A characterization of Poissonian domains

Christopher J. Bishop

Abstract. We give a characterization of Poissonian domains in $\mathbf{R}^n$, i.e., those domains for which every bounded harmonic function is the harmonic extension of some function in L^∞ of harmonic measure. We deduce several properties of such domains, including some results of Mountford and Port. In two dimensions we give an additional characterization in terms of the logarithmic capacity of the boundary. We also give a necessary and sufficient condition for the harmonic measures on two disjoint planar domains to be mutually singular.

1. Introduction

Suppose Ω is a domain in $\mathbf{R}^n$ for which harmonic measure ω is defined. Given a (real valued) function $f \in L^\infty(d\omega)$ we can define a bounded harmonic function u on Ω by

$$u(z) = \int_{\partial\Omega} f(x)\, d\omega_z(x).$$

If $n > 2$ and Ω is unbounded, ω may contain a point mass at ∞. We say Ω is Poissonian if every bounded harmonic function on Ω is of this form. For example, the unit ball is Poissonian, and a non-Poissonian domain in $\mathbf{R}^2$ can be constructed by removing the line segment $[0, 1]$ from the unit disk, $\mathbf{D}$. There are bounded harmonic functions on this domain which have different boundary values "above" and "below" the slit, and these are not of the form described above. However, using results of [8] it is possible to build a Jordan arc Γ connecting 0 and 1 in $\mathbf{D}$ such that $\mathbf{D} \setminus \Gamma$ is Poissonian. Thus more than just topology is important in determining whether a domain is Poissonian.

The purpose of this paper is to give a characterization of Poissonian domains and derive some of their properties. Our characterization is in terms of harmonic measures of subdomains of Ω. Since the harmonic measures corresponding to

---

The author is partially supported by a NSF Postdoctoral Fellowship.

two different points in a domain are mutually absolutely continuous (i.e., they have the same null sets) and our conditions only concern whether a set has zero or positive measure we will refer to "the" harmonic measure $\omega(E, \Omega)$ (or just $\omega(E)$ if the domain is clear from context).

Theorem 1.1. *$\Omega\subset\mathbf{R}^n$ is Poissonian iff for every pair of disjoint subdomains Ω_1 and Ω_2 of Ω with $\partial\Omega_1\cap\partial\Omega_2\subset\partial\Omega$, the harmonic measures ω_1 and ω_2 of Ω_1 and Ω_2 are mutually singular.*

By the measures being singular we mean that there exists $E\subset\partial\Omega_1\cap\partial\Omega_2$ with $\omega_1(E)=0$ and $\omega_2((\partial\Omega_1\cap\partial\Omega_2)\setminus E)=0$ and we write this as $\omega_1\perp\omega_2$. It is not clear how easy this condition is to check in practice, but it is sufficient to prove the following results (the necessary definitions will be given in Section 3):

Corollary 1.2, [23]. *Each component of the intersection of two Poissonian domains is Poissonian.*

Corollary 1.3, [23]. *If Ω_1 and Ω_2 are Poissonian and $\omega(\partial\Omega_1\cap\partial\Omega_2, \Omega_1\cup\Omega_2)=0$ then $\Omega=\Omega_1\cup\Omega_2$ is Poissonian.*

Corollary 1.4. *If $E\subset\mathbf{R}^n$ is closed and has zero $n-1$ dimensional measure, then $\Omega=\mathbf{R}^n\setminus E$ is Poissonian.*

Corollary 1.5. *If $E\subset\mathbf{R}^n$ is a closed subset of a Lipschitz graph, then $\Omega=\mathbf{R}^n\setminus E$ is Poissonian iff E has zero $n-1$ dimensional measure.*

Some hypothesis is needed in Corollary 1.3 since it is easy to see that the union of Poissonian domains need not be Poissonian, e.g. $\Omega_1=\{0<|z|<1, 0<\arg(z)<3\pi/2\}$ and $\Omega_2=\{0<|z|<1, \pi/2<\arg(z)<2\pi\}$ whose union is $\mathbf{D}\setminus[0, 1]$. Corollary 1.5 was proven in [23] when E is a subset of a $n-1$ hyperplane and this case also follows from results in [4]. Also see [2]. Mountford and Port [23] also gave a characterization of Poissonian domains in terms of the Martin boundary Δ of Ω. It says that a domain is Poissonian iff there is a measurable mapping $\varphi\colon \partial\Omega\to\Delta$ which takes harmonic measure on $\partial\Omega$ to the harmonic measure μ on Δ, i.e., iff $(\partial\Omega, \omega)$ and (Δ, μ) are equivalent as measure spaces.

For domains in $\mathbf{R}^2$, the characterization in Theorem 1.1 can be restated in terms of a Wiener type condition involving the logarithmic capacity of $\partial\Omega$, as follows. For $x\in\mathbf{R}^2$, $\delta>0$, $\varepsilon>0$ and $\theta\in[0, 2\pi)$ we define the cone and wedge

$$C(x, \delta, \varepsilon, \theta) = \{x+re^{i\psi}\colon 0<r<\delta, |\psi-\theta|<\varepsilon\}$$

$$W(x, \delta, \varepsilon, \theta) = C(x, \delta, \varepsilon, \theta)\cap\{z\colon \delta/2\leqq|z-x|\}.$$

We also let $\operatorname{cap}(E)$ denote the logarithmic capacity of E (to be defined in Section 4).

For a fixed x, ε and θ let

$$\gamma_i(k) = \operatorname{cap}\left(2^{k-2}(W(x, 2^{-k}, \varepsilon, (-1)^{i+1}\theta)\setminus\Omega_i)\right),$$

i.e., $\gamma_i(k)$ is capacity of $\Omega_i^c\cap W(x, 2^{-k}, \varepsilon, \theta)$ after we have dilated it to have diameter about 1/2. We say a point $x\in\partial\Omega_1\cap\partial\Omega_2$ satisfies a weak double cone condition (WDCC) with respect to the pair Ω_1, Ω_2 if there exist ε and θ such that

$$\sum_{n=1}^{\infty}(\gamma_1(n)+\gamma_2(n)) < \infty. \tag{1.1}$$

If $\Omega_1=\Omega_2=\Omega$ we simply say x satisfies a WDCC with respect to Ω. We refer to this as a "weak" condition because it generalizes the double cone condition stated in [6] and [8] which requires that

$$C(x, \delta, \varepsilon, (-1)^{i+1}\theta) \subset \Omega_i.$$

It is clear that this condition implies the WDCC since all but finitely many of the terms in (1.1) will be zero.

Theorem 1.6. *A domain $\Omega\subset\mathbf{R}^2$ is Poissonian iff the set of points $x\in\partial\Omega$ which satisfy a weak double cone condition with respect to Ω has zero 1 dimensional measure.*

The proof will also show:

Theorem 1.7. *Suppose Ω_1 and Ω_2 are disjoint subdomains in $\mathbf{R}^2$ and let ω_1 and ω_2 be their harmonic measures. Then $\omega_1\perp\omega_2$ iff the set of points in $\partial\Omega_1\cap\partial\Omega_2$ satisfying a weak double cone condition with respect to Ω_1, Ω_2 has zero 1 dimensional measure, Λ_1. Moreover, if ω_1 and ω_2 are mutually absolutely continuous on a set E then there is Besicovitch regular $F\subset E$ with $\omega_i(F)=\omega_i(E)$ and ω_i mutually absolutely continuous with Λ_1 on F for $i=1, 2$.*

In the case when the domains are simply connected, these results follow easily from the results of [8] and the fact that the capacity of a connected set can be estimated in terms of its diameter. We can also characterize the disjoint planar domains for which the two harmonic measures are mutually absolutely continuous. This happens iff $\Omega_i=\tilde{\Omega}_i\setminus E_i$ for $i=1, 2$ where $\operatorname{cap}(E_i)=0$ and $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are disjoint simply connected domains with mutually absolutely continuous harmonic measures $\tilde{\omega}_1$ and $\tilde{\omega}_2$. Such domains are characterized in [8] (also see [6]): $\tilde{\omega}_1\ll\tilde{\omega}_2\ll\tilde{\omega}_1$ iff for every $\varepsilon>0$ there are subdomains $D_i\subset\tilde{\Omega}_i$ with rectifiable boundaries Γ_i such that $\tilde{\omega}_i(\Gamma_1\cap\Gamma_2)\geqq 1-\varepsilon$ for $i=1, 2$.

Theorem 1.7 implies that if Ω is not Poissonian then $\partial\Omega$ contains a Besicovitch regular set of positive length. Thus we obtain

Corollary 1.8. *If $E\subset\mathbf{R}^2$ is a closed, Besicovitch irregular set, then $\Omega=\mathbf{R}^2\setminus E$ is Poissonian.*

In the next section we shall prove Theorem 1.1. In Section 3 we will prove its corollaries and in Section 4 we prove Theorems 1.6 and 1.7 and Corollary 1.8. In Section 5 we prove a lemma used in Section 4 and we conclude in Section 6 with some remarks concerning the Martin boundary and possible generalizations of Theorem 1.7 to higher dimensions.

I would like to thank Tom Mountford and Sidney Port for discussing their work on Poissonian domains with me and for their comments on this paper. I am also indebted to Mike Cranston, John Garnett and Tom Wolff for many helpful conversations on harmonic measure and the Martin boundary.

2. Proof of Theorem 1.1

Before starting the proof we recall a few basic facts of potential theory (see [10], [12]). By the Newtonian kernel on $\mathbf{R}^n$ we mean

$$K(|x|) = \begin{cases} \log \dfrac{1}{|x|}, & n = 2 \\ |x|^{2-n}, & n > 2 \end{cases}$$

and given a positive measure μ on $\mathbf{R}^n$ we define its potential by

$$u_\mu(x) = \int K(|x-y|)\,d\mu(y)$$

and its energy by

$$I(\mu) = \iint K(|x-y|)\,d\mu(x)\,d\mu(y).$$

For a set $E \subset \mathbf{R}^n$, we let $\Pr(E)$ denote the set of probability measures on E. We define the capacity of E as

$$\operatorname{cap}(E) = \big(\inf_{\mu \in \Pr(E)} I(\mu)\big)^{-1}.$$

There exists a unique probability measure $\bar{\mu}$, called the equilibrium measure, for which the inf is attained. Moreover, the potential of this measure satisfies

$$u_{\bar{\mu}}(x) = I(\bar{\mu})$$

for every $x \in E$ except possibly a subset of capacity zero. In 2 dimensions the kernel is not positive, so a large E may have infinite or negative capacity, and for this reason the capacity is sometimes defined as $\exp(-\operatorname{cap}(E)^{-1})$ to make it positive and monotonic. However, we will only consider sets of diameter less that 1, so our capacities will always be positive.

If Ω is a domain with $\operatorname{cap}(\partial\Omega) > 0$ then Ω has a Green's function and the harmonic measure for Ω exists. A set E with $\operatorname{cap}(E) = 0$ is called polar. A property which holds everywhere of $\partial\Omega$ except possibly on a polar set is said to hold p.p.

("presque partout") on $\partial\Omega$. For example, the Green's function of Ω always tends to 0 p.p. on $\partial\Omega$. The points of $\partial\Omega$ where it does so are called regular for the Dirichlet problem on Ω. If $E\subset\partial\Omega$ is polar then E has zero harmonic measure in Ω. If f is continuous on $\partial\Omega$ and we extend it to a harmonic function u on Ω via the Perron process such that u extends to be continuous and agree with f p.p. on $\partial\Omega$. The mapping $f\to u(z)$ turns out to be a continuous linear functional on $C(\partial\Omega)$ so by the Riesz representation theorem there is a probability measure ω_z on $\partial\Omega$ such that $u(z)=\int f d\omega_z$. This is the harmonic measure for Ω with respect to z. For a fixed $E\subset\partial\Omega$, $\omega_z(E)$ is a nonnegative, harmonic function in z, so is either 0 for all z or for none if Ω is connected. Thus the harmonic measures for different points of Ω are mutually absolutely continuous, as mentioned before.

If $F\subset\partial\Omega$ is closed then the harmonic function $u(z)=\omega_z(F)$ tends to zero p.p. on $\partial\Omega\setminus F$. Furthermore, the maximum principle states that if u is a bounded subharmonic function on Ω such that

$$\limsup_{z\to x,\, z\in\Omega} u(z) \leqq M$$

for p.p. $x\in\partial\Omega$ then $u\leqq M$ on Ω. I shall also use the phrase "maximum principle" to refer to the following fact: if $\tilde{\Omega}\subset\Omega$ and $E\subset\partial\tilde{\Omega}\cap\partial\Omega$ then $\omega(E, \tilde{\Omega})\leqq\omega(E, \Omega)$.

Now we begin the proof of Theorem 1.1. First we will show the stated condition implies Ω is Poissonian. Let HB(Ω) denote the Banach space of bounded harmonic functions on Ω with the "sup" norm. Then there is a continuous linear mapping P: $L^\infty(\omega)\to$ HB(Ω) given by

$$P(f)(z) = \int f(x)\, d\omega_z.$$

In order to show Ω is Poissonian we want to show this mapping is onto. It suffices to show that for any $u\in$ HB(Ω) with $\|u\|_\infty\leqq 1$ there exists an $f\in L^\infty(\omega)$ with $\|f\|_\infty\leqq 1$ and such that $\|u-P(f)\|_\infty\leqq 3/4$. Since P is bounded and linear a standard successive approximation argument gives a g with $P(g)=u$.

So fix a $u\in$ HB(Ω) with $\|u\|_\infty\leqq 1$. We may assume that $u(\Omega)=(-1, 1)$. Let

$$\Omega_1 = \left\{z\in\Omega:\ u(z) > \frac{1}{3}\right\};$$

$$\Omega_2 = \left\{z\in\Omega:\ u(z) < -\frac{1}{3}\right\}.$$

Note that $\partial\Omega_1\cap\partial\Omega_2$ must lie in $\partial\Omega$ and that Ω_1 and Ω_2 may have countable many components. By hypothesis the harmonic measure for any component of Ω_1 is singular to harmonic measure of any component of Ω_2. Therefore, given compact subsets K_i of Ω_i $(i=1, 2)$ and $\varepsilon>0$ we can find disjoint closed sets $E_i\subset\partial\Omega_1\cap$

$\partial\Omega_2 \subset \partial\Omega$ such that

$$\omega(z, E_i, \Omega_i) \geqq \omega(z, \partial\Omega \cap \partial\Omega_i, \Omega_i) - \varepsilon$$

for every $z \in K_i$ and $i = 1, 2$.

Now define a function g by setting $g(z) = 1/3$ for $z \in E_1$, $g(z) = -1/3$ on E_2 and extending g to be continuous on all of $\mathbf{R}^n$ and satisfying $-1/3 \leqq g \leqq 1/3$. Now restrict g to $\partial\Omega$ and let $v = P(g)$. By the maximum principle $-1/3 \leqq v \leqq 1/3$ so for $z \in \Omega \setminus (\Omega_1 \cup \Omega_2)$, $|u(z) - v(z)| \leqq 2/3$. For $z \in \partial\Omega_1 \cap \Omega$ we have $u(z) = 1/3$, so $|u(z) - v(z)| \leqq 2/3$. Also, for p.p. $x \in E_1$ $v(z) \to 1/3$ as $z \to x$ in Ω_1. Thus for p.p. $x \in E_1$

$$\limsup_{z \to x, z \in \Omega_1} \left(u(z) - v(z)\right) \leqq 1 - 1/3 = 2/3$$

and for every $x \in (\partial\Omega_1 \cap \partial\Omega) \setminus E_1$,

$$\limsup_{z \to x, z \in \Omega_1} u(z) - v(z) \leqq 2.$$

From this and the maximum principle we obtain

$$|u(z) - v(z)| \leqq 2/3 + 2\omega(z, \partial\Omega \setminus E_1, \Omega_1) \leqq 2/3 + 2\varepsilon, \quad z \in K_1.$$

Similarly for Ω_2. Thus if we take an exhaustion of Ω_1 and Ω_2 by compact sets and a sequence of ε's tending to zero, we obtain a sequence $\{g_n\} \in L^\infty(\omega)$ with $|P(g_n)(z) - u(z)| \leqq 3/4$ for every $z \in \Omega$ and $n = n(z)$ large enough. By passing to a subsequence we may assume the $\{g_n\}$ converge weakly in $L^\infty(\omega)$ and the limit g clearly satisfies $\|u - P(g)\|_\infty \leqq 3/4$. Thus Ω is Poissonian.

Next we prove the converse. We will show that if the condition in Theorem 1.1 fails then Ω is not Poissonian. So suppose Ω_1 and Ω_2 are subdomains of Ω, that $\partial\Omega_1 \cap \partial\Omega_2 \subset \partial\Omega$ and that ω_1 and ω_2 are not singular. Then there is a set $E \subset \partial\Omega_1 \cap \partial\Omega_2$ such that $\omega_1(E) > 0$, $\omega_2(E) > 0$ and ω_1 and ω_2 are mutually absolutely continuous on E.

Let $\mathscr{F}$ be the family of subharmonic functions v on Ω which satisfy

$$v(z) \leqq w(z) = \begin{cases} 1 - 2\omega(z, E, \Omega_2), & z \in \Omega_2 \\ 1, & z \in \Omega \setminus \Omega_2. \end{cases}$$

Note w is a superharmonic function on Ω, and so $\mathscr{F}$ is a Perron family (see [1, page 248]). Thus

$$-1 \leqq u \equiv \sup_{\mathscr{F}} v \leqq w$$

exists and is harmonic. We claim that u cannot be uniformly approximated on Ω by functions of the form $P(f)$, $f \in L^\infty(\omega)$.

First note that if we define v on Ω by

$$v(z) = \begin{cases} 2\omega(z, E, \Omega_1) - 1, & z \in \Omega_1 \\ -1, & z \in \Omega \setminus \Omega_1 \end{cases}$$

then v is subharmonic on Ω and is in $\mathcal{F}$. Thus $v \leqq u$. Let $D_i \subset \Omega_i$ be defined by

$$D_i = \{z \in \Omega_i\colon\ \omega(z, E, \Omega_i) > 3/4\}$$

for $i=1, 2$. Note that $u \geqq 1/2$ on D_1 and $u \leqq -1/2$ on D_2.

Now suppose there exists $f \in L^\infty(\omega)$ such that $\|u-P(f)\|_\infty < 1/4$. Then $P(f) \geqq 1/4$ on D_1. Therefore $f \geqq 1/4$ a.e. (ω_1) on E, for if $F=\{f<1/4-\varepsilon\}$ and $\omega_1(F)>0$ for some $\varepsilon>0$, there would be a sequence $\{z_n\} \subset D_1$ with $\omega(z_n, F, \Omega_1) \to 1$ and hence $P(f)(z_n) < 1/4$ for some n, a contradiction.

The same argument shows that $P(f) \leqq -1/4$ on D_2 and therefore $f \leqq -1/4$ a.e. (ω_2) on E. But $f \geqq 1/4$ a.e. ω_1 and ω_1 and ω_2 are mutually absolutely continuous on E! This is a contradiction, and so no such f can exist. Hence Ω is not Poissonian.

3. Proof of the corollaries

Before proving the corollaries it is convenient to record the following simple results.

Lemma 3.1. *Suppose* $\Omega \subset \mathbf{R}^n$ *and* $E \subset \partial\Omega$ *has positive harmonic measure. Fix* $0<a<1$ *and set* $\tilde{\Omega}=\{z \in \Omega\colon\ \omega(z, E, \Omega)>a\}$. *Then* $F \subset E$ *has positive harmonic measure in* Ω *iff it has positive harmonic measure in some component of* $\tilde{\Omega}$.

Lemma 3.2. *Suppose* Ω_1 *and* Ω_2 *are disjoint domains with mutually continuous harmonic measures* ω_1 *and* ω_2 *on a compact subset* $E \subset \partial\Omega_1 \cap \partial\Omega_2$ *of positive measure. Then there exist* $\tilde{\Omega}_i \subset \Omega_i$ *for* $i=1, 2$ *with mutually continuous harmonic measures* $\tilde{\omega}_i$ *on a subset* $F \subset E$ *of positive* $\tilde{\omega}_i$ *measure and* $\partial\tilde{\Omega}_1 \cap \partial\tilde{\Omega}_2 \subset E$. *Moreover, given any open neighborhood* U *of* E, *we may take* $\tilde{\Omega}_i \subset U$ *for* $i=1, 2$.

To prove the first lemma, let $F \subset E$ be compact and note by the maximum principle that for each component Ω_j of $\tilde{\Omega}$, $\omega(F, \Omega_j) \leqq \omega(F, \Omega)$, so $\omega(F)=0$ implies $\omega_j(F)=0$ for every j. On the other hand if $\omega(F)>0$ then there exists $z_0 \in \tilde{\Omega}$ with $\omega(z_0, F, \Omega)>(1+a)/2$. Suppose z_0 is in a component Ω_j of $\tilde{\Omega}$. First note that $\omega(z, F, \Omega)$ is harmonic on Ω_j but less than a on $\partial\Omega_j \cap \Omega$. Also observe that $\operatorname{cap}(\partial\Omega_j \cap \partial\Omega) \setminus F)=0$ (this holds since $\omega(z, F, \Omega) \to 0$ p.p. on $\partial\Omega \setminus F$). Thus $\omega(z_0, F, \Omega_j)>0$. This proves Lemma 3.1.

To prove Lemma 3.2 let $\tilde{\Omega}_i$ be components of $\{z \in \Omega_i\colon\ \omega(z, E, \Omega_i)>1/2\}$ for $i=1, 2$. By Lemma 3.1 they can be chosen so they have mutually absolutely continuous harmonic measures on some subset $F \subset E$ of positive measure. Now suppose $x \in \partial\tilde{\Omega}_1 \cap \partial\tilde{\Omega}_2$ but not in E. Then $x \in \partial\Omega_1 \cap \partial\Omega_2$. After the proof of Lemma 3.3 we shall observe that a point of the common boundary to two disjoint domains must be regular for the Dirichlet problem for at least one of the domains. So assume x is regular for Ω_1. Then since $x\notin E$, $\omega(z, E, \Omega_1)\to 0$ as $z\to x$ in Ω_1. Since $\omega(z, E, \Omega_1)>1/2$ for $z\in\tilde{\Omega}_1$, $x\notin\tilde{\Omega}_1$, a contradiction. Thus $x\in E$, as required.

To prove the last claim we take $\tilde{\Omega}_i$ to be components of

$$\{z\in\Omega_i\colon\ \omega(z, E, \Omega_1) > 1-\varepsilon\}$$

for small ε. To show $\tilde{\Omega}_i\subset U$ if ε is small enough observe

$$\omega(z, E, \Omega_1) \leqq \frac{u_{\bar{\mu}}(z)}{I(\bar{\mu})},$$

where $\bar{\mu}$ is the equilibrium measure for E. This inequality holds by the maximum principle since both sides are 1 p.p. on E and the the right-hand side is positive on $\partial\Omega_1\setminus E$ while the left-hand side is 0 p.p. on $\partial\Omega_1\setminus E$. The right-hand side is strictly less than 1 off E and so is $\leqq 1-\varepsilon$ on the complement of U. This argument needs to be slightly modified in 2 dimensions because the potential is not positive. However in this case we may assume that Ω_1 is bounded (use a Möbius transformation to put $\infty\in\Omega_2$) and then compare $\omega(z, E, \Omega_1)$ to $1-\eta\big(1-u_{\bar{\mu}}(z)/I(\bar{\mu})\big)$ which is positive on Ω_1 if η is small enough. This proves the second lemma.

To prove Corollary 1.2 suppose Ω is a component of $\Omega_1\cap\Omega_2$ and that Ω is not Poissonian. Then there exist disjoint domains $D_1, D_2\subset\Omega$ whose harmonic measures ω_1 and ω_2 are not singular. Thus there is a subset $E\subset\partial D_1\cap\partial D_2$ on which ω_1 and ω_2 are mutually absolutely continuous and which has positive measure. Since $\partial\Omega\subset\partial\Omega_1\cup\partial\Omega_2$ either $E_1=E\cap\partial\Omega_1$ or $E_2=E\cap\partial\Omega_2$ must have positive measure with respect to ω_1. Without loss of generality suppose it is E_1. Then D_1, D_2 are disjoint subdomains of Ω_1 whose harmonic measures (restricted to E_1) are not singular. We may not have $\partial D_1\cap\partial D_2\subset\partial\Omega_1$, but we can fix this by using Lemma 3.2. Thus Ω_1 is not Poissonian.

To prove Corollary 1.3 suppose Ω_1 and Ω_2 are domains, and define $\Omega=\Omega_1\cup\Omega_2$. Suppose also that $\omega(\partial\Omega_1\cap\partial\Omega_2, \Omega)=0$, but that Ω is not Poissonian. Then there exist disjoint subdomains D_1 and D_2 and a set $E\subset\partial D_1\cap\partial D_2$ as above. Since $\omega_1(E)>0$, $\omega(E, \Omega)>0$ by the maximum principle. Since $\omega(\partial\Omega_1\cap\partial\Omega_2, \Omega)=0$ either $E_1=E\cap\partial\Omega_1\setminus\partial\Omega_2$ or $E_2=E\cap\partial\Omega_2\setminus\partial\Omega_1$ must have positive harmonic measure in Ω. Assume it is E_1. We may also assume E_1 is compact and so does not meet $\bar{\Omega}_2$. Therefore there is an open neighborhood U of E_1 which also does not meet Ω_2. By the second part of Lemma 3.2 we can find subdomains of $U\cap\Omega_1$ which have mutually continuous harmonic measures on a set of positive measure. This proves Corollary 1.3.

Before proving Corollary 1.4 we we recall the definition of Hausdorff measure. For a set $E\subset\mathbf{R}^n$ we let

$$\Lambda_s(E) = \lim_{\delta\to 0}\big(\inf\big\{\sum (r_j)^s\colon\ E\subset\cup B(x_j, r_j),\ r_j\leqq\delta\big\}\big).$$

Here $B(x, r)$ denotes a solid ball of radius r and center x. We call Λ_s the s dimensional Hausdorff measure. See [10] or [13] for further details. We also need the following estimate on harmonic measure.

Lemma 3.3. *Suppose Ω_1 and Ω_2 are disjoint domains in $\mathbf{R}^n$. Fix points $z_i \in \Omega_i$ for $i=1, 2$. There is a $C>0$ so that for $x \in \partial\Omega_1 \cap \partial\Omega_2$ and*

$$r < \min\left(\operatorname{dist}(z_1, \partial\Omega_1), \operatorname{dist}(z_2, \partial\Omega_2)\right),$$

$$\omega\left(z_1, B(x, r)\cap\partial\Omega_1, \Omega_1\right)\omega\left(z_2, B(x, r)\cap\partial\Omega_2, \Omega_2\right) \leqq Cr^{2(n-1)}.$$

The constant C depends only on $\operatorname{dist}(z_1, \partial\Omega_1)$ *and* $\operatorname{dist}(z_2, \partial\Omega_2)$.

First normalize so that $\operatorname{dist}(z_i, \partial\Omega_i) \geqq 1$ for $i=1, 2$. For domains in $\mathbf{R}^2$ this result is proven in [8]. For higher dimensions it follows from estimates of Huber [17] and Friedland and Hayman [14]. Suppose u is positive and subharmonic on $\mathbf{R}^n$ and vanishes on $\partial\Omega$, and for $r>0$ let $S(x, r) = \partial B(x, r)$ and define

$$m_r(u) = \left(\int_{S(x,r)} u^2\, d\sigma\right)^{1/2}$$

where σ is surface measure on the sphere normalized to have mass 1. Then we have

$$m_r(u) \leqq C m_1(u) \exp\left(-\int_r^{1/2} \alpha(t) \frac{dt}{t}\right) \tag{3.1}$$

where $\alpha(t)$ is the characteristic constant of the $n-1$ dimensional set $\Omega(t)$ which is the radial projection of $\Omega \cap S(x, t)$ onto the unit sphere. This can be defined by $\alpha(\alpha+n-2) = \lambda$ where

$$\lambda(\Omega(t)) = \inf \frac{\int |\nabla_S f|^2\, d\sigma}{\int |f|^2\, d\sigma},$$

where the "inf" is over all Lipschitz, nonnegative functions vanishing off $\Omega(t)$ and $\nabla_S f$ denotes the spherical gradient of f. The constant $-\lambda$ is also the first eigenvalue for the Dirichlet problem with vanishing boundary conditions, at least if $\Omega(t)$ is smooth enough. If f is the eigenfunction corresponding to λ then $u(x) = |x|^\alpha f(x/|x|)$ is harmonic in the cone defined by $\Omega(t)$ iff $\alpha(\alpha+n-2) = \lambda$. This is because for a homogeneous function u the spherical Laplacian is given by $\Delta_S = -u_{rr} - (n-1)u_r$ on the unit sphere. Thus $\Delta_S f = (-\alpha(\alpha-1) - (n-1)\alpha) f = -\alpha(\alpha+n-1) f$. See [14] for details.

To deduce our estimate from this result, let $\tilde{\Omega} = (\Omega \setminus \overline{B(x, r)}) \cup \{1/2 < |z-x| < 2\}$ and let $v(z) = \omega(z, S(x, r), \tilde{\Omega})$ for $z \in \tilde{\Omega}$ and $v(z) = 0$ elsewhere. Then v is subharmonic on $\mathbf{R}^n \setminus B(x, r)$ and an easy application of the maximum principle shows $\omega(z, B(x, r), \Omega) \leqq v(z)$ for $z \in \Omega$. Without loss of generality, suppose $x=0$ and define $u(z) = v(z/|z|^2)|z|^{2-d}$. Since u is obtained from v by a Kelvin transformation (reflection across a sphere) u is harmonic at z iff v is harmonic at $z^*=z/|z|^2$. Therefore u *is positive and subharmonic on* $B(0, R)$ $(R=1/r)$ and is zero on $\partial\Omega^*$ and equals R^{2-d} on $S(0, R)\cap\Omega^*$. Applying the result from [14] gives

$$m_1(u) \leqq C m_R(u) \exp\left(-\int_1^{R/2} \alpha(1/t)\frac{dt}{t}\right).$$

Note that $m_1(u)=m_1(v)$, $m_R(u)\leqq R^{2-d}$ and by Harnack's inequality

$$m_1(v) \sim \min_{S(0,1)} v \sim \max_{S(0,1)} v.$$

Thus for $|z|>1$ the maximum principle gives

$$\omega(z, B(0, r)\cap\partial\Omega, \Omega) \leqq \max_{S(0,1)} v \leqq C r^{d-2} \exp\left(-\int_{2r}^{1} \alpha(t)\frac{dt}{t}\right). \tag{3.2}$$

To obtain Lemma 3.3, we use Theorem 3 of [14] which states that $\alpha_i(t)\geqq 2(1-S_i(t))$ where $S_i(t)$ is the $(n-1)$ dimensional surface area of $\Omega_i(t)$ (normalized so the whole sphere has area 1). This is proven using a result of Sperner [25] that among all domains on the sphere with equal area, the spherical cap is the one with smallest characteristic constant and then estimating α for a spherical cap. Since $\Omega_1(t)$ and $\Omega_2(t)$ are disjoint we have $S_1(t)+S_2(t)\leqq 1$, and hence $\alpha_1(t)+\alpha_2(t)\geqq 2$. Multiplying the two estimates in (3.2) and using $\exp\left(-\int_r^1 dt/t\right)=r$ proves Lemma 3.3.

If we apply the arguments of the above paragraphs to the Green's functions G_1 and G_2 of two disjoint domains we see that

$$\left(\max_{B(x,r)} G_1(z)\right)\left(\max_{B(x,r)} G_2(z)\right) \leqq C r^{2(n-1)}.$$

This implies that at least one of the two functions tends to 0 at x and hence x is regular for the Dirichlet problem on at least one of the domains. This is a fact which we used earlier. A similar argument shows that if Ω_1 and Ω_2 are disjoint, unbounded domains, their harmonic measures cannot both have point masses at infinity. Therefore the point at infinity is not important in deciding whether a domain is Poissonian or not.

To prove Corollary 1.4 suppose E has zero $(n-1)$ dimensional measure and let $\Omega=\mathbf{R}^n\setminus E$. Suppose Ω_1 and Ω_2 are disjoint subdomains with $\partial\Omega_1\cap\partial\Omega_2\subset E$. Fix ε small and let $\mathscr{D}=\{B_j\}=\{B(x_j, r_j)\}$ be a covering of E with $\sum r_j^{n-1}<\varepsilon$. If $\omega_1(E)=0$ we are done so suppose $\omega_1(E)>0$. Let $\mathscr{C}\subset\mathscr{D}$ be the subcollection of balls such that $\omega_1(B_j)\geqq r_j^{n-1}$. By Lemma 3.3 $\omega_2(B_j)\leqq C r^{n-1}$ for $B_j\in\mathscr{C}$. Thus $\mathscr{C}$ covers a subset F of E with $\omega_1(F)\geqq\omega_1(E)-\varepsilon$ and

$$\omega_2(F) \leqq \sum\nolimits_{\mathscr{C}} \omega_2(B_j) \leqq C \sum\nolimits_{\mathscr{C}} r_j^{n-1} \leqq C\varepsilon.$$

Taking $\varepsilon\to 0$ we see that $\omega_1\perp\omega_2$. Thus Ω is Poissonian.

Recall that a real valued function A on $\mathbf{R}^n$ is called Lipschitz if there is a constant $C>0$ such that $|A(x)-A(y)|\leq C|x-y|$ for all $x, y\in\mathbf{R}^n$. A Lipschitz graph in $\mathbf{R}^n$ is a set of the form $\{(x, A(x)): x\in\mathbf{R}^{n-1}\}$ where A is a Lipschitz function on $\mathbf{R}^{n-1}$.

If $\Lambda_{n-1}(E)=0$ then Ω is Poissonian by Corollary 1.4. To prove the other direction of Corollary 1.5, suppose $E=\{(x, A(x)): x\in F\}$ and let Ω_1 be the region above the graph of the Lipschitz function $A(x)$ and Ω_2 the region below it. Then the harmonic measures are mutually absolutely continuous because of Dahlberg's theorem that harmonic measure on a Lipschitz domain and $(n-1)$ dimensional measure are mutually absolutely continuous ([11], [18]). Using Lemma 3.2 we may assume $\partial\Omega_1\cap\partial\Omega_2\subset E$ and so Ω is not Poissonian.

4. Poissonian domains in $\mathbf{R}^2$

We start by reviewing some related material from [6] and [8]. Suppose Ω_1 and Ω_2 are domains and that $x\in\partial\Omega_1\cap\partial\Omega_2$. We say x satisfies a double cone condition with respect to the pair Ω_1, Ω_2 if there exists δ, $\varepsilon>0$ and $\theta\in[0, 2\pi)$ such that

$$C(x, \delta, \varepsilon, \theta) = \{x+re^{i\psi}: 0<r<\delta, |\theta-\psi|<\varepsilon\}\subset\Omega_1$$

and $C(x, \delta, \varepsilon, -\theta)\subset\Omega_2$. The point x is called a tangent point if there is a fixed θ for which can take ε as close to π as we wish (if δ is small enough). Up to a set of Λ_1 measure zero, the set of tangent points and the set of points satisfying the DCC are the same. From [8] we have

Lemma 4.1. *If Ω_1 and Ω_2 are simply connected domains with harmonic measures ω_1 and ω_2 then $\omega_1\perp\omega_2$ iff the set of points satisfying a DCC with respect to Ω_1, Ω_2 has zero Λ_1 measure.*

Simply connected Poissonian domains in $\mathbf{R}^2$ were considered by Glicksberg in [15] in connection with certain function algebras. He called them "nicely connected" and defined them as those domains for which the Riemann mapping from the unit disk to Ω is 1-1 on a full measure subset of $\mathbf{T}$. For further details see [7] and its references.

It follows from the lemma that if ω_1 and ω_2 are not mutually singular then $\partial\Omega_1\cap\partial\Omega_2$ "looks like" a Lipschitz graph. More precisely, if ω_1 and ω_2 are not singular then for any $\varepsilon>0$ we can find real valued functions f_1 and f_2 on $[-1, 1]$ such that

(1) f_i are Lipschitz with constant ε, $i=1, 2$.
(2) $f_1=f_2$ except on a set of length $\leq\varepsilon$.

Moreover, if

$$D_1 = \{z\colon |z|<1,\ \mathrm{Im}\,(z)>f_1(\mathrm{Re}\,(z))\}$$

and

$$D_2 = \{z\colon |z| < 1,\ \mathrm{Im}\,(z) < f_2(\mathrm{Re}\,(z))\}$$

then after translating, rotating and dilating $\Omega_1\cup\Omega_2$ we have $D_i\subset\Omega_i$ for $i=1,2$. The subarc of ∂D_i corresponding to the graph of f_i will be denoted Γ_i. See [6] for details.

Note in particular that if ω_1 and ω_2 are not mutually singular then $\partial\Omega_1\cap\partial\Omega_2$ contains a positive length subset of a Lipschitz graph.

Given a set $E\subset\mathbf{R}^2$ with $0<\Lambda_1(E)<\infty$ we call a point $x\in E$ Besicovitch regular if

$$\lim_{r\to 0}\frac{\Lambda_1(E\cap B(x,r))}{r} = 1$$

and irregular otherwise. The set E is called Besicovitch regular if a.e. (Λ_1) point of E is regular. E is called irregular if a.e. point is irregular. One can show that a set is regular iff it consists of a subset of zero Λ_1 measure plus a subset of a countable union of rectifiable curves. Conversely, the intersection of an irregular set with any rectifiable curve has Λ_1 measure zero. Furthermore, any set E with $\Lambda_1(E)<\infty$ can be divided into two sets, one of which is regular and the other irregular. For the proofs of these facts and further details see [13].

We now prove Corollary 1.8. Suppose Ω_1 and Ω_2 are two general domains whose harmonic measures ω_1 and ω_2 are not singular. Since Ω_2 is connected it is contained in exactly one of Ω_1's complementary components. Let $\tilde\Omega_1$ be the simply connected domain containing Ω_1 obtained by removing all the complementary components of Ω_1 except the one containing Ω_2. Similarly we define $\tilde\Omega_2$ by removing all of Ω_2's complementary components except the one containing Ω_1. (More precisely, let F_1 be the component of $\partial\Omega_1$ separating Ω_1 from Ω_2. Define F_2 similarly. Then $F_1\cap F_2=\partial\Omega_1\cap\partial\Omega_2\neq\emptyset$, so $F_1\cup F_2$ is connected. Thus the components $\tilde\Omega_1$ and $\tilde\Omega_2$ of $(F_1\cup F_2)^c$ containing Ω_1 and Ω_2 are disjoint, simply connected domains and $\partial\tilde\Omega_1\cap\partial\tilde\Omega_2=F_1\cap F_2$.) The harmonic measures $\tilde\omega_1$ and $\tilde\omega_2$ for the new domains cannot be mutually singular (if they were, then so would ω_1 and ω_2 by the maximum principle). Therefore $\partial\tilde\Omega_1\cap\partial\tilde\Omega_2$ hits a Lipschitz graph in positive length. Thus $\partial\Omega_2\cap\partial\Omega_2$ contains a Besicovitch regular set of positive length. This proves Corollary 1.8.

Let $K(x)=\log\frac{1}{|x|}$. Recall from Section 2 that

$$I(\mu) = \int K(|x-y|)\,d\mu(y)\,d\mu(x)$$

$$\mathrm{cap}\,(E) = (\inf_{\Pr(E)} I(\mu))^{-1}$$

and that there exists a unique probability measure $\bar\mu$ which minimizes the energy integral. It also has the property that for p.p. $z \in E$

$$u_{\bar{\mu}}(z) = \int \log |z-w|^{-1} d\bar{\mu}(w) = I(\bar{\mu}) = \operatorname{cap}(E)^{-1}.$$

If Ω is a domain a theorem of Wiener [27] says that $x \in \partial\Omega$ is regular for the Dirichlet problem on Ω iff

$$\sum_{n=1}^{\infty} n \cdot \operatorname{cap}\left(\partial\Omega \cap \{2^{-n} \leqq |z-x| \leqq 2^{-n+1}\}\right) < \infty.$$

We will not need this result, but the computations we will do are quite similar to those in the proof of Wiener's theorem. (Note however that the series in Wiener's theorem is not quite the same as in Theorem 1.6.)

Recall from the introduction that given δ, $\varepsilon > 0$ and $\theta \in [0, 2\pi)$ we define a cone and wedge

$$C(x, \delta, \varepsilon, \theta) = \{x + re^{i\psi}:\ 0 < r < \delta,\ |\theta - \psi| < \varepsilon\}$$

$$W(x, \delta, \varepsilon, \theta) = C(x, \delta, \varepsilon, \theta) \setminus C(x, \delta/2, \varepsilon, \theta).$$

Given an open sets Ω_1, Ω_2 we define

$$\gamma_i(k) = \operatorname{cap}\left(2^{k-1}(W(x, 2^{-k}, \varepsilon, (-1)^{i+1}\theta) \setminus \Omega_i)\right)$$

and say $x \in \partial\Omega_1 \cap \partial\Omega_2$ satisfies a weak double cone condition if there exists ε and θ such that

$$\sum_{k=1}^{\infty} \left(\gamma_1(k) + \gamma_2(k)\right) < \infty.$$

Now we start the proof of Theorem 1.6. We will start by proving that if Ω is not Poissonian then the set of points satisfying the WDCC must have positive Λ_1 measure. So let Ω_1 and Ω_2 be, as usual, two disjoint subdomains with non-singular harmonic measures ω_1 and ω_2. As in the paragraph above we construct simply connected domains $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ containing Ω_1, Ω_2 and then the corresponding Lipschitz subdomains D_1 and D_2. Let $\tilde{D}_i = \Omega_i \cap D_i$ for $i = 1, 2$. Let Φ_1 denote a Riemann mapping of D_1 onto the upper half plane, say with Γ_1 going to the interval $[-1, 1]$. Since D_1 is a Lipschitz domain the mapping Φ is conformal almost everywhere on ∂D_1, i.e., if θ is the inward normal angle at a boundary point x and r is small enough then the image of a cone $C(x, r, \varepsilon, \theta)$ contains and is contained in a cone at $\Phi(x) \in \mathbf{R}$, centered on a vertical line segment (see e.g., [24]). Moreover the series for C diverges if the corresponding series for a cone inside $\Phi(C)$ diverges, and it converges if the series for a cone containing $\Phi(C)$ converges (we are using the fact that the capacity is changed by at most a constant factor under a smooth mapping. Since the mapping in question is conformal, the Koebe 1/4 theorem provides the necessary uniform bound on the derivative).

Now let $E \subset \Gamma_1 \cap \Gamma_2$ be a set where the harmonic measures for Ω_1 and Ω_2 are mutually continuous. We have seen that on this set these measures are also mutually absolutely continuous with Λ_1. Therefore E has positive length and $\Phi(E)\subset\mathbf{R}$ has positive length and positive harmonic measure in $U_1=\Phi(\tilde{D}_1)$. If we can show that a.e. $x\in\Phi_1(E)$ is the vertex of a "vertical cone" with convergent Wiener series, we deduce the same for a.e. $x\in E$ with cones in the direction normal to Γ_1 at x. Applying the same argument to D_2 and using the fact that Γ_1 and Γ_2 agree on a set of positive length (and hence have opposite inward normals on a set of positive length) finishes the proof of this direction.

Thus we need only the first part of:

Lemma 4.2. *Suppose Ω is a subdomain of* $\mathbf{H}$, *the upper half-plane, and let ω denote harmonic measure on $\partial\Omega$. Suppose $E\subset\mathbf{R}$ and $\Lambda_1(E)>0$. Then $\omega(E)=0$ if for a.e.(Λ_1) $x\in E$ there is a cone $C(x, 1, \varepsilon, \theta)$ in the upper half-plane for which the corresponding series diverges. Conversely, if a.e. $x\in E$ is the vertex of a cone in* $\mathbf{H}$ *with a convergent series then $\omega(E)>0$.*

This will be proven in the next section. Note that if a.e. $x\in E$ has some "convergent cone" then every cone is convergent for a.e. $x\in E$. The other direction of Theorem 1.6 will follow from the second claim in Lemma 4.2. We want to show that if the set of points satisfying a WDCC has positive Λ_1 measure then Ω is not Poissonian. Our first step is to prove:

Lemma 4.3. *Suppose Ω is a domain, and the WDCC is satisfied on a set E with $\Lambda_1(E)<\infty$. Then E is Besicovitch regular.*

The proof of this just uses a few basic facts about regular and irregular sets from [13]. The first fact is that if $\Lambda_1(E)<\infty$ then for Λ_1 a.e. $x\in E$ ([13, Corollary 2.5])

$$\frac{1}{2} \leqq \limsup_{r\to 0} \frac{\Lambda_1(E\cap D(x, r))}{2r} \leqq 1. \tag{4.1}$$

The second fact is that if E is an irregular set then for Λ_1 a.e. $x\in E$ and any $\varepsilon>0$ and $\theta\in[0, 2\pi)$ ([13, Corollary 3.30])

$$\limsup_{r\to 0} \frac{\Lambda_1(E\cap C(x, r, \varepsilon, \theta))}{2r} + \limsup_{r\to 0} \frac{\Lambda_1(E\cap C(x, r, \varepsilon, -\theta))}{2r} \geqq \frac{\varepsilon}{10}. \tag{4.2}$$

Choose a set F with $0<\Lambda_1(F)<\infty$ where a WDCC is satisfied. Suppose F is Besicovitch irregular. By passing to a positive measure subset if necessary, we may assume (4.1) and (4.2) hold on F and that

$$\Lambda_1(F\cap B(x, r)) \leqq 4r \tag{4.3}$$

for all $x \in F$ and $r \leqq r_0$. Now fix $x \in F$ and let ε and θ be as in the WDCC at x. By (4.2) (and replacing θ by $-\theta$ if necessary) we can choose a small $r \leqq r_0$ such that

$$\Lambda_1(E \cap C(z, r, \varepsilon, \theta)) \geqq \frac{\varepsilon r}{40}.$$

Then by (4.3)

$$\Lambda_1(E \cap (C(x, r, \varepsilon, \theta) \setminus B(x, r\varepsilon/320))) \geqq \frac{\varepsilon r}{80}$$

if r is small enough. But this implies

$$\Lambda_1(E \cap W(x, 2^{-n}, \varepsilon, \theta)) \geqq \left(\frac{r\varepsilon}{160}\right) \Big/ \log_2(\varepsilon/320)$$

for some $r\varepsilon/320 \leqq 2^{-n} \leqq r$. The capacity of $2^{n-2}(E \cap W(x, 2^{-n}, \varepsilon, \theta))$ is easily seen to be bounded below by some absolute constant A by using (4.3) to estimate the potential for $\Lambda_1|_E$. Thus the corresponding term in the series is bounded below by A. Since this happens infinitely often, the series must diverge, a contradiction. Thus F must be regular and the Lemma 4.3 is proven.

Now suppose $E \subset \partial\Omega$ is a set of positive Λ_1 measure where the WDCC holds. By dividing E into a countable number of subsets and choosing one of positive measure we may also assume we have the same ε and θ for every $x \in E$. By the previous lemma we may assume E lies on a rectifiable curve, even on a Lipschitz graph Γ with small constant. Since almost every point of E is a point of density, the cones must be disjoint from Γ for a.e. x. By taking a small neighborhood around a point of density of E and recalling we may assume Γ is the graph of a Lipschitz function on $[-1, 1]$. Let D_1 be the part of $B(0, 1)$ lying above Γ and D_2 the part lying below. Let Φ_1 be a Riemann mapping from D_1 to the upper half-plane which maps $\Gamma \cap \partial D_1$ to $[-1, 1]$. Since each point of E has a cone in D_1 for which the Wiener series converges, a.e. point of $\Phi_1(E)$ has a cone in the upper half-plane with a convergent series. If $\tilde{D}_1 = D_1 \cap \Omega$ and $U_1 = \Phi_1(\tilde{D}_1)$ then the second part of Lemma 4.2 implies $\Phi_1(E)$ has positive harmonic measure in U_1, hence in $\tilde{D}_1$. In fact the harmonic measure for $\tilde{D}_1$ is mutually absolutely continuous with Λ_1 on E. The same argument applies to $\tilde{D}_2$ so the harmonic measures for $\tilde{D}_1$ and $\tilde{D}_2$ are not singular. Hence Ω is not Poissonian.

The proof of Theorem 1.7 is clearly similar.

5. Proof of Lemma 4.2

First we will show that if every point of E is the vertex of a cone in $\mathbf{H}$ with a divergent series, then the harmonic measure of E in Ω is zero. Suppose $i\in\Omega$ and fix an $x\in E$. We will show that

$$\frac{\omega(i, B(x, r)\cap\partial\Omega, \Omega)}{r} \to 0$$

as $r\to 0$. Since E has finite length and $\Omega\subset\mathbf{H}$, this implies $\omega(E)=0$. In this section, if $E\subset\mathbf{R}$ we will let $|E|=\Lambda_1(E)$ denote its Lebesgue measure. The notation $a\sim b$ will mean the ratio a/b is bounded and bounded away from zero by some absolute constant.

Fix a $x\in E$ with a cone $C(x)=C(x, 1, \varepsilon, \theta)$ for which the associated series diverges. For each integer $k>0$ let $A(k)=\partial\Omega\cap C(x)\cap\{2^{-k-1}\leqq|z-x|\leqq 2^{-k}\}$ and let $B(k)=2^{k-2}A(k)$ (since x is fixed we will omit it from our notation to simplify matters). Let $\gamma(k)=\operatorname{cap}(B_k)$. Then our hypothesis is that $\sum_k \gamma(k)=\infty$.

Let μ_k be the equilibrium measure for $B(k)$. This is the unique probability measure supported on $B(k)$ which minimizes the energy integral discussed in the last section. It also has the property that its potential

$$u_{\mu_k}(z) = \int \log|z-w|^{-1}\, d\mu_k(w)$$

is equal to $\operatorname{cap}(B(k))^{-1}$ p.p. on $B(k)$. Now set

$$\begin{aligned}\tilde{F}_k(z) &= \int \log\left|\frac{z-\bar{w}}{z-w}\right| d\mu_k(w)\\ &= \int \log|z-w|^{-1}\, d\mu|k(w) + \int \log|z-\bar{w}|\, d\mu_k(w)\\ &= u_{\mu_k}(z) + \int \log|z-\bar{w}|\, d\mu_k(w).\end{aligned}$$

Then $\tilde{F}_k$ is positive and harmonic on $\mathbf{H}\setminus B(k)$, is zero on $\mathbf{R}$ and near ∞, so takes its maximum on $B(k)$. For $z, w\in B(k)$, $|z-\bar{w}|\sim 1$, so the second term in the last line above is uniformly bounded (depending only on the cone) whereas the first term (the potential) is like $\gamma(k)^{-1}$ on $B(k)$. By replacing $B(k)$ by a subset whose capacity is smaller by a fixed constant we may arrange for the series to still diverge, but also

$$\frac{1}{2}\gamma(k)^{-1} \leqq \tilde{F}_k(z) \leqq 2\gamma(k)^{-1} \tag{5.1}$$

for $z\in B(k)$.

We also need to estimate $\tilde{F}_k(z)$ for z far away from $B(k)$. Let ϱ denote the hyperbolic metric on **H**. Note that if $w\in B(k)$ and $\varrho(z, A(k))\geqq 1$, then

$$\tilde{F}_k(z)\sim \|\mu_k\| \log\left|\frac{z-\bar{w}}{z-w}\right|.$$

Since $\|\tilde{\mu}\|=1$, if either $|z|\geqq 2^n$ or $\operatorname{Im}(z)\leqq 2^{-n}$ we have $\tilde{F}_k(z)\leqq C2^{-n}$ for some absolute constant C. Now let

$$F_k(z) = \tilde{F}_k(2^{k-2}z).$$

Then F_k is positive and harmonic on $\mathbf{H}\setminus A(k)$, is approximately $\gamma(k)^{-1}$ on $A(k)$ and for $z\in A(n)$, $F_k(z)\sim 2^{-|n-k|}$. For $|z|=1$, $F_k(z)\sim 2^{-k}$.

Now choose an integer m such that

$$1 \leqq \sum_{k=1}^{m} \gamma(k) \leqq 2. \tag{5.2}$$

This can be done since each term in the series is less than $|\log(\operatorname{diam}(B(k)))|<1$ and the series diverges. Define F by

$$F(z) = \sum_{k=1}^{m} 2^{k-m}\gamma(k)F_k(z).$$

Then F is positive and harmonic on $D=\mathbf{H}\setminus\bigcup_1^m A(k)$, is zero on **R**, is greater than 2^{k-m-1} on each $A(k)$ (since $F_k(z)\geqq\gamma(k)^{-1}/2$ there). Next we want to find an upper bound for F on each $A(k)$.

So fix a $1\leqq n\leqq m$ and a $z\in A(n)$. Write

$$\begin{aligned} F(z) &= \sum_{k=1}^{n-2} 2^{k-m}\gamma(k)F_k(z)+\sum_{k=n-1}^{n+1} \gamma(k)2^{k-m}F_k(z)+\sum_{k=n+2}^{m} 2^{k-m}\gamma(k)F_k(z) \\ &= I_1+I_2+I_3. \end{aligned}$$

By (5.1), each term of I_2 is bounded by $C2^{n-m}$ so I_2 is also bounded by this. Next,

$$I_1 \leqq \sum_1^{n-2} 2^{k-m}\gamma(k)F_k(z) \leqq C\sum_1^{n-2}\gamma(k)2^{k-m+k-n} \leqq C2^{n-m}\sum_1^m\gamma(k) \leqq C2^{n-m}$$

since $2k-n\leqq n$ and by (5.2). Finally,

$$I_3 \leqq \sum_{n+2}^{m} 2^{k-m}\gamma(k)F_k(z) \leqq C\sum_{n+2}^{m} 2^{k-m+n-k}\gamma(k) \leqq C2^{n-m}.$$

Thus $F(z)\sim 2^{n-m}$ for $z\in A(n)$. Also note if $|z|=1$ then,

$$\begin{aligned} F(z) &\geqq \sum_1^m 2^{k-m}\gamma(k)F_k(z) \geqq C^{-1}\sum_1^m 2^{k-m-k}\gamma(k) \\ &\geqq C^{-1}2^{-m}\sum_1^m\gamma(k) \geqq C^{-1}2^{-m}. \end{aligned}$$

by our choice of m. Let $J=(x-2^{-m}, x+2^{-m})$. Since the harmonic measure in **H** of this interval looks like 2^{-m} at i a more careful version of the above argument gives a $M>0$ such that

$$\omega(z, J, \mathbf{H}) \leqq MF(z), \quad |z| = 1.$$

Note that if $z\in A(n)$ then $\omega(z, J, \mathbf{H})\sim 2^{n-m}$. Since the same is true for F, we may also assume M satisfies $F(z)\leqq M\omega(z, J, \mathbf{H})$ for $z\in A(n)$. Now let

$$u(z) = \omega(z, J, \mathbf{H}) - M^{-1}F(z).$$

Then u is harmonic on D, equals 1 on J and 0 on $\mathbf{R}\setminus J$ and is positive on every $A(k)$. Thus by the maximum principle,

$$\omega(z, J, D) \leqq u(z)$$

for $z\in D$ and in particular, for $|z|=1$,

$$\omega(z, J, D) \leqq \omega(z, J, \mathbf{H}) - M^{-1}F(z) \leqq (1-M^{-2})\,\omega(x, J, \mathbf{H}).$$

The same argument works for any interval J with $|J|\leqq 2^{-m}$ if replace the coefficient 2^{k-m} in the definition of F by $|J|2^k$.

Now we define a new domain D_2 by $D_2 = \mathbf{H}\setminus\bigcup_1^{m_2} A(k)$ where m_2 has been chosen so

$$1 \leqq \sum_{k=m+2}^{m_2} \gamma(k) \leqq 2.$$

The argument above can be easily modified to show that if $J_2 = (x-2^{-m_2}, x+2^{-m_2})$

$$\omega(z, J_2, D_2) \leqq (1-M^{-2})\,\omega(z, J_2, D)$$

for $|z-x| = 2^{-m-1}$ and hence everywhere in $D\cap\{|z-x|>2^{-m-1}\}$ (the inequality elsewhere on ∂D is obvious). By the maximum principle,

$$\omega(i, J_2, D_2) \leqq (1-M^{-2})\,\omega(i, J_2, D) \leqq (1-M^{-2})^2\,\omega(i, J_2, \mathbf{H}).$$

The obvious induction argument gives us intervals $\{J_n\}$ shrinking down to x such that

$$\omega(i, J_n, \Omega) \leqq (1-M^{-2})^n |J_n|.$$

Doing this for every $x\in E$ and using Vitali's covering theorem, we see that for every $\delta>0$ we can cover almost all of E by intervals $\{J_j\}$ such that

$$\sum_j \omega(J_j) \leqq \delta \sum_j |J_j| \leqq 2\delta|E|$$

and hence $\omega(E)=0$ as desired.

Now we turn to the proof of the other direction of Lemma 4.2: if $|E|>0$ and every point of E has a cone for which the Wiener series converges, then E has positive harmonic measure in Ω. For $x\in E$ and $k>0$ an integer we define $A(x, k)$, $\gamma(x, k)$, $F_{x,k}(z)$ as before and set

$$F_x(z) = \sum_{k=1}^{m} 2^k \gamma(x, k) F_{x,k}(z).$$

By dividing E up into a countable number of pieces, we may assume the same ε and θ work for all points of E. We may also assume

$$\sum_{k=1}^{\infty} \gamma(x, k) \leqq A < \infty$$

for some fixed A and every $x \in E$. Furthermore, it is easy to see that E has positive harmonic measure in Ω iff it does in $\Omega_\lambda = \Omega \cup \{\mathrm{Im}(z) > \lambda\}$ for all small $\lambda > 0$ (use Lemma 3.2 and the maximum principle. Thus we need only show that E has positive harmonic measure in $\Omega = \Omega_\lambda$ for some λ. Therefore we may assume that $\partial\Omega \subset \{0 \leqq \mathrm{Im}(z) \leqq \lambda\}$. This also means

$$\sum_{k=1}^{\infty} \gamma(x, k) \leqq \delta, \quad x \in E$$

may be assumed as small as we wish. Since $\gamma(x, k)$ is small, we have inequalities $F_{x,k}(z) \sim \gamma(x, k)^{-1}$ for $z \in A(x, k)$ as before.

Note that $F_x(z) \geqq M^{-1} P_x(z)$ for $z \in A(x, k)$ and some fixed $M > 0$ where $P_x(z)$ is the Poisson kernel on $\mathbf{H}$ with a pole at x. This holds because $P_x(z) \sim 2^k$ on $A(x, k)$ and $F_x(z) \geqq 2^k \gamma(x, k) F_{x,k}(z) \geqq 2^{k-1}$ there. Now define $u_x(z) = P_x(z) - MF_x(z)$ and let $v(z) = \omega(z, E, \mathbf{H})$. Then by (5.1)

$$u(z) = \int_E u_x(z)\,dx = v(z) - M \int_E F_x(z)\,dx.$$

Then u is harmonic on Ω and is 0 on $\mathbf{R} \setminus E$. Note that $u_x(z)$ is negative if $z \in A(x, k)$ by our choice of M and this happens iff $x \in I(z)$ where $I(z)$ is an interval with $|I(z)| \sim \mathrm{Im}(z)$ and $\mathrm{dist}(z, I(z)) \sim \mathrm{Im}(z)$. The constants in these "$\sim$"'s depend only on the fixed ε and θ we are considering. Thus if $z \in A(x, k)$ for some x and k

$$u(z) \leqq \int_{E \setminus I(z)} P_x(z)\,dx \leqq \omega(z, E \setminus I(z), \mathbf{H}) \leqq 1 - \eta.$$

for some $\eta > 0$ which only depends on ε and θ.

On the other hand $u(i)$ must be very close to $\omega(i, E, \mathbf{H})$. This is because for each $x \in E$

$$F_x(i) \leqq \sum_{k=1}^{m} 2^k \gamma(x, k) F_{x,k}(i) \leqq C \sum_{k=1}^{m} 2^{k-k} \gamma(x, k) \leqq C\delta$$

by our earlier remarks. Thus $u(i) \geqq v(i) - C\delta$ with $\delta \to 0$ as $\lambda \to 0$. Choose a compact $E_0 \subset E$ of positive measure such that v has non-tangential limit 1 uniformly on E_0 (we can do this since v has non-tangential limit 1 a.e. on E). Let D be the union of all the cones under consideration with vertices in E_0. D is a Lipschitz domain with E_0 in its boundary and E_0 has positive harmonic measure in D since it has positive length (by the F. and M. Riesz theorem). Let $F_1 = \partial\Omega \cap D$ and $F_2 = \partial\Omega \cap \mathbf{H} \setminus D$. Since $u(z) \leqq v(z)$ for $z \in F_2$ and $u(z) \leqq 1 - \eta$ for $z \in F_1$ we have by the maximum principle

$$u(z) \leqq v(z) + \sup_{F_1}(1 - v) - \eta\omega(z, F_1, \Omega_\lambda).$$

Using $z = i$, $\delta \to 0$ and $v(i) - C\delta \leqq u(i)$ we have

$$\omega(i, F_1, \Omega_\lambda) \leqq \eta^{-1}\big(\sup_{F_1}(1 - v) + C\delta\big).$$

η is fixed and the right-hand side goes to 0 as λ does, so $\omega(z, F_1, \Omega_\lambda) \to 0$ as $\lambda \to 0$.

On the other hand E_0 has positive harmonic measure in D, say $>\nu$ (with respect to i), and hence in $\mathbf{H}\setminus F_2$. Therefore if λ is chosen so small that $\omega(i, F_1, \Omega_\lambda) \leqq \nu/2$ we must have by the maximum principle

$$\omega(i, E_0, \Omega_\lambda) \geqq \omega(i, E_0, \mathbf{H}\setminus F_2) - \omega(i, F_1, \Omega_\lambda) \geqq \nu/2.$$

Since $E_0 \subset E$, this completes the proof of Lemma 4.2.

6. Remarks

In this final section I would like to make a few remarks about the Martin boundary of Ω. A minimal harmonic function φ on Ω is a positive harmonic function with the property that if ψ is another positive harmonic function on Ω such that $\psi \leqq \varphi$ then $\psi = \lambda\varphi$ for some $\lambda > 0$. For example, on the unit disk the minimal harmonic functions are just the Poisson kernels corresponding to different points on the unit circle. On a general domain, just as on the disk, any positive harmonic function has a unique integral representation in terms of the minimal harmonic functions and, in particular, every bounded harmonic function u can be represented as

$$u = \int_{\Delta_1} K_x f(x)\, d\mu(x)$$

where Δ_1 is the set of minimal functions, μ is the measure representing the constant function 1 and $f \in L^\infty(\mu)$. Moreover, μ a.e. minimal function is unbounded in every neighborhood of exactly one point of $\partial\Omega$ (although this is not true for every minimal function). Thus there is a "projection" $P\colon \Delta_1 \to \partial\Omega$ defined μ a.e. on Δ_1.

Poissonian domains are simply those for which there is a full measure subset of Δ_1 on which P is 1-1 (see e.g. [23]). What we wish to point out here is that for domains in $\mathbf{R}^2$ there is always a set of full measure in Δ_1 on which P is at most 2 to 1. This is because given three subsets of Δ_1, X_1, X_2, X_3, we can form the three harmonic functions u_1, u_2, u_3 corresponding to the characteristic functions of the sets. Since $\sum u_i \leqq 1$, the domains $\Omega_i = \{u_i > 1/2\}$ are disjoint subdomains on Ω. If $E \subset \partial\Omega_1 \cap \partial\Omega_2$ is a set where ω_1 and ω_2 are mutually continuous then we know Ω_1 and Ω_2 look like Lipschitz domains on near E. From this it is easy to see $\omega_3(E) = 0$. Thus we can find sets $E_i \subset \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_3$ such that $\omega_i(E_i) = 0$ and $\omega_j(\partial\Omega_j \setminus E_i) = 1$ for $j \neq i$. Then let $F_i = X_i \cap P^{-1}(P(X_i) \setminus E_i)$. Then $F_i \subset X_i$ has the same measure as X_i but P is at most 2 to 1 on $F_1 \cup F_2 \cup F_3$.

We should point out that it is possible for 3 disjoint planar domains to have the same boundary, e.g., the so-called "Wada lakes" (see [16, pages 143—144]). Also, the "2 to 1" nature of harmonic measures on planar domains had previously been observed by John Garnett and Peter Jones using the Green's lines of Ω and

Moore's triod theorem. (A Green's line is a path orthogonal to every level line of Green's function, i.e., a line of steepest descent. A triod is any continuum consisting of three Jordan arcs with a common endpoint. Moore's theorem states that any disjoint collection of triods in the plane must be countable).

The same result should be true in higher dimensions, but I don't know how to prove it because it is unknown under what conditions two disjoint domains in $\mathbf{R}^n$ can have mutually absolutely continuous harmonic measures. If mutual absolute continuity of the measures implied that they look "$(n-1)$ dimensional" then the 2 to 1 property would also hold. In 2 dimensions harmonic measure for any domain Ω always gives full measure to a set of dimension $\leqq 1$ [19] (in fact to a set of sigma finite Λ_1 measure) and it had been conjectured that for domains in $\mathbf{R}^n$ the same would be true with sets of dimension $n-1$. Tom Wolff has shown this is false by building a domain for which $\omega(E)=0$ for every $E\subset\partial\Omega$ with $\Lambda_{n-1+\varepsilon}(E)=0$ and some ε [28]. It should be possible to build such a domain so that the complement of its closure also has this property, but it is not clear whether the measures will be mutually absolutely continuous (probably they will not).

We can prove something weaker than 2 to 1 by using the estimate mentioned in Section 3 from [14]. Suppose $\Omega_1, \ldots, \Omega_m$ are disjoint domains in $\mathbf{R}^n$ and $x\in\bigcap_j \partial\Omega_j$. Then the estimates imply

$$\Pi_{j=i}^{m}\, \omega_j(B(x,r)) \leqq Cr^{m(n-2)} \exp\left(-\int_r^1 \sum_{j=1}^{m} \alpha_j(t)\frac{dt}{t}\right)$$

where α_i are the characteristic constants for the Dirichlet problem on the domains $\Omega_j(t)$. If $S_j(t)$ denotes the $n-1$ area of $\Omega_j(t)$ (normalized so the sphere has area 1), [14] contains the estimates (dropping the j's) $\alpha(\Omega)\geqq\varphi_\infty(S)$

$$\varphi_\infty(S) = \begin{cases} \frac{1}{2}\log\left(\frac{1}{4S}\right)+\frac{3}{2}, & 0<S\leqq\frac{1}{4} \\ 2(1-S), & \frac{1}{4}\leqq S<1 \end{cases}$$

independent of the dimension. They also give estimates depending on the dimension, which I will repeat only for $n=3$. We have $\alpha(\Omega)\geqq\varphi_3(S)$ where

$$\varphi_3(S) = \max\left(\varphi_\infty(S), \frac{1}{2}j_0\left(\frac{1}{S}-\frac{1}{2}\right)^{1/2}-\frac{1}{2}\right)$$

where $j_0=2.4048\ldots$ is the first zero of Bessel's function of order 0. The two dimensional version of this harmonic measure estimate is called Tsuji's estimate [26] (see [3] for the history of such estimates).

For a bounded domain in $\mathbf{R}^n$ and a.e. (ω) point $x\in\partial\Omega$ we have an estimate $\Omega(B(x,r))\geqq Cr^n$ for some $C>0$ (since $\partial\Omega$ has finite n measure). (In fact a result of Bourgain improves this to $r^{n-\varepsilon_n}$ [9].) Therefore if x is a common boundary point of $\Omega_1, \ldots, \Omega_m$ where this estimate holds for every domain we must have

$$mn \geqq m(n-2)+\sum_{j=1}^{m} \alpha_j .$$

Using the fact that the domains are disjoint, we see that $\sum_j S_i(t) \leqq 1$. Since the functions φ_n are convex we get

$$mn \geqq m(n-2)+m\varphi_n\left(\frac{1}{m}\right)$$

$$2 \geqq \varphi_n\left(\frac{1}{m}\right).$$

Plugging in the formulas for φ_n we see that $m \leqq 4$ for $n=3$ ($\varphi_3(1/4) \approx 1.749$ and $\varphi_3(1/5) \approx 2.045$) and $m \leqq 10$ for $n=\infty$ ($\varphi_\infty(1/10) \approx 1.958$ and $\varphi_\infty(1/11) \approx 2.005$). Thus Δ_1 always has a finite to 1 projection onto the topological boundary (a.e. with respect to harmonic measure). This is essentially just a restatement of results in [14].

Of course the estimates we have used are not sharp. The estimates on the characteristic constants involve replacing each domain $\Omega_j(t)$ by a spherical cap of the same area. But three disjoint domains on S^2, for example, need not be spherical caps. We would get $m \leqq 2$ for $n=3$ if we knew $\alpha_1+\alpha_2+\alpha_3 \geqq 6$ for any three disjoint domains on S^2. Unfortunately, this is not true. Think of $\mathbf{R}^3=(x,y,z)$ in the polar coordinates (r,θ,z) and let Ω_i $i=1,2,3$ be the domains corresponding to $0<\theta<2\pi/3$, $2\pi/3<\theta<4\pi/3$ and $4\pi/3<\theta<2\pi$. On Ω_1 $u(x,y,z)=\operatorname{Im}(x+iy)^{3/2}$ is a positive harmonic function which vanishes on $\partial\Omega_1$ and is homogeneous of degree $3/2$. Thus the characteristic constant α of $\Omega_1(t)$ is equal to $3/2$ for all t. Similarly for Ω_2 and Ω_3 since they are just rotations of Ω_1. Therefore $\alpha_1+\alpha_2+\alpha_3=9/2=4\frac{1}{2}$.

We could also get $m=2$ from following improvement of Bourgain's theorem: For any $\Omega \subset \mathbf{R}^3$ $\omega(B(x,r)) \geqq r^\beta$ for a.e. (ω) $x \in \partial\Omega$, all $r>0$ small enough and $\beta \leqq \varphi_3(1/3)+1 \approx 2.4011$. This is probably false since the critical β for this estimate is conjectured to be 2.5. However if we knew that the example in the last paragraph was extremal, i.e., $\alpha_1+\alpha_2+\alpha_3 \geqq 9/2$ for any three disjoint domains on the sphere, then we would only need the previous estimate with $\beta=2.5-\varepsilon$ for any $\varepsilon>0$. Moreover, we may also assume that the harmonic measures for Ω_1, Ω_2 and Ω_3 are pairwise mutually absolutely continuous on a set E and that x is a point of density of this set. The estimate should certainly be true with this additional hypothesis.

I will finish the paper with one last conjecture concerning harmonic measure in $\mathbf{R}^2$. Tom Wolff has proven that

$$F = \left\{x \in \partial\Omega \colon \limsup_{r\to 0} \frac{\omega(B(x,r)\cap\partial\Omega)}{r} > 0\right\}$$

has sigma finite length and full harmonic measure for any planar domain Ω (unpublished). We have also seen in Section 5 that if $E\subset\partial\Omega$ has positive length and if every point of E is a vertex of a single cone with convergent series (as in (1.1) but with only one cone) then

$$\lim_{r\to 0}\frac{\omega(B(x,r)\cap\partial\Omega)}{r}<\infty$$

Λ_1 a.e. on E. It seems possible that the converse is also true, i.e., if $0<\Lambda_1(E)<\infty$ and no point of E has a such a "convergent cone" then for ω a.e. $x\in E$

$$\limsup_{r\to 0}\frac{\omega(B(x,r)\cap\partial\Omega)}{r}=\infty$$

so that there exists $F\subset E$ with $\omega(F)=\omega(E)$, but $\Lambda(F)=0$. This is consistent with what is known in the simply connected case. It also has the following consequence which is of interest in its own right: if $\Omega\subset\mathbf{R}^2$ is a domain and $E\subset\partial\Omega$ is Besicovitch irregular then there exists $F\subset E$ with $\omega(F)=\omega(E)$ and $\Lambda_1(F)=0$. Peter Jones has pointed out that this is true in the case when $E=\partial\Omega$ satisfies a capacitary "thickness" condition: there exists $\varepsilon>0$ such that for every $x\in\partial\Omega$ and $0<r<r_0$, $\operatorname{cap}(r^{-1}(B(x,r/4)\cap\partial\Omega))\geqq\varepsilon$.

References

1. Ahlfors, L. V., *Complex Analysis*, McGraw-Hill, New York, 1973.
2. Ancona, A., Une propriété de la compactification de Martin d'un domaine euclidien, *Ann. Inst. Fourier (Grenoble)* **29** (1979), 71—90.
3. Baernstein, A., Ahlfors and conformal invariants, *Ann. Acad. Sci. Fenn.* **13** (1989), 289—312.
4. Benedicks, M., Positive harmonic functions vanishing on the boundary of certain domains in $\mathbf{R}^n$, *Ark. Mat.* **18** (1980), 53—72.
5. Beurling, A., *Études sur un problème de majoration*, thesis, Uppsala 1933; in *The collected works of Arne Beurling, I*, 1—107, Birkhäuser, Boston, 1989.
6. Bishop, C. J., *Harmonic measures supported on curves*, thesis, University of Chicago, 1987.
7. Bishop, C. J., Constructing continuous functions holomorphic off a curve, *J. Funct. Anal.* **82** (1989), 113—137.
8. Bishop, C. J., Carleson, L., Garnett, J. B. and Jones, P. W., Harmonic measures supported on curves, *Pacific J. Math.* **138** (1989), 233—236.
9. Bourgain, J., On the Hausdorff dimension of harmonic measure in higher dimensions, *Invent. Math.* **87** (1987), 477—483.
10. Carleson, L., *Selected problems on exceptional sets*, Van Nostrand, New York, 1967.
11. Dahlberg, B., Estimates of harmonic measure, *Arch. Rational Mech. Anal.* **65** (1977), 275—288.
12. Doob, J. L., *Classical potential theory and its probabilistic counterpart*, Springer-Verlag, New York, Berlin, 1984.

13. FALCONER, K. J., *The geometry of fractal sets*, Cambridge University Press, New York, 1985.
14. FRIEDLAND, S. and HAYMAN, W. K., Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions, *Comment. Math. Helv.* **51** (1976), 133—161.
15. GLICKSBERG, I., A remark on analyticity of function algebras, *Pacific J. Math.* **13** (1963), 1181—1185.
16. HOCKING, J. G. and YOUNG, G. S., *Topology*, Addison-Wesley, Reading, Massachusetts, 1961.
17. HUBER, A., Über Wachstumseigenschaften gewisser Klassen von subharmonischen Funktionen, *Comment. Math. Helv.* **26** (1952), 81—116.
18. JERISON, D. S. and KENIG, C. E., An identity with applications to harmonic measure, *Bull. Amer. Math. Soc.* **2** (1980), 447—451.
19. JONES, P. W. and WOLFF, T. H., Hausdorff dimension of harmonic measures in the plane, *Acta Math.* **161** (1988), 131—144.
20. MAKAROV, N. G., On distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.* **51** (1985), 369—384.
21. MARTIN, R. S., Minimal positive harmonic functions, *Trans. Amer. Math. Soc.* **49** (1941), 137—172.
22. MOORE, R. L., Concerning triods in the plane and junction points of plane continua, *Proc. Nat. Acad. Sci. USA* **14** (1928), 85—88.
23. MOUNTFORD, T. S. and PORT, S. C., Representations of bounded harmonic functions, *Ark. Mat.* **29** (1991), 107—126.
24. POMMERENKE, CH., *Univalent functions*, Vanderhoeck and Ruprecht, Göttingen, 1975.
25. SPERNER, E., Zur Symmetrisierung von Funktionen auf Sphären, *Math. Z.* **134** (1973), 317—327.
26. TSUJI M., *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.
27. WIENER, N., The Dirichlet problem, *J. Math. Phys.* 3 (1924), 127—147; in *Norbert Wiener: Collected works, I*, 394—413, MIT Press, Cambridge, Massachusetts, 1976.
28. WOLFF, T. H., Counterexamples with harmonic gradients, preprint, 1987.

Received Nov. 20, 1989

Christopher J. Bishop
Department of Mathematics
University of California, Los Angeles
405 Hilgard Avenue
Los Angeles, CA 90024, USA