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On a class of Diophantine equations

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Hurwitz [1][1] has studied the Diophantine equation

$$x_1^2+x_2^2+\cdots+x_n^2=x\cdot x_1 x_2\cdots x_n. \tag{1}$$

For $n=2$ more general equations of this type have been investigated by Barnes [2] and Mills [3, 4]. This paper deals with the equation

$$\sum_{i=1}^{n}(x_i+a_i)^2+c=x\cdot x_1 x_2\cdots x_n, \tag{2}$$

where the numbers a_i and c are rational integers, $c\geqq 0$. We show that with some alterations the method used by Hurwitz on equation (1) also applies to equation (2). The results obtained are analogous to those of Hurwitz.

We note that if any of the polynomials

$$(x_i+a_i)^2+c=0$$

is reducible, equation (2) is solvable for every value of x. Similarly if there exists a solution in which one at least of the numbers $x_1, x_2, \ldots, x_n$ is zero.

In the following lines we consider only solutions in which the numbers $x_1, x_2, \ldots, x_n$ are all $\neq 0$.

Starting from a solution

$$A=(x, x_1, x_2, \ldots, x_n)$$

we can find a new solution $A^{(k)}$ in the following way. In (2) we solve for x_k. Since the equation considered is of the second degree and since it is satisfied by x_k, the other root is

$$x_k'=x\cdot x_1\cdots x_{k-1}x_{k+1}\cdots x_n-2a_k-x_k. \tag{3}$$

It is obvious that x_k' is an integer. Thus

$$A^{(k)}=(x, x_1, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_n)$$

is a solution of (2). The solutions $A^{(k)}, k=1, 2, \ldots, n$ are said to be *associated* with the solution A.

[1] Numbers in brackets refer to the bibliography at the end of this paper.

A sequence of solutions $A_1, A_2, \ldots,$ of (2) with the property that every solution is associated with the preceding one forms a *chain of solutions*. It is evident that x has the same value for every solution in a chain.

The number

$$\sum_{i=1}^{n} |x_i| \qquad i=1, 2, \ldots, n$$

is called the *weight* of the solution $(x, x_1, \ldots, x_n)$.

A solution of weight h is said to be *fundamental* if the weight of all the solutions associated with it is $\geqq h$.

A fundamental solution is said to *generate* the chain to which it belongs.

We have the following result:

Theorem 1. *Every solution of (2)*

$$(x, x_1, x_2, \ldots, x_n)$$

with $x_1, x_2, \ldots, x_n$ *all* $\neq 0$, *belongs to a chain generated by a fundamental solution.*

Proof. Suppose that a solution A of (2) is not fundamental. Then it has an associated solution whose weight is less than the weight of A. If the new solution is not fundamental we can repeat the process. Since the weight is a natural number we must find a fundamental solution after a finite number of steps.

To solve (2) completely we have only to determine the fundamental solutions. Our main result will be that the number of different fundamental solutions is finite and furthermore we will give inequalities for these solutions.

If $(x, x_1, \ldots, x_n)$ is a solution of (2) in which the numbers $x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_r}$ are negative, $(x, |x_1|, |x_2|, \ldots, |x_n|)$ is obviously a solution of the equation derived from (2) by changing signs for $a_{\nu_1}, a_{\nu_2}, \ldots, a_{\nu_r}$. Furthermore it is clear that if we can prove for this new equation that the number of different fundamental solutions is finite, the same result holds for (2). Hence we suppose in the sequel that all the numbers $x_1, x_2, \ldots, x_n$ are >0.

If the weight of the solution A is not greater than that of $A^{(k)}$, we get from (3) on multiplication with x_k

$$x_k x_k' + x_k^2 + 2 a_k x_k = x \prod_{i=1}^{n} x_i.$$

From (2) we see that x_k and x_k' have the same sign and therefore

$$2 x_k^2 + 2 a_k x_k \leqq x \prod_{i=1}^{n} x_i, \tag{4}$$

for $k = 1, 2, \ldots, n$.

Let us consider the expression

$$(x \cdot x_1 x_2 \cdots x_{k-1} x_{k+1} \cdots x_n - 2(x_k + a_k))^2 = x^2 x_1^2 \cdots x_{k-1}^2 x_{k+1}^2 \cdots x_n^2 + \\ + 4\left[x \cdot x_1 x_2 \cdots x_n \left(1 + \frac{a_k}{x_k}\right) - (x_k + a_k)^2\right]$$

which can be written

$$x \cdot x_1 x_2 \cdots x_n - 2(x_k^2 + a_k x_k) = x_1 x_2 \cdots x_n \sqrt{x^2 - 4t_k} \tag{5}$$

where

$$t_k = \frac{\left(1 + \frac{a_k}{x_k}\right)\left[\sum_{i=1}^{n} (x_i + a_i)^2 + c\right] - (x_k + a_k)^2}{x_1^2 x_2^2 \cdots x_n^2} x_k^2. \tag{6}$$

For some index μ we have

$$\sqrt{x^2 - 4t_\mu} \leqq \sqrt{x^2 - 4t_i}, \qquad i = 1, 2, \ldots, n. \tag{7}$$

From (5) we get on addition

$$n x \prod_{i=1}^{n} x_i - 2 \sum_{i=1}^{n} (x_i^2 + a_i x_i) = \prod_{i=1}^{n} x_i \cdot \left(\sum_{i=1}^{n} \sqrt{x^2 - 4t_i} \right),$$

or

$$(n-2) x \prod_{i=1}^{n} x_i + 2T = \prod_{i=1}^{n} x_i \cdot \left(\sum_{i=1}^{n} \sqrt{x^2 - 4t_i} \right), \tag{8}$$

where

$$T = \sum_{i=1}^{n} (x_i a_i + a_i^2) + c.$$

We distinguish two cases according as T is $\leqq 0$ or > 0.

Suppose first that $T \leqq 0$. Then we get from (8)

$$(n-2) x \geqq \sum_{i=1}^{n} \sqrt{x^2 - 4t_i}.$$

As is easily seen, it is sufficient to consider the case $t_\mu > 0$. In fact, if $t < 0$ we must have $t_i < 0$ for all i. It then follows from the left-hand side of (5) that

$$x_k (a_k + x_k) < 0, \qquad k = 1, 2, \ldots, n.$$

This, however, means that $x_k < |a_k|$ for all k.

If t_μ is $= 0$, we must have $t_i \leqq t_\mu$ for all i. Thus we get as before $x_k \leqq |a_k|$.

On squaring and introducing t_μ instead of t_i we get

$$n^2 t_\mu \geqq (n-1) x^2.$$

Substituting in this inequality the expression for t_μ given by (6) we find

$$n^2 \left[\left(1 + \frac{a_\mu}{x_\mu}\right) \left(\sum_{i=1}^{n} (x_i + a_i)^2 + c - (x_\mu + a_\mu)^2 \right) \right] \geqq (n-1) \frac{x^2}{x_\mu^2} \prod_{i=1}^{n} x_i^2. \tag{9}$$

We now distinguish two cases according as a_μ is > 0 or $\leqq 0$. If $a_\mu \leqq 0$ we get from (9)

$$n^2\left[\sum_{i\neq\mu}(x_i+a_i)^2+c\right]\geqq(n-1)\frac{x^2}{x_\mu^2}\prod_{i=1}^{n}x_i^2.$$

Suppose now that $(x_m+a_m)^2\geqq(x_i+a_i)^2,\ i\neq\mu.$

Then we have

$$n^2\left[(n-1)(a_m^2+1)+c\right]\geqq(n-1)\frac{x^2x_1^2\cdots x_n^2}{x_\mu^2x_m^2}\cdot \tag{10}$$

If on the other hand $a_\mu>0$ we have

$$n^2\left[(n-1)(a_m^2+1)x_m^2+c+\frac{a_\mu}{x_\mu}x\prod_{i=1}^{n}x_i\right]\geqq(n-1)\frac{x^2}{x_\mu^2}\prod_{i=}^{n}x_i^2.$$

Here either

$$x_\mu x_m^2\geqq x\prod_{i=1}^{n}x_i\quad\text{or}\quad x_\mu x_m^2<x\prod_{1}^{n}x_i.$$

However, in both cases we have the inequality

$$\prod_{i=1}^{n}x_i\cdot\frac{x}{x_\mu x_m}\leqq\frac{n^2}{n-1}\left[(n-1)(a_m^2+1)+|a_\mu|+c\right]. \tag{11}$$

Comparing with (10) we find that if $T\leqq0$, (11) is always true.

We turn now to the case $T>0$, and start with a few observations. If $\sum_{i=1}^{n}(x_i^2+a_ix_i)$ is $\leqq0$, we must have $x_i\leqq\sum_{i=1}^{n}|a_i|$. Thus we can assume that this sum is >0. From (2) it follows that we have either

$$\sum_{i=1}^{n}(x_i^2+a_ix_i)\leqq\sum_{i=1}^{n}(a_ix_i+a_i^2)+c \tag{12}$$

or

$$\sum_{i=1}^{n}(x_i^2+a_ix_i)>\sum_{i=1}^{n}(a_ix_i+a_i^2)+c. \tag{13}$$

If (12) is true

$$\sum_{i=1}^{n}x_i^2\leqq\sum_{i=1}^{n}a_i^2+c,$$

and thus all the numbers x_i are bounded. If on the other hand (13) is true we deduce from (8) and (7)

$$(n-1)x\geqq n\sqrt{x^2-4t_\mu}.$$

On squaring we get

$$n^2t_\mu\geqq\frac{2n-1}{4}x^2.$$

We now proceed exactly as in the case $T\leqq0$. The result is

$$\frac{x}{x_\mu x_m}\prod_{i=1}^{n}x_i\leqq\frac{4n^2}{2n-1}\left[(n-1)(a_m^2+1)+|a_\mu|+c\right]. \tag{14}$$

Combining this inequality with our previous results we have that every fundamental solution of (2) satisfies the inequality

$$\frac{x}{x_\mu x_m} \prod_{i=1}^{n} x_i \leqq \frac{4n^2}{2n-1} [(n-1)a^2 + a + n - 1 + c] = C, \tag{15}$$

where $a = \max |a_i|$.

The following theorem is an immediate consequence of (15):

Theorem 2. *The number of different fundamental solutions (chains of solutions) is finite.*

(15) gives at once an upper bound for x. Furthermore it is possible to determine explicit bounds for all the x_i. In fact, the following inequality is easily derived from (4):

$$|x^2 - (x_m + a_m)^2| \leqq a^2 + a_m^2 + \sum_{i \neq \mu, m} (x_i + a_i)^2 + c.$$

Since, according to (15), the product

$$\frac{1}{x_\mu x_m} \prod_{i=1}^{n} x_i$$

is bounded the same is true for the sum $\sum_{i \neq \mu, m} (x_i + a_i)^2$.

In fact, we have

$$\sum_{i \neq \mu, m} (x_i + a_i)^2 \leqq (C + a)^2 + (n-3)(a+1)^2.$$

Thus we have for $x, x_i, i = 1, 2, \ldots, n$ the inequalities

$$\begin{cases} x \leqq C, \\ |x_i| < \sqrt{(C+a)^2 + (n-1)(a+1)^2 + c} + a. \end{cases}$$

Equation (2) with $c < 0$ may be treated in the same way as when $c \geqq 0$. However, the details are more complicated. I shall return to this case in a following paper.

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