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On the uniform convexity of L^p and l^p

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CLARKSON defined in 1936 the uniformly convex spaces [2]. The uniform convexity asserts that there is a function $\delta(\varepsilon)$ of $\varepsilon>0$ such that $\|x\|=1$, $\|y\|=1$, and $\|x-y\|\geqq\varepsilon$ imply $\|\frac{1}{2}(x+y)\|\leqq 1-\delta(\varepsilon)$, where x and y are elements of the space. CLARKSON proved that the well-known spaces L^p and l^p are uniformly convex if $p>1$. The purpose of this note is to give the best possible function $\delta(\varepsilon)$ for these spaces, i.e. to find for each $p>1$ and $\varepsilon>0$

$$\inf\left(1-\left\|\frac{x+y}{2}\right\|\right)$$

under the conditions $\|x\|=1$, $\|y\|=1$, $\|x-y\|\geqq\varepsilon$. We need two inequalities, which are given in Theorem 1, formula (1). I have been informed that the left-hand side inequality of this formula was proved by BEURLING at a seminar in Uppsala in 1945, but it does not seem to be in print. The right-hand side inequality is proved by CLARKSON ([2] p. 400) and BOAS ([1] p. 305). We give here a reconstruction of BEURLING's proof and also for completeness a simple proof of the other inequality.

Let the functions in L^p be defined over $0\leqq t\leqq 1$. The norm of $x=x(t)$ is then given by

$$\|x\|^p=\int_0^1 |x(t)|^p\,dt.$$

In l^p the norm of $x=(x_1, x_2, \ldots)$ is given by

$$\|x\|^p=\sum_{i=1}^{\infty}|x_i|^p.$$

Theorem 1. *For $p>2$ the following inequalities hold*

$$(\|x\|+\|y\|)^p+|\,\|x\|-\|y\|\,|^p\geqq\|x+y\|^p+\|x-y\|^p\geqq 2\|x\|^p+2\|y\|^p. \qquad (1)$$

For $1<p<2$ these inequalities hold in the reverse sense.

The equality sign holds for L^p [for l^p] in the left-hand side of (1) if and only if $x=0$, or $y=0$, or there is a number $a>0$ such that $(x(t)-a\,y(t))\,(x(t)+a\,y(t))=0$ for almost every t [such that $(x_i-a\,y_i)\,(x_i+a\,y_i)=0$ for every i], and in the right-hand side of (1) if and only if $x(t)\,y(t)=0$ for almost every t [$x_i\,y_i=0$ for every i].

It is easy to show that for given $\|x\|$ and $\|y\|$ each of these conditions for equality can be satisfied by suitable x and y. Hence the inequalities in Theorem 1 give the maximum and the minimum of $\|x+y\|^p+\|x-y\|^p$ for fixed $\|x\|$ and $\|y\|$.

Remark. For $p=2$ the three terms in (1) are equal for any x and y. This is the relationship

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$$

well known in the theory of Hilbert spaces.

Proof. A. The left-hand side of (1). Let $1<p<2$ and consider L^p. We have to prove that

$$\int_0^1 |x(t)+y(t)|^p+|x(t)-y(t)|^p\,dt \geqq (\|x\|+\|y\|)^p+|\,\|x\|-\|y\|\,|^p. \tag{2}$$

Let us first show that it is sufficient to prove (2) for non-negative functions. Consider

$$d=|z_1+z_2|^p+|z_1-z_2|^p, \tag{3}$$

where z_1 and z_2 are complex numbers. Let $|z_1|$ and $|z_2|$ be fixed and let us calculate the minimum of d. If $z_1=0$ this minimum is $2|z_2|^p$ and if $z_2=0$ this minimum is $2|z_1|^p$. Take $|z_1|=a>0$ and $z_2=z_1a^{-1}b\,e^{i\varphi}$, $b>0$. Then

$$d(\varphi)=|a+b\,e^{i\varphi}|^p+|a-b\,e^{i\varphi}|^p=(a^2+b^2+2ab\cos\varphi)^{\frac{p}{2}}+(a^2+b^2-2ab\cos\varphi)^{\frac{p}{2}}.$$

The minimum of $d(\varphi)$ is $(a+b)^p+|a-b|^p$ and is reached for $\varphi=0,\pi$. Thus

$$|z_1+z_2|^p+|z_1-z_2|^p \geqq (|z_1|+|z_2|)^p+|\,|z_1|-|z_2|\,|^p, \tag{4}$$

where equality holds if and only if z_1 and z_2 have a real quotient or one of them is zero. Let $x^*(t)=|x(t)|$ and $y^*(t)=|y(t)|$. Put $z_1=x(t)$ and $z_2=y(t)$ in (4) and integrate.

$$\int_0^1 |x(t)+y(t)|^p+|x(t)-y(t)|^p\,dt \geqq \int_0^1 |x^*(t)+y^*(t)|^p+|x^*(t)-y^*(t)|^p\,dt. \tag{5}$$

Here equality holds if and only if for almost every t such that $x(t)\neq 0$ and $y(t)\neq 0$, the quotient of $x(t)$ and $y(t)$ is real. Because of (5), since $\|x\|=\|x^*\|$ and $\|y\|=\|y^*\|$, we only have to prove (2) for the non-negative functions $x^*(t)$ and $y^*(t)$.

Now introduce

$$\zeta(u,v)=(u^{\frac{1}{p}}+v^{\frac{1}{p}})^p+|u^{\frac{1}{p}}-v^{\frac{1}{p}}|^p, \qquad u\geqq 0,\ v\geqq 0,$$

and let $f(t)=(x^*(t))^p$ and $g(t)=(y^*(t))^p$. Then (2) may be written

$$\int_0^1 \zeta(f(t),\, g(t))\, dt \geqq \zeta\left(\int_0^1 f(t)\, dt,\ \int_0^1 g(t)\, dt\right). \tag{6}$$

We shall show below that ζ is convex. (6) is an immediate consequence of this fact. For the three integrals in (6) are the w-, u-, and v-coordinates of the center of gravity for the distribution of mass given by $u=f(t)$, $v=g(t)$, $0\leqq t\leqq 1$ on the surface $w=\zeta(u, v)$. Hence we only have to prove the convexity of ζ. We have

(a) $\zeta(u, v)=\zeta(v, u)$,

(b) $\zeta(0, 0)=0$,

(c) $\zeta(tu, tv)=t\,\zeta(u, v)$ for $t\geqq 0$.

Thus $w=\zeta(u, v)$ is a cone with its center in the origin. The convexity of $w=\zeta(u, v)$ will therefore follow from the convexity of $w=\zeta(u, 1)$. But for

$$\eta(u)=\zeta(u, 1)=(1+u^{\frac{1}{p}})^p+|1-u^{\frac{1}{p}}|^p$$

the second derivative is

$$\eta''(u)=\frac{p-1}{p}\, u^{\frac{1}{p}-2}\left(|1-u^{\frac{1}{p}}|^{p-2}-|1+u^{\frac{1}{p}}|^{p-2}\right),$$

which is strictly positive for every $u>0$. For $u=1$ we have $\eta''=\infty$, but η' is continuous. Thus $\eta(u)$, and therefore also $\zeta(u, v)$, are convex. This proves (2).

In order to get equality in (2) we must have equality in both (5) and (6). Since η'' is never 0, equality in (6) holds if and only if the point $(f(t), g(t))$ for almost every t lies on one single line through the origin in the uv-plane, i.e. $f(t)=0$ for almost every t, $g(t)=0$ for almost every t, or there is a positive number, say a^p, such that for almost every t we have $f(t)=a^p g(t)$, i.e. $x^*(t)=$ $=a\,y^*(t)$. Combined with the condition for equality in (5) this gives the condition in Theorem 1.

The case $p>2$ is proved similarly. For these p-values $d(\varphi)$ reaches its maximum for $\varphi=0, \pi$ and ζ is concave.

The proof for l^p is analogous.

B. The right-hand side of (1). Let $p>2$ and consider L^p. We have to prove that

$$\int_0^1 |x(t)+y(t)|^p+|x(t)-y(t)|^p\, dt \geqq \int_0^1 2\,|x(t)|^p+2\,|y(t)|^p\, dt. \tag{7}$$

Introduce as before d by (3). Then the minimum of $d(\varphi)$ is $2(a^2+b^2)^{\frac{p}{2}}$ and is reached for $\varphi=\pm\frac{\pi}{2}$. But, since $t^{\frac{p}{2}}$ is a convex function $(a>0,\ b>0)$,

$$(a^2+b^2)^{\frac{p}{2}}>a^p+b^p.$$

Hence

$$|z_1+z_2|^p+|z_1-z_2|^p \geqq 2|z_1|^p+2|z_2|^p, \tag{8}$$

where equality holds if and only if at least one of z_1 and z_2 is 0. Put in (8) $z_1=x(t)$ and $z_2=y(t)$ and integrate. This proves (7). It also shows, that equality in (7) holds if and only if $f(t)\,g(t)=0$ for almost every t.

The remaining cases are proved similarly.

Theorem 2. *Let x and y be two elements of L^p or of l^p. Suppose that*

$$\|x\|=1, \qquad \|y\|=1, \qquad \|x-y\|\geqq \varepsilon,$$

where $0<\varepsilon<2$. *Then*

$$\left\|\frac{x+y}{2}\right\| \leqq 1-\delta(\varepsilon), \tag{9}$$

where $\delta=\delta(\varepsilon)$ is determined in the following way:

when $1<p<2$: $\qquad \left(1-\delta+\dfrac{\varepsilon}{2}\right)^p+\left|1-\delta-\dfrac{\varepsilon}{2}\right|^p=2,$

when $p\geqq 2$: $\qquad \delta=1-\left(1-\left(\dfrac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$

For each ε, we can chose x and y such that equality holds in (9).

Proof. Put $x^*=\frac{1}{2}(x+y)$ and $y^*=\frac{1}{2}(x-y)$. Thus

$$x=x^*+y^* \text{ and } y=x^*-y^*. \tag{10}$$

A. $1<p<2$. Let

$$\xi(u,v)=(u+v)^p+|u-v|^p$$

for $u\geqq 0$, $v\geqq 0$. Then ξ is symmetric in the variables u and v, and if one of these remains fixed, ξ is strictly increasing in the other one. The left-hand side inequality of Theorem 1 may be written

$$\|x+y\|^p+\|x-y\|^p\geqq \xi(\|x\|,\|y\|). \tag{11}$$

Apply this formula on x^* and y^*. Then

$$2\geqq \xi(\|x^*\|,\|y^*\|)\geqq \xi\left(\|x^*\|,\frac{\varepsilon}{2}\right). \tag{12}$$

Since $\xi(1,0)=2$, we get $\xi\left(1,\dfrac{\varepsilon}{2}\right)>2$ and $\xi\left(0,\dfrac{\varepsilon}{2}\right)<2$. Thus there is a positive uniquely determined solution δ of

$$\xi\left(1-\delta,\frac{\varepsilon}{2}\right)=2.$$

Hence, because of (12),

$$\|x^*\| \leqq 1-\delta.$$

The last two formulas prove formula (9) for $1<p<2$.

To get equality in (9) we may take in L^p

$$x^*(t)=1-\delta \quad \text{for} \quad 0 \leqq t \leqq 1,$$

$$y^*(t)=\frac{\varepsilon}{2} \quad \text{for} \quad 0 \leqq t \leqq \tfrac{1}{2},$$

$$=-\frac{\varepsilon}{2} \quad \text{for} \quad \tfrac{1}{2}<t \leqq 1.$$

Then $\|x^*\|=1-\delta$, $\|y^*\|=\frac{\varepsilon}{2}$. Let x and y be defined by (10). Hence $\|x\|=\|y\|$. By Theorem 1 (or by simple calculation) we have equality in (11) for these x^* and y^*, and we get

$$\|x\|^p=\|y\|^p=\tfrac{1}{2}\xi\,(\|x^*\|,\ \|y^*\|)=\tfrac{1}{2}\xi\left(1-\delta,\ \frac{\varepsilon}{2}\right)=1.$$

In l^p we may take

$$x^*=2^{-\frac{1}{p}}(1-\delta,\ 1-\delta,\ 0,\ 0,\ 0,\ \ldots),$$

$$y^*=2^{-\frac{1}{p}}\left(\ \frac{\varepsilon}{2},\ -\frac{\varepsilon}{2},\ 0,\ 0,\ 0,\ \ldots\right).$$

B. $p \geqq 2$. We have by Theorem 1 (and the remark following the theorem)

$$\|x^*+y^*\|^p+\|x^*-y^*\|^p \geqq 2\|x^*\|^p+2\|y^*\|^p. \tag{13}$$

Hence

$$2 \geqq 2\|x^*\|^p+2\|y^*\|^p \geqq 2\|x^*\|^p+2\left(\frac{\varepsilon}{2}\right)^p,$$

$$\|x^*\|^p \leqq 1-\left(\frac{\varepsilon}{2}\right)^p.$$

Put

$$\delta=1-\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

Then

$$\|x^*\|^p \leqq (1-\delta)^p,$$

which implies (9).

To get equality in (9) we may take in L^p

$$x^*(t)=2^{\frac{1}{p}}(1-\delta) \quad \text{for} \quad 0 \leqq t \leqq \tfrac{1}{2},$$

$$=0 \quad \text{for} \quad \tfrac{1}{2}<t \leqq 1,$$

$$y^*(t) = 0 \qquad \text{for } 0 \leq t \leq \tfrac{1}{2},$$
$$= 2^{\frac{1}{p}} \frac{\varepsilon}{2} \qquad \text{for } \tfrac{1}{2} < t \leq 1.$$

Then $\|x^*\| = 1 - \delta$, $\|y^*\| = \frac{\varepsilon}{2}$. Let x and y be defined by (10). Hence $\|x\| = \|y\|$. By Theorem 1 (or by simple calculation) we have equality in (13) for these x^* and y^*. Thus

$$\|x\|^p = \|y\|^p = \tfrac{1}{2}\left(2(1-\delta)^p + 2\left(\frac{\varepsilon}{2}\right)^p\right) = 1.$$

In l^p we may take

$$x^* = (1-\delta,\ 0,\ 0,\ 0,\ \ldots),$$
$$y^* = \left(\quad 0,\ \frac{\varepsilon}{2},\ 0,\ 0,\ \ldots\right).$$

Remark. For fixed ε $\lim_{p\to 1} \delta(\varepsilon) = 0$ and $\lim_{p\to\infty} \delta(\varepsilon) = 0$. For small $\varepsilon > 0$ we have

$$\delta(\varepsilon) = \frac{p-1}{2}\left(\frac{\varepsilon}{2}\right)^2 + \cdots \qquad \text{for } 1 < p < 2,$$
$$\delta(\varepsilon) = \frac{1}{p}\left(\frac{\varepsilon}{2}\right)^p + \cdots \qquad \text{for } p \geq 2.$$

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