Moment functions on real algebraic sets

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1. Introduction

The well-known Haviland's solution of the multiparameter moment problem for a closed subset V of $\mathbf{R}^d$ states that a function $\varphi\colon \mathbf{N}^d \to \mathbf{R}$ is a moment function on V if and only if an appropriate linear functional Λ_φ on real polynomials in d indeterminates is positive on $C(V)$, the set of all polynomials which are non-negative on V. This characterization of moment functions is not satisfactory, because the cone $C(V)$ could not be described in terms of the pure algebra. In particular for every $d \geqq 2$, $C(\mathbf{R}^d)$ is essentially larger than Σ_d^2, the set of all finite sums of squares of polynomials in d indeterminates (cf. [2], [5] and [13]). On the other hand positiveness of Λ_φ on Σ_d^2 (read: positive definiteness of φ) is not sufficient for φ to be a moment function on $\mathbf{R}^d$ (cf. [1], [2] and [16]). So the question is what else we have to assume to get the solution of the multiparameter moment problem on V. Some answers to the question have been found in the case of V being compact (cf. [3], [4], [6], [19] and [20]).

If V is a (real) algebraic set induced by a polynomial p, then each moment function φ on V satisfies the following condition

(A) $$\Lambda_\varphi|_{(p)} = 0,$$

where (p) is the principal ideal of p. Recently in [18] Szafraniec and the author, inspired by [14], distinguished a few classes of algebraic sets on which moment functions could be completely characterized by positive definiteness and condition (A). In the present paper we show, among other things, that this is not the case for an arbitrary (real) algebraic curve in $\mathbf{R}^2$. Using the terminology from [2] we can say that there are positive definite functions $\varphi\colon \mathbf{N}^2 \to \mathbf{R}$ which are neither strictly positive definite nor moment functions. To achieve this fact, we consider another condition

(B) $$\Lambda_\varphi|_{I(V)} = 0.$$

where $I(V)$ is the ideal of polynomials vanishing on V. It turns out that conditions (A) and (B) are equivalent for all functions $\varphi\colon \mathbf{N}^d \to \mathbf{R}$ if and only if $I(V)=(p)$. Unfortunately we do not know if there exist an algebraic set V of type B, which is not of type A, where "V of type A" (resp. "V of type B") means that all positive definite functions $\varphi\colon \mathbf{N}^d \to \mathbf{R}$ satisfying (A) (resp. (B)) are moment functions on V.

2. Necessary conditions of type A and B

Let $\mathscr{P}_d=\mathbf{R}[X_1, \ldots, X_d]$ be the ring of all polynomials in d indeterminates $X_1, \ldots, X_d$ with real coefficients and let $\mathscr{P}_{d,k}$ be the vector space of all polynomials from $\mathscr{P}_d$ of degree less or equal to k $(k=0, 1, \ldots)$. Σ_d^2 stands for the convex cone of all finite sums of squares of elements of $\mathscr{P}_d$. Denote by $\mathscr{X}^*$ the vector space of all linear functionals on a real vector space $\mathscr{X}$. The space of all real valued functions on $\mathbf{N}^d$ can be identified with $\mathscr{P}_d^*$ via one-to-one linear correspondence $\varphi \leftrightarrow \Lambda_\varphi$ determined by

$$\varphi(i) = \Lambda_\varphi(X_1^{i_1} \ldots X_d^{i_d}), \quad i = (i_1, \ldots, i_d) \in \mathbf{N}^d,$$

where $\mathbf{N}=\{0, 1, \ldots\}$.

Let V be a closed subset of $\mathbf{R}^d$, the real d-dimensional Euclidean space. A function $\varphi\colon \mathbf{N}^d \to \mathbf{R}$ is said to be a *V-moment function* if there is a positive Borel measure μ on $\mathbf{R}^d$ such that

$$\int_V |x|^i \, d\mu(x) < \infty, \quad i \in \mathbf{N}^d$$

and

$$\varphi(i) = \int_V x^i \, d\mu(x), \quad i \in \mathbf{N}^d.$$

Denote by $M(V)$ the class of all V-moment functions. Due to Haviland (cf. [11] and [12]), $\varphi \in M(V)$ if and only if $\Lambda_\varphi(f) \geqq 0$ for every $f \in \mathscr{P}_d$ which is non-negative on V. It is clear that each V-moment function $\varphi\colon \mathbf{N}^d \to \mathbf{R}$ is *positive definite*, i.e. $\Lambda_\varphi(f) \geqq 0$ for every $f \in \Sigma_d^2$. Denoting by $P(\mathbf{N}^d)$ the class of all positive definite functions on $\mathbf{N}^d$, we can write $M(V) \subseteqq M(\mathbf{R}^d) \subseteqq P(\mathbf{N}^d)$ for every $d \geqq 1$. It is well-known that $P(\mathbf{N}) = M(\mathbf{R})$ and that $M(\mathbf{R}^d)$ is a proper subset of $P(\mathbf{N}^d)$ for any $d \geqq 2$ (cf. [2]).

Recall that a subset V of $\mathbf{R}^d$ is said to be *(real) algebraic set*, if there exists $p \in \mathscr{P}_d$ such that $V = V(p) := \{x \in \mathbf{R}^d \colon p(x) = 0\}$. In such a case we say that V is induced by p.

All algebraic sets we consider in this paper are assumed to be properly included in $\mathbf{R}^d$ (the empty set is not excluded).

The ideal of all polynomials from $\mathscr{P}_d$ vanishing on V will be denoted by $I(V)$. As usual (f) stands for the principal ideal generated by f in $\mathscr{P}_d$ (all facts concerning algebraic sets we need in this paper can be found in [5] and [10]).

The following result presents some necessary conditions for $\varphi\colon \mathbf{N}^d \to \mathbf{R}$ to be a moment function on an algebraic set V in $\mathbf{R}^d$.

Proposition 2.1. *Let V be an algebraic set in $\mathbf{R}^d$ induced by $p\in\mathscr{P}_d$. If $\varphi\in M(\mathbf{R}^d)$, then the following conditions are equivalent*

(A) *for every $f\in(p)$, $\Lambda_\varphi(f)=0$,*
(B) *for every $f\in I(V)$, $\Lambda_\varphi(f)=0$.*
(M) *$\varphi\in M(V)$.*

Proof. The implications (M)$\Rightarrow$(B) and (B)$\Rightarrow$(A) are obvious. If $\Lambda_\varphi(f)=0$ for $f\in(p)$, then $\int_{\mathbf{R}^d}|p(x)|^2 d\mu(x)=\Lambda_\varphi(p^2)=0$, which means that the closed support of the measure μ is contained in V.

A question of characterization of algebraic sets for which conditions (A) and (B) are equivalent has a simple algebraic solution which is more carefully investigated in Section 3.

Proposition 2.2. *Let V be an algebraic set in $\mathbf{R}^d$ induced by a polynomial $p\in\mathscr{P}_d$. Then conditions* (A) *and* (B) *are equivalent for all functions $\varphi\colon \mathbf{N}^d\to\mathbf{R}$ if and only if $I(V)=(p)$.*

Proof. Assume that conditions (A) and (B) are equivalent for every function $\varphi\colon \mathbf{N}^d\to\mathbf{R}$. This means that each functional $\Lambda\in\mathscr{P}_d^*$ vanishing on (p) vanishes on $I(V)$. Suppose for a moment that there is $f_0\in I(V)\setminus(p)$. Since (p) is a linear subspace of $\mathscr{P}_d$, there exists a functional $\Lambda\in\mathscr{P}_d^*$ vanishing on (p) and not vanishing at f_0. This leads to contradiction.

Notice that there are infinite bounded algebraic curves V in $\mathbf{R}^2$ induced by polynomials $p\in\mathscr{P}_2$, for which conditions (A) and (B) are equivalent within $P(\mathbf{N}^2)$, though $I(V)\setminus(p)\neq\emptyset$.

Example 2.1. Consider the polynomial $p=(X_1^2+X_2^2)(X_1^2+X_2^2-1)$. Then the algebraic curve $V=V(p)$ in $\mathbf{R}^2$ has the irreducible algebraic component $\{(0,0)\}$ and, by Proposition 3.4, $I(V)\setminus(p)\neq\emptyset$. It follows from Proposition 5.1 that if $\varphi\in P(\mathbf{N}^2)$ satisfies (A), then $\varphi\in M(V)$ and consequently φ satisfies (B).

3. When $I(V)=(p)$?

In this section we present some necessary and sufficient conditions for an algebraic set $V=V(p)$ in $\mathbf{R}^d$ to satisfy equality $I(V)=(p)$. To begin with consider the case of p being irreducible. Notice that then equality $I(V)=(p)$ implies algebraic irreducibility of V (cf. [10], Proposition 1.1) but not conversely.

Theorem 3.1. ([5], Théorème 4.5.1). *If V is an algebraic set in $\mathbf{R}^d$ induced by an irreducible polynomial $p\in\mathcal{P}_d$, then the following conditions are equivalent*

(i) *the ideal (p) is real,*

(ii) $I(V)=(p)$,

(iii) *there are $i\in\{1, ..., d\}$ and $x\in V$ such that* $\frac{\partial p}{\partial X_i}(x)\neq 0$,

(iv) *there are $x, y\in\mathbf{R}^d$ such that $p(x)p(y)<0$,*

(v) $\dim(V(p))=d-1$.

The following proposition enables us to reduce a question of when equality $I(V)=(p)$ holds to lower dimensional case.

Proposition 3.2. *If V is an algebraic set in $\mathbf{R}^d$ $(d\geqq 2)$ induced by an irreducible polynomial $p\in\mathcal{P}_d$, then $I(V)=(p)$ if and only if either $I(V)\cap\mathcal{P}_{d-1}=(0)$ or $p\in\mathcal{P}_{d-1}$ and $I(V_{d-1}(p))=p\cdot\mathcal{P}_{d-1}$, where $V_{d-1}(p):=\{x\in\mathbf{R}^{d-1}\colon p(x)=0\}$.*

Proof. Suppose that $I(V)=(p)$. If there exists a nonzero f in $I(V)\cap\mathcal{P}_{d-1}$, then p divides f and consequently $p\in\mathcal{P}_{d-1}$. Therefore $I(V_{d-1}(p))=p\cdot\mathcal{P}_{d-1}$, which proves the "only if" part of the conclusion.

To prove the "if" part of the conclusion assume that $p\in\mathcal{P}_{d-1}$ and $I(V_{d-1}(p))=p\cdot\mathcal{P}_{d-1}$. Take $f\in I(V)$. Since f can always be written as the sum $\sum_{j=0}^k h_j\cdot X_d^j$ with $h_j\in\mathcal{P}_{d-1}$ and because $V=V_{d-1}(p)\times\mathbf{R}$, we get $h_j\in I(V_{d-1}(p))=p\cdot\mathcal{P}_{d-1}$ for every $j=0, ..., k$. Thus $f\in(p)$.

Suppose now that $I(V)\cap\mathcal{P}_{d-1}=(0)$. Let g be a nonzero polynomial in $I(V)$ such that $\deg_{x_d} g=n:=\min\{\deg_{x_d} h\colon h\in I(V), h\neq 0\}$ ($\deg_{x_d} h$ is the degree of $h\in\mathcal{P}_d$ with respect to the indeterminate X_d). Since $I(V)\cap\mathcal{P}_{d-1}=(0)$, it must be $n\geqq 1$. We show that $\deg_{x_d} p=n$. Applying a division algorithm ([21], Theorem I.17.9) to polynomials p and g in $\mathcal{P}_{d-1}[X_d]$, we get $q, r\in\mathcal{P}_d$ and an integer $k\geqq 1$ such that $\deg_{x_d} r<n$ and $a^k p=qg+r$, where $a\in\mathcal{P}_{d-1}$ is the leading coefficient of g in $\mathcal{P}_{d-1}[X_d]$. This implies that $r\in I(V)$. Since $\deg_{x_d} r<n$, r must be the zero polynomial. Thus $a^k p=qg$. Since p is prime and $\deg_{x_d} a^k=0<\deg_{x_d} g$, p divides g and consequently, $n=\deg_{x_d} p$.

Take now a nonzero f in $I(V)$. It follows from the previous paragraph that $\deg_{x_d} f\geqq n$. Applying again a division algorithm to polynomials f and p in $\mathcal{P}_{d-1}[X_d]$, we get $q, r\in\mathcal{P}_d$ and an integer $k\geqq 1$ such that $\deg_{x_d} r<n$ and $b^k f=qp+r$, where

$b \in \mathcal{P}_{d-1}$ is the leading coefficient of p in $\mathcal{P}_{d-1}[X_d]$. This implies that $r \in I(V)$. Since $\deg_{x_d} r < n$, r must be the zero polynomial. Thus $b^k f = qp$. Since p is prime and $\deg_{x_d} b = 0 < \deg_{x_d} p$, p divides f and consequently $f \in (p)$.

Let V be an algebraic set in $\mathbf{R}^d$ induced by the product $p_1 \dots p_k$ of irreducible polynomials in $\mathcal{P}_d$. We say that the product $p_1 \dots p_k$ is *V-irredundant* if no p_i is superfluous in the representation $V = V(p_1 \dots p_k)$. It is well-known that any algebraic set V in $\mathbf{R}^d$ is induced by some V-irredundant product of polynomials.

Proposition 3.3. *Let V be an algebraic set in $\mathbf{R}^d$ induced by a V-irredundant product $p = p_1 \dots p_k \in \mathcal{P}_d$. Then $I(V) = (p)$ if and only if $I(V(p_j)) = (p_j)$ for every $j = 1, \dots, k$.*

Proof. Only the case $V \neq \emptyset$ needs a justification. The "if" part follows from the fact that $p_1, \dots, p_k$ are pairwise relatively prime.

To prove the "only if" part, suppose that $I(V) = (p)$, $I(V(p_1)) \neq (p_1)$ and $k \geqq 2$. Take f in $I(V(p_1)) \setminus (p_1)$. Then $f p_2 \dots p_k \in I(V) = (p)$. Thus $f p_2 \dots p_k = q p_1 \dots p_k$ with some $q \in \mathcal{P}_d$. Consequently $f \in (p_1)$, which leads to contradiction.

Proposition 3.4. *Let V be an algebraic set in $\mathbf{R}^2$ induced by a V-irredundant product $p = p_1 \dots p_k \in \mathcal{P}_2$ and let $V_1, \dots, V_m$ be the irreducible algebraic components of V. Then the following conditions are equivalent*

(i) $I(V) = (p)$,

(ii) *for every* $j = 1, \dots, k$, $V(p_j)$ *is infinite,*

(iii) *for every* $j = 1, \dots, m$, V_j *is infinite.*

Moreover if $I(V) = (p)$, *then* $m = k$ *and* $\{V_1, \dots, V_m\} = \{V(p_1), \dots, V(p_m)\}$[1].

Proof. Without loss of generality we can assume that $V \neq \emptyset$. Since each V_j is induced by an irreducible polynomial $q_j \in \mathcal{P}_2$, the product $q = q_1 \dots q_m$ is V-irredundant. It follows from Proposition 3.2 (also from Corollaries 1.1 and 1.2 in [10]) that for every irreducible polynomial $r \in \mathcal{P}_2$, $V(r)$ is infinite; if and only if $I(V(r)) = (r)$. Thus (i)$\Leftrightarrow$(ii), due to Proposition 3.3.

If $I(V) = (p)$, then, by Proposition 3.3, $I(V(p_j)) = (p_j)$ for all j. Thus $V(p_1), \dots, V(p_k)$ are infinite irreducible components of V. Consequently $\{V_1, \dots, V_m\} = \{V(p_1), \dots, V(p_m)\}$, which shows (i)$\Rightarrow$(iii).

Conversely if all V_j are infinite, then (again by Proposition 3.3) we have $I(V) = I(V(q)) = (q)$. Since $p \in I(V)$, we conclude that q divides p and, because both products $p_1 \dots p_k$ and $q_1 \dots q_m$ are V-irredundant, $(q) = (p)$. Thus $I(V) = (p)$.

[1] The last-mentioned statement is also true for algebraic sets in $\mathbf{R}^d$ for any $d > 2$.

4. Characterizations of algebraic sets of type A and B

Motivated by Proposition 2.1 we say that an algebraic set V in $\mathbf{R}^d$ induced by a polynomial $p\in\mathscr{P}_d$ is of type A (resp. B) if each positive definite function $\varphi\colon \mathbf{N}^d\to\mathbf{R}$ which satisfies condition (A) (resp. (B)) is a V-moment function. It is obvious that each algebraic set of type A is automatically of type B. However we do not know whether the converse implication is true in general. Also we do not know whether the statement "V is of type A" is independent of the choice of $p\in\mathscr{P}_d$ inducing V.

In this section we present necessary and sufficient conditions of algebraic and topological nature for V to be of type A (resp. B). To begin with fix some notations. Let $\mathscr{F}$ be a set of real valued functions on a given subset V of $\mathbf{R}^d$ and let $\mathscr{X}$ be a linear subspace of $\mathscr{P}_d$. Denote by $\mathscr{F}^+$ the set of all non-negative elements of $\mathscr{F}$ and by $\Pi_{\mathscr{X}}$ the quotient linear mapping from $\mathscr{P}_d$ onto the quotient linear space $\mathscr{P}_d/\mathscr{X}$. Notice that $(\mathscr{P}_d/\mathscr{X})^*$ is linearly isomorphic to the space $\{\Lambda\in\mathscr{P}_d^*\colon \Lambda|_{\mathscr{X}}=0\}$ via one-to-one correspondence $\Xi\leftrightarrow\Lambda$ determined by $\Lambda=\Xi\circ\Pi_{\mathscr{X}}$. Put $\Pi_V:=\Pi_{I(V)}$ and $\mathscr{P}_d|V:=\mathscr{P}_d/I(V)$. Writing $(\mathscr{P}_d|V)^+$, we regard $\mathscr{P}_d|V$ as the ring of restrictions of polynomials to the set V. More generally $(\mathscr{P}_d/\mathscr{X})^+$ is understood as the set of all $\Pi_{\mathscr{X}}(f)$ such that $\Pi_V(f)\in(\mathscr{P}_d|V)^+$. It is clear that $\Pi_{\mathscr{X}}(\Sigma_d^2)\subseteqq\Pi_{\mathscr{X}}(\mathscr{P}_d^+)\subseteqq(\mathscr{P}_d/\mathscr{X})^+$. However the last-mentioned inclusion could not be replaced by equality even for $\mathscr{X}=I(V)$.

Example 4.1. Let $d=2$ and $p=X_2(X_2-X_1^2)$. We show that

$$\Pi_V(X_2)\in(\mathscr{P}_d|V)^+\setminus\Pi_V(\mathscr{P}_d^+).$$

Suppose that there exists $f\in\mathscr{P}_d^+$ such that $\Pi_V(X_2)=\Pi_V(f)$. Since $I(V)=(p)$ (use Proposition 3.4), $f-X_2=h\cdot p$ for some $h\in\mathscr{P}_d$. Thus $x_2(1+(x_2-x_1^2)h(x_1,x_2))\geqq 0$ for $x_1,x_2\in\mathbf{R}$. In particular, substituting x_2 for $-x_1^2$, we get $x_1^2(1-2x_1^2h(x_1,-x_1^2))\leqq 0$ for $x_1\neq 0$. This in turn implies that $1\leqq 2x_1^2h(x_1,-x_1^2)$ for $x_1\neq 0$, which leads to contradiction.

A locally convex topology τ on a vector space $\mathscr{H}$ is said to be *admissible* if all linear functionals on $\mathscr{H}$ are τ-continuous. It is well-known ([15], Theorem 3.12) that if $\mathscr{C}$ is a convex subset of $\mathscr{H}$, then the closure $\overline{\mathscr{C}}$ of $\mathscr{C}$ in any admissible topology on $\mathscr{H}$ is equal to its $\sigma(\mathscr{H},\mathscr{H}^*)$-closure. Notice that for each vector subspace $\mathscr{X}$ of $I(V)$, the convex cone $(\mathscr{P}_d/\mathscr{X})^+$ is closed in any admissible topology on $\mathscr{P}_d/\mathscr{X}$. This follows from the fact that for every $z\in V$, the linear mapping which sends $\Pi_{\mathscr{X}}(f)$ into $f(z)$ is well defined on $\mathscr{P}_d/\mathscr{X}$.

Lemma 4.1. *Let V be an algebraic set in $\mathbf{R}^d$ induced by $p\in\mathscr{P}_d$ and let $\mathscr{X}$ be a vector subspace of $I(V)$ such that $\mathbf{R}p^2\subseteqq\mathscr{X}$. If $\mathscr{C}$ is a convex cone such that $\Pi_{\mathscr{X}}(\Sigma_d^2)\subseteqq\mathscr{C}\subseteqq(\mathscr{P}_d/\mathscr{X})^+$, then $(\mathscr{P}_d/\mathscr{X})^+=\overline{\mathscr{C}}$ if and only if the following condition holds*

(i) *for every functional $\Xi\in(\mathscr{P}_d/\mathscr{X})^*$, which is positive on $\mathscr{C}$, there is a positive Borel* *measure* μ *on* $\mathbf{R}^d$ *such that*

$$\Xi(\Pi_{\mathscr{X}}(f)) = \int f\,d\mu, \quad f\in\mathscr{P}_d. \tag{4.1}$$

Here the closure refers to any admissible topology on $\mathscr{P}_d/\mathscr{X}$.

Proof. To show the "only if" part, assume that $(\mathscr{P}_d/\mathscr{X})^+=\overline{\mathscr{C}}$. Take a functional $\Xi\in(\mathscr{P}_d/\mathscr{X})^*$ which is positive on $\mathscr{C}$. Since Ξ is continuous in any admissible topology on $\mathscr{P}_d/\mathscr{X}$, we get that Ξ is positive on $(\mathscr{P}_d/\mathscr{X})^+$. Thus $\Xi(\Pi_{\mathscr{X}}(f))\geqq 0$ for every $f\in\mathscr{P}_d$ which is non-negative on V. Consequently, by the Haviland criterion, the functional $\Xi\circ\Pi_{\mathscr{X}}$ is represented by a measure μ via (4.1).

To show the "if" part, suppose that (i) holds and there exists $g\in\mathscr{P}_d$ such that $\Pi_{\mathscr{X}}(g)\in(\mathscr{P}_d/\mathscr{X})^+\setminus\overline{\mathscr{C}}$. Then, by the Hahn—Banach theorem ([15], Theorem 3.4), there exists a linear functional $\Xi\in(\mathscr{P}_d/\mathscr{X})^*$ such that

$$\Xi(\Pi_{\mathscr{X}}(g)) < 0 \leqq \Xi(h), \quad h\in\mathscr{C}. \tag{4.2}$$

It follows from (i) that the functional $\Xi\circ\Pi_{\mathscr{X}}$ is represented by a measure μ via (4.1). Since $\Pi_{\mathscr{X}}(p^2)=0$, we get $\int p^2 d\mu=\Xi(\Pi_{\mathscr{X}}(p^2))=0$. This implies that the measure μ is supported by V. On the other hand $\Pi_V(g)\in(\mathscr{P}_d|V)^+$, so $\Xi(\Pi_{\mathscr{X}}(g))=\int_V g\,d\mu\geqq 0$, which contradicts a part of condition (4.2).

Proposition 4.2. *Let V be an algebraic set in $\mathbf{R}^d$ induced by $p\in\mathscr{P}_d$ and let $\mathscr{X}$ be a vector subspace of $I(V)$ such that $\mathbf{R}p^2\subseteqq\mathscr{X}$. Then $\Pi_{\mathscr{X}}(\mathscr{P}_d^+)$ is dense in $(\mathscr{P}_d/\mathscr{X})^+$ in any admissible topology on $\mathscr{P}_d/\mathscr{X}$.*

Proof. It follows from the Haviland criterion on $\mathbf{R}^d$ that condition (i) of Lemma 4.1 is satisfied for $\mathscr{C}:=\Pi_{\mathscr{X}}(\mathscr{P}_d^+)$, which completes the proof.

Now we can characterize algebraic sets of type A and B.

Theorem 4.3. *An algebraic set V in $\mathbf{R}^d$ induced by $p\in\mathscr{P}_d$ is of type* A *(resp.* B*) if and only if one of the following two conditions holds with $\mathscr{X}=(p)$ (resp. $\mathscr{X}=I(V)$)*

(i) $(\mathscr{P}_d/\mathscr{X})^+ = \overline{\Pi_{\mathscr{X}}(\Sigma_d^2)}$,

(ii) $\Pi_{\mathscr{X}}(\mathscr{P}_d^+) \subseteqq \overline{\Pi_{\mathscr{X}}(\Sigma_d^2)}$,

where the closure refers to any admissible topology on $\mathscr{P}_d/\mathscr{X}$.

Proof. It follows from Proposition 4.2 that conditions (i) and (ii) are equivalent. Noting that V is of type A (resp. B) if and only if the convex cone $\mathscr{C}:=\Pi_{(p)}(\Sigma_d^2)$ (resp. $\mathscr{C}:=\Pi_V(\Sigma_d^2)$) satisfies condition (i) of Lemma 4.1, we infer the other part of the conclusion directly from Lemma 4.1.

It is worthwhile to notice that both characterizations of algebraic sets of type A appearing in Theorem 4.3 remain true replacing the vector space $\mathscr{X}=(p)$ by any other one such that $\mathbf{R}p^2\subseteqq\mathscr{X}\subseteqq(p)$.

In all characterizations of algebraic sets of type A and of type B, offered by Theorem 4.3, the knowledge of the closure of the set $\Pi_{\mathscr{X}}(\Sigma_d^2)$ is required. So it is of special interest to know whether the cone $\Pi_{\mathscr{X}}(\Sigma_d^2)$ is closed. This question is especially important for $\mathscr{X}=I(V)$. In Sections 5 and 6 we answer it in the affirmative for some algebraic curves in $\mathbf{R}^2$. Unfortunately we have not been able to answer the question in its full generality.

We finish this section with the following useful observation.

Proposition 4.4. *Let T be an affine isomorphism of $\mathbf{R}^d$ and let $p\in\mathscr{P}_d$. Then $V(p)$ is of type* A (resp. B) *if and only if $V(p\circ T)$ is of type* A *(resp.* B*).*

Proof. The proof follows from the fact that the mapping $\Phi_T\colon \mathscr{P}_d\to\mathscr{P}_d$ defined by $\Phi_T(f)=f\circ T$, $f\in\mathscr{P}_d$, is a ring isomorphism such that $\Phi_T(\Sigma_d^2)=\Sigma_d^2$, $\Phi_T((p))=(p\circ T)$ and $\Phi_T(I(V(p)))=I(V(p\circ T))$.

5. Examples of algebraic sets of type *A*

In this section we present examples of algebraic curves in $\mathbf{R}^2$ of type A. We begin with bounded ones. In [3] Berg and Maserick consider the following question:

(Q) Let $p\in\mathscr{P}_d$ be such that $\{p\geqq 0\}$ is compact. Is it true that $\varphi\colon \mathbf{N}^d\to\mathbf{R}$ is a moment function on $\{p\geqq 0\}$ if and only if φ and $p(E)\varphi$ are positive definite, where $p(E)\varphi(n)=\Lambda_\varphi(p\cdot X^n)$, $n\in\mathbf{N}^d$.

Notice that if the question (Q) is answered in the affirmative, then the set $V(p)$ is of type A. Indeed if φ is a positive definite function satisfying (A), then $p(E)\varphi=0$ is positive definite and, consequently $\varphi\in M(\{p\geqq 0\})$. It follows from Proposition 2.1 that $\varphi\in M(V)$, which means that V is of type A.

In particular using Theorem 5 in [6], which answers the problem (Q) in the affirmative for a class of polynomials, we get

Proposition 5.1. *Let V be an algebraic curve in $\mathbf{R}^2$ induced by $p\in\mathscr{P}_2$. If the homogeneous part of p of the highest degree is strictly negative on $\mathbf{R}^2\setminus\{0\}$, then V is of type* A.

Now we can pass to unbounded algebraic curves in $\mathbf{R}^2$ of parabolic and hyperbolic type. The case of parabolic ones has been treated in another way in [18].

Proposition 5.2. *Let V be an algebraic curve in $\mathbf{R}^2$ induced by a polynomial $p=X_2-q(X_1)$ (resp. $p=X_2q(X_1)-1$), where $q\in\mathscr{P}_1$ (resp. $q\in\mathscr{P}_1\setminus\{0\}$). Then V is of type* A.

Proof. In virtue of Theorem 4.3 it is enough to show that in both cases $I(V)=(p)$ and $(\mathscr{P}_2|V)^+=\Pi_V(\Sigma_2^2)$. The first equality follows from Proposition 3.4 and the fact that polynomials $X_2-q(X_1)$ and $X_2\cdot q(X_1)-1$ are irreducible in $\mathscr{P}_2$. To prove the other one take $g\in(\mathscr{P}_2|V)^+$ of the form $g=\Pi_V(f)$, where $f\in\mathscr{P}_2$. In the case $p=X_2-q(X_1)$, we have $h:=f(X_1, q(X_1))\in\mathscr{P}_1^+$. Consequently, by Lemma 6.2.1 in [2], there are polynomials $h_1, h_2\in\mathscr{P}_1$ such that $h=h_1^2+h_2^2$. This implies that $g=\Pi_V(h_1^2+h_2^2)\in\Pi_V(\Sigma_2^2)$. Consider now the polynomial $p=X_2\cdot q(X_1)-1$. Then one can find $k\in\mathbf{N}$ and a polynomial $r\in\mathscr{P}_1$ such that $r(\lambda)=q(\lambda)^{2k}f(\lambda, q(\lambda)^{-1})$ for $\lambda\in\mathbf{R}\setminus V(q)$. Since $f(\lambda, q(\lambda)^{-1})\geqq 0$ for $\lambda\in\mathbf{R}\setminus V(q)$ and $V(q)$ is finite, we have $r\in\mathscr{P}_1^+$. Thus, again by Lemma 6.2.1 in [2], there are two polynomials $r_1, r_2\in\mathscr{P}_1$ such that $r=r_1^2+r_2^2$. Define new polynomials $h_1, h_2\in\mathscr{P}_2$ by $h_j:=r_jX_2^k$ for $j=1, 2$. Then it is easy to see that $g=\Pi_V(h_1^2+h_2^2)\in\Pi_V(\Sigma_2^2)$.

The union of two arbitrary real lines in $\mathbf{R}^2$ gives another example of an algebraic set of type A.

Proposition 5.3. *Let V be an algebraic curve in $\mathbf{R}^2$ induced by a polynomial $p=p_1p_2$, where p_1 and p_2 are polynomials of degree 1 in $\mathscr{P}_2$. Then V is of type* A.

Proof. Assume that $V(p_1)\neq V(p_2)$. Since $I(V)=(p)$ (use Proposition 3.4), the conclusion of Proposition 5.3 will follow from Theorem 4.3 provided we show that $(\mathscr{P}_2|V)^+=\Pi_V(\Sigma_2^2)$. To prove this take $g\in(\mathscr{P}_2|V)^+$ of the form $g=\Pi_v(f)$, where $f\in\mathscr{P}_2$. First we consider the case when the lines $V(p_1)$ and $V(p_2)$ are parallel. Without loss of generality we may assume that $p_1=X_2$ and $p_2=X_2-1$ (use Proposition 4.4). Since $f(X_1, 0)$ and $f(X_1, 1)$ belong to $\mathscr{P}_1^+$, there are polynomials $r_{01}, r_{02}, r_{11}, r_{12}\in\mathscr{P}_1$ such that $f(X_1, 0)=r_{01}^2+r_{02}^2$ and $f(X_1, 1)=r_{11}^2+r_{12}^2$. Define new polynomials $h_1, h_2\in\mathscr{P}_2$ by $h_j(X_1, X_2)=r_{0j}(X_1)\cdot(1-X_2)+r_{1j}(X_1)\cdot X_2$, $j=1, 2$. Then it is easy to see that $g=\Pi_V(f)=\Pi_V(h_1^2+h_2^2)\in\Pi_V(\Sigma_2^2)$.

Consider now the other case when the lines $V(p_1)$ and $V(p_2)$ are not parallel. Without loss of generality we may assume that $p_1=X_1$ and $p_2=X_2$ (use again Proposition 4.4). Since $f(X_1, 0)\in\mathscr{P}_1^+$ and $f(0, X_1)\in\mathscr{P}_1^+$, there are polynomials $u_1, u_2, w_1, w_2\in\mathscr{P}_1$ of the form $u_j=\sum_{m=0}^n\alpha_{jm}X_1^m$ and $w_j=\sum_{m=0}^n\beta_{jm}X_1^m$ $(j=1, 2)$ such that $f(X_1, 0)=u_1^2+u_2^2$ and $f(0, X_1)=w_1^2+w_2^2$. Define two sequences $\{\xi_m\}_{m=0}^n$ and $\{\zeta_m\}_{m=0}^n$ of vectors in $\mathbf{R}^2$ by $\xi_m=(\alpha_{1m}, \alpha_{2m})$ and $\zeta_m=(\beta_{1m}, \beta_{2m})$. Then $f(X_1, 0)=\sum_{m=0}^{2n}\left(\sum_{k+l=m}\langle\xi_k, \xi_l\rangle\right)X_1^m$ and $f(0, X_1)=\sum_{m=0}^{2n}\left(\sum_{k+l=m}\langle\zeta_k, \zeta_l\rangle\right)X_1^m$, where $\langle\cdot, -\rangle$ stands for the canonical inner product of $\mathbf{R}^2$. Since $f(0, 0)=\langle\xi_0, \xi_0\rangle=\langle\zeta_0, \zeta_0\rangle$, there exists a unitary operator T on $\mathbf{R}^2$ such that $T\zeta_0=\xi_0$. Define polynomials $r_1, r_2\in\mathscr{P}_1$ by $r_j=\sum_{m=0}^n\gamma_{jm}X_1^m$ $(j=1, 2)$, where $T\zeta_m=(\gamma_{1m}, \gamma_{2m})$. Since $\langle\zeta_k, \zeta_l\rangle=\langle T\zeta_k, T\zeta_l\rangle$ for all k, l, we have $f(0, X_1)=\sum_{m=0}^{2n}\left(\sum_{k+l=m}\langle T\zeta_k, T\zeta_l\rangle\right)X_1^m=r_1^2+r_2^2$ and $u_j(0)=r_j(0)$ $(j=1, 2)$. Define new polynomials $h_1, h_2\in\mathscr{P}_2$ by $h_j=u_j(X_1)+r_j(X_2)-u_j(0)$ $(j=1, 2)$. Then one can check that

$$g=\Pi_V(f)=\Pi_V(h_1^2+h_2^2)\in\Pi_V(\Sigma_2^2).$$

Assume now that $V(p_1)=V(p_2)$. Then, again by Proposition 4.4, we may assume (without loss of generality) that $p=X_2^2$. To show that $V(X_2^2)$ is of type A, take $\varphi\in P(\mathbf{N}^2)$ satisfying (A) with $p=X_2^2$. Then, applying the Cauchy—Schwarz inequality to the sesquilinear form $\mathscr{P}_2\times\mathscr{P}_2\ni(f,g)\to\Lambda_\varphi(f\cdot g)\in\mathbf{R}$, we get

$$0\leqq\Lambda_\varphi(f\cdot X_2)^2\leqq\Lambda_\varphi(f^2)\Lambda_\varphi(X_2^2)=0,\quad f\in\mathscr{P}_2.$$

Thus φ satisfies (A) with $p=X_2$. Since the algebraic set $V(X_2)$ is of type A (use Proposition 5.2), we get $\varphi\in M(V)$.

In Section 6 we show that there exist a polynomial $p\in\mathscr{P}_2$ of degree 3 such that $V(p)$ is not of type A. Below we prove that each algebraic curve in $\mathbf{R}^2$ induced by a polynomial of degree less than or equal to 2 is of type A.

Theorem 5.4. *Each algebraic curve V in $\mathbf{R}^2$ induced by a polynomial $p\in\mathscr{P}_2$ of degree less than or equal to 2 is of type* A.

Proof. Basing on the affine classification of algebraic plane curves of order 2 and using Proposition 4.4 one can show that it is enough to consider the following eight cases: (i) $p=X_1^2+X_2^2-1$ (a circle), (ii) $p=X_1X_2-1$ (a hyperbola), (iii) $p=X_2-X_1^2$ (a parabola), (iv) $p=p_1p_2$, where p_1 and p_2 are polynomials of degree 1 and $V(p_1)\neq V(p_2)$ (a sum of two distinct real lines), (v) $p=X_2^2$ (a real line), (vi) $p=X_2$ (a real line), (vii) $p=X_1^2+X_2^2$ (a one point set) and (viii) $p=1+\alpha_1X_1^2+\alpha_2X_2^2$, where $\alpha_1,\alpha_2\geqq 0$ (the empty set). It follows from Propositions 5.1, 5.2 and 5.3 that $V(p)$ is of type A in either of the first seven cases.

Assume now that p is as in (viii) and take $\varphi\in P(\mathbf{N}^2)$ satisfying (A). Since φ is positive definite we have

$$0\leqq\Lambda_\varphi(1)\leqq\Lambda_\varphi(1+\alpha_1X_1^2+\alpha_2X_2^2)=0.$$

This and the Cauchy—Schwarz inequality imply that

$$0\leqq\Lambda_\varphi(f)^2\leqq\Lambda_\varphi(f^2)\Lambda_\varphi(1)=0,\quad f\in\mathscr{P}_2.$$

Consequently $\varphi=0\in M(V)$, which completes the proof.

6. Algebraic curves which are not of type *B*

This section deals with the following question.

Do there exist algebraic sets in $\mathbf{R}^d$ which are not of type B?

It turns out that the answer is in the negative for $d=1$ and in the affirmative for every $d\geqq 2$. More precisely, each algebraic set in $\mathbf{R}$ is of type A (this follows from equality $P(\mathbf{N})=M(\mathbf{R})$ and Proposition 2.1). On the other hand the algebraic set V in $\mathbf{R}^3$ induced by a polynomial $p=X_3$ is not of type B. Indeed if we take any function $\psi\in P(\mathbf{N}^2)\setminus M(\mathbf{R}^2)$ (cf. [1], [16], [2] and [8]), then the function $\varphi\colon \mathbf{N}^3\to\mathbf{R}$ defined by

$$\varphi(i,j,k)=\begin{cases}0 & \text{if } i,j\in\mathbf{N} \text{ and } k>0\\ \psi(i,j) & \text{if } i,j\in\mathbf{N} \text{ and } k=0\end{cases}$$

belongs to $P(\mathbf{N}^3)\setminus M(V)$ and has property (B).

The case $d=2$ needs more attention. We start with some indispensable lemmas.

Lemma 6.1. *Let V be an algebraic curve in $\mathbf{R}^2$ induced by a polynomial $p\in\mathscr{P}_2$ of the form $p=X_2\cdot q$, where $q\in\mathscr{P}_2$. Assume that $(0,0)\in V(q)$, $\Pi_{V(q)}(X_2)\in(\mathscr{P}_2|V(q))^+$, $I(V(q))=(q)$ and polynomials X_2 and q are relatively prime. Then $I(V)=(p)$ and $\Pi_V(X_2)\in(\mathscr{P}_2|V)^+\setminus\Pi_V(\Sigma_2^2)$.*

Proof. Since $I(V(q))=(q)$ and polynomials X_2 and q are relatively prime, we have $I(V)=(p)$. Suppose that there exist polynomials $f_1,\dots,f_n\in\mathscr{P}_2$ such that $\Pi_V(X_2)=\Pi_V\left(\sum_{j=1}^n f_j^2\right)$. Since $I(V)=(p)$, there exists a polynomial $h\in\mathscr{P}_2$ such that

$$\sum_{j=1}^n f_j^2-X_2=X_2\cdot q\cdot h. \tag{6.1}$$

This implies that $\sum_{j=1}^n f_j(X_1,0)^2=0$. Thus $f_j(X_1,0)=0$ for all j and consequently, there are polynomials $g_1,\dots,g_n\in\mathscr{P}_2$ such that $f_j=X_2\cdot g_j$ for all j. It follows from (6.1) that $X_2\left(\sum_{j=1}^n g_j^2\right)-1=q\cdot h$, which contradicts our assumption $q(0,0)=0$.

To show that $\Pi_V(X_2)\in(\mathscr{P}_2|V)^+$ take $(x_1,x_2)\in V=V(X_2\cdot q)$. Then either $x_2=0$ or $(x_1,x_2)\in V(q)$. The latter implies that $x_2\geqq 0$, because $\Pi_{V(q)}(X_2)\in(\mathscr{P}_2|V(q))^+$. This completes the proof.

In the next lemma we show that $\Pi_V(\Sigma_2^2)$ is closed for some algebraic curves in $\mathbf{R}^2$.

Lemma 6.2. *Let V be an algebraic curve in $\mathbf{R}^2$ induced by a polynomial $p\in\mathscr{P}_2$ of the form $p=X_2(X_2-r)$, where $r\in\mathscr{P}_1$. Then $\Pi_V(\Sigma_2^2)$ is closed in the finest locally convex topology on $\mathscr{P}_2|V$.*

Proof. (In the proof we use some arguments from [1] and [2].) Since for $r\in\mathbf{R}$, $(\mathscr{P}_2|V)^+=\Pi_V(\Sigma_2^2)$ (see proofs of Propositions 5.2 and 5.3), we may assume that r is nonconstant. Set $\mathscr{X}_k:=\Pi_V(\mathscr{P}_{2,k})$. Notice that $\dim\mathscr{X}_k<\infty$ and $\mathscr{P}_2|V=\bigcup_{k=0}^\infty\mathscr{X}_k$. Thus, in virtue of Lemma 6.3.3 in [2], we have only to prove that for every $k\in\mathbf{N}$, $\mathscr{X}_k\cap\Pi_V(\Sigma_2^2)$ is closed in the canonical topology on $\mathscr{X}_k$. We split the proof of this fact into a few parts.

Step 1. For each $g\in\Pi_V(\Sigma_2^2)$, there are polynomials $a_j,b_j\in\mathscr{P}_1$, $j=1,\dots,n$, such that $g=\sum_{j=1}^n\Pi_V(a_jX_2+b_j)^2$.

Take $f\in\mathcal{P}_2$. Applying division algorithm to polynomials f and p in $\mathcal{P}_1[X_2]$, we get $a, b\in\mathcal{P}_1$ and $h\in\mathcal{P}_2$ such that $f=hp+aX_2+b$. Thus $\Pi_V(f)=\Pi_V(aX_2+b)$, which implies conclusion of Step 1.

Step 2. If $g=\sum_{j=1}^{n}\Pi_V(a_jX_2+b_j)^2\in\mathcal{X}_k$, *where* $a_j, b_j\in\mathcal{P}_1$, *then*

$$\max\{\deg(a_j), \deg(b_j)\} \leqq 2^{-1}k\cdot\deg(r),$$

where $\deg(a)$ *stands for the degree of* $a\in\mathcal{P}_1$.

Let $g=\Pi_V(f)$ with some $f\in\mathcal{P}_{2,k}$. Then we have

$$\deg(b_m) \leqq 2^{-1}\deg\left(\sum_{j=1}^{n}b_j^2\right) = 2^{-1}\deg\left(f(X_1, 0)\right) \leqq 2^{-1}k. \tag{6.2}$$

To prove the other part of conclusion of Step 2, notice that

$$\deg(a_mr+b_m) \leqq 2^{-1}\deg\left(\sum_{j=1}^{n}(a_jr+b_j)^2\right) \tag{6.3}$$
$$= 2^{-1}\deg\left(f(X_1, r(X_1))\right) \leqq 2^{-1}k\cdot\deg(r), \quad m = 1, \ldots, n.$$

If $\deg(a_mr)\leqq\deg(b_m)$, then, by (6.2), we have

$$\deg(a_m) \leqq \deg(b_m) \leqq 2^{-1}k\cdot\deg(r).$$

If $\deg(a_mr)>\deg(b_m)$, then (6.3) implies

$$\deg(a_m) \leqq \deg(a_mr) = \deg(a_mr+b_m) \leqq 2^{-1}k\cdot\deg(r), \quad m = 1, \ldots, n,$$

which finishes the proof of Step 2.

Step 3. Let $\iota:=k\cdot\deg(r)+2$ *and* $\varkappa:=\dim(\mathcal{X}_\iota)$. *If* $g\in\mathcal{X}_k\cap\Pi_V(\Sigma_2^2)$, *then there exist* $a_j, b_j\in\mathcal{P}_1$ *such that* $g=\sum_{j=1}^{\varkappa}\Pi_V(a_jX_2+b_j)^2$.

It follows from Step 1 that the function g can be written as the sum $\sum_{j=1}^{n}\Pi_V(a_jX_2+b_j)^2$ with the smallest number of summands. Step 2 implies that $\Pi_V(a_jX_2+b_j)^2\in\mathcal{X}_\iota$ for $j=1, \ldots, n$. Suppose that $n>\dim(\mathcal{X}_\iota)$. Then there are $\alpha_1, \ldots, \alpha_n\in\mathbf{R}$ such that $\sum_{j=1}^{n}\alpha_j\Pi_V(a_jX_2+b_j)^2=0$. Rearranging, if necessary, we may assume that $|\alpha_1|\leqq|\alpha_2|\leqq\ldots\leqq|\alpha_n|$. Thus $g=\sum_{j=1}^{n-1}\Pi_V(\tilde{a}_jX_2+\tilde{b}_j)^2$, where $\tilde{a}_j=a_j\sqrt{1-\alpha_j/\alpha_n}$ and $\tilde{b}_j=b_j\sqrt{1-\alpha_j/\alpha_n}$. This contradicts minimality of n.

Step 4. Let $\omega:=\deg(r)\iota$ *and let* $\lambda_1, \ldots, \lambda_{\omega+1}$ *be distinct real numbers. Then the topology* τ_Q *on* $\mathcal{X}_m$ *of pointwise convergence on the set* $Q=\{(\lambda_1, 0), \ldots, (\lambda_{\omega+1}, 0)\}\cup\{(\lambda_1, r(\lambda_1)), \ldots, (\lambda_{\omega+1}, r(\lambda_{\omega+1}))\}$ *coincides with the canonical topology on* $\mathcal{X}_m$ *for every* $m\leqq\iota$.

It is enough to show that τ_Q is a Hausdorff topology on $\mathcal{X}_m$. Take $f\in\mathcal{P}_{2,m}$ such that $f(x)=0$ for each $x\in Q$. Then $f(\lambda_j, 0)=0$ and $f(\lambda_j, r(\lambda_j))=0$ for every $j=1, \ldots, \omega+1$. Since $\deg\left(f(X_1, 0)\right)\leqq m\leqq\omega$ and $\deg\left(f(X_1, r(X_1))\right)\leqq m\cdot\deg(r)\leqq\omega$

(compare with (6.3)), we get $f(X_1, 0)=0$ and $f(X_1, r(X_1))=0$, which precisely means that $\Pi_V(f)=0$.

Now we show that $\mathscr{X}_k \cap \Pi_V(\Sigma_2^2)$ is closed in the canonical topology on $\mathscr{X}_k$. Take a sequence $\{g_n\}_{n=1}^{\infty} \subseteq \mathscr{X}_k \cap \Pi_V(\Sigma_2^2)$ converging in $\mathscr{X}_k$ to $h \in \mathscr{X}_k$. By Step 3, each g_n can be written in the form $g_n = \sum_{j=1}^{\varkappa} g_{jn}^2$, where $g_{jn} = \Pi_V(a_{jn} X_2 + b_{jn})$ and $a_{jn}, b_{jn} \in \mathscr{P}_1$. Let Q be as in Step 4. Then the sequence $\{g_n(x)\}_{n=1}^{\infty}$ is bounded for every $x \in Q$. Consequently, the sequence $\{g_{jn}(x)\}_{n=1}^{\infty}$ is bounded for all $j=1, \ldots, \varkappa$ and $x \in Q$. Passing to subsequences, if necessary, we may assume that $\{g_{jn}(x)\}_{n=1}^{\infty}$ is convergent in $\mathbf{R}$ for all j and $x \in Q$. Since, by Steps 2 and 4, each $\{g_{jn}\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete space $\mathscr{X}_{[\iota/2]}$, there exist functions $h_1, \ldots, h_{\varkappa} \in \mathscr{X}_{[\iota/2]}$ such that each $\{g_{jn}^2\}_{n=1}^{\infty}$ converges pointwise on V to h_j^2. Thus the sequence $\{g_n\}_{n=1}^{\infty}$ converges pointwise on V to $\sum_{j=1}^{\varkappa} h_j^2$. Consequently, $h = \sum_{j=1}^{\varkappa} h_j^2 \in \Pi_V(\Sigma_2^2)$. This completes the proof.

The following theorem, which is a simple consequence of Lemma 6.1, Lemma 6.2 and Theorem 4.3, shows that the sum of a real line and a parabolic curve in $\mathbf{R}^2$ is not of type B. Thus unions of algebraic curves of type B need not be of type B.

Theorem 6.3. *Let V be an algebraic curve in $\mathbf{R}^2$ induced by a polynomial $p \in \mathscr{P}_2$ of the form $p = X_2(X_2 - r)$, where $r \in \mathscr{P}_1$. If $r \in \mathscr{P}_1^+$, $r \neq 0$ and $V(r) \neq \emptyset$, then V is not of type* B.

Notice that none of the assumptions we have imposed on the polynomial r in Theorem 6.3 could be omitted. Appropriate examples can be obtained with help of Proposition 5.3.

Some arguments of this section can be used to obtain similar results for some other algebraic plane curves. Below we state only the simplest one.

Theorem 6.4. *Let V be an algebraic curve in $\mathbf{R}^2$ induced by a polynomial $p \in \mathscr{P}_2$ of the form $p = X_2(X_2 - r)(X_2 - s)$, where $r, s \in \mathscr{P}_1$. If $r, s \in \mathscr{P}_1^+$, $rs \neq 0$,* $\deg(r) \neq \deg(s)$ *and $V(rs) \neq \emptyset$, then V is not of type* B.

7. Applications

To begin with, consider one parameter complex moment problem. The general observation is that any answer to 2-parameter real moment problem has an implication in solving one parameter complex moment problem. In particular Proposition 5.2 and Theorem 5.4 answer in the affirmative the complex moment problem studied in [18] for algebraic curves in $\mathbf{R}^2$ of parabolic and hyperbolic type as well as for those of order 2. On the other hand Theorem 6.3 leads to the negative answer to the problem for an unbounded algebraic curve in $\mathbf{R}^2$.

Proposition 7.1. *There exists a function* $\psi\colon \mathbf{N}^2 \to \mathbf{C}$ *satisfying the following three conditions*

(i) ψ *is not a complex moment function,*

(ii) *for any function* $\varrho\colon \mathbf{N}^2 \to \mathbf{C}$ *of finite support*

$$\sum_{i,j}\sum_{k,l}\psi(i+l, j+k)\varrho(i,j)\overline{\varrho(k,l)} \geqq 0,$$

(iii) *for all* $k, l \in \mathbf{N}$

$$\psi(3+k, l)+\psi(2+k, 1+l)+2i\psi(2+k, l)+2i\psi(k, 2+l)$$
$$= \psi(1+k, 2+l)+\psi(k, 3+l)+4i\psi(1+k, 1+l).$$

Proof. Let $p = X_2(X_2 - X_1^2)$. It follows from Theorem 6.3 that there exists $\varphi \in P(\mathbf{N}^2) \setminus M(\mathbf{R}^2)$ such that $\Lambda_\varphi|_{(p)} = 0$. Define a function $\psi\colon \mathbf{N}^2 \to \mathbf{C}$ by

$$\psi(m, n) = \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l} i^{m-k}(-i)^{n-l}\varphi(k+l, m+n-(k+l)), \quad m, n \in \mathbf{N}.$$

Then ψ satisfies conditions (i), (ii) and (iii).

It is well-known (cf. [9]) that operator theory could be applied to solve moment problems. Below the reverse influence is indicated.

Proposition 7.2. *Let* $\mathscr{H}$ *be a complex Hilbert space,* ξ — *a vector in* $\mathscr{H}$ *and* (A, B) — *a pair of (algebraically) commuting symmetric operators in* $\mathscr{H}$ *with common invariant domain* $\mathscr{D} = \{f(A, B)\xi\colon f \in \mathbf{C}[X_1, X_2]\}$. *Suppose that either of the following two conditions holds*

(i) $q(A)B$ *is the identity operator on* $\mathscr{D}$ *for some* $q \in \mathscr{P}_1$,

(ii) $p(A, B) = 0$ *for some* $p \in \mathscr{P}_2 \setminus \{0\}$ *of degree less than or equal to* 2.

Then the operator $A+iB$ *is subnormal i.e. it has a normal extension in some Hilbert space* $\mathscr{K} \supseteqq \mathscr{H}$.

Proof. Suppose that (i) holds. Define a function $\varphi\colon \mathbf{N}^2 \to \mathbf{R}$ by $\varphi(j, k) = \langle A^j B^k \xi, \xi\rangle$ for $j, k \in \mathbf{N}$. Then $\varphi \in P(\mathbf{N}^2)$ and $\Lambda_\varphi|_{(p)} = 0$, where $p = X_2 \cdot q(X_1) - 1$. Thus, by Proposition 5.2, φ is a moment function on $V(p)$. It follows from Theorem 5 in [9] that there is a pair $(\tilde{A}, \tilde{B})$ of commuting self-adjoint operators in some Hilbert space $\mathscr{K} \supseteqq \mathscr{H}$ such that $\tilde{A}$ and $\tilde{B}$ extend A and B, respectively. Therefore the operator $\tilde{A}+i\tilde{B}$ is normal and extends $A+iB$. Similarly, using Theorem 5.4, we can prove that (ii) implies subnormality of $A+iB$.

Contrary to Proposition 5.2 and Theorem 5.4, Theorem 6.3 leads to examples of formally normal operators with pathological properties. A sample of such an operator is contained in the following proposition (compare with Proposition 7.1).

Proposition 7.3. *There exists a formally normal operator S in a complex Hilbert space $\mathcal{H}$ such that*

(i) *the domain $\mathcal{D}(S)$ of S is invariant for S and S^*,*

(ii) $\mathcal{D}(S)=\{f(S, S^*)\xi: f\in \mathbf{C}[X_1, X_2]\}$ *for some* $\xi\in\mathcal{D}(S)$,

(iii) $S^3+S^2S^*+2iS^2+2iS^{*2}\subseteq SS^{*2}+S^{*3}+4iSS^*$

and

(iv) *S is not subnormal.*

Notice that the well-known examples of formally normal operators S fulfilling conditions (i) and (iv) of Proposition 7.3 (cf. [7] and [17]) do not satisfy condition (iii). It seems that they do not satisfy the equation $f(S, S^*)=0$ for any nonzero f in $\mathbf{C}[X_1, X_2]$.

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* The problem (Q) has been solved by K. Schmüdgen in [22]. In particular his result implies that each compact algebraic set in $\mathbf{R}^d$ is of type A (compare with Proposition 5.1).