defined on $C_0^\infty(\mathbf{R})$ with values in the space of measurable functions and $1 \le \lambda < \infty$. If the operator satisfies*

$$\|\omega S f\|_\infty \le C_\omega \|\omega f\|_\infty$$

for every ω such that $\omega^{-\lambda} \in A_1$ and $(\nu\omega)^{-\lambda} \in A_1$, then

$$\|Sf\|_{L^p(\omega)} \le C_\omega \|f\|_{L^p(\omega)}$$

holds for every $\omega \in A^{(\nu)}_{p,\lambda}$ and $p > \lambda$.

(1.6) Theorem. *Let S be a sublinear operator defined on C_0^∞ with values in the space of measurable functions. Let $1\leq\lambda\leq r<\infty$. If the operator satisfies*

$$\|Sf\|_{L^r(\omega)}\leq C_\omega\|f\|_{L^r(\omega)},$$

for every $\omega\in A_{r,\lambda}^{(\nu)}$, then for every p, $\lambda<p<\infty$,

$$\|Sf\|_{L^p(\omega)}\leq C_\omega\|f\|_{L^p(\omega)}$$

holds for every $\omega\in A_{p,\lambda}^{(\nu)}$.

2. Singular integrals and commutators

Let E be a Banach space, b an E-valued measurable function, and ν a positive function. We shall say that b is in $\mathrm{BMO}_E(\nu)$ if for any interval I

$$\int_I \|b(x)-b_I\|_E\,dx\leq C\nu(I)=C\int_I \nu(x)\,dx,$$

where $b_I=(1/|I|)\int_I b(y)\,dy$.

The following lemma is an easy consequence of the definition of $\mathrm{BMO}_E(\nu)$.

(2.1) Lemma. *Let I be an interval and $I_k=2^kI$. Then, if $b\in\mathrm{BMO}_E(\nu)$ it follows that*

$$\|b_I-b_{I_k}\|_E\leq Ck\nu_{I_{i(k)}},$$

where $I_{i(k)}$ is the interval I_i such that

$$\nu_{I_{i(k)}}=\max_{1\leq i\leq k}\nu_{I_i}.$$

(2.2) Theorem. *Let E,F be Banach spaces. Let T be a bounded linear operator from $L_E^p(\mathbf{R})$ into $L_F^p(\mathbf{R})$, for $s'<p<\infty$, $s>1$. Assume that there exists an $\mathcal{L}(E,F)$-valued kernel $K(x,y)$ satisfying:*

(K.1) *For any compactly supported f, we have,*

$$Tf(x)=\int K(x,y)f(y)\,dy,\quad \text{for } x\notin\operatorname{supp} f.$$

(K.2) *There exists a sequence $\{c_m\}_{m=1}^\infty$ such that*

$$\sum_{m=1}^\infty mc_m<+\infty,$$

and

$$\left(\int_{x\in I_m(y,z)} \|\langle a, K(y,x)-K(z,x)\rangle\|_{E^*}^s\,dx\right)^{1/s} \le C c_m \|a\|_{F^*} |I_m(y,z)|^{-1/s'}.$$

for any integer $m\geq 1$ *and any* $y,z\in\mathbf{R}$*, where*

$$I_m(y,z)=\{\,x: 2^m|y-z|<|x-z|\le 2^{m+1}|y-z|\,\}.$$

Let $l\to\tilde{l}$ *be a bounded linear mapping from* $\mathcal{L}(E,E)$ *into* $\mathcal{L}(F,F)$*, such that*

$$\tilde{l}Tf(x)=T(lf)(x),\quad l\in\mathcal{L}(E,E),\ x\in\mathbf{R},$$

and

$$K(x,y)l=\tilde{l}K(x,y),\quad l\in\mathcal{L}(E,E).$$

Then, if ν *is an* A_2 *weight and* $b\in\mathrm{BMO}_{\mathcal{L}(E,E)}(\nu)$*, the commutator*

$$C_bf(x)=\tilde{b}(x)Tf(x)-T(bf)(x),$$

is bounded from $L^p_E(\alpha)$ *into* $L^p_F(\beta)$ *for* $\alpha=\nu^p\beta$ *and* $\beta\in A^{(\nu)}_{p,s'}$.

Proof. The main idea is to obtain the estimate

$$(C_bf)^{\#}(x)\le U_1(\|Tf\|_F)(x)+U_2(\|f\|_E)(x) \tag{2.3}$$

where the operators $S_1(g)=U_1(\nu^{-1}g)$ and $S_2(g)=U_2(\nu^{-1}g)$ are sublinear operators satisfying Theorem (1.5). Then, by the sharp maximal function theorem for vector-valued function (see [RFRT]), we have,

$$\begin{aligned}
\|C_bf\|_{L^p_F(\beta)} &\le C\|(C_bf)^{\#}\|_{L^p(\beta)}\\
&\le C\left\{\big\|U_1(\|Tf\|_F)\big\|_{L^p(\beta)}+\big\|U_2(\|f\|_E)\big\|_{L^p(\beta)}\right\}\\
&= C\left\{\big\|S_1(\nu\|Tf\|_F)\big\|_{L^p(\beta)}+\big\|S_2(\nu\|f\|_E)\big\|_{L^p(\beta)}\right\}\\
&\le C\left\{\big\|\nu\|Tf\|_F\big\|_{L^p(\beta)}+C_2\big\|\|f\|_{E^\nu}\big\|_{L^p(\beta)}\right\}\\
&= C\left\{\|Tf\|_{L^p_F(\alpha)}+\|f\|_{L^p_F(\alpha)}\right\}\le C\|f\|_{L^p_F(\alpha)}.
\end{aligned}$$

In the last inequality we have used the fact that T is bounded from $L^p_E(\alpha)$ into $L^p_F(\alpha)$ for $\alpha\in A_{p,s'}$ (see [RFRT]).

Now we shall show how to obtain (2.3). Let x_0 be a point in $\mathbf{R}$ and I be an interval with center at x_0. Given a smooth and compactly supported function, f, we define

$$f_1(x)=f(x)\mathcal{X}_{2I}(x), \quad \text{and} \quad f_2=f-f_1.$$

Let $c_I=T((b_I-b)f_2)(x_0)$. Then if $x\in I$, we have,

$$\begin{aligned}\|C_b f(x)-c_I\|_F &\le \|(\tilde{b}(x)-\tilde{b}_I)Tf(x)\|_F+\|T((b_I-b)f_1)(x)\|_F\\ &\quad +\|T((b_I-b)f_2)(x)-T((b_I-b)f_2)(x_0)\|_F\\ &=\sigma_1(x)+\sigma_2(x)+\sigma_3(x).\end{aligned}$$

If we denote

$$N_r(g)(z)=\sup_{z\in I}\left(\frac{1}{|I|}\int_I(\|b(x)-b_I\|g(x))^r\,dx\right)^{1/r},$$

it is clear that

$$\frac{1}{|I|}\int_I\sigma_1(x)\,dx\le\frac{1}{|I|}\int_I\|b(x)-b_I\|\,\|Tf\|_F\,dx\le N_1(\|Tf\|_F)(x_0).$$

On the other hand, by the boundedness properties of T, we have for $r'>s'$ that

$$\begin{aligned}\frac{1}{|I|}\int_I\sigma_2(x)\,dx&\le\left(\frac{1}{|I|}\int_I\|T((b-b_I)f_1)(x)\|_F^{r'}\,dx\right)^{1/r'}\\ &\le C\left(\frac{1}{|I|}\int_I(\|b(x)-b_I\|\,\|f(x)\|_E)^{r'}\,dx\right)^{1/r'}\\ &\le C(N_{r'}(\|f\|_E)(x_0)).\end{aligned}$$

Now we shall estimate $\sigma_3(x)$.

Let $g(x)$ be an arbitrary F^*-valued function with $\|g(x)\|_{F^*}\le 1$ for all $x\in I$. Then we have

$$\sigma_3(x)=\sup_g\left|\left\langle g(x),\int_{y\notin 2I}(K(x,y)-K(x_0,y))(b(y)-b_I)f(y)\,dy\right\rangle\right|.$$

Given $x\in I$, there exists a j, depending on x, such that

$$2^{-j-1}|I|<|x-x_0|\le 2^{-j}|I|.$$

Therefore,

$$\begin{aligned}I_m(x,x_0)&=\{\,y:2^m|x-x_0|<|y-x_0|\le 2^{m+1}|x-x_0|\,\}\\ &\subset\{\,y:2^{m-j-1}|I|<|y-x_0|\le 2^{m-j+1}|I|\,\}\subset I_{m-j+1};\end{aligned}$$

in particular, $|I_m(x,x_0)|\sim 2^{m-j}|I|\sim|I_{m-j+1}|$.

Now, for each g and each $x\in I$, we use condition (K.2) and we get

$$\int_{y\notin 2I}\left|\langle g(x),(K(x,y)-K(x_0,y))((b(y)-b_I)f(y))\rangle\right| dy$$
$$\leq \int_{y\notin 2I}\|\langle g(x),K(x,y)-K(x_0,y)\rangle\|_{E^*}\|(b(y)-b_I)f(y)\|_E\,dy.$$

If $y\notin 2I$ then $|y-x_0|>|I|\geq 2^j|x-x_0|$, therefore the last integral is less than

$$\int_{|y-x_0|>2^j|x-x_0|}\|\langle g(x),K(x,y)-K(x_0,y)\rangle\|_{E^*}\|(b(y)-b_I)f(y)\|_E\,dy$$
$$\leq \sum_{m\geq j}\int_{I_m(x,x_0)}\|\langle g(x),K(x,y)-K(x_0,y)\rangle\|_{E^*}\|(b(y)-b_I)f(y)\|_E\,dy$$
$$\leq \sum_{m\geq j}\left(\int_{I_m(x,x_0)}\|\langle g(x),K(x,y)-K(x_0,y)\rangle\|_{E^*}^s\,dy\right)^{1/s}$$
$$\times\left(\int_{I_m(x,x_0)}\|(b(y)-b_I)f(y)\|_E^{s'}\,dy\right)^{1/s'}$$
$$\leq C\sum_{m\geq j}c_m|I_m(x,x_0)|^{-1/s'}\left(\int_{I_m(x,x_0)}\|(b(y)-b_I)f(y)\|_E^{s'}\,dy\right)^{1/s'}$$
$$\leq C\sum_{m\geq j}c_m|I_{m-j+1}|^{-1/s'}\left(\int_{I_{m-j+1}}(\|b(y)-b_I\|\,\|f(y)\|_E)^{s'}\,dy\right)^{1/s'}$$
$$\leq C\sum_{m\geq j}c_m\left(|I_{m-j+1}|^{-1}\int_{I_{m-j+1}}(\|b(y)-b_{I_{m-j+1}}\|\,\|f(y)\|_E)^{s'}\,dy\right)^{1/s'}$$
$$+C\sum_{m\geq j}c_m\|b_I-b_{I_{m-j+1}}\|\left(|I_{m-j+1}|^{-1}\int_{I_{m-j+1}}\|f(y)\|_E^{s'}\,dy\right)^{1/s'}.$$

By Lemma (2.1) this is less than

$$C\sum_{m\geq j}c_mN_{s'}(\|f\|_E)(x_0)$$
$$+\sum_{m\geq j}c_m(m-j+1)\nu_{I_{i(m-j+1)}}\left(|I_{m-j+1}|^{-1}\int_{I_{m-j+1}}\|f(y)\|_E^{s'}\,dy\right)^{1/s'}.$$

But

$$\left(|I_{m-j+1}|^{-1}\int_{I_{m-j+1}}\|f(y)\|_E^{s'}\,dy\right)^{1/s'}\leq \inf_{x\in I_{i(m-j+1)}}(M(\|f\|_E^{s'})(x))^{1/s'},$$

where M is the Hardy–Littlewood maximal operator, so that

$$\begin{aligned}
&\sum_{m\geq j} c_m(m-j+1)\nu_{I_{i(m-j+1)}}\left(|I_{m-j+1}|^{-1}\int_{I_{m-j+1}}\|f(y)\|_E^{s'}\,dy\right)^{1/s'}\\
&\quad\leq \sum_{m\geq j} c_m\cdot m\frac{1}{|I_{i(m-j+1)}|}\int_{I_{i(m-j+1)}}(M(\|f\|_E^{s'})(x))^{1/s'}\nu(x)\,dx\\
&\quad\leq \sum_{m\geq j} c_m\cdot mM((M(\|f\|_E^{s'}))^{1/s'}\nu)(x_0)\\
&\quad\leq CM((M(\|f\|_E^{s'}))^{1/s'}\nu)(x_0).
\end{aligned}$$

Therefore (2.3) is proved if we take

$$\begin{aligned}
U_1(g)(x)&=N_1(g)(x), \text{ and}\\
U_2(g)(x)&=C_1N_{r'}(g)(x)+C_2N_{s'}(g)(x)\\
&\quad+C_3M((M(g^{s'}))^{1/s'}\nu)(x).
\end{aligned}$$

These operators appeared in [ST1]. Then it follows that the operators S_i, $(i=1,2)$, defined at the beginning of this proof, satisfy Theorem (1.5).

(2.4) *Remark.* If the condition (K.2) is substituted by:
(K.2′) if $|x-y|>2|y-z|$ then

$$\|K(y,x)-K(z,x)\|\leq C\frac{|x-y|}{|y-x|^2},$$

then the conclusion of Theorem (2.2) remains valid for all $\nu\in A_2$, $\alpha=\nu^p\beta$ and $\beta=A_p^{(\nu)}$, $1<p<\infty$, (see [ST2]).

(2.5) *Remark.* The theory of vector-valued Calderón–Zygmund operators can be applied in Theorem (2.2), and T is a bounded operator from $L^p_E(\alpha)$ into $L^p_F(\alpha)$ for $\alpha\in A_{p/s'}$, (for $\alpha\in A_p$ in the case of Remark (2.4)). See [RFRT].

3. Application to Littlewood–Paley theory

For every interval $I\subset\mathbf{R}$ we denote by S_I the partial sum operator:

$$(S_If)^\wedge = f\mathcal{X}_I.$$

Given a sequence of disjoint intervals I_j, we form the square function

$$\Delta f(x)=\left(\sum_j |S_{I_j}f(x)|^2\right)^{1/2}.$$

When I_j is the sequence of dyadic intervals

$$\{[2^j,2^{j+1}],\ -[2^j,2^{j+1}],\ j\in\mathbf{Z}\}$$

it is well known that, for $1<p<\infty$, the following inequality holds, (see [LP]):

$$\|\Delta f\|_p\leq C_p\|f\|_p. \tag{3.0}$$

When all the intervals have the same length, then inequality (3.0) holds for $2\leq p<\infty$, and this is the best possible result, see [C].

Rubio de Francia proved in [RF2] that for every sequence $\{I_j\}$ of disjoint intervals, the inequality (3.0) holds for $2\leq p<\infty$. The constant C_p is an absolute constant not depending on the sequence $\{I_j\}$.

(3.1) *Remark.* Weighted versions of inequality (3.0) are also known. In the case of dyadic intervals Δ maps $L^p(\omega)$ into $L^p(\omega)$ for $\omega\in A_p$, $1<p<\infty$. See [K]. In the general case Δ maps $L^p(\omega)$ into $L^p(\omega)$ for $\omega\in A_{p/2}$, $2<p<\infty$. See [RF1], [RF2]. It is also well-known that for any sequence of intervals $\{I_j\}$ and any p, $1<p<\infty$, the following inequality is true

$$\left\|\left(\sum|S_{I_j}f_j|^2\right)^{1/2}\right\|_{L^p(\omega)}\leq C_p\left\|\left(\sum|f_j|^2\right)^{1/2}\right\|_{L^p(\omega)},\quad \omega\in A_p.$$

The main results of this section are Theorems (3.2) and (3.5). Before stating them we introduce a definition. Given a weight ν, a function $b\in\mathrm{BMO}(\nu)$ and a sequence of disjoint intervals we form the square function

$$[\Delta,b]f(x)=\left(\sum_j|[S_{I_j},b]f(x)|^2\right)^{1/2},$$

where

$$[S_{I_j},b]f(x)=b(x)S_{I_j}f(x)-S_{I_j}(bf)(x).$$

(3.2) Theorem. *Let ν,α,β be positive functions, such that $\nu\in A_2$ and $\alpha=\nu^p\beta$. Let b be a function in* $\mathrm{BMO}(\nu)$ *and $\mathcal{J}$ be a family of disjoint intervals. Then the inequality*

$$\|[\Delta,b]f\|_{L^p(\beta)}\leq C_p\|f\|_{L^p(\alpha)}$$

holds in the following cases:

(3.3) *$\mathcal{J}$ is the family of dyadic intervals, $1<p<\infty$, $\beta\in A_p^{(\nu)}$.*

(3.4) *$\mathcal{J}$ is an arbitrary family, $2<p<\infty$, $\beta\in A_{p,2}^{(\nu)}$.*

This theorem has the following consequence:

(3.5) Theorem. *Given a weight ν in A_2 and a function b, the following conditions are equivalent:*

(i) $b \in \mathrm{BMO}(\nu)$.

(ii) *For the family of dyadic intervals, we have,*

$$\big\|[\Delta, b]f\big\|_{L^p(\beta)} \le C_p \|f\|_{L^p(\alpha)},$$

for $1<p<\infty$, $\alpha=\nu^p\beta$ *and* $\beta \in A_p^{(\nu)}$.

(iii) *For any family of disjoint intervals, we have,*

$$\|[\Delta, b]f\|_{L^p(\beta)} \le C'_p \|f\|_{L^p(\alpha)},$$

where $2<p<\infty$, $\alpha=\nu^p\beta$ *and* $\beta \in A_{p,2}^{(\nu)}$.

(iv) *If H denotes the Hilbert transform, then*

$$\|[H, b]f\|_{L^p(\beta)} \le C''_p \|f\|_{L^p(\alpha)}$$

for $1<p<\infty$, $\alpha=\nu^p\beta$, *and* $\beta \in A_p^{(\nu)}$.

The constants C_p, C'_p and C''_p depend on α, ν and $\|b\|_{\mathrm{BMO}(\nu)}$.

Proof of Theorem (3.5). That (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) are contained in Theorem (3.2). On the other hand (iv) $\Longrightarrow$ (i) is known and due to Bloom, see [B]. Let us prove first that (iii) implies

(iv')
$$\|[H, b]f\|_{L^p(\beta)} \le C_p \|f\|_{L^p(\alpha)},$$

for $2<p<\infty$, $\alpha=\nu^p\beta$ and $\beta \in A_{p,2}^{(\nu)}$.

We shall use the following fact:

(3.6) If h is a function in $L^1(\mathbf{R})$ and its Fourier transform $\hat{h}$ is compactly supported in $[-R, R]$, denoting $S_R=S_{[-R,R]}$, we have

$$Hh(x) = -i\big\{e^{2\pi iRx} S_R(e^{-2\pi iR\cdot} h(\cdot))(x) - e^{-2\pi iRx} S_R(e^{2\pi iR\cdot} h(\cdot))(x)\big\}.$$

In particular

$$[H, b]h(x) = -i\big\{e^{2\pi iRx}[S_R, b](e^{-2\pi iR\cdot} h(\cdot))(x) - e^{-2\pi iRx}[S_R, b](e^{2\pi iR\cdot} h(\cdot))(x)\big\}.$$

Therefore, if we assume (iii), we have

$$\begin{aligned}\|[H, b]h\|_{L^p(\beta)} &\le \big\|[S_R, b](e^{-2\pi iR\cdot} h(\cdot))\big\|_{L^p(\beta)} + \big\|[S_R, b](e^{2\pi iR\cdot} h(\cdot))\big\|_{L^p(\beta)} \\ &\le C_p \big\|e^{-2\pi iR\cdot} h(\cdot)\big\|_{L^p(\alpha)} + C_p \big\|e^{2\pi iR\cdot} h(\cdot)\big\|_{L^p(\alpha)} \\ &\le C'_p \|h\|_{L^p(\alpha)}.\end{aligned}$$

Since $A_{p,2}^{(\nu)} \subset A_p^{(\nu)}$, then by Bloom's theorem, see [B], we get that (iv′) implies (i). Therefore, (iii) implies (i).

On the other hand, if we assume (ii) and I is any dyadic interval, we have

$$\|[S_I, b]\|_{L^p(\beta)} \leq C_p \|f\|_{L^p(\alpha)}.$$

If we denote by $I+R$, $R>0$, the interval $\{x : x-R \in I\}$, then

$$S_{I+R} f = e^{2\pi i R x} S_I(e^{-2\pi i R \cdot} f(\cdot))(x),$$

and therefore

$$\|S_{I+R} f\|_{L^p(\beta)} \leq C_p \|f\|_{L^p(\alpha)}$$

holds for $1<p<\infty$, any dyadic interval I and any $R>0$. Now we can continue as in the case (iii) $\Longrightarrow$ (iv′), showing that (ii) $\Longrightarrow$ (iv) and therefore (ii) $\Longrightarrow$ (i).

Now we state some lemmas that we shall need for the proof of Theorem (3.2).

(3.7) Lemma. *Let $1<p<\infty$ and ν be an A_2-weight. Given an arbitrary sequence of intervals $\{I_j\}$ and a function $b \in \mathrm{BMO}(\nu)$, we have*

$$\left\| \left(\sum_j |[S_j, b] f_j|^2 \right)^{1/2} \right\|_{L^p(\beta)} \leq C_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(\alpha)} \tag{3.8}$$

provided that $\alpha = \nu^p \beta$ and $\beta \in A_p^{(\nu)}$.

Proof. The operator $[H, b]$ is bounded from $L^p_{l^2}(\alpha)$ into $L^p_{l^2}(\beta)$ see [ST2] for details. Therefore (3.8) holds due to the following fact:

(3.9) If $I_j = (a_j, c_j)$, we have,

$$S_{I_j} f(x) = \frac{1}{2i} \left\{ e^{2\pi i a_j x} H(e^{-2\pi i a_j \cdot} f(\cdot))(x) - e^{-2\pi i c_j x} H(e^{2\pi i c_j \cdot} f(\cdot))(x) \right\}.$$

(3.10) *Definition.* Given an interval I we shall say that a function φ_I is adapted to I if φ_I is a Schwartz function with Fourier transform $\hat{\varphi}_I$ such that $\hat{\varphi}_I(\xi) = 1$, $\xi \in I$ and $\widehat{\varphi}_I(\xi) \equiv 0$, $\xi \notin NI$, for some fixed natural number N.

(3.11) *Definition.* Given an interval I and an adapted function φ_I, let us denote

$$\widetilde{S}_I f = \varphi_I * f.$$

Given a family of disjoint intervals I_j, we define

$$\mathcal{G} f(x) = \left(\sum_j |\widetilde{S}_{I_j} f(x)|^2 \right)^{1/2}$$

and

$$[\mathcal{G},b]f(x)=\Big(\sum_j |[\widetilde{S}_{I_j},b]f(x)|^2\Big)^{1/2},$$

where as usual

$$[\widetilde{S}_{I_j},b]f(x)=b(x)\widetilde{S}_{I_j}f(x)-\widetilde{S}_{I_j}(bf)(x).$$

(3.12) Lemma. *Given a weight ν and a function b in* BMO(ν)*, we have that for any smooth function f and any interval I, the equalities*

$$[S_I,b]f=S_I([\widetilde{S}_I,b]f)+[S_I,b](\widetilde{S}_If) \tag{3.13}$$

and

$$[S_I,b]f=S_I([S_I,b]f)+[S_I,b](S_If) \tag{3.14}$$

hold.

Proof. Observe that $S_I\widetilde{S}_I=S_IS_I=S_I$.

(3.15) Lemma. *Let ν,α, and β be positive functions, such that $\nu\in A_2$ and $\alpha=\nu^p\beta$. Let b be a function in* BMO(ν) *and $\mathcal{J}$ be a family of disjoint intervals.*

The inequality

$$\|[\mathcal{G},b]f\|_{L^p(\beta)}\le C_p\|f\|_{L^p(\alpha)}$$

holds in the following cases:

(3.16) *$\mathcal{J}$ is the family of dyadic intervals, $1<p<\infty$, $\beta\in A_p^{(\nu)}$.*

(3.17) *$\mathcal{J}$ satisfies $\sum_{I\in\mathcal{J}}\mathcal{X}_{8I}(x)\le C$, $2<p<\infty$, $\beta\in A_{p,2}^{(\nu)}$.*

Proof. The proof of (3.16) can be found in [ST2]; we shall give here a brief idea.

Let φ be a test function, $\varphi\in\mathcal{S}(\mathbf{R})$, such that its Fourier transform satisfies $\widehat{\varphi}(0)=0$ and $\widehat{\varphi}(\xi)=1$, $\xi\in[\frac12,1]$; define $\varphi_j(x)=2^j\varphi(2^jx)$, and

$$Tf(x)=\{\varphi_j*f(x)\}_j.$$

Then T is a bounded linear operator from $L^p(\mathbf{R})$ into $L^p_{l^2}(\mathbf{R})$ with kernel satisfying (K.2′) of Remark (2.4).

Consider the linear map $l\to\tilde{l}$ from $\mathbf{R}$ into $\mathcal{L}(l^2,l^2)$ given by $\tilde{l}\{t_j\}=\{lt_j\}$. Then, by Remark (2.4), the operator

$$C_bf(x)=b(x)(\varphi_j*f)(x)-\varphi_j(bf)(x),$$

is bounded from $L^p(\alpha)$ into $L^p_{l^2}(\beta)$, that is to say $[\mathcal{G},b]$ maps $L^p(\alpha)$ into $L^p(\beta)$.

The proof of (3.17) runs parallel to the proof of (3.16). It is enough to show that a family φ_j of functions adapted to the intervals I_j of $\mathcal{J}$ can be found in such a way that the operator

$$Tf(x)=\{\varphi_j * f(x)\}_j$$

be a bounded linear operator from $L^2(\mathbf{R})$ into $L^2_{l^2}(\mathbf{R})$ with a kernel satisfying condition (K.2). This was done by Rubio de Francia in his celebrated paper, (see [RF2]), where he showed that we may take $c_m=2^{-5/6m}$. Therefore Theorem (2.2) can be applied, which finishes the proof of the lemma.

Now we can prove part (3.3) of Theorem (3.2). We observe that part (3.4) can be proved at this moment assuming that the family $\mathcal{J}$ satisfies

$$\sum \mathcal{X}_{2I}(x)\le C.$$

If this is the case we divide each interval I into seven consecutive intervals of equal length

$$I=I^{(1)}\cup I^{(2)}\cup\ldots\cup I^{(7)}, \qquad |I^{(i)}|=|I|/7,$$

so that $8I^{(i)}\subset 2I$. It suffices to prove the theorem for each one of the families,

$$\{\,I^{(i)}\,:\,I\in \text{ initial sequence}\,\},\quad 1\le i\le 7.$$

Therefore, we can assume that the family satisfies $\sum \mathcal{X}_{8I}(x)\le C$. Now the proofs for the two parts are the same. By using Lemma (3.13), we have,

$$\begin{aligned}\|[\Delta,b]f\|_{L^p(\beta)} &\le \left\|\left(\sum_j |S_j([\tilde S_j,b]f)|^2\right)^{1/2}\right\|_{L^p(\beta)} + \left\|\left(\sum_j |[S_j,b]\tilde S_j f|^2\right)^{1/2}\right\|_{L^p(\beta)}\\ &= \mathrm{I}+\mathrm{II}.\end{aligned}$$

Now we apply Remark (3.1) and Lemma (3.15), and we get

$$\mathrm{I}\le C_p\left\|\left(\sum_j |[\tilde S_j,b]f|^2\right)^{1/2}\right\|_{L^p(\beta)}\le C_p\|f\|_{L^p(\alpha)}.$$

On the other hand applying Lemma (3.7) and Remark (2.5) we get

$$\mathrm{II}\le C_p\left\|\left(\sum_j |\tilde S_j f|^2\right)^{1/2}\right\|_{L^p(\alpha)}\le C_p\|f\|_{L^p(\alpha)}.$$

We need to do some additional work in order to prove part (3.4) of Theorem (3.2) completely. We follow closely the ideas of Rubio de Francia, see [RF2].

Given an interval I, we define the Whitney decomposition $W(I)$ of I to be the construction, invariant under translations and dilations, such that if $I=[0,1]$, then $W(I)$ consists of the intervals

$$\{[a_{k+1},a_k]\}_{k=0}^{\infty},\quad \left[\tfrac{1}{3},\tfrac{2}{3}\right],\quad \{[1-a_k,1-a_{k+1}]\}_{k=0}^{\infty},$$

where $a_k=2^{-k}/3$.

The intervals of $W(I)$ form a disjoint covering of I, $2H\subset I$ for every $H\in W(I)$ and $\sum_{H\in W(I)}\mathcal{X}_{2H}(x)\leq 5$ for all x.

(3.18) Lemma. *Let ν, α, and β be weights such that $\alpha=\nu^p\beta$, and b be a function in* BMO(ν). *Let $\{I_j\}$ be an arbitrary family of disjoint intervals. Then for $1<p<\infty$ and $\beta\in A_p^{(\nu)}$, we have*

$$\left\|\left(\sum_j|[S_{I_j},b]f|^2\right)^{1/2}\right\|_{L^p(\beta)}\leq C_p\left\|\left(\sum_j\sum_{H^j\in W(I_j)}|[S_{H_j},b]f|^2\right)^{1/2}\right\|_{L^p(\beta)}$$
$$+C_p'\left\|\left(\sum_j\sum_{H^j\in W(I_j)}|S_{H_j}f|^2\right)^{1/2}\right\|_{L^p(\alpha)}.$$

Taking this lemma for granted, observe that the proof of (3.4) can be easily deduced since the family $\{\{H^j\}_{H^j\in W(I_j)}\}_j$ satisfies $\sum\mathcal{X}_{2H^j}\leq 5$, and then we can apply to the first summand in the lemma the reduced and proved part of (3.4) and to the second summand, the result of Rubio de Francia mentioned in Remark (3.1).

Proof of Lemma (3.18). Observe that $[S_{I_j},b]=\sum_{H^j\in W(I_j)}[S_{H^j},b]$. Then by using (3.14), we have

$$\left\|\left(\sum_j|[S_{I_j},b]f|^2\right)^{1/2}\right\|_{L^p(\beta)}\leq\left\|\left(\sum_j\left|\sum_{H^j\in W(I_j)}S_{H^j}[S_{H^j},b]f\right|^2\right)^{1/2}\right\|_{L^p(\beta)}$$
$$+\left\|\left(\sum_j\left|\sum_{H^j\in W(I_j)}[S_{H^j},b]S_{H^j}f\right|^2\right)^{1/2}\right\|_{L^p(\beta)}=\mathrm{I}+\mathrm{II}.$$

Now for any j we consider the sequence of l^2-valued functions

$$F_j=\{f_{H^j}\}_{H^j\in W(I_j)},$$

and we define the operator

$$T_j^* F_j = \sum_{H^j \in W(I_j)} S_{H^j} f_{H^j}.$$

This operator is the transpose of an l^2-valued operator

$$T_j f = \{S_{H^j} f\}_{H^j \in W(I_j)},$$

that can be handled as the case of $Tg = \{S_{I_k} g\}_k$ when $\{I_k\}$ are the dyadic intervals; in particular, for each I_j the operator T_j is bounded from $L^2(\omega)$ into $L^2_{l^2}(\omega)$, $\omega \in A_2$. The operators T_j are uniformly bounded in j, therefore by the extrapolation theorem for A_p weights (see [RF3], [GC]) we have,

$$\left\| \left(\sum_j \|T_j f_j\|_{l^2}^2 \right)^{1/2} \right\|_{L^p(\omega)} \le C_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(\omega)},$$

for $1<p<\infty$, $\omega \in A_p$.

On the other hand, by (3.3) the operator $[T_j, b]$ is bounded from $L^2(\alpha)$ into $L^2_{l^2}(\beta)$, $\alpha = \nu^2 \beta$, $\beta \in A_2^{(\nu)}$. The operators $[T_j, b]$ are uniformly bounded in j, therefore, by the extrapolation theorem for $A_p^{(\nu)}$ weights, see Theorem (1.6), we have,

$$\left\| \left(\sum_j \|[T_j, b] f_j\|_{l^2}^2 \right)^{1/2} \right\|_{L^p(\beta)} \le C_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(\alpha)},$$

for $1<p<\infty$, $\beta \in A_p^{(\nu)}$, $\alpha = \nu^p \beta$.

Therefore the following inequalities are true:

(3.19) Given $1<p<\infty$, $\omega \in A_p$, we have

$$\left\| \left(\sum_j |T_j^* F_j|^2 \right)^{1/2} \right\|_{L^p(\omega)} \le C_p \left\| \left(\sum_j \|F_j\|_{l^2}^2 \right)^{1/2} \right\|_{L^p(\omega)}.$$

(3.20) Given $1<p<\infty$, $\beta \in A_p^{(\nu)}$, $\alpha = \nu^p \beta$, we have

$$\left\| \left(\sum_j |[T_j^*, b] f_j|^2 \right)^{1/2} \right\|_{L^p(\beta)} \le C_p \left\| \left(\sum_j \|F_j\|_{l^2}^2 \right)^{1/2} \right\|_{L^p(\alpha)}.$$

Now, if we take $F_j = \{[S_{H^j}, b] f\}_{H^j}$ in (3.19), we get that

$$\mathrm{I} \le C_p \left\| \left(\sum_j \sum_{H^j} |[S_{H^j}, b] f|^2 \right)^{1/2} \right\|_{L^p(\beta)};$$

and if we take $F_j = \{S_{H^j} f\}_{H^j}$ in (3.20) we have

$$\mathrm{II} \le C_p \left\| \left(\sum_j \sum_{H^j} |S_{H^j} f|^2 \right)^{1/2} \right\|_{L^p(\alpha)}.$$

4. Generalizations to sequences of commutators

Given a sequence of arbitrary intervals $\{I_j\}$ and a sequence of functions $\mathbf{b}=\{b_k\}$ we shall study the boundedness of the operator

$$[\Delta, \mathbf{b}]f(x)=\left(\sum_{k,j}\left|[S_{I_j}, b_k]f(x)\right|^2\right)^{1/2}.$$

We shall need the following theorem

(4.1) Theorem. *Let E_i and $F_i, i=1,2$ be Banach spaces. Let $T_i, i=1,2$ be bounded linear operators from $L^p_{E_i}(\mathbf{R})$ into $L^p_{F_i}(\mathbf{R})$ for $s'<p<\infty$, $s>1$. Assume that there exist $\mathcal{L}(E_i,F_i)$-valued kernels $K_i(x,y)$, $i=1,2$ satisfying* (K.1) *and* (K.2) *of Theorem* (2.2). *Let $l\to\tilde{l}$ be a bounded linear mapping from $\mathcal{L}(E_1,E_2)$ into $\mathcal{L}(F_1,F_2)$, such that*

$$\tilde{l}T_1 f(x)=T_2(lf)(x)$$

and

$$\tilde{l}K_1(x,y)=K_2(x,y)l.$$

If ν is a weight in A_2, and $b\in\mathrm{BMO}_{\mathcal{L}(E_1,E_2)}$, then the operator

$$B_b f(x)=\tilde{b}(x)T_1 f(x)-T_2(bf)(x),$$

is bounded from $L^p_{E_1}(\alpha)$ into $L^p_{F_2}(\beta)$ for $\alpha=\nu^p\beta$ and $\beta\in A^{(\nu)}_{p,s'}$.

Proof. The proof follows the lines of the proof of Theorem (2.2) with some technical changes. Let $x_0\in\mathbf{R}$ and let I be an interval with center at x_0. Given an E_1-valued function with compact support we define f_1 and f_2 as in the proof of Theorem (2.2).

Let $c_I=T_2((b_I-b)f_2)(x_0)$. Then if $x\in I$, we have

$$\begin{aligned}B_b f(x)&=\tilde{b}(x)T_1 f(x)-T_2(bf)(x)\\&=(\tilde{b}(x)-\tilde{b}_I)T_1 f(x)+T_2(b_I f)(x)-T_2(bf)(x)\\&=(\tilde{b}(x)-\tilde{b}_I)T_1 f(x)+T_2((b_I-b)f)(x).\end{aligned}$$

Therefore for $x\in I$, we have,

$$\begin{aligned}\|B_b f(x)-c_I\|_{F_2}\le{}&\|(\tilde{b}(x)-\tilde{b}_I)T_1 f(x)\|_{F_2}+\|T_2((b_I-b)f_1)(x)\|_{F_2}\\&+\|T_2((b_I-b)f_2)(x)-T_2((b_I-b)f_2)(x_0)\|_{F_2}.\end{aligned}$$

Now the proof follows that of Theorem (2.2).

An application of this result will be the following theorem:

(4.2) Theorem. *Let ν, α, and β be positive functions, such that $\nu \in A_2$ and $\alpha = \nu^p \beta$. Let $\mathbf{b} = \{b_k\}$ be a sequence of functions in $\mathrm{BMO}_{l^2}(\nu)$ and $\mathcal{J}$ a family of disjoint intervals. The inequality*

$$\left\| [\Delta, \mathbf{b}] f \right\|_{L^p(\beta)} \leq C_p \|f\|_{L^p(\alpha)}$$

holds in the cases (3.3) *and* (3.4) *of Theorem* (3.2).

For the proof we shall need some versions of Lemmas (3.7), (3.15) and (3.18).

(4.3) Lemma. *Let $1 < p < \infty$ and ν be an A_2-weight. Given an arbitrary sequence of intervals $\{I_j\}$ and a sequence of functions $\mathbf{b} = \{b_k\} \in \mathrm{BMO}_{l^2}(\nu)$, we have that*

$$\left\| \left(\sum_{j,k} |[S_j, b_k] f_j|^2 \right)^{1/2} \right\|_{L^p(\beta)} \leq C_p \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(\alpha)} \tag{4.4}$$

holds, provided $\alpha = \nu^p \beta$ and $\beta \in A_p^{(\nu)}$.

Given a sequence of functions $\mathbf{b} = \{b_k\}$ and a family of intervals I_j, we shall write

$$[\mathcal{G}, \mathbf{b}] f(x) = \left(\sum_{j,k} |[\tilde{S}_{I_j}, b_k] f(x)|^2 \right)^{1/2},$$

where $\widetilde{S}_I$ is defined in (3.11).

(4.5) Lemma. *Let ν, α, and β be positive functions, such that $\nu \in A_2$ and $\alpha = \nu^p \beta$. Let $\mathbf{b} = \{b_k\}$ be a sequence of functions in $\mathrm{BMO}_{l^2}(\nu)$ and $\mathcal{J}$ be a family of disjoint intervals. The inequality*

$$\|[\mathcal{G}, \mathbf{b}] f\|_{L^p(\beta)} \leq C_p \|f\|_{L^p(\alpha)}$$

holds in the cases (3.16) *and* (3.17).

Proof of Lemma (4.3). Let T_1 be the extension of the Hilbert transform to a bounded operator from $L^p_{l^2(j)}$ into $L^p_{l^2(j)}$, defined as $T_1(\{f_j(x)\}) = \{Hf_j(x)\}$, and T_2 the extension of the Hilbert transform to a bounded operator from $L^p_{l^2(j,k)}$ into $L^p_{l^2(j,k)}$, defined by $T_2(\{f_{j,k}(x)\}) = \{Hf_{j,k}(x)\}$; let $\mathbf{b}(x) = \{b_k(x)\} \in \mathrm{BMO}_{l^2(k)}(\nu)$.

The space $\mathrm{BMO}_{l^2(k)}(\nu)$ can be considered as a subspace of the space $\mathrm{BMO}_{\mathcal{L}(l^2(j), l^2(j,k))}(\nu)$ by the identification $\mathbf{b}(x)\,(\{\lambda_j\}) = \{b_k(x)\lambda_j\}$. Then, if we apply Theorem (4.1) with the already defined T_1 and T_2, $E_1 = E_2 = l^2(j)$, $F_1 = F_2 = l^2(j,k)$, and $\widetilde{\mathbf{b}}(x) = \mathbf{b}(x)$, we get the lemma.

Proof of Lemma (4.5). Let T_1 be the operator

$$T_1 f(x) = \{\varphi_j * f(x)\},$$

bounded from L^p into $L^p_{l^2(j)}$, and T_2 be the operator

$$T_2 f_k(x) = \varphi_j * f_k(x),$$

bounded from $L^p_{l^2(k)}$ into $L^p_{l^2(j,k)}$.

The mapping that sends $\{\lambda_k\} \in l^2(k)$ to the element of $\mathcal{L}(\mathbf{C}, l^2(k))$ defined by $\lambda \to \{\lambda_k \lambda\}$ is a Banach space isomorphism. Therefore, if

$$\{b_k(x)\} = \mathbf{b}(x) \in \mathrm{BMO}_{l^2(k)}(\nu)$$

then it can be considered as an element of $\mathrm{BMO}_{\mathcal{L}(\mathbf{C}, l^2(k))}$. Let $\widetilde{\mathbf{b}}$ be the element of $\mathrm{BMO}_{\mathcal{L}(l^2(j), l^2(j,k))}$ given by

$$\widetilde{\mathbf{b}}\{\lambda_j\} = \{b_k(x)\lambda_j\}.$$

Then by Theorem (4.1) with $E_1 = \mathbf{C}$, $F_1 = l^2(j)$, $E_2 = l^2(k)$, and $F_2 = l^2(j,k)$ we obtain the lemma.

(4.6) Lemma. *Let ν, α, and β be weights such that $\alpha = \nu^p \beta$, and $\mathbf{b} = \{b_k\}$ a sequence of functions, $\mathbf{b} \in \mathrm{BMO}_{l^2}(\nu)$. Let $\{I_j\}$ be an arbitrary family of disjoint intervals. Then for $1 < p < \infty$ and $\beta \in A_p^{(\nu)}$ we have*

$$\left\| \left(\sum_{j,k} \left| [S_{I_j}, b_k] f \right|^2 \right)^{1/2} \right\|_{L^p(\beta)} \le C_p \left\| \left(\sum_{j,k} \sum_{H^j \in W(I_j)} \left| [S_{H^j}, b_k] f \right|^2 \right)^{1/2} \right\|_{L^p(\beta)} + C_p \left\| \left(\sum_{j} \sum_{H^j \in W(I_j)} |S_{H^j} f|^2 \right)^{1/2} \right\|_{L^p(\alpha)}.$$

Proof. As in the proof of Lemma (3.21), we have

$$\left\| \sum_{j,k} \left| [S_{I_j} b_k] f \right|^2 \right)^{1/2} \right\|_{L^p(\beta)} \le \left\| \left(\sum_{j,k} \left| \sum_{H^j \in W(I_j)} \widetilde{S}_{H^j} [S_{h^j}, bk] f \right|^2 \right)^{1/2} \right\|_{L^p(\beta} + \left\| \left(\sum_{j,k} \left| \sum_{H^j \in W(I_j)} [\widetilde{S}_{H^j} b_k] S_{H^j} f \right|^2 \right)^{1/2} \right\|_{L^p(\beta)} = \mathrm{I} + \mathrm{II}.$$

Now for each j we consider the operator

$$U_j F_j = \sum_{H^j} \widetilde{S}_{H^j} f_{H^j}, \quad F_j = \{f_{H^j}\}_{H^j}.$$

For each j, U_j is a vector-valued Calderón–Zygmund operator defined on $l^2(H^j)$-valued functions, and U_j is bounded from $L^p_{l^2(H^j)}(\omega)$ into $L^p(\omega)$, $1<p<\infty$, $\omega\in A_p$. We consider also the $l^2(k)$-valued extension of U_j, that is

$$V_j G_j = \left\{ \sum_{H^j} \widetilde{S}_{H^j} g^k_{H^j} \right\}_k, \quad \text{where } G_j = \{g^k_{H^j}\}_{H^j,k}.$$

V_j are vector-valued Calderón–Zygmund operators bounded from $L^p_{l^2(H^j,k)}(\omega)$ into $L^p_{l^2(k)}(\omega)$, $1<p<\infty$, $\omega\in A_p$, uniformly on j. Therefore by the extrapolation theorem for A_p weights (see [GC]) we have that

$$\left\| \left(\sum_j \|V_j G_j\|^2_{l^2(k)} \right)^{1/2} \right\|_{L^p(\omega)} \le C_p \left\| \left(\sum_j \|G_j\|^2_{l^2(H^j,k)} \right)^{1/2} \right\|_{L^p(\omega)}$$

for $1<p<\infty$ and $\omega\in A_p$. This means that

$$\left\| \left(\sum_j \sum_k \left| \sum_{H^j} \widetilde{S}_{H^j} g^k_{H^j} \right|^2 \right)^{1/2} \right\|_{L^p(\omega)} \le C_p \left\| \left(\sum_j \sum_k \sum_{H^j} |g^k_{H^j}|^2 \right)^{1/2} \right\|_{L^p(\omega)}, \tag{4.7}$$

for $1<p<\infty$, $\omega\in A_p$.

On the other hand, for each j, if we apply the first part of Theorem (4.2) (case (3.3)) taking $T_1=U_j$, $T_2=V_j$ and

$$(b_k) \in \mathrm{BMO}_{\mathcal{L}(\mathbb{C},l^2(k))} \subset \mathrm{BMO}_{\mathcal{L}(l^2(H^j),l^2(H^j,k))},$$

we get that the operators

$$[U_j, \mathbf{b}] = \{b_k(U_j F_j) - U_j(b_k F_j)\}_k,$$

are bounded, uniformly in j, from $L^p_{l^2(H^j)}(\alpha)$ into $L^p_{l^2(k)}(\beta)$, $1<p<\infty$, $\alpha=\nu^p\beta$, and $\beta\in A^{(\nu)}_p$. Therefore, by the extrapolation theorem for $A^{(\nu)}_p$ (see Theorem (1.6)) we have that

$$\left\| \left(\sum_j \|[U_j, \mathbf{b}]F_j\|^2_{l^2(k)} \right)^{1/2} \right\|_{L^p(\beta)} \le C_p \left\| \left(\sum_j \|F_j\|^2_{l^2(H^j)} \right)^{1/2} \right\|_{L^p(\alpha)}$$

holds for $1<p<\infty$, $\alpha=\nu^p\beta$, $\beta\in A^{(\nu)}_p$; that is

$$\left\| \left(\sum_j \sum_k \left| \sum_{H^j} [\tilde{S}_{H^j}, b_k] f_{H^j} \right|^2 \right)^{1/2} \right\|_{L^p(\beta)} \le C_p \left\| \left(\sum_j \sum_{H^j} |f_{H^j}|^2 \right)^{1/2} \right\|_{L^p(\alpha)}, \tag{4.8}$$

for $1<p<\infty$, $\alpha=\nu^p\beta$ and $\beta\in A_p^{(\nu)}$.

Now if we take $g^k_{H^j}=[S_{H^j},b_k]f$ in (4.7) we get that

$$\mathrm{I}\le\left\|\left(\sum_{j,k}\sum_{H^j}|[S_{H^j},b_k]f|^2\right)^{1/2}\right\|_{L^p(\beta)},$$

and taking $f_{H^j}=S_{H^j}f$ in (4.8), we get that

$$\mathrm{II}\le C_p\left\|\left(\sum_{j}\sum_{H^j}|S_{H^j}f|^2\right)^{1/2}\right\|_{L^p(\alpha)}.$$

Now, the proof of (4.2) continues as the proof of Theorem (3.2).

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