Proper holomorphic maps between balls in one co-dimension

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Introduction

We will prove that there exists a proper holomorphic map from B^N $(N\geqq 2)$, the unit ball of $\mathbf{C}^N$, to B^{N+1} that cannot be extended in a C^∞ way to an open non-empty subset of the boundary. This map can be extended to a continuous map from $\bar{B}^N$ to $\bar{B}^{N+1}$.

E. Løw [16] and F. Forstnerič [10] found such a map from B^N $(N\geqq 2)$ to B^M when $M\gg 2N$. Josip Globevnik [12] proved that for any $N\geqq 1$ there exists $M_0\gg 2N$ so that if $E\subset bB^N$ is an interpolation set for $A(B^N)$, $M\geqq M_0$, and $f\colon E\to bB^M$ is continuous, then there exists a continuous extension $F\colon \bar{B}^N\to\bar{B}^M$ which is a proper holomorphic map from B^N to B^M. He showed that f can be chosen so that $F(bB^N)=bB^M$.

In the second and third section we will prove this result (for $N\geqq 2$) with $M_0=N+1$. This will give a positive answer to an open question by Globevnik [12].

A proper holomorphic map can not lower the dimension (Rudin [18], 15.1.3) and when $f\colon B^N\to B^N$ is a proper holomorphic map, then Alexander [1] proved that f is an inner automorphism of B^N and therefore (see [18], 2.2.5) f can be extended holomorphically to RB^N for some $R>1$. So if $f\colon B^N\to B^M$ is a "bad" proper holomorphic map (which means a map that cannot be extended holomorphically past an open non empty part of the boundary) then $M>N$. In the above mentioned examples M is always very big relative to N. So the question remained if there exists a bad proper holomorphic from B^N to B^{N+1}.

When $f, g\colon B^N\to B^M$ are proper holomorphic maps they are said to be equivalent if there exist $\varphi\in\mathrm{Aut}(B^N)$, $\psi\in\mathrm{Aut}(B^M)$ such that $\psi\circ f\circ\varphi=g$.

If $f\colon B^N\to B^{N+1}$ is a proper holomorphic map and $f\in C^3(\bar{B}^N)$ then Webster [21] proved that when $N\geqq 3$ f is equivalent to $z\to(z, 0)$ (for $z\in B^N$) and when $N=2$

Faran [7] proved that f is equivalent to one of the following maps:

(i) $z \to (z, 0)$

(ii) $z \to (z_1, z_1 z_2, (z_2)^2)$

(iii) $z \to ((z_1)^2, 2^{1/2} z_1 z_2, (z_2)^2)$

(iv) $z \to ((z_1)^3, 3^{1/2} z_1 z_2, (z_2)^3)$.

Cima and Suffridge [3] showed that the Webster and Faran results hold under the weaker assumption that $f \in C^2(B^N \cup G)$ where G is a nonempty, relatively open subset of bB^N.

In a recent paper [4] Cima and Suffridge proved the Faran result using completely elementary means. Our paper shows that there exists a proper holomorphic map from B^2 to B^3 which is not equivalent to one of the above mentioned four maps and a proper holomorphic map from B^N to B^{N+1} $(N \geqq 3)$ which is not equivalent to a linear map.

A classification of all polynomial proper holomorphic maps between balls was done in J. D'Angelo [6]. He also gave examples of polynomial proper holomorphic maps between B^2 and B^4 which are not equivalent to monomial maps [5].

A proper holomorphic map from a ball to a polydisc (in finite dimensional complex spaces) was first discovered by E. Løw who proved [16] that if $D \subset\subset \mathbf{C}^N$ $(N \geqq 2)$ is a strongly pseudoconvex domain with a smooth boundary then there exists $M \gg 2N$ so that there exists a proper holomorphic map from D to Δ^M (the unit polydisc in $\mathbf{C}^M$).

Earlier it was proved that there is no proper holomorphic map from Δ^N to B^M or from B^N to Δ^N for any $M, N \geqq 2$ (see [18]).

So the question remained if there exists a proper holomorphic map from B^N to Δ^{N+1}. A positive answer to this question in the case $N=2$ was given by Berit Stensønes [19]. This paper uses methods and ideas from her construction. I would like to thank her for describing in detail the ideas of her construction of proper holomorphic maps from a strongly pseudoconvex domain in $\mathbf{C}^2$ to Δ^3. I also thank J. Chaumat and A.-M. Chollet for helpful discussions during their visit to the Institut Mittag-Leffler.

It should be noted that all the above mentioned constructions (including this work) of irregular proper holomorphic maps are based on ideas that were originated by Hakim—Sibony [13] and E. Løw [17] in the construction of an inner function in the unit ball of $\mathbf{C}^N$ $(N \geqq 2)$.

In Sections 1 and 2 of our paper the fundamental construction lemmas (Lemma 1 and Lemma 2) are proved for dimension $N=2$ which is the relatively simple case. In Section 3 an additional tool is developed (Lemma 3) to allow generalization of the results obtained in Sections 1, 2 to dimension $N \geqq 2$.

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Section 1

Theorem 1. *Let $D=B^N$, $\varepsilon>0$, $K\subset D$ compact and $f\colon bD\to B^{N+1}$ a continuous map, $|f|>0$. Then there exists a map $g\colon \bar{D}\to \mathbf{C}^{N+1}$ continuous in $\bar{D}$ and holomorphic in D so that for all $z\in bD$ $|f(z)+g(z)|=1$ and for all $z\in K$, $|g(z)|<\varepsilon$.*

The proof of Theorem 1 shows that when a domain D admits a solution of $\bar\partial$ in the L^∞ norm then a construction of a map described in Theorem 1 depends only on the local behaviour of local peak functions on D (see Sublemma 1 and proof of Lemma 1). Our proof is almost elementary, the only non elementary result that is used is the L^∞ solution of $\bar\partial$ in B^N. The proof of Theorem 1 clearly holds when the target ball is B^M, for any $M>N$.

Using Theorem 1 we prove:

Theorem 2. *There exists a continuous map $F\colon \bar{B}^N\to\bar{B}^{N+1}$ which is holomorphic in B^N such that $F(bB^N)\subseteq bB^{N+1}$ and there is no nonempty open subset G of bB^N such that F extends to a C^2 map on $B^N\cup G$. So F is a proper holomorphic map from B^N to B^{N+1} that does not have a C^2 extension to any open subset of bB^N.*

Proof of Theorem 2. It follows from [3] that if $F\colon B^N\to B^{N+1}$ is a proper holomorphic map and there exists a nonempty open subset $G\subset bB^N$ such that F extends to a C^2 map on $B^N\cup G$ then F is rational, and so by Theorem 2 in the same paper, F takes affine hyperplanes of $\mathbf{C}^N$ into affine hyperplanes of $\mathbf{C}^{N+1}$. So we look for a proper holomorphic map, $F\colon B^N\to B^{N+1}$, which does not take affine hyperplanes into affine hyperplanes.

Define for $z=(z_1, \dots, z_N)\in\mathbf{C}^N$:

$$f(z) = (2N)^{-1}(1+z_1,\ 1+(z_1)^2,\ 1+(z_1)^3, \dots, 1+(z_1)^{N+1}).$$

Let $w_1, w_2, \dots, w_{N+1}\in\mathbf{C}$, where $0<|w_i|<1/2$ and $w_i\neq w_j$ for $i\neq j$ $(1\leq i,j\leq N+1)$.

It is clear that:

$$\operatorname{Det}\big(f(w_1,0)-f(0),\ f(w_2,0)-f(0), \dots, f(w_{N+1},0)-f(0)\big)\neq 0$$

(1) Let $\varepsilon>0$ be so that for any $v_1, \dots, v_{N+1}\in 2\varepsilon B^{N+1}$

$$\operatorname{Det}\big(f(w_1,0)-f(0)+v_1,\ f(w_2,0)-f(0)+v_2, \dots, f(w_{N+1},0)-f(0)+v_{N+1}\big)\neq 0.$$

Theorem 1 implies that there exists $g\colon \bar{B}^N \to \mathbf{C}^{N+1}$ continuous and holomorphic in B^N so that for all $z\in bB^N$, $|f(z)+g(z)|=1$ and for all $z\in(1/2)B^N$, $|g(z)|<\varepsilon$. If $F=f+g$ then (1) implies that

$$\operatorname{Det}\big(F(w_1,0)-F(0),\ F(w_2,0)-F(0),\ \ldots,\ F(w_{N+1},0)-F(0)\big)\neq 0.$$

So $F(\Delta\times\{0\})$ is not contained in any affine hyperplane of $\mathbf{C}^{N+1}$ and since $\Delta\times\{0\}$ is a subset of a hyperplane of $\mathbf{C}^N$, F fulfils the requirements of Theorem 2.

Since $B^{N+1}\subseteq B^M$ when $M\geqq N+1$, it is clear that Theorem 2 holds when we replace B^{N+1} with B^M for any $M\geqq N+1$.

Before we procede with the proof of Theorem 1 we will introduce the peak functions that will be used in the approximation process.

0.1. Let $D=B^N$, $Z_0\in bD$, $0<d_1<1/N$, and let $\{e_1, e_2, \ldots, e_N\}\subseteq\mathbf{C}^N$ be an orthonormal set of vectors in $\mathbf{C}^N$ such that $Z_0=(1-(N-1)(d_1)^2)^{1/2}e_N+d_1\sum_{1\leqq j\leqq N-1}e_j$.

0.2. Let $0<d<d_1\cdot 10^{-10N}$ and define:

$$\begin{aligned}
U' &= [d_1-2d,\ d_1+2d]^{N-1}\times[-2d,\ 2d]^N\times[0,\ 2d],\\
V' &= [d_1-2d,\ d_1+2d]^{N-1}\times[-2d,\ 2d]^N\times\{0\},\\
U\ &= [d_1-d,\ d_1+d]^{N-1}\times[-d,\ d]^N\times[0,\ d],\\
V\ &= [d_1-d,\ d_1+d]^{N-1}\times[-d,\ d]^N\times\{0\}.
\end{aligned}$$

We define for $X=(x_1, \ldots, x_{N-1}, y_1, \ldots, y_{N-1}, z, w)\in U'$:

0.3.
$$\begin{aligned}
Z = Z(X) &= \textstyle\sum_{1\leqq j\leqq N-1} x_j(1-w)\exp\big(i(y_j+z)\big)e_j\\
&\quad +\big(1-\textstyle\sum_{1\leqq j\leqq N-1} x_j^2\big)^{1/2}(1-w)\exp(iz)e_N.
\end{aligned}$$

(Throughout this paper $Z=(Z_1, \ldots, Z_N)$ (in capital letters) will be used to describe complex variables whenever there is a possibility of confusion with the real coordinate z.)

We will view X as a coordinate system on a neighborhood of Z_0 in $\bar{D}$ which is prescribed by 0.2, 0.3. This coordinate system is a slight variation of the standard polar coordinate system.

The choice of U' above (i.e. the choice of d, d_1 in the definition of U') is not necessary for obtaining the properties of the function that we will introduce in 0.5, but it will be important in our later construction.

If $Z\in\bar{D}$ and there is $X\in U'$ so that $Z=Z(X)$ (as defined in 0.3) then we will define $X(Z)=X$ and we define:

0.4. $W'=Z(U')$ and $W=Z(U)$.

This definition is correct since the correspondence $X\to Z(X)$ is one to one in U'. Note that W is a neighborhood of Z_0 in the topology of $\bar{D}$ and $Z(U')\cap bD=Z(V')$, $Z(U)\cap bD=Z(V)$.

Fix $\bar{X}=(\bar{x}_1, \ldots, \bar{x}_{N-1}, \bar{y}_1, \ldots, \bar{y}_{N-1}, \bar{z}, 0)\in V'$ and let $Z\in\mathbf{C}^N$ we define:

0.5.

$$u_X(Z) = 1/2\Big(1-\big(\textstyle\sum_{1\le j\le N-1}(Z_j\exp(-i(\bar y_j+\bar z))-(Z_j\exp(-i(\bar y_j+\bar z))-\bar x_j)^4)^2$$
$$+(Z_N\exp(-i\bar z)-(Z_N\exp(-i\bar z)-(1-\textstyle\sum_{1\le j\le N-1}(\bar x_j)^2)^{1/2})^4)^2\big)\Big).$$

Throughout this paper when $Z\in W'$ and $X=X(Z)$ then $u_X(X)\overset{\text{def}}{=}u_X(Z)$.

The holomorphic polynomial u_X is zero on $Z(\bar X)$ and has a positive real part elsewhere in W' (see Sublemma 1 below). In the construction of proper holomorphic maps from a strongly pseudoconvex domain in $\mathbf{C}^2$ to the unit polydisc of $\mathbf{C}^3$ by *B.* Stensønes [19] a different type of peak functions is used which have similar properties in the process of approximation (in the proof of Lemma 1). The properties of the peak function developed by B. Stensønes in [19] has motivated the search and use (in Lemma 1 and Lemma 2) of the local peak function defined by 0.5.

Sublemma 1. *Let* $Z\in W'$ *and let* $X=X(Z)$, *where*

$$X = (x_1, \ldots, x_{N-1}, y_1, \ldots, y_{N-1}, z, w),$$

then:

$$u_X(Z) = w+\textstyle\sum_{1\le j\le N-1}(x_j)^2(y_j-\bar y_j+z-\bar z)^2+x_j(x_j-\bar x_j)^4$$
$$+\big(1-\textstyle\sum_{1\le j\le N-1}(x_j)^2\big)(z-\bar z)^2+\big(1-\textstyle\sum_{1\le j\le N-1}(x_j)^2\big)^{-3/2}\big(\textstyle\sum_{1\le j\le N-1}x_j(\bar x_j-x_j)\big)^4$$
$$-i\textstyle\sum_{1\le j\le N-1}(x_j)^2(1-w)^2(y_j-\bar y_j)-i(1-w)^2(z-\bar z)+R_X(Z),$$

where the remainder term $R_X(Z)$ *is bounded in the following way:*

$$|R_X(Z)| \le 10^5\big(w+|z-\bar z|^2+\textstyle\sum_{1\le j\le N-1}|y_j-\bar y_j+z-\bar z|^2+|x_j-\bar x_j|^4\big)$$
$$\times\big(|z-\bar z|^{1/2}+\textstyle\sum_{1\le j\le N}|y_j-\bar y_j+z-\bar z|^{1/2}+w+\textstyle\sum_{1\le j\le N-1}|\bar x_j-x_j|\big)$$

and:

$$\big|\mathrm{Im}\,(R_X(Z))\big| \le 10^4\big(w+|z-\bar z|^2+\textstyle\sum_{1\le j\le N-1}|y_j-\bar y_j+z-\bar z|^2+|x_j-\bar x_j|^4\big)$$
$$\times\big(|z-\bar z|^{1/2}+\textstyle\sum_{1\le j\le N}|y_j-\bar y_j+z-\bar z|^{1/2}\big).$$

We will conclude that the remainder is small enough in W' so that:

$$\mathrm{Re}\,(u_X(Z)) \ge 99/100\big(w+\textstyle\sum_{1\le j\le N-1}((x_j)^2(y_j-\bar y_j+z-\bar z)^2+x_j(x_j-\bar x_j)^4)$$
$$+\big((1-\textstyle\sum_{1\le j\le N-1}(x_j)^2)(z-\bar z)^2+(1-\textstyle\sum_{1\le j\le N-1}(x_j)^2)^{-3/2}\big)\big(\textstyle\sum_{1\le j\le N-1}(x_j)(\bar x_j-x_j)\big)^4\big).$$

Note that $\mathrm{Re}\,(u_X)>0$ on $W'\setminus\{Z(\bar X)\}$, $u_X(Z(\bar X))=0$ and for $Z\in W'$, $R_X(Z)=o(\mathrm{Re}\,(u_X(Z)))$. This will imply that R_X is essentially negligible in the construction process of the proofs of Lemma 1, Lemma 2. The remainder term, R_X, will be obtained as a sum of five "smaller" remainders $R_{X,1}, \ldots, R_{X,5}$.

Proof. Let $Z\in W'$ and let $X=X(Z)$ where

$$X=(x_1,\dots,x_{N-1},y_1,\dots,y_{N-1},z,w)$$

then

$$Z=\sum_{1\leq j\leq N-1}x_j(1-w)\exp\big(i(y_j+z)\big)e_j+\big(1-\sum_{1\leq j\leq N-1}(x_j)^2\big)^{1/2}(1-w)\exp(iz)e_N.$$

0.6. To simplify our next calculations we define for $1\leq j\leq N-1$ $t_j=x_j(1-w)$, $\theta_j=y_j+z$ and

$$t_N=\big(1-\sum_{1\leq j\leq N-1}(x_j)^2\big)^{1/2}(1-w),\quad \theta_N=z.$$

So we have: $Z=\sum_{1\leq j\leq N}t_j\exp(i\theta_j)e_j$, here $(t_1,\theta_1,\dots,t_N,\theta_N)$ are the standard polar coordinates of Z with respect to the basis $e_1,e_2,\dots,e_N$.

Similarly we define for $1\leq j\leq N-1$:

$$\bar{t}_j=\bar{x}_j,\quad \bar{\theta}_j=\bar{y}_j+\bar{z}$$

and

$$\bar{t}_N=\big(1-\sum_{1\leq j\leq N-1}(\bar{x}_j)^2\big)^{1/2},\quad \bar{\theta}_N=\bar{z}.$$

(So for $Z^*=Z(\bar{X})$ we have $Z^*=\sum_{1\leq j\leq N}\bar{t}_j\exp(i\bar{\theta}_j)e_j$.)

0.7. Using these notations we have:

$$\begin{aligned}2u_{\bar{X}}(Z)&=1-\sum_{1\leq j\leq N}\big(t_j\exp\big(i(\theta_j-\bar{\theta}_j)\big)-\big(t_j\exp\big(i(\theta_j-\bar{\theta}_j)\big)-\bar{t}_j\big)^4\big)^2\\&=1-\sum_{1\leq j\leq N}\big((t_j)^2\exp\big(2i(\theta_j-\bar{\theta}_j)\big)\\&-2t_j\exp\big(i(\theta_j-\bar{\theta}_j)\big)\cdot\big(t_j\exp\big(i(\theta_j-\bar{\theta}_j)\big)-\bar{t}_j\big)^4+\big(t_j\exp\big(i(\theta_j-\bar{\theta}_j)\big)-\bar{t}_j\big)^8\big)\\&=2w-w^2+(1-w)^2+\sum_{1\leq j\leq N}\big(-(t_j)^2\big(\cos 2(\theta_j-\bar{\theta}_j)+\sin 2(\theta_j-\bar{\theta}_j)i\big)\\&+2t_j\big(\cos(\theta_j-\bar{\theta}_j)+\sin(\theta_j-\bar{\theta}_j)i\big)\cdot\big((t_j-\bar{t}_j)+t_j\big(\cos(\theta_j-\bar{\theta}_j)-1\big)+t_j\sin(\theta_j-\bar{\theta}_j)i\big)^4\\&-\big((t_j-\bar{t}_j)+t_j\big(\cos(\theta_j-\bar{\theta}_j)-1\big)+t_j\sin(\theta_j-\bar{\theta}_j)i\big)^8\big).\end{aligned}$$

0.8. Since (by 0.6) $\sum_{1\leq j\leq N}(t_j)^2=(1-w)^2$ then 0.7 implies:

$$\begin{aligned}2u_{\bar{X}}(Z)&=2w-w^2+\sum_{1\leq j\leq N}\big(-(t_j)^2\big(\cos 2(\theta_j-\bar{\theta}_j)-1+\sin 2(\theta_j-\bar{\theta}_j)i\big)\\&+2t_j\big(\cos(\theta_j-\bar{\theta}_j)+\sin(\theta_j-\bar{\theta}_j)i\big)\cdot\big((t_j-\bar{t}_j)+t_j\big(\cos(\theta_j-\bar{\theta}_j)-1\big)+t_j\sin(\theta_j-\bar{\theta}_j)i\big)^4\\&-\big((t_j-\bar{t}_j)+t_j\big(\cos(\theta_j-\bar{\theta}_j)-1\big)+t_j\sin(\theta_j-\bar{\theta}_j)i\big)^8\big).\end{aligned}$$

0.9. Using the trigonometric identity: $1-\cos 2A=2(\sin A)^2$ 0.8 yields:

$$\begin{aligned}2u_{\bar{X}}(Z)&=2w-w^2+\sum_{1\leq j\leq N}\big((t_j)^2\big(1-\cos 2(\theta_j-\bar{\theta}_j)-\sin 2(\theta_j-\bar{\theta}_j)i\big)\\&+2t_j\big(1-2\big(\sin(\theta_j-\bar{\theta}_j)/2\big)^2+\sin(\theta_j-\bar{\theta}_j)i\big)\\&\times\big((t_j-\bar{t}_j)-2t_j\big(\sin(\theta_j-\bar{\theta}_j)/2\big)^2+t_j\sin(\theta_j-\bar{\theta}_j)i\big)^4\\&-\big((t_j-\bar{t}_j)-2t_j\big(\sin(\theta_j-\bar{\theta}_j)/2\big)^2+t_j\sin(\theta_j-\bar{\theta}_j)i\big)^8\big)\end{aligned}$$

(after a few trivial manipulations)

$$= 2w-w^2+\sum_{1\leq j\leq N}\big((t_j)^2(2(\theta_j-\bar\theta_j)^2-2(\theta_j-\bar\theta_j)i)+2t_j(t_j-\bar t_j)^4$$
$$+(t_j)^2(1-\cos 2(\theta_j-\bar\theta_j)-2(\theta_j-\bar\theta_j)^2+2(\theta_j-\bar\theta_j)i-\sin 2(\theta_j-\bar\theta_j)i)$$
$$+2t_j(1-2(\sin(\theta_j-\bar\theta_j)/2)^2+\sin(\theta_j-\bar\theta_j)i)$$
$$\times(((t_j-\bar t_j)-2t_j(\sin(\theta_j-\bar\theta_j)/2)^2+t_j\sin(\theta_j-\bar\theta_j)i)^4-(t_j-\bar t_j)^4)$$
$$+(2t_j(-2(\sin(\theta_j-\bar\theta_j)/2)^2+\sin(\theta_j-\bar\theta_j)i)\cdot(t_j-\bar t_j)^4)$$
$$-((t_j-\bar t_j)-2t_j(\sin(\theta_j-\bar\theta_j)/2)^2+t_j\sin(\theta_j-\bar\theta_j)i)^8\big).$$

0.10. Let us define the following remainder terms:

$$R_{\bar X,1}(Z)=1/2\sum_{1\leq j\leq N}(t_j)^2(1-\cos 2(\theta_j-\bar\theta_j)-2(\theta_j-\bar\theta_j)^2+2(\theta_j-\bar\theta_j)i-\sin 2(\theta_j-\bar\theta_j)i)$$

and

$$R_{\bar X,2}(Z)=1/2\sum_{1\leq j\leq N}(2t_j(1-2(\sin(\theta_j-\bar\theta_j)/2)^2+\sin(\theta_j-\bar\theta_j)i)$$
$$\times(((t_j-\bar t_j)-2t_j(\sin(\theta_j-\bar\theta_j)/2)^2+t_j\sin(\theta_j-\bar\theta_j)i)^4-(t_j-\bar t_j)^4)$$
$$+2t_j(-2(\sin(\theta_j-\bar\theta_j)/2)^2+\sin(\theta_j-\bar\theta_j)i)(t_j-\bar t_j)^4$$
$$-((t_j-\bar t_j)-2t_j(\sin(\theta_j-\bar\theta_j)/2)^2+t_j\sin(\theta_j-\bar\theta_j)i)^8).$$

Then by 0.9:

0.11. $$2u_{\bar X}(Z)=2w-w^2+\sum_{1\leq j\leq N}(t_j)^2(2(\theta_j-\bar\theta_j)^2-2(\theta_j-\bar\theta_j)i)+2t_j(t_j-\bar t_j)^4$$
$$+2R_{\bar X,1}(Z)+2R_{\bar X,2}(Z).$$

First let us evaluate $R_{\bar X,1}(Z)$. When we look at the Taylor expansion of sine and cosine it is apparent that:

$$|2(\theta_j-\bar\theta_j)-\sin 2(\theta_j-\bar\theta_j)|\leq 2|\theta_j-\bar\theta_j|^3$$

and

$$|1-\cos 2(\theta_j-\bar\theta_j)-2(\theta_j-\bar\theta_j)^2|\leq 2|\theta_j-\bar\theta_j|^4\quad(\text{for } 1\leq j\leq N).$$

0.12. We conclude that: $|R_{\bar X,1}(Z)|\leq 2\sum_{1\leq j\leq N}|\theta_j-\bar\theta_j|^3$.

Before evaluating $R_{\bar X,2}(Z)$ we need the following simple inequality (which is designed for this evaluation).

Let $1\geq a, b\geq 0$ and take $n\geq 1$, $k\geq 0$ so that $n+k=4$ then:

0.13. $$(a^2+b^4)a^{1/2}\geq a^n b^k.$$

Proof. Put $c=a^{1/2}$, 0.13 is then equivalent to:

$$c^5+cb^4\geq c^{2n}b^k.$$

If $b\geq c$ then $cb^4\geq c^{2n}b^k$ and if $c\geq b$ then $c^5\geq c^{2n}b^k$, and 0.13 is proved.

0.14. It follows that if $1\geqq a, b\geqq 0$ and $n\geqq 1$, $k\geqq 0$, $n+k\geqq 4$ then:

$$(a^2+b^4)a^{1/2}(a+b)^{n+k-4}\geqq a^n b^k.$$

After opening brackets and collecting terms we can look at $R_{\bar{X},2}(Z)$ as a polynomial in the variables $(t_j-\bar{t}_j)$, $\sin(\theta_j-\bar{\theta}_j)/2$, $\sin(\theta_j-\bar{\theta}_j)$ $(1\leqq j\leqq N)$, where each monomial is of total degree $\geqq 4$ and the coefficients are polynomials in t_j, $\bar{t}_j$. So $R_{\bar{X},2}(Z)$ can be expressed in the following way:

0.15. $R_{\bar{X},2}(Z)=\sum_{1\leqq j\leqq N}\sum_{0\leqq k,l,m\leqq 16}a^j_{klm}(t_j-\bar{t}_j)^k(\sin(\theta_j-\bar{\theta}_j)/2)^l(\sin(\theta_j-\bar{\theta}_j))^m.$

One can easily calculate that $|a^j_{klm}|<10^8$ and, as we mentioned above, if $a^j_{klm}\neq 0$ then $k+l+m\geqq 4$. Since the term:

$$((t_j-\bar{t}_j)-2t_j(\sin(\theta_j-\bar{\theta}_j)/2)^2+t_j\sin(\theta_j-\bar{\theta}_j)i)^4-(t_j-\bar{t}_j)^4$$

does not contain a monomial which is free from $\sin(\theta_j-\bar{\theta}_j)/2$ and $\sin(\theta_j-\bar{\theta}_j)$, then the only monomial of $R_{\bar{X},2}(Z)$ which is free from $\sin(\theta_j-\bar{\theta}_j)/2$ and $\sin(\theta_j-\bar{\theta}_j)$ is $-(1/2)(t_j-\bar{t}_j)^8$. In other words, (for all $1\leqq j\leqq N$) if $a^j_{klm}\neq 0$ and $l=m=0$ then $k=8$.

Since $|\sin A|\leqq|A|$ (for all $A\in\mathbf{R}$) we have:

0.16. $|R_{\bar{X},2}(Z)|\leqq\sum_{1\leqq j\leqq N}\sum_{0\leqq k,l,m\leqq 16}|a^j_{klm}|\,|t_j-\bar{t}_j|^k|\theta_j-\bar{\theta}_j|^{l+m}.$

0.17. *Claim.* For every $1\leqq j\leqq N$ and $0\leqq k,l,m\leqq 16$ if $a^j_{klm}\neq 0$ then:

$$|t_j-\bar{t}_j|^k|\theta_j-\bar{\theta}_j|^{l+m}\leqq(|\theta_j-\bar{\theta}_j|^2+|t_j-\bar{t}_j|^4)(|\theta_j-\bar{\theta}_j|+|t_j-\bar{t}_j|)^{k+l+m-4}|\theta_j-\bar{\theta}_j|^{1/2}$$

or $l=m=0$, $k=8$.

Proof. If $l+m=0$ then (as we mentioned before) $k=8$. If $m+l\geqq 1$ then since $(l+m)+k\geqq 4$ the claim follows from 0.14.

Using the fact that for every $1\leqq j\leqq N$:

$$\sum_{0\leqq k,l,m\leqq 16,\,k+l+m=4}|a^j_{klm}|<20$$

and if $0\leqq k,l,m\leqq 16$, $k+l+m>4$ then $|a^j_{klm}|<10^8$.

0.18. We obtain that:

$$\begin{aligned}|R_{\bar{X},2}(Z)| &\leqq \sum_{0\leqq k,l,m\leqq 16}\sum_{1\leqq j\leqq N}|a^j_{klm}|\,|t_j-\bar{t}_j|^k|\theta_j-\bar{\theta}_j|^{l+m}\\ &\leqq \sum_{1\leqq j\leqq N}20(|\theta_j-\bar{\theta}_j|^2+|t_j-\bar{t}_j|^4)|\theta_j-\bar{\theta}_j|^{1/2}\\ &+\sum_{0\leqq k,l,m\leqq 16}\sum_{1\leqq j\leqq N}10^8(|\theta_j-\bar{\theta}_j|^2+|t_j-\bar{t}_j|^4)(|\theta_j-\bar{\theta}_j|+|t_j-\bar{t}_j|)|\theta_j-\bar{\theta}_j|^{1/2}\\ &\leqq 25\Big(\sum_{1\leqq j\leqq N}|\theta_j-\bar{\theta}_j|^2+|t_j-\bar{t}_j|^4\Big)\Big(\sum_{1\leqq j\leqq N}|\theta_j-\bar{\theta}_j|^{1/2}\Big)\end{aligned}$$

(0.2, 0.6 (i.e. the smallness of $(t_j-\bar{t}_j)$ and $(\theta_j-\bar{\theta}_j)$) were used in the last inequality).

Combining 0.12 and 0.18 we obtain:

0.19. $|R_{X,1}(Z)+R_{X,2}(Z)| \leqq 30\left(\sum_{1\leqq j\leqq N}|\theta_j-\bar\theta_j|^2+|t_j-\bar t_j|^4\right)\left(\sum_{1\leqq j\leqq N}|\theta_j-\bar\theta_j|^{1/2}\right).$

Now by 0.11:

0.20. $$u_X(Z) = w-(1/2)w^2+\sum_{1\leqq j\leqq N}(t_j)^2(\theta_j-\bar\theta_j)^2+t_j(t_j-\bar t_j)^4 \\ -i\sum_{1\leqq j\leqq N}(t_j)^2(\theta_j-\bar\theta_j)+R_{X,1}(Z)+R_{X,2}(Z).$$

Since $\sum_{1\leqq j\leqq N}(t_j)^2=(1-w)^2$ then

0.21. $$\sum_{1\leqq j\leqq N}(t_j)^2(\theta_j-\bar\theta_j) \\ = \left(\sum_{1\leqq j\leqq N-1}(t_j)^2(\theta_j-\bar\theta_j)\right)+\left((1-w)^2-\sum_{1\leqq j\leqq N-1}(t_j)^2\right)(\theta_N-\bar\theta_N) \\ = \left(\sum_{1\leqq j\leqq N-1}(t_j)^2\left((\theta_j-\bar\theta_j)-(\theta_N-\bar\theta_N)\right)\right)+(1-w)^2(\theta_N-\bar\theta_N).$$

This form of the imaginary part reduces the dimension of the target ball, in the construction process, by one.

We obtain from 0.20, 0.21 (see also the definitions at 0.6):

0.22.
$$u_X(Z) = w-(1/2)w^2+\sum_{1\leqq j\leqq N-1}(t_j)^2(y_j-\bar y_j+z-\bar z)^2+t_j(t_j-\bar t_j)^4 \\ +(t_N)^2(z-\bar z)^2+t_N(t_N-\bar t_N)^4-i\sum_{1\leqq j\leqq N-1}(t_j)^2(y_j-\bar y_j)-i(1-w)^2(z-\bar z) \\ +R_{X,1}(Z)+R_{X,2}(Z) \\ = w-(1/2)w^2+\sum_{1\leqq j\leqq N-1}(x_j)^2(1-w)^2(y_j-\bar y_j+z-\bar z)^2+x_j(1-w)(x_j-\bar x_j-wx_j)^4 \\ +(t_N)^2(z-\bar z)^2+t_N(t_N-\bar t_N)^4 \\ -i\sum_{1\leqq j\leqq N-1}(x_j)^2(1-w)^2(y_j-\bar y_j)-i(1-w)^2(z-\bar z)+R_{X,1}(Z)+R_{X,2}(Z).$$

Our main task is done now as we obtained a useful expression for the imaginary part of $u_X(Z)$, and we are very close to the desired expression of the real part. We also evaluated the size of the first two remainder terms which are complex functions.

Next we want to find a simple expression for $\operatorname{Re}(u_X(Z))$ in terms of the X-coordinates. We will need again to select a principle part and to separate from it remainders that will prove to be marginal. All our next remainders $R_{X,3}(Z), \ldots, R_{X,5}(Z)$ will be purely real. The most laborious part of our calculations will be the work to simplify the term $t_N(t_N-\bar t_N)^4$.

0.23. $$t_N-\bar t_N = ((t_N)^2-(\bar t_N)^2)/(t_N+\bar t_N)$$

and

$$(t_N)^2-(\bar t_N)^2 = \left(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\right)(1-w)^2-\left(1-\sum_{1\leqq j\leqq N-1}(\bar x_j)^2\right) \\ = \sum_{1\leqq j\leqq N-1}(\bar x_j+x_j)(\bar x_j-x_j)+(w^2-2w)\left(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\right) \\ = 2\sum_{1\leqq j\leqq N-1}x_j(\bar x_j-x_j)+(w^2-2w)\left(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\right)+\sum_{1\leqq j\leqq N-1}(\bar x_j-x_j)^2.$$

0.24. Let

$$R_{\bar{X},3}(Z) = t_N(t_N-\bar{t}_N)^4 - t_N(t_N+\bar{t}_N)^{-4}\left(2\sum_{1\leq j\leq N-1} x_j(\bar{x}_j-x_j)\right)^4.$$

0.25. Then:

$$R_{\bar{X},3}(Z) = t_N(t_N+\bar{t}_N)^{-4}\left(((t_N)^2-(\bar{t}_N)^2)^4 - \left(2\sum_{1\leq j\leq N-1} x_j(\bar{x}_j-x_j)\right)^4\right).$$

The choice of U' in 0.2 and the definition 0.6 imply that

$1/2 < |t_N|, |\bar{t}_N| < 1, \quad |t_N-\bar{t}_N| < 10^{-5N}$ and (for $1 \leq j \leq N-1$) $|\bar{x}_j|, |x_j| < 1/N$,

therefore we can see from 0.23 that:

0.26. $|R_{\bar{X},3}(Z)| < 100\left(w+\sum_{1\leq j\leq N-1}|\bar{x}_j-x_j|^4\right)\left(w+\sum_{1\leq j\leq N-1}|\bar{x}_j-x_j|\right).$

Let us now define:

0.27.
$$R_{\bar{X},4}(Z) = t_N(t_N+\bar{t}_N)^{-4}\left(2\sum_{1\leq j\leq N-1} x_j(\bar{x}_j-x_j)\right)^4 - t_N(2t_N)^{-4}\left(2\sum_{1\leq j\leq N-1} x_j(\bar{x}_j-x_j)\right)^4.$$

We need the following calculation to evaluate $R_{\bar{X},4}(Z)$

$(t_N+\bar{t}_N)^{-1} = (2t_N)^{-1}+(t_N+\bar{t}_N)^{-1}-(2t_N)^{-1} = (2t_N)^{-1}+((t_N)^2-(\bar{t}_N)^2)(2t_N(t_N+\bar{t}_N)^2)^{-1}.$

So

$$R_{\bar{X},4}(Z) = t_N\left(2\sum_{1\leq j\leq N-1} x_j(\bar{x}_j-x_j)\right)^4\left((t_N+\bar{t}_N)^{-4}-(2t_N)^{-4}\right)$$
$$= t_N\left(2\sum_{1\leq j\leq N-1} x_j(\bar{x}_j-x_j)\right)^4\left(((2t_N)^{-1}+((\bar{t}_N)^2-(t_N)^2)(2t_N(t_N+\bar{t}_N)^2)^{-1})^4-(2t_N)^{-4}\right).$$

When we combine this with 0.23 and the facts that $1/2<|t_N|, |\bar{t}_N|<1, |t_N-\bar{t}_N|<10^{-5N}$, we obtain the following (nonsharp) estimate:

0.28. $|R_{\bar{X},4}(Z)| < 100\left(\sum_{1\leq j\leq N-1}|\bar{x}_j-x_j|^4\right)\left(w+\sum_{1\leq j\leq N-1}|\bar{x}_j-x_j|\right).$

We conclude the following from 0.20—0.27:

0.29.

$$u_{\bar{X}}(Z) = w-(1/2)w^2$$
$$+\sum_{1\leq j\leq N-1}\left((x_j)^2(1-w)^2(y_j-\bar{y}_j+z-\bar{z})^2+x_j(1-w)(x_j-\bar{x}_j-wx_j)^4\right)$$
$$+\left(1-\sum_{1\leq j\leq N-1}(x_j)^2\right)(1-w)^2(z-\bar{z})^2+\left(1-\sum_{1\leq j\leq N-1}(x_j)^2\right)^{-3/2}(1-w)^{-3}$$
$$\times\left(\sum_{1\leq j\leq N-1}x_j(\bar{x}_j-x_j)\right)^4-i\sum_{1\leq j\leq N-1}(x_j)^2(1-w)^2(y_j-\bar{y}_j)-i(1-w)^2(z-\bar{z})$$
$$+R_{\bar{X},1}(Z)+R_{\bar{X},2}(Z)+R_{\bar{X},3}(Z)+R_{\bar{X},4}(Z).$$

0.30. Let us define:

$$R_{\bar{X},5}(Z) = -(1/2)w^2$$
$$+\Big(\sum_{1\leqq j\leqq N-1}(x_j)^2(1-w)^2(y_j-\bar{y}_j+z-\bar{z})^2-(x_j)^2(y_j-\bar{y}_j+z-\bar{z})^2$$
$$+x_j(1-w)(x_j-\bar{x}_j-wx_j)^4-x_j(x_j-\bar{x}_j)^4\Big)$$
$$+\Big((1-\sum_{1\leqq j\leqq N-1}(x_j)^2)(1-w)^2(z-\bar{z})^2-(1-\sum_{1\leqq j\leqq N-1}(x_j)^2)(z-\bar{z})^2\Big)$$
$$+\Big(\big((1-\sum_{1\leqq j\leqq N-1}(x_j)^2)^{-3/2}(1-w)^{-3}\big)\big(\sum_{1\leqq j\leqq N-1}x_j(\bar{x}_j-x_j)\big)^4$$
$$-(1-\sum_{1\leqq j\leqq N-1}(x_j)^2)^{-3/2}\big(\sum_{1\leqq j\leqq N-1}x_j(\bar{x}_j-x_j)\big)^4\Big)$$

then

0.31.

$$u_{\bar{X}}(Z) = w+\sum_{1\leqq j\leqq N-1}(x_j)^2(y_j-\bar{y}_j+z-\bar{z})^2+x_j(x_j-\bar{x}_j)^4$$
$$+(1-\sum_{1\leqq j\leqq N-1}(x_j)^2)(z-\bar{z})^2+(1-\sum_{1\leqq j\leqq N-1}(x_j)^2)^{-3/2}\big(\sum_{1\leqq j\leqq N-1}x_j(\bar{x}_j-x_j)\big)^4$$
$$-i\sum_{1\leqq j\leqq N-1}(x_j)^2(1-w)^2(y_j-\bar{y}_j)-i(1-w)^2(z-\bar{z})+R_{\bar{X},1}(Z)+R_{\bar{X},2}(Z)$$
$$+R_{\bar{X},3}(Z)+R_{\bar{X},4}(Z)+R_{\bar{X},5}(Z).$$

This is true simply because the definition of $R_{\bar{X},5}(Z)$ (and 0.29) imply that we add and subtract the same terms.

Looking again at 0.30 we see when we look at the terms within each set of bold brackets separately, that:

0.32. $|R_{\bar{X},5}(Z)| \leqq 10^3 w\big(|z-\bar{z}|^2+\sum_{1\leqq j\leqq N-1}|\bar{x}_j-x_j|^3+|\bar{y}_j-y_j+z-\bar{z}|^2+w\big).$

When we combine 0.19 with 0.6, 0.24—0.26 we obtain:

0.33.

$$|R_{\bar{X},1}(Z)+R_{\bar{X},2}(Z)|$$
$$\leqq 10^4\big(w+|z-\bar{z}|^2+\sum_{1\leqq j\leqq N-1}|y_j-\bar{y}_j+z-\bar{z}|^2+|x_j-\bar{x}_j|^4\big)$$
$$\times\big(|z-\bar{z}|^{1/2}+\sum_{1\leqq j\leqq N}|y_j-\bar{y}_j+z-\bar{z}|^{1/2}\big).$$

0.34. Define: $R_{\bar{X}}(Z)=R_{\bar{X},1}(Z)+R_{\bar{X},2}(Z)+R_{\bar{X},3}(Z)+R_{\bar{X},4}(Z)+R_{\bar{X},5}(Z).$

0.35. Combining 0.26, 0.28, 0.32, 0.33 we have:

$$|R_{\bar{X}}(Z)| \leqq 10^5\big(w+|z-\bar{z}|^2+\sum_{1\leqq j\leqq N-1}|y_j-\bar{y}_j+z-\bar{z}|^2+|x_j-\bar{x}_j|^4\big)$$
$$\times\big(|z-\bar{z}|^{1/2}+\sum_{1\leqq j\leqq N}|y_j-\bar{y}_j+z-\bar{z}|^{1/2}+w+\sum_{1\leqq j\leqq N-1}|\bar{x}_j-x_j|\big).$$

0.36. Since $\operatorname{Im}(R_{\bar{X}}(Z))=\operatorname{Im}(R_{\bar{X},1}(Z)+R_{\bar{X},2}(Z))$ 0.33 yields that:

$$|\operatorname{Im}(R_{\bar{X}}(Z))| \leqq 10^4\big(w+|z-\bar{z}|^2+\sum_{1\leqq j\leqq N-1}|y_j-\bar{y}_j+z-\bar{z}|^2+|x_j-\bar{x}_j|^4\big)$$
$$\times\big(|z-\bar{z}|^{1/2}+\sum_{1\leqq j\leqq N}|y_j-\bar{y}_j+z-\bar{z}|^{1/2}\big).$$

Now if we look at 0.34, 0.31, 0.2 we can see that the neighborhood we have chosen is small enough so that the effect of the remainder term is so limited that:

$$\operatorname{Re}\big(u_{\bar X}(Z)\big) \geqq 99/100\big(w+\textstyle\sum_{1\leqq j\leqq N-1}(x_j)^2(y_j-\bar y_j+z-\bar z)^2+x_j(x_j-\bar x_j)^4$$
$$+\big(1-\textstyle\sum_{1\leqq j\leqq N-1}(x_j)^2\big)(z-\bar z)^2+\big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\big)^{-3/2}\big(\sum_{1\leqq j\leqq N-1}x_j(\bar x_j-x_j)\big)^4\big).$$

We have completed the proof of Sublemma 1.

The following Lemma is the first step toward constructing the map in Theorem 1 ($\beta=\beta(N)$ is a positive constant that will be defined for $N>2$ in the third section, $\beta(2)=1$).

Lemma 1. *Let* $D=B^N$, $R_0>0$, $\varepsilon_0=10^{-(10N)!/(R_0\cdot\beta)}$ *and* $Z_0\in bD$. *There exists an open neighborhood* (in the topology of $\bar D$) $W\subseteq\bar D$ *of* Z_0, *so that when* $f\colon bD\to B^{N+1}$ *is continuous,* $|f|>R_0$, $K\subset D$ *is compact and* $\varepsilon_0>\varepsilon>0$, $\varepsilon'>0$ *then there is a smooth map* $g\colon \bar D\to \mathbf{C}^{N+1}$, *holomorphic on D such that the following hold:*

(a) *when* $Z\in bD$: $\quad |f(Z)+g(Z)|<1+\varepsilon^{45}$

(b) *when* $Z\in bD\cap W$: $\quad |f(Z)+g(Z)|^2-|f(Z)|^2>\varepsilon^2\big(1-|f(Z)|^2\big)-\varepsilon^{45}$

(c) *when* $Z\in bD$: $\quad |f(Z)+g(Z)|^2-|f(Z)|^2>-3\varepsilon^{100}$

(d) *when* $Z\in K$: $\quad |g(Z)|<\varepsilon'$

(e) *when* $Z\in bD$: $\quad |g(Z)|^2<\varepsilon^{1/2}\big(1-|f(Z)|^2\big)+\varepsilon^{45}$.

Proof of Lemma 1 (for $N=2$):

We will present here a proof of Lemma 1 when $N=2$. The proof for $\infty>N\geqq 2$ (which is based on Lemma 3) will be presented in Section 3. The ideas of the proof in the case $N=2$ originate from *B.* Stensønes [19].

1.1. Let $X=(x,y,z,w)$ be the coordinate system in a neighborhood of Z_0 described by 0.1—0.3 and U', V', U; V, W, W' described by 0.2, 0.4 where $d_1=10^{-1}$, $d=10^{-40}$. For $\bar X=(\bar x,\bar y,\bar z,0)\in U'$ we will define the polynomial $u_{\bar X}$ as in 0.5. Sublemma 1 implies then that $u_{\bar X}$ has the following properties (we suppress the distinction between X and $Z(X)$):

1.2. (i) $\operatorname{Re}(u_{\bar X})>0$ on $W'\setminus\{Z(\bar X)\}$ and $u_{\bar X}(Z(\bar X))=0$

(ii) Let $X=(x,y,z,w)\in X(W')$ then:

$$u_{\bar X}(X)=w+x^2(y-\bar y+z-\bar z)^2+x(x-\bar x)^4+(1-x^2)(z-\bar z)^2+(1-x^2)^{-3/2}\big(x(\bar x-x)\big)^4$$
$$-ix^2(1-w)^2(y-\bar y)-i(1-w)^2(z-\bar z)+R_{\bar X}(Z)$$

where:

$$|R_{\bar X}(X)|\leqq 10^5(w+|z-\bar z|^2+|y-\bar y+z-\bar z|^2+|x-\bar x|^4)$$
$$\times(|z-\bar z|^{1/2}+|y-\bar y+z-\bar z|^{1/2}+w+|\bar x-x|)$$

and:

$$|\mathrm{Im}\,(R_X(X))| \leqq 10^4(w+|z-\bar{z}|^2+|y-\bar{y}+z-\bar{z}|^2+|x-\bar{x}|^4)\cdot(|z-\bar{z}|^{1/2}+|y-\bar{y}+z-\bar{z}|^{1/2}).$$

It was also proved in Sublemma 1 that for $X=(x, y, z, w)\in X(W')$:

1.3. $\mathrm{Re}\,(u_X(X)) \geqq 99/100\big(w+x^2(y-\bar{y}+z-\bar{z})^2+x(x-\bar{x})^4+(1-x^2)(z-\bar{z})^2$
$+(1-x^2)^{-3/2}(x(\bar{x}-x))^4\big).$

1.4. For $X\in X(W')$, the following notation for the coordinates of X will be used: $X=(X_1, X_2, X_3, X_4)=(x, y, z, w)$.

1.5. Since f is continuous there exists an r, $\varepsilon^{1/\varepsilon}>r>0$, so that when $X, X'\in V'$, $|X-X'|<(\log(1/r))^{-1}$ we have:

$$|f(X)-f(X')| < \varepsilon^{100}.$$

1.6. Define $c_1=r^{1/4}$, $c_2=10r^{1/2}$ and $c_3=r^{1/2}$.

We can assume that r is chosen so that $(\log\varepsilon)/(r^{1/2}\cdot 4\pi)$ is an integer (this assumption is possible since $\varepsilon^{1/\varepsilon}>r>0$ and there is no positive lower bound on the choice of r). This assumption combined with the form of $\mathrm{Im}\,(u_X)$ in 1.2 above, "saves" one dimension of the target ball. An equivalent assumption (for the same purpose) can be found in [19].

1.7. For $a=(a_1, a_2, a_3)\in \mathbf{Z}^3$ we define $X_a=(a_1c_1, a_2c_2, a_3c_3, 0)$ and define:

$$L = \{a\in \mathbf{Z}^3\colon d(X_a, V) < r^{1/5}\},\quad L' = \{a\in\mathbf{Z}^3\colon X_a\in V'\},$$

$\{X_a\colon a\in L'\}$ forms a lattice-like set that "covers" V' $(=Z(W'\cap bD))$. $\{X_a\colon a\in L\}$ is a smaller lattice that "covers" $V(=Z(W\cap bD))$ with a very small margins around it.

1.8. Define for $a\in L$:

$$u_a = u_{X_a} \quad\text{and}\quad R_a = R_{X_a} \quad \text{(the remainder term)}.$$

By 1.2 when $a\in L$, $X=(X_1, \dots, X_4)=(x, y, z, w)\in U'$ and $t_i=X_i/c_i$ $1\leqq i\leqq 3$ then:

1.9. $$u_a(X) = w+x^2(c_2(t_2-a_2)+c_3(t_3-a_3))^2+(1-x^2)(c_3(t_3-a_3))^2+x(c_1(t_1-a_1))^4$$
$$+(1-x^2)^{-3/2}(xc_1(t_1-a_1))^4-ix^2(1-w)^2(c_2(t_2-a_2))-ic_3(1-w)^2(t_3-a_3)+R_a(X)$$
$$= w+x^2r(10(t_2-a_2)+(t_3-a_3))^2+(1-x^2)r(t_3-a_3)^2+r(x+(1-x^2)^{-3/2}x^4)(t_1-a_1)^4$$
$$-10ir^{1/2}x^2(1-w)^2(t_2-a_2)-ir^{1/2}(1-w)^2(t_3-a_3)+R_a(X).$$

We define for $Z\in\bar{D}$ and $a\in L$:

1.10. $$p_a(Z) = \exp\big(u_a(Z)\cdot(\log\varepsilon)/2r\big)$$

and for $a\in L'\setminus L$ we define $p_a\equiv 0$.

Capital Z (for an element in $\mathbf{C}^N$) is used in this work whenever there is a possibility of confusion with the real coordinate z. Property (i) in 1.2 implies that $\{p_a\colon a\in L\}$ is a set of local peak functions that peak on the lattice $\{X_a\colon a\in L\}$.

Throughout the proof of Lemma 1 and Lemma 2 we will suppress the distinction between Z and $X(Z)$ when $Z\in W'$. Let $a\in L$ and $X=(X_1, \dots, X_4)=(x, y, z, w)\in U'$ $(t_i=X_i/c_i\ 1\leqq i\leqq 3)$ then 1.9 and 1.10 imply:

1.11. (We will use the notation $\mathscr{E}\overset{\text{def}}{=\!=}\varepsilon$.)

$$p_a(X) = \mathscr{E}^{(1/2)(w/r+x^2(10(t_2-a_2)+(t_3-a_3))^2+(1-x^2)(t_3-a_3)^2+(x+(1-x^2)^{-3/2}x^4)(t_1-a_1)^4+\mathrm{Re}(R_a(X)/r))}$$

$$\times\exp\big(-i((\log\varepsilon)/2)\big(10r^{-1/2}x^2(1-w)^2(t_2-a_2)+r^{-1/2}(1-w)^2(t_3-a_3)-r^{-1}\,\mathrm{Im}\big(R_a(X)\big)\big)\big).$$

Thus if we add the assumption that $X\in V'$ (i.e. $w=0$) then:

$$p_a(X) = \mathscr{E}^{(1/2)(x^2(10(t_2-a_2)+(t_3-a_3))^2+(1-x^2)(t_3-a_3)^2+(x+(1-x^2)^{-3/2}x^4)(t_1-a_1)^4+\mathrm{Re}(R_a(X)/r))}$$

$$\times\exp\big(-i((\log\varepsilon)/2)\big(10r^{-1/2}x^2(t_2-a_2)+r^{-1/2}t_3\big)\big)\cdot\theta_a(X)$$

where $\theta_a(X)=\exp\big((i/2)(\log\varepsilon)\cdot r^{-1}\cdot\mathrm{Im}\,(R_a(X))\big)$. Note that $|\theta_a|\equiv 1$.

The second equality holds since we assumed (1.6) that $(\log\varepsilon)/(r^{1/2}\cdot 4\pi)$ is an integer.

1.12. We need an estimate for $\theta_a(X)$ when $X\in V'$ is close to X_a (in a sense that will be soon explained).

Let $a\in L$, $X\in V'$ $(t_i=X_i/c_i\ 1\leqq i\leqq 3)$. We have from 1.2, 1.6 and a calculation like the one in 1.9:

$$\big|((\log\varepsilon)/r)\,\mathrm{Im}\,(R_a(X))\big|$$

$$\leqq 10^4(\log\varepsilon)\big((10(t_2-a_2)+(t_3-a_3))^2+(t_3-a_3)^2+(t_1-a_1)^4\big)$$

$$\times r^{1/4}\big(|t_3-a_3|^{1/2}+|10(t_2-a_2)+(t_3-a_3)|^{1/2}\big).$$

If we assume (for example) that for all $1\leqq i\leqq 3$, $|t_i-a_i|<10^6$ then since $10^{-10}>\varepsilon_0>\varepsilon>0$ and $\varepsilon^{1/\varepsilon}>r>0$ (see 1.5) we have that:

$$\big|(\log\varepsilon)r^{-1}\,\mathrm{Im}\,(R_a(X))\big| < 10^{-10},$$

$$|\theta_a(X)-1| < 10^{-10}$$

and, as mentioned after 1.11, $|\theta_a(X)|=1$.

1.13.

$$p_a(X)\cdot\bar{p}_b(X)$$

$$= \exp\big((\log\varepsilon)r^{-1}\big(\mathrm{Re}\,u_a(X)+\mathrm{Re}\,u_b(X)+i\big(\mathrm{Im}\,u_a(X)-\mathrm{Im}\,u_b(X)\big)\big)\big)$$

$$= \mathscr{E}^{x^2((10(t_2-a_2)+(t_3-a_3))^2+(10(t_2-b_2)+(t_3-b_3))^2)/2}\cdot\mathscr{E}^{(1-x^2)((t_3-a_3)^2+(t_3-b_3)^2)/2}$$

$$\times\mathscr{E}^{(1/2)((x+(1-x^2)^{-3/2}x^4)((t_1-a_1)^4+(t_1-b_1)^4)+(\mathrm{Re}(R_a(X)+R_b(X))/r))}$$

$$\times\exp\big(-i(\log\varepsilon)\big(5r^{-1/2}x^2(b_2-a_2)\big)\big)\cdot\theta_a(X)\cdot\bar{\theta}_b(X).$$

It follows that if $a_2=b_2$ then:

$$p_a(X)\cdot\bar{p}_b(X)=|p_a(X)\cdot\bar{p}_b(X)|\cdot\theta_a(X)\cdot\bar{\theta}_b(X).$$

The fact that $\theta_a(X)$, $\theta_b(X)$ are very close to 1 when X_a, X_b are close enough to X (see precise computation in 1.12) will prove to be crucial later (following 1.19).

Let $v_1, v_2\colon V'\to\mathbf{C}^3$ be continuous maps such that for every $X\in V'$

$$\{f(X), v_1(X), v_2(X)\}$$

are mutually orthogonal and $|v_1(X)|^2=|v_2(X)|^2=1$. Such maps exist since f is continuous and $|f|>R_0>0$. By shrinking r further (in 1.6) we can assume that for $X, X'\in V'$ such that $|X-X'|<(\log(1/r))^{-1}$ we have (for $i=1, 2$): $|v_i(X)-v_i(X')|<\varepsilon^{60}$. For $a\in L'$ we define $i(a)=1$ whenever a_2 is odd and $i(a)=2$ whenever a_2 is even and we denote $t_a=(2\varepsilon(1-|f(X_a)|^2))^{1/2}$. We define for $a\in L'$: $v_a=t_a v_{i(a)}(X_a)$, (so when a_2 progresses our choice between $t_a v_1(X_a)$ and $t_a v_2(X_a)$ alternates).

The set $\{v_a\colon a\in L'\}$ has the following properties:

1.14.

(i) $(v_a, f(X_a))=0$.

(ii) $|v_a|^2=2\varepsilon(1-|f(X_a)|^2)$.

(iii) If $a, b\in L'$, $|a-b|<1000$, and b_2-a_2 is odd, then:

$$|(v_a, v_b)|<\varepsilon^{60}.$$

(iv) If $a, b\in L'$, $|a-b|<1000$, and b_2-a_2 is even, then:

$$\left|(v_a, v_b)-|v_a|^2\right|<\varepsilon^{60}.$$

Properties (i) and (ii) are obvious from the definition of $\{v_a\}$. To prove property (iii) and (iv) notice that when $|a-b|<1000$ (or even when $|a-b|<-\log r$) then by 1.6 and 1.7 $|X_a-X_b|<(-\log r)^{-1}$ and thus $|v_i(X_a)-v_i(X_b)|<\varepsilon^{60}$ $(i=1, 2)$ and (by 1.5) $|f(X_a)-f(X_b)|<\varepsilon^{100}$. Using $(v_1, v_2)\equiv 0$, $|v_i|\equiv 1$ we have for $a, b\in L'$, $|a-b|<1000$:

$$(1)\qquad \left|(t_a v_i(X_a), t_b v_i(X_b))-|t_a v_i(X_a)|^2\right|=\left|(t_b t_a v_i(X_b)-t_a t_a v_i(X_a), v_i(X_a))\right|$$
$$\leqq 2|(t_a)^2-(t_b)^2|+2|v_i(X_b)-v_i(X_a)|(t_a+t_b)<\varepsilon^{60}$$

and

$$(2)\qquad \left|(t_a v_1(X_a), t_b v_2(X_b))\right|\leqq\left|(v_1(X_a), v_2(X_b))\right|$$
$$=\left|(v_1(X_a)-v_1(X_b), v_2(X_b))\right|\leqq|v_1(X_a)-v_1(X_b)|\cdot|v_2(X_b)|<\varepsilon^{60}.$$

Now if b_2-a_2 is even then $i(a)=i(b)$ and (iv) follows from (1). If b_2-a_2 is odd then $i(a)\neq i(b)$ we can assume that $i(a)=1$ and $i(b)=2$ and then (iii) follows from (2).

Properties (iii) and (iv) will play a very important role in our construction.
1.15. We define for $Z\in\bar{D}$

$$h(Z)=\sum_{a\in L}p_a(Z)v_a=\sum_{a\in L'}p_a(Z)v_a.$$

The map h is the first and most important stage in the construction of g of Lemma 1. The map g will correct h only slightly on $W'\cap bD$ (so that it will not have a significant effect on its properties in $W'\cap bD$) and will make it very small out of W'. The motivation behind this definition of h is to add to f a holomorphic function h which is almost perpendicular to $f(X)$ at each point X of $W'\cap bD$. This will add to $|f(X)|^2$ almost $|h(X)|^2$ so we need to show that $|h(X)|^2$ is sufficiently large (i.e. uniformly bounded from below) on $W\cap bD$ but not too large on W'. It is important to realize that in this correcting method we can not have a full control on the direction of h. It is impossible (for example) to provide a correction function h that will be almost in the same direction as f at each $X\in W'$ and will be of sufficient size everywhere in W. Such a construction seems to be impossible regardless of the co-dimension. When the co-dimension is 1, then the best control we can have on the direction of h is that it will be almost perpendicular to f. Our construction is based on the fact that in the evaluation of $|h(X)|^2$, $|(f(X),h(X))|$ the dominant part of $h(X)$ is $\sum_{(X_a \text{ is close to } X)}p_a(X)v_a$ and the rest sum up to a small proportion of it plus a small error term.

The evaluation of $|h(X)|^2$, $|(f(X),h(X))|$ will require a division of L, L' into appropriate subsets. Fix $X=(X_1,\dots,X_4)\in U'$, and $t_i=X_i/c_i$ $1\leqq i\leqq 3$. We choose $a_3(X)$ to be an integer so that $|a_3(X)-t_3|\leqq 1/2$. Next we choose $a_2(X)$ to be an integer so that $|10(t_2-a_2(X))+t_3-a_3(X)|\leqq 5$ (note that $a_3(X)$ is chosen first and then $a_2(X)$) and we choose $a_1(X)$ to be an integer so that $|a_1(X)-t_1|\leqq 1/2$. We define $a(X)=(a_1(X),a_2(X),a_3(X))$.

1.16. Define:

$$L'(x,0)=\{a=(a_1,a_2,a_3)\in L'\colon\ |a_i-a_i(X)|\leqq 1 \text{ for all } 1\leqq i\leqq 3\}$$

and for $n\geqq 1$:

$$L'(x,n)=\{a\in L'\setminus L'(x,0)\colon\ n^2<(1/2)\big(x^2(10(t_2-a_2)+(t_3-a_3))^2+(1-x^2)(t_3-a_3)^2 +(x+(1-x^2)^{-3/2}x^4)(t_1-a_1)^4\big)\leqq(n+1)^2\}$$

and we define $L(x,n)=L'(x,n)\cap L$, for $n\geqq 0$. The positive numbers

$$\{(1/2r)(\mathrm{Re}\,(u_a(X)-R_a(X))\colon\ a\in L'(X,0))\}$$

are the smallest in the set $\{(1/2r)(\mathrm{Re}\,(u_a(X)-R_a(X))\colon\ a\in L')\}$,

$$\{(1/2r)(\mathrm{Re}\,(u_a(X)-R_a(X))\colon\ a\in L'(X,1))\}$$

are the next smallest, etc. There is a computational convenience in this division of L', L into subsets as we will see later.

Let $X \in V'$, then:

$$|f(X)+h(X)|^2 = |f(X)|^2+|h(X)|^2+2\,\mathrm{Re}\,(f(X), h(X))$$
$$\geqq |f(X)|^2+|h(X)|^2-2\left|(f(X), h(X))\right|.$$

We will prove the following for $X \in V'$.

1.17. (A) $\left|(f(X), h(X))\right| < \varepsilon^{100}$

(B) $|h(X)|^2 < \varepsilon^{1/2}(1-|f(X)|^2)+\varepsilon^{50}$

(C) when $X \in V$ then $|h(X)|^2 > \varepsilon^2(1-|f(X)|^2)-\varepsilon^{50}$.

The proof of 1.17 is the main step in the proof of Lemma 1. Careful considerations will be needed to prove (C).

1.18. We will freely use the following facts. When $a \in L(X, n)$, $n \geqq 0$ then:

(1) $\mathrm{card}\,(L(X, n)) \leqq (10n+10)^3$

(2) $|p_a(X)| \leqq \varepsilon^{n^2/2}$.

Proof. Since $|x-10^{-1}|<10^{-40}$ (see 1.1) then (1) is a simple consequence of Definition 1.16.

It follows from 1.3 that

$$\mathrm{Re}\,(u_a(X))$$
$$\geqq (99/100)\,r\left(x^2(10(t_2-a_2)+(t_3-a_3))^2+(1-x^2)(t_3-a_3)^2+(x+(1-x^2)^{-3/2}x^4)(t_1-a_1)^4\right)$$

and then Definitions 1.10, 1.16 immediately imply that (2) is true.

These are by no means sharp estimates, but they are sufficient. To simplify our calculations we will not try to obtain the sharpest estimates with the smallest error terms but rather estimates that are sufficient for our needs and are easy to work with. When $X \in U'$ is fixed and $a \in L'(X, n)$ we define $[a]=n$ (there is only one such n).

Proof of (A) (using 1.14 (i), (ii) and 1.5, 1.6):

$$\left|(f(X), h(X))\right| = \left|\sum_{a\in L}(f(X), v_a)\bar{p}_a(X)\right|$$
$$= \left|\sum_{0\leqq n\leqq 100}\sum_{a\in L(X,n)}(f(X)-f(X_a), v_a)\bar{p}_a(X)\right|+\left|\sum_{100<n}\sum_{a\in L(X,n)}(f(X), v_a)\bar{p}_a(X)\right|$$
$$\leqq \sum_{0\leqq n\leqq 100}\sum_{a\in L(X,n)}|f(X)-f(X_a)|\cdot|v_a|+\sum_{100<n}\sum_{a\in L(X,n)}|p_a(X)|$$
$$< 2\sum_{0\leqq n\leqq 100}\varepsilon^{100}\cdot\varepsilon^{1/2}(10n+10)^3+\sum_{100<n}\varepsilon^{n^2/2}\cdot(10n+10)^3 < \varepsilon^{100}.$$

Now (A) is proved.

Proof of (B). Fix $X\in V'$: We will use 1.18 and the fact that when $a, b\in \cup \{L(n, X)\colon 0\leqq n\leqq 100\}$ then (by 1.6, 1.7, 1.16) $|X_a-X|, |X_b-X|<r^{1/5}$, and 1.5 with 1.14 (ii) imply:

$$|(v_a, v_b)| \leqq (|v_a|^2+|v_b|^2)/2 = \varepsilon\big(2-|f(X_a)|^2-|f(X_b)|^2\big)$$
$$= 2\varepsilon\big(1-|f(X)|^2\big)+\varepsilon\big(|f(X)|^2-|f(X_a)|^2\big)+\varepsilon\big(|f(X)|^2-|f(X_b)|^2\big)$$
$$< 2\varepsilon\big(1-|f(X)|^2\big)+\varepsilon^{100}.$$
$$|h(X)|^2 = \big|\sum\nolimits_{a,b\in L}(v_a, v_b)p_a(X)\bar{p}_b(X)\big|$$
$$\leqq \sum\nolimits_{0\leqq m\leqq 100}\sum\nolimits_{a,b\in L,\,[a]+[b]=m}|(v_a, v_b)p_a(X)\bar{p}_b(X)|$$
$$+\sum\nolimits_{100<m}\sum\nolimits_{a,b\in L,\,[a]+[b]=m}|p_a(X)\bar{p}_b(X)|$$
$$< \sum\nolimits_{0\leqq m\leqq 100}\big(2\varepsilon\big(1-|f(X)|^2\big)+\varepsilon^{100}\big)\cdot(10m+10)^6\cdot\varepsilon^{m^2/4}$$
$$+\sum\nolimits_{100<m}(10m+10)^6\cdot\varepsilon^{m^2/4} < 10^6\varepsilon\big(1-|f(x)|^2\big)+\varepsilon^{60}.$$

(B) is proved.

Proof of (C). Fix $X\in V$,

1.19. $$|h(X)|^2 = \sum\nolimits_{a,b\in L}(v_a, v_b)p_a(X)\bar{p}_b(X)$$
$$= \sum\nolimits_{a,b\in L(X,0)}(v_a, v_b)p_a(X)\bar{p}_b(X)+\sum\nolimits_{m\geqq 1}\sum\nolimits_{[a]+[b]=m}(v_a, v_b)p_a(X)\bar{p}_b(X)$$
$$\geqq \operatorname{Re}\big(\sum\nolimits_{a,b\in L(X,0)}(v_a, v_b)p_a(X)\bar{p}_b(X)\big)-\big|\sum\nolimits_{m\geqq 1}\sum\nolimits_{[a]+[b]=m}(v_a, v_b)p_a(X)\bar{p}_b(X)\big|.$$

We will prove that the second term is a small proportion of the first term (plus a marginal error term), and the lower bound of the first term is close to $|v_{a(X)}|^2|p_{a(X)}(X)|^2$. We will first estimate the first term, and divide the summands to four cases.

When $a, b\in L(x, 0)$ and $a=b$, then

(1) $$(v_a, v_b)p_a(X)\bar{p}_b(X) = |v_a|^2|p_a(X)|^2 > 0.$$

When $a, b\in L(X, 0)$, $a\neq b$, and $a_2=b_2$ then by 1.13:

$$p_a(X)\cdot\bar{p}_b(X) = |p_a(X)\cdot\bar{p}_b(X)|\cdot\theta_a(X)\cdot\bar{\theta}_b(X),$$

and by 1.12: $|\theta_a(X)-1|, |\theta_b(X)-1|<10^{-10}$, therefore it follows from 1.14 (iv) that:

(2) $$\operatorname{Re}\big((v_a, v_b)p_a(X)\bar{p}_b(X)\big) > -\varepsilon^{60}.$$

If $a, b\in L(X, 0)$ and $a_2\neq b_2$, then if $a_2=b_2\pm 1$ 1.14 (iii) implies that:

(3) $$|(v_a, v_b)| < \varepsilon^{60}.$$

The remaining case is of $a, b\in L(X, 0)$ and $a_2=b_2\pm 2$. After possibly interchanging a and b, we have $a_2=a_2(X)-1$ and $b_2=a_2(X)+1$ (a, b will be fixed until

1.20). Since $-1\leqq a_3-a_3(X)\leqq 1$ the following unsharp estimate follows from the definition of $a(X)$ before 1.16:

$$\big((10(t_2-a_2)+(t_3-a_3))^2+(10(t_2-b_2)+(t_3-b_3))^2\big) \\ -2\big((10(t_2-a_2(X))+(t_3-a_3(X)))^2\big) > 80.$$

This and the definition of $a(X)$ (and the fact that $1-10^{-20}<x/10<1+10^{-20}$ (see 1.1)) imply that:

$$(1/2)\big(x^2\big((10(t_2-a_2)+(t_3-a_3))^2+(10(t_2-b_2)+(t_3-b_3))^2\big) \\ +(1-x^2)\big((t_3-a_3)^2+(t_3-b_3)^2\big)+\big(x+(1-x^2)^{-3/2}x^4\big)\big((t_1-a_1)^4+(t_1-b_1)^4\big)\big) \\ -\big(x^2(10(t_2-a_2(X))+(t_3-a_3(X)))^2+(1-x^2)(t_3-a_3(X))^2 \\ +\big(x+(1-x^2)^{-3/2}x^4\big)(t_1-a_1(X))^4\big) > 0.3.$$

We obtain from this and the estimate of the remainder term in 1.12 and from 1.13 that

$$|p_a(X)\bar{p}_b(X)| < \varepsilon^{1/4}|p_{a(X)}(X)|^2.$$

It follows from 1.14 and from 1.5 that $\big|(v_a, v_b)-|v_{a(X)}|^2\big|\leqq\big|(v_a, v_b)-|v_b|^2\big|+\big||v_b|^2-|v_{a(X)}|^2\big|<2\cdot\varepsilon^{60}$ so we conclude that:

(4) $$|(v_a, v_b)p_a(X)\bar{p}_b(X)| < \varepsilon^{1/4}|v_{a(X)}|^2|p_{a(X)}(X)|^2+\varepsilon^{60}.$$

Since $a(X)\in L(X, 0)$ (and $\operatorname{car}(L(X, 0))=27$), then combining (1)—(4) yields:

1.20. $$\operatorname{Re}\big(\textstyle\sum_{a,b\in L(X,0)}(v_a, v_b)p_a(X)\bar{p}_b(X)\big) > (3/4)|v_{a(X)}|^2|p_{a(X)}(X)|^2-\varepsilon^{55}.$$

Looking again at 1.13, 1.16 we can calculate (using an argument as in (4)) that when $a, b\in L$ and $[a]+[b]=m\geqq 1$, then:

1.21. $$|p_a(X)\bar{p}_b(X)| \leqq \varepsilon^{1/4}|p_{a(X)}(X)|^2,$$

$$|p_a(X)\bar{p}_b(X)| < \varepsilon^{m^2/4}$$

and when $[a]+[b]=m\geqq 2$, then

$$|p_a(X)\bar{p}_b(X)| < \varepsilon^{(m^2/4-1/2)}\cdot|p_{a(X)}(X)|^2.$$

An important part of the proof of Lemma 1 is contained in (1.20), (1.21). We are now ready to estimate the second term in 1.19. We will be using (1.20), (1.21), the fact that $\{(a, b)\in L\times L\colon [a]+[b]=m\}$ has less than $(10m+10)^6$ elements and the fact that (see 1.5, 1.14 (ii)) for $a\in L$ such that $[a]\leqq 100$, $\left||v_a|^2-|v_{a(X)}(X)|^2\right|=2\varepsilon\cdot\left||f(X_a)|^2-|f(X_{a(X)})|^2\right|<\varepsilon^{100}$.

1.22.
$$\begin{aligned}&\left|\sum_{1\leqq m}\sum_{a,b\in L,\,[a]+[b]=m}(v_a,v_b)p_a(X)\bar p_b(X)\right|\\ &\leqq \sum_{1\leqq m\leqq 100}(1/2)\sum_{a,b\in L,\,[a]+[b]=m}(|v_a|^2+|v_b|^2)|p_a(X)\bar p_b(X)|\\ &\quad+\sum_{100<m}\sum_{a,b\in L,\,[a]+[b]=m}|p_a(X)\bar p_b(X)|\\ &<\sum_{1\leqq m\leqq 100}\sum_{a,b\in L,\,[a]+[b]=m}(|v_{a(X)}|^2+\varepsilon^{100})\varepsilon^{1/4}|p_{a(X)}(X)|^2\\ &\quad+\sum_{100<m}\sum_{a,b\in L,\,[a]+[b]=m}\varepsilon^{m^2/4}\\ &\leqq (|v_{a(X)}|^2+\varepsilon^{100})|p_{a(X)}(X)|^2\Big(2\sum_{1\leqq m\leqq 100}(10m+10)^6\Big)\varepsilon^{1/4}\\ &\quad+\sum_{100<m}(10m+10)^6\cdot\varepsilon^{m^2/4}<|v_{a(X)}|^2|p_{a(X)}(X)|^2\cdot\varepsilon^{1/5}+\varepsilon^{80}.\end{aligned}$$

Combining 1.19, 1.20, 1.22 we obtain:

1.23.
$$\begin{aligned}|h(X)|^2&>(3/4)|v_{a(X)}|^2|p_{a(X)}(X)|^2-\varepsilon^{55}-\left(|v_{a(X)}|^2|p_{a(X)}(X)|^2\cdot\varepsilon^{1/5}+\varepsilon^{80}\right)\\ &>|v_{a(X)}|^2|p_{a(X)}(X)|^2(3/4-\varepsilon^{1/5})-\varepsilon^{54}>\left(1-|f(X)|^2\right)\varepsilon^2-\varepsilon^{50}.\end{aligned}$$

We used $|v_{a(X)}|^2=2\varepsilon(1-|f(X_{a(X)})|^2)$ (see 1.14 (ii)) and 1.5 which implies:
$$\left||v_{a(X)}|^2-2\varepsilon\left(1-|f(X)|^2\right)\right|<\varepsilon^{100}.$$

We also used the fact that $|p_{a(X)}(X)|^2>\varepsilon$ (follows from 1.13 and the definition of $a(X)$).

(C) is now proved.

We will now obtain an estimate for $|h(X)|^2$ away from V. We will prove the following:

(D) For every $\delta>0$, $\mu>0$ there exists $r_0>0$, $r_0=r_0(\delta,\mu)$ so that if $0<r<r_0$ in the definition of h (at 1.5, 1.6) then for every $X=(x,y,z,w)\in U'$ (U' is defined in 0.2) such that $d(X,V)>\delta$ (distance in the coordinates) we have $|h(X)|<\mu$.

Proof. Define:

1.24.
$$\begin{aligned}q(X)&=d^4(x,[d_1-d-r^{0.1},d_1+d+r^{0.1}])\\ &\quad+d^2(y,[-d-r^{0.1},d+r^{0.1}])+d^2(z,[-d-r^{0.1},d+r^{0.1}])\end{aligned}$$

(see 0.2 for the definition of V. Here the first term is the distance, to the power 4, of x from the interval $[d_1-d-r^{0.1},d_1+d+r^{0.1}]$ etc.). When $1\leqq n$ and $a\in L(X,n)$ we have by 1.16 that (we put $t_i=X_i/c_i$, $1\leqq i\leqq 3$):

1.25.
$$\begin{aligned}&x^2(10(t_2-a_2)+(t_3-a_3))^2+(1-x^2)(t_3-a_3)^2\\ &+(x+(1-x^2)^{-3/2}x^4)(t_1-a_1)^4\leqq 2(n+1)^2\end{aligned}$$

by 1.6 this is the same as:

1.26. $$x^2(y-c_2a_2+z-c_3a_3)^2+\big((1-x^2)(z-c_3a_3)\big)^2 + \big((x+(1-x^2)^{-3/2}x^4)(x-c_1a_1)\big)^4 \leqq 2r(n+1)^2.$$

We need the following trivial (and non sharp) inequality:

1.27. For $\mathscr{X},\mathscr{Y}\in\mathbf{R}$: $(1/200)(\mathscr{X}+\mathscr{Y})^2+(1/2)\mathscr{Y}^2 \geqq (1/400)\mathscr{X}^2+(1/400)\mathscr{Y}^2$

this is equivalent to

$$(1/400)\mathscr{X}^2+(1/2+1/400)\mathscr{Y}^2+(1/100)\mathscr{X}\mathscr{Y} \geqq 0$$
$$\Leftrightarrow (1/400)(\mathscr{X}+2\mathscr{Y})^2+(1/2-3/400)\mathscr{Y}^2 \geqq 0.$$

We obtain from 1.26 and from $x\in[10^{-1}-2\cdot 10^{-40},\ 10^{-1}-2\cdot 10^{-40}]$ (see 1.1):

$$2r(n+1)^2 \geqq (1/200)(y-c_2a_2+z-c_3a_3)^2+\big((1/2)(z-c_3a_3)\big)^2+(1/10)(x-c_1a_1)^4$$
$$\text{(using 1.27)} \geqq (1/400)\big((y-c_2a_2)^2+(z-c_3a_3)^2+(x-c_1a_1)^4\big) \geqq (1/400)q(X)$$

(the last inequality is a simple consequence of the definition of L in 1.7 and the definition of $q(X)$).

1.28. We obtain that when $X\in U'$ and $L(X,n)\neq\emptyset$ then $2r(n+1)^2 \geqq (1/400)q(X)$ so $n\geqq(1/30)\big(q(X)/r\big)^{1/2}-1$. We define now

1.29. $$M(X) = (1/30)\big(q(X)/r\big)^{1/2}-1.$$

Fix $X=(x,y,z,w)\in U'$ such that $d(X,V)>\delta$ then:

1.30. $$|h(X)|^2 = \sum_{a,b\in L}(v_a, v_b)p_a(X)\bar{p}_b(X)$$
$$= \sum_{M(X)\leqq m}\sum_{a,b\in L,\,[a]+[b]=m}(v_a,v_b)p_a(X)\bar{p}_b(X) \leqq \sum_{M(X)\leqq m}(10m+10)^6\varepsilon^{m^2/4}\varepsilon^{2w/r}.$$

It is evident that by shrinking $r>0$ we can make this sum as small as we want, uniformly on $\{X\in U'\colon d(X,V)>\delta\}$.

Let $A\subseteqq\bar{D}$, $t>0$ we define: $A^t=\{z\in\bar{D}\,|\,d(z,A)<t\}$ where the metric is the one of $\mathbf{C}^2$ (not of the coordinates system).

1.31. Fix (until the end of the proof) $\delta>0$ so that if $z\in(W\cap bD)^{2\delta}$ then

(i) $X(z)\in[d_1-1.5d,\ d_1+1.5d]\times[-1.5d,\ 1.5d]^2[0,\ 0.5d]$
and
(ii) $z\notin K$.

1.32. Let $0<\varepsilon'<\varepsilon^{1000}$. By (D) if $r>0$ is small enough in the definition of h, then for all $z\in W'\setminus(W\cap bD)^\delta$ we have $|h(z)|<(\varepsilon')^2$. We will assume it is so from now on.

Summarizing our results and assumptions so far yields that h has the following properties (compare with the desired properties of g in the statement of Lemma 1):

1.33.

(A) and (B) imply:

(a)′ when $z \in W' \cap bD$:

$$|f(z)+h(z)|^2 \leqq |f(z)|^2+|h(z)|^2+2\left|(f(z), h(z))\right|$$

$$< |f(z)|^2+\varepsilon^{1/2}\left(1-|f(z)|^2\right)+\varepsilon^{50}+2\varepsilon^{100} < 1+\varepsilon^{49}$$

(C) and (A) imply:

(b)′ when $z \in bD \cap W$:

$$|f(z)+h(z)|^2-|f(z)|^2 \geqq |h(z)|^2-2\left|(f(z), h(z))\right|$$

$$> \varepsilon^2\left(1-|f(z)|^2\right)-\varepsilon^{50}-2\varepsilon^{100} > \varepsilon^2\left(1-|f(z)|^2\right)-\varepsilon^{49}$$

(A) implies:

(c)′ when $z \in W' \cap bD$:

$$|f(z)+h(z)|^2-|f(z)|^2 \geqq |h(z)|^2-2\left|(f(z), h(z))\right| > -2\varepsilon^{100}$$

1.31, 1.32 imply:

(d)′ when $z \in K \cap W'$:

$$|h(z)| < (\varepsilon')^2$$

from (B):

(e)′ when $z \in W' \cap bD$:

$$|h(z)|^2 < \varepsilon^{1/2}\left(1-|f(z)|^2\right)+\varepsilon^{50}.$$

So the function h satisfies the requirements of Lemma 1 (with smaller error terms) in the domain W', but we do not have control over h out of W'. The following claim is necessary to construct a map g that will fulfil the requirements of Lemma 1 globally in $\bar{D}$. To do so we have g that differs very little from h in W' and is very small in $\bar{D} \setminus W'$. The globalization process that will follow is essentially the one in Stensønes [19].

1.34. *Claim.* There exists $g: \bar{D} \to \mathbf{C}^3$ a C^∞ function, holomorphic in D, such that:

(i) for all $z \in W'$:

$$|g(z)-h(z)| < (\varepsilon')^{1.5}$$

(ii) for all $z \in \bar{D} \setminus W'$:

$$|g(z)| < (\varepsilon')^{1.5}.$$

Proof. Let χ be a C^∞ function on $\bar{D}$ with the following properties:

1.35. (1) $\chi \equiv 1$ on $(W\cap bD)^\delta$

(2) $\chi \equiv 0$ on $\bar{D}\setminus(W\cap bD)^{2\delta}$

(3) $|\bar{\partial}\chi(z)| < C/\delta$ for $z\in\bar{D}$

where $C>0$ is a universal constant.

Since $\bar{\partial}(\chi h)\equiv\bar{\partial}(\chi)h$ on $\bar{D}$, and (by 1.35) $\bar{\partial}(\chi)\equiv 0$ on $(\bar{D}\setminus(W\cap bD)^{2\delta})\cup(W\cap bD)^\delta$; also (by 1.32) $|h|<(\varepsilon')^2$ on $(W\cap bD)^{2\delta}\setminus(W\cap bD)^\delta$ (note that by 1.31 $(W\cap bD)^{2\delta}\subset W'$) therefore:

1.36. $$|\bar{\partial}(\chi h)| < (\varepsilon')^2 C/\delta \quad \text{on} \quad \bar{D}.$$

There exists a C^∞ function h_1 on $\bar{D}$ so that (see [14])

1.37. $$\bar{\partial}h_1 \equiv \bar{\partial}(\chi h) \quad \text{on} \quad \bar{D}, \quad \text{and} \quad \|h_1\|_\infty^D \leqq C_1\|\bar{\partial}(\chi h)\|_\infty^D$$

where $C_1>0$ is a constant.

1.38. Define now $g=\chi h-h_1$. Then g is C^∞ on $\bar{D}$ and holomorphic in D.

(1) For all $z\in W'$ we have (using 1.32, 1.35 (1)):

$$|g(z)-h(z)| \leqq |h_1(z)|+|\chi(z)-1||h(z)| < \|h_1\|_\infty^D+(\varepsilon')^2 \leqq (C_1C/\delta)(\varepsilon')^2+(\varepsilon')^2.$$

(2) For all $z\in\bar{D}\setminus W'$ (since by 1.31, 1.35 $\chi\equiv 0$ on $\bar{D}\setminus W'$):

$$|g(z)| = |h_1(z)| \leqq (C_1C/\delta)(\varepsilon')^2.$$

We can assume that ε' was chosen small enough relative to the constants C, C_1, δ so that claim 1.34 would follow from (1) and (2). Note that we needed here the fact that h decreases very rapidly as we move (in W') away from V.

When we look again at 1.33, 1.34 we can see that g (defined in 1.38) fulfils the requirements of Lemma 1. To do this we have to check that (a)—(e) in the statement of Lemma 1 hold for g. We first check for $Z\in W'$. In 1.33 we proved (in W') (a)—(e) for h with smaller error terms. Since $|g-h|<(\varepsilon')^{1.5}$ in W' it is clear (from looking at the error terms in 1.33 and in (a)—(e)) that (a)—(e) in the statement of Lemma 1 hold for g in W'. We do the same check in $\bar{D}\setminus W'$ and there since $|g|<(\varepsilon')^{1.5}$, (a)—(e) trivially hold.

So Lemma 1 is now proved, for $N=2$.

Proof of Theorem 1.

2.1. Let $W_1, \dots, W_M$ be open subsets of $\bar{D}$ so that $\cup\{W_i\colon 1\leqq i\leqq M\}\supseteq bD$, and W_i $(1\leqq i\leqq M)$ has the properties of W in Lemma 1. Such a cover is possible since bD is compact. As in the statement of Theorem 1 $f\colon bD\to B^{N+1}$ is a continuous map such that $|f|>0$, K is a compact subset of D and $\varepsilon>0$. We choose

$R_0>0$ so that $|f|>R_0$. Define:

2.2. $$K_n = \{z\in D\colon d(z, bD) \geqq d(K, bD)/2^n\}, \quad 1 \leqq n < \infty,$$

then K_n is compact and $K\subset K_n\subset K_{n+1}\subset D$ $(1\leqq n<\infty)$.

2.3. We will use the following notation, when n is an integer there are unique integers a, b so that $aM+b=n$ and $1\leqq b\leqq M$. We define:

$$\bar{n} = b.$$

2.4. Define for $n\geqq 1$ $\varepsilon_n=(100M/(n+A))^{1/2}$ where the constant $A>0$ is to be chosen later. There is no upper bound in the choice of A but there is a lower bound. We require that A is large enough so that (i)—(iii) below are satisfied.

(i) $\varepsilon_1<\varepsilon_0$ where ε_0 is defined in the statement of Lemma 1 with respect to $R_0/2$.

(ii) $\sum_{1\leqq n}(\varepsilon_n)^{45}<\varepsilon/2$

(iii) $A/M^2>100$.

The constant A might have to be magnified later to provide for additional properties.

2.5. Let $S=\max\{|f(z)|\colon z\in bD\}$ so $0<R_0<S<1$.

2.6. We define $b_0=1$, $b_k=1-(\varepsilon_k)^{45}$ for $1\leqq k<\infty$ and $B_0=1$,

$$B_n = b_0 b_1 b_2 \ldots b_{n-1}, \quad 1 \leqq n < \infty \quad \text{and} \quad B = \lim_{n\to\infty} B_n.$$

We will assume that the constant A is large enough so that $S<B<1$.

2.7. Define $g_0\equiv 0$ and $f_1=B^{-1}f$, then $\|f_1\|_\infty<1$.

2.8. The induction hypothesis; let $n\geqq 1$ and assume that $f_1, \ldots, f_n$ and $g_0, \ldots, g_{n-1}$ are defined so that the following hold:

(i) $f_i\colon bD\to B^{N+1}$ is continuous, $1\leqq i\leqq n$

(ii) $g_i\colon \bar{D}\to \mathbf{C}^{N+1}$ is continuous and holomorphic in D

(iii) for $z\in bD$:

$$f_n(z) = (B_n/B)f(z)+(B_n/B_0)g_0(z)+(B_n/B_1)g_1(z)+\ldots+(B_n/B_{n-1})g_{n-1}(z).$$

By Lemma 1 there exists a C^∞ map $g_n\colon \bar{D}\to\mathbf{C}^{N+1}$ which is holomorphic in D such that:

2.9. (a) when $z\in bD$:

$$|f_n(z)+g_n(z)| < 1+(\varepsilon_n)^{45}$$

(b) when $z\in bD\cap W_{\bar{n}}$:

$$|f_n(z)+g_n(z)|^2-|f_n(z)|^2 > (\varepsilon_n)^2(1-|f_n(z)|^2)-(\varepsilon_n)^{45}$$

(c) when $z\in bD$:

$$|f_n(z)+g_n(z)|^2-|f_n(z)|^2 > -(\varepsilon_n)^{45}$$

(d) when $z \in K_n$:

$$|g_n(z)| < (\varepsilon_n)^{45}$$

(e) when $z \in bD$:

$$|g_n(z)|^2 < (\varepsilon_n)^{1/2}(1-|f_n(z)|^2)+(\varepsilon_n)^{45}.$$

2.10. We define for $z \in bD$: $f_{n+1}(z)=b_n(f_n(z)+g_n(z))$. We can see from property (a) that the induction hypothesis 2.8 (i)—(iii) now holds for $n+1$. Property (b) implies: when $z \in bD \cap W_{\bar{n}}$:

$$|f_{n+1}(z)|^2-|f_n(z)|^2 > (\varepsilon_n)^2(1-|f_n(z)|^2)-(\varepsilon_n)^{40}$$

$$\Rightarrow (1-|f_n(z)|^2)-(1-|f_{n+1}(z)|^2) > (\varepsilon_n)^2(1-|f_n(z)|^2)-(\varepsilon_n)^{40}.$$

2.11. We conclude the following:

(A) for all $z \in bD \cap W_{\bar{n}}$:

$$(1-|f_{n+1}(z)|^2) < (1-(\varepsilon_n)^2)(1-|f_n(z)|^2)+(\varepsilon_n)^{40}.$$

(B) for all $z \in bD$:

$$|f_{n+1}(z)|^2-|f_n(z)|^2 > -(\varepsilon_n)^{40}.$$

For the inductive process to work we need to have $|f_n|>R_0/2$ for $\infty>n\geqq 1$ so that we can choose ε_0 with respect to $R_0/2$ and then Lemma 1 can be applied to f_n, ε_n for every $n\geqq 1$ (as $\varepsilon_n<\varepsilon_0$). By choosing A large enough in 2.4, we can assume that $\sum_{k\geqq 1}(\varepsilon_k)^{40}<(R_0)^2/2$.

2.11. (B) implies that for $n>1$, $z \in bD$:

$$|f_n(z)|^2 > |f_1(z)|^2-\sum\nolimits_{n-1\geqq k\geqq 1}(\varepsilon_k)^{40} > (R_0)^2-\sum\nolimits_{\infty>k\geqq 1}(\varepsilon_k)^{40} > (R_0)^2/2$$

so $|f_n(z)|>R_0/2$.

2.12. Define for $z \in \bar{D}$, $n>1$: $G_n(z)=\sum_{0\leqq i\leqq n-1}(B_n/B_i)g_i(z)$. Then 2.9 (d) implies that G_n converges uniformly on compact subsets of D.

2.13. We define for $z \in D$ $g(z)=\lim_{n\to\infty}G_n(z)$, g is then holomorphic on D.

2.14. Define for $n\geqq 1$, $z \in bD$: $\varrho_n(z)=1-|f_n(z)|^2$. The next claim is the main step in proving Theorem 1 from Lemma 1.

Claim 1. There exists a constant $C>0$ so that for all $z \in bD$ and a positive integer s:

$$0 < \varrho_s(z) < C/s^{10}$$

(C does not depend on s and z).

Proof. We define for $z \in bD$.

2.15. $$m(z) = \min\{1 \leqq j \leqq M:\ z \in W_j\}.$$

Let $z \in bD$ and $k > 1$ and $n = kM + m(z)$, then $z \in W_{\bar{n}} = W_{m(z)}$, so:

2.16. $$|f_{n+M}(z)|^2 - |f_n(z)|^2 = \left(\sum_{1 \leqq j \leqq M-1} |f_{n+j+1}(z)|^2 - |f_{n+j}(z)|^2\right)$$
$$+ |f_{n+1}(z)|^2 - |f_n(z)|^2 > \sum_{0 \leqq j \leqq M-1} -(\varepsilon_{n+j})^{40} + (\varepsilon_n)^2 \left(1 - |f_n(z)|^2\right)$$
$$> -M(\varepsilon_n)^{40} + (\varepsilon_n)^2 \left(1 - |f_n(z)|^2\right).$$

(A) and (B) (of 2.11) were used here. 2.16 implies:

2.17. $$M(\varepsilon_n)^{40} + \left(1 - (\varepsilon_n)^2\right)\left(1 - |f_n(z)|^2\right) > 1 - |f_{n+M}(z)|^2$$

or

$$M(\varepsilon_n)^{40} + \left(1 - (\varepsilon_n)^2\right) \varrho_n(z) > \varrho_{n+M}(z).$$

Fix $z \in bD$ and define for $k > 1$

2.18. $$a_k = \varrho_{kM+m(z)}(z).$$

Definition 2.4 and 2.17 imply now that for $k > 1$ (we denote $A_1 = 2A/M$):

2.19. $$a_{k+1} \leqq a_k\left(1 - 100M/(kM + m(z) + A)\right)$$
$$+ M\left(100M/(kM + m(z) + A)\right)^{20} < a_k\left(1 - 100/(k + A_1)\right) + M10^{40}/k^{20}.$$

Define $C_1 = 2M10^{40}A_1 = 4 \cdot 10^{40}A$.

Claim 2. For $k \geqq 1$ $a_k \leqq (C_1)^{10}/k^{10}$.

Proof. of Claim 2. It will be proved by induction. For $k \leqq C_1$ it is trivial since $a_k \leqq 1$ (for all $k \geqq 1$). We will assume that it is true for k, $k \geqq C_1$ and prove that it is true for $k+1$.

By 2.19 and our induction assumption (of Claim 2), we have:

$$a_{k+1} \leqq a_k\left(1 - 100/(k + A_1)\right) + M10^{40}/k^{20}$$
$$\leqq \left((C_1)^{10}/k^{10}\right)\left(1 - 100/(k + A_1)\right) + M10^{40}/k^{20}$$
$$= \left((C_1)^{10}/k^{10}\right)\left(1 - 50/(k + A_1)\right) + \left(M10^{40}/k^{20} - 50(C_1)^{10}/((k + A_1)k^{10})\right)$$
$$< \left((C_1)^{10}/k^{10}\right)\left(1 - 50/(k + A_1)\right) < \left((C_1)^{10}/k^{10}\right)(1 - 25/k)$$
$$= (1 - 25/k)\left((k+1)/k\right)^{10}\left((C_1)^{10}/(k+1)^{10}\right) < (1 - 25/k)(1 + 25/k)\left((C_1)^{10}/(k+1)^{10}\right)$$
$$< (C_1)^{10}/(k+1)^{10}.$$

So Claim 2 is now proved.

We have proved in Claim 2 that for all $k > 1$ and $z \in bD$

2.20. $$\varrho_{kM+m(z)}(z) \leqq (C_1)^{10}/k^{10}.$$

Let us fix $z \in bD$ and $k>1$ and let $n=kM+m(z)$ and fix $0 \leqq j \leqq M-1$ then:

2.21.
$$\begin{aligned}\varrho_{n+j}(z) &= \varrho_n(z)+\sum\nolimits_{0\leqq i\leqq j-1} \varrho_{n+i+1}(z)-\varrho_{n+i}(z)\\ &= \varrho_n(z)+\sum\nolimits_{0\leqq i\leqq j-1} |f_{n+i}(z)|^2-|f_{n+i+1}(z)|^2\\ &\leqq (C_1)^{10}/k^{10}+\sum\nolimits_{0\leqq i\leqq j-1} (\varepsilon_{n+i})^{40} < (C_1)^{10}/k^{10}+M(\varepsilon_n)^{40} < 2(C_1)^{10}/k^{10}.\end{aligned}$$

Let us assume (as we may) that $M>100$ so we have $k>(n+j)/M^2$ 2.21 then implies: $\varrho_{n+j}(z)<2(C_1M^2)^{10}/(n+j)^{10}$. It follows that for every integer $s \geqq 1$ $\varrho_s(z)<C/s^{10}$ where $C=2(C_1M^2)^{10}$.

Claim 1 is now proved, and with 2.9 (e) it implies that G_n converges uniformly on bD and therefore on $\bar{D}$. We will call its limit g which is a continuous extension of the g that we defined at 2.13.

Claim 1 implies that $\lim_{n\to\infty} |f_n(z)|=1$, for $z \in bD$, and by 2.8 (iii) and the uniform convergence of G_n on $\bar{D}$ $\lim_{n\to\infty} f_n(z)=f(z)+g(z)$ uniformly on bD, so for $z \in bD$, $|f(z)+g(z)|=1$.

To complete the proof of Theorem 1 we have to show that $|g(z)|<\varepsilon$ for $z \in K$. 2.4 (ii) and 2.9 (d) imply that for $z \in K$:

$$|g(z)| = \left|\sum\nolimits_{n\geqq 0} (B/B_n)\, g_n(z)\right| < \sum\nolimits_{n\geqq 1} (\varepsilon_n)^{45} < \varepsilon.$$

Theorem 1 is now proved.

It is apparent that the proof above is independent of the dimension of the target ball, as long as the co-dimension allows Lemma 1 to hold. Thus Theorem 1 is proved with target ball B^M, for any $M>N$.

Section 2

Throughout Sections 2, 3 $A(B^N)$ will denote the algebra of all the functions that are continuous on $\bar{B}^N$ and holomorphic in B^N. Let $E \subset bB^N$ be compact, E is called a peak set if there exists $\varphi \in A(B^N)$ so that:

for every $z \in E$: $\varphi(z)=1$

for every $z \in \bar{B}^N \setminus E$: $|\varphi(z)|<1$.

We will call the function φ, a peak function on E. A compact set $E \subset bB^N$ is called an interpolation set if every complex continuous function on E extends to a member of $A(B)$. These two definitions are equivalent, see [18], Chap. 10 for discussion of these and other equivalent definitions.

Theorem 3. *Let* $E \subset bB^N$ $(N \geqq 2)$ *be an interpolation set for* $A(B^N)$ *and* $\hat{f}$: $E \to bB^{N+1}$ *continuous,* $K \subset B^N$ *compact and* $\varepsilon>0$. *Then there exists* $F: \bar{B}^N \to \bar{B}^{N+1}$, *a continuous extension of* $\hat{f}$ *which is holomorphic in* B^N, *so that* $F(bB^N)\subseteqq bB^{N+1}$ *and* $|F(z)|<\varepsilon$ *for all* $z\in K$.

Like the proofs of Theorems 1, 2 the proof of this theorem clearly holds if the target ball is any B^M, where $N<M$. In order to prove this theorem we will use the following Lemma 2, that will play the same role that Lemma 1 has in the proof of Theorem 1. But additional control is now needed on g_n, the correction function of f_n. We must have $|f_n+g_n|\leqq 1$, so that we can maintain $f_n=\hat{f}$ on E for all $\infty>n\geqq 1$. Thus we should have $g_n\equiv 0$ on E and we need to control the growth of g_n as we move away from E. We are able to do so if we maintain in the induction process that $1-|f_n|^2\leqq C_n|1-\varphi|^2$, for all $n\geqq 1$, where $C_n>0$ is a constant and φ is a fixed function in $A(B^2)$, φ is zero on E and has a positive real part elsewhere in $\bar{B}^2$.

The difference between the proof of Lemma 2 and Lemma 1 is a consequence of this additional control. To provide for such a control Globevnik [12] used a different method which is based on a topological observation.

Throughout this section, E will be a fixed interpolation set and $\varphi\in A(B^N)$ will be a fixed peak function on E such that $\mathrm{Re}\,(\varphi)\geqq 0$; thus $\varphi(z)=1$ for all $z\in E$ and $|\varphi(z)|<1$ for all $z\in\bar{B}^N\setminus E$. The constant $\beta=\beta(N)>0$ will be defined for $N>2$ in the third section, $\beta(2)=1$.

Lemma 2. *Let* $R_0>0$, $\varepsilon_0=10^{-(10N)!/(R_0\beta)}$ *and* $z_0\in bB^N$. *There exists* W, *an open neighborhood of* z_0 *in the topology of* $\bar{B}^N$, *such that the following holds.*

(0) *Take* $f\colon \bar{B}^N\to\bar{B}^{N+1}$, $|f|>R_0$, *a continuous map, holomorphic in* B^N *such that:*

(1) $|f(z)|=1\Leftrightarrow z\in E$

(2) *there exists* $C>0$ *so that for all* $z\in bB^N$: $1-|f(z)|^2\geqq C|1-\varphi(z)|^2$.
Take also $\varepsilon_0>\varepsilon>0$, $\varepsilon'>0$ *so that:*

(3) $\varepsilon^{1000}C/(C+1)>\varepsilon'$

(4) $\{z\in\bar{B}^N\colon |1-\varphi(z)|<\varepsilon'\}\subseteqq\{z\in\bar{B}^N\colon 1-|f(z)|^2<\varepsilon^{100}\}$,

and let $K\subset B^N$ *be compact.*

Then there exists a continuous map $g\colon \bar{B}^N\to\mathbf{C}^{N+1}$ *which is holomorphic in* B^N *that has the following properties:*

(a) *There exist* $C'>0$ *so that for all* $z\in bB^N$: $1-|f(z)+g(z)|^2\geqq C'|1-\varphi(z)|^2\geqq 0$ *(it is true for* $C'=C/4$*)*

(b) *for all* $z\in W\cap bB^N$: $|f(z)+g(z)|^2-|f(z)|^2>(1-|f(z)|^2)\varepsilon^2-\varepsilon^{50}$

(c) *for all* $z\in bB^N$: $|f(z)+g(z)|^2-|f(z)|^2>-\varepsilon'$

(d) *for all* $z\in K$: $|g(z)|<\varepsilon'$

(e) *for all* $z\in bB^N$: $|g(z)|^2<\varepsilon^{1/2}(1-|f(z)|^2)+\varepsilon'$

(f) *for all* $z\in E$: $g(z)=0$.

The neighborhood W is the one chosen in the proof of Lemma 1. A significant difference from Lemma 1 is in (a) and (f). Property (2) enables us to prove (a) which preserves property (2) in the induction process and implies $|f+g|\leqq 1$. This is vital in our construction.

Proof of Lemma 2.

In this section we will present a proof for the case $N=2$. The case $N\geqq 2$ will be proved on Section 3 when the necessary tools for it have been developed. Our construction is similar to the one in the proof of Lemma 1 but a few changes will take place to provide for the additional control mentioned above.

The set up in the proof of Lemma 1 until 1.4 will be adopted here unchanged where $D=B^2$, U', U, V', V, W, W', u_X and the coordinate system will be the same as there, and thus, they have the same properties.

3.1. Let $(\varepsilon')^{1/\varepsilon'}>r>0$ be so that when $X, Y\in V'$ and

$$|X-Y|<\left(\log(1/r)\right)^{-1}$$

then:

(i) $|f(X)-f(Y)|<(\varepsilon')^{100}$,

(ii) $\left|1/(|1-\varphi(X)|^4+(\varepsilon')^{10})-1/(|1-\varphi(Y)|^4+(\varepsilon')^{10})\right|<(\varepsilon')^{100}$,

(iii) $\sum_{m\geqq -l(r)}(1/2)^{m^2/4}<(\varepsilon')^{100}$

where $l(r)\overset{\text{def}}{=}-\log r$,

(iv) $(\log\varepsilon)/(r^{1/2}\cdot 4\pi)$ is an integer.

We might have to shrink $r>0$ later to have additional properties.

3.2. Next $c_1, c_2, c_3, L, L', X_a, u_a$ $(a\in L')$ are defined by 1.6, 1.7, 1.8. Hence u_a is described by 1.8, 1.9 (where the remainder term is described in Sublemma 1). Also p_a $(a\in L')$ are defined by 1.10 and have the properties that are mentioned in 1.11, 1.12, 1.13.

We will now define for every $a\in L'$, $v_a\in\mathbf{C}^3$ in a way that is based on exactly the same principal but is somewhat different in its details then in 1.14, a difference that will be understood later. The set $\{v_a\colon a\in L'\}$ will have the following properties:

3.3. For $a, b\in L'$

(i) $(v_a, f(X_a))=0$

(ii) $|v_a|^2=2\varepsilon\left(1-|f(X_a)|^2\right)/\left(|1-\varphi(X_a)|^4+(\varepsilon')^{10}\right)$

(iii) when $a, b\in L'$, $|a-b|<1000$ and b_2-a_2 is odd then:

$$|(v_a, v_b)|<(\varepsilon')^{50}$$

(iv) when $a, b \in L'$, $|a-b|<1000$ and b_2-a_2 is even then:

$$\left|(v_a, v_b)-|v_a|^2\right| < (\varepsilon')^{50}.$$

The proof that a set $\{v_a\colon a\in L'\}$ with the properties (i)—(iv) exists, is essentially the same proof as in 1.14.

Define as in Lemma 1:

$$h(z) = \sum_{a\in L} p_a \cdot v_a = \sum_{a\in L'} p_a \cdot v_a.$$

The sets $L(x, n)$, $L'(x, n)$ are defined as in 1.16.

We will now prove the following (compare with 1.17 (A), (B), (C) in the proof of Lemma 1). For $X\in V'$ (V', V are defined in 0.2):

3.4. (A) $\left|(f(X), h(X))\right| < (\varepsilon')^4$,

(B) $|h(X)|^2 < \varepsilon^{1/2}(1-|f(X)|^2)/(|1-\varphi(X)|^4+(\varepsilon')^{10})+(\varepsilon')^4$,

(C) when $X\in V$, then

$$|h(X)|^2 > \varepsilon^2(1-|f(X)|^2)/(|1-\varphi(X)|^4+(\varepsilon')^{10})-(\varepsilon')^4.$$

We will freely use the facts that $|v_a|^2<(\varepsilon')^{-10}$ for all $a\in L$, and when $X\in V'$, $a\in L(X,n)$ for $n\leq l(r)\overset{\text{def}}{=\!=}-\log r$ then since $|X-X_a|, |X_a-X_{a(X)}|<r^{1/5}$ it follows from 3.1:

$$\left||v_a|^2-|v_{a(X)}|^2\right| < (\varepsilon')^{80}, \quad \text{and} \quad |f(X)-f(X_a)| < (\varepsilon')^{100}.$$

We will also use (without mention) the facts stated in 1.18 and assumption 3.1 (iii).

Proof of (A):

$$\left|(f(X), h(X))\right| = \left|\sum_{a\in L}(f(X), v_a)\bar{p}_a(X)\right|$$

$$= \left|\sum_{0\leq n\leq l(r)} \sum_{a\in L(X,n)} (f(X)-f(X_a), v_a)\bar{p}_a(X)\right| + \left|\sum_{l(r)<n} \sum_{a\in L(X,n)} (f(X), v_a)\bar{p}_a(X)\right|$$

$$\leq \sum_{0\leq n\leq l(r)} \sum_{a\in L(X,n)} |f(X)-f(X_a)|\cdot|v_a|\,|p_a(X)| + \sum_{l(r)<n} \sum_{a\in L(X,n)} |p_a(X)|\,|v_a|$$

$$< \sum_{0\leq n\leq l(r)} (10n+10)^3(\varepsilon')^{100}\cdot(\varepsilon')^{-5}\cdot\varepsilon^{n^2} + \sum_{l(r)<n} (10n+10)^3\varepsilon^{n^2}\cdot(\varepsilon')^{-5} < (\varepsilon')^4.$$

Proof of (B): Fix $X\in V'$:

$$|h(X)|^2 = \left|\sum_{a,b\in L}(v_a, v_b)p_a(X)\bar{p}_b(X)\right|$$

$$< \sum_{0\leq m\leq l(r)} \sum_{a,b\in L,\,[a]+[b]=m} |(v_a, v_b)p_a(X)\bar{p}_b(X)|$$

$$+ \sum_{l(r)<m} \sum_{a,b\in L,\,[a]+[b]=m} |p_a(X)\bar{p}_b(X)|\cdot(\varepsilon')^{-10}$$

$$< \sum_{0\leq m\leq l(r)} \left(|v_{a(X)}|^2+(\varepsilon')^{50}\right)(10m+10)^6\cdot\varepsilon^{m^2/4} + \sum_{l(r)<m} (\varepsilon')^{-10}(10m+10)^6\cdot\varepsilon^{m^2/4}$$

$$< 10^7|v_{a(X)}|^2+(\varepsilon')^5 < \varepsilon^{1/2}(1-|f(X)|^2)/(|1-\varphi(X)|^4+(\varepsilon')^{10})+(\varepsilon')^4.$$

3.1 was used in the last two inequalities, (B) is now proved.

Proof of (C):
Fix $X\in V$ then:

3.5. $$|h(X)|^2=\sum_{a,b\in L}(v_a, v_b)p_a(X)\bar{p}_b(X)$$
$$=\sum_{a,b\in L(X,0)}(v_a, v_b)p_a(X)\bar{p}_b(X)+\sum_{m\geqq 1}\sum_{a,b\in L,\,[a]+[b]=m}(v_a, v_b)p_a(X)\bar{p}_b(X)$$
$$\geqq \operatorname{Re}\big(\sum_{a,b\in L(X,0)}(v_a, v_b)p_a(X)\bar{p}_b(X)\big)$$
$$-\sum_{m\geqq 1}\sum_{a,b\in L,\,[a]+[b]=m}(|v_a|^2+|v_b|^2)|p_a(X)\bar{p}_b(X)|.$$

Let us look at the first term on the righ side of 3.5. This estimate is essentially the one that follows 1.19, we will use the calculations done there, and 1.13, 3.3

$$\operatorname{Re}\big(\sum_{a,b\in L(X,0)}(v_a, v_b)p_a(X)\bar{p}_b(X)\big)\geqq\sum_{a\in L(X,0)}|v_a|^2|p_a(X)|^2$$
$$+\operatorname{Re}\big(\sum_{a,b\in L(X,0),\,a\neq b,\,a_2=b_2}|v_a|^2p_a(X)\bar{p}_b(X)\big)$$
$$+\operatorname{Re}\big(\sum_{a,b\in L(X,0),\,a\neq b,\,a_2=b_2}((v_a, v_b)-|v_a|^2)p_a(X)\bar{p}_b(X)\big)$$
$$-\sum_{a,b\in L(X,0),\,a\neq b,\,a_2=b_2\pm 1}|(v_a, v_b)p_a(X)\bar{p}_b(X)|$$
$$-\sum_{a,b\in L(X,0),\,a\neq b,\,a_2=b_2\pm 2}|(v_a, v_b)p_a(X)\bar{p}_b(X)|>(3/4)|v_{a(X)}|^2|p_{a(X)}(X)|^2-(\varepsilon')^{40}.$$

Let us estimate the second term of 3.5. As mentioned in 1.21, when $a, b\in L$, $[a]+[b]=m\geqq 1$ then:

$$|p_a(X)\bar{p}_b(X)|<\varepsilon^{m^2/4},\quad |p_a(X)\bar{p}_b(X)|\leqq\varepsilon^{1/4}|p_{a(X)}(X)|^2$$

and when $[a]+[b]=m\geqq 2$, then

$$|p_a(X)\bar{p}_b(X)|<\varepsilon^{(m^2/4-1/2)}\cdot|p_{a(X)}(X)|^2.$$

3.6. $$\sum_{m\geqq 1}\sum_{a,b\in L,\,[a]+[b]=m}(|v_a|^2+|v_b|^2)|p_a(X)\bar{p}_b(X)|$$
$$<\sum_{1\leqq m\leqq l(r)}\sum_{a,b\in L,\,[a]+[b]=m}(|v_a|^2+|v_b|^2)|p_a(X)\bar{p}_b(X)|$$
$$+\sum_{m>l(r)}\sum_{a,b\in L,\,[a]+[b]=m}2(\varepsilon')^{-10}\cdot\varepsilon^{m^2/4}$$
$$<\sum_{1\leqq m\leqq l(r)}\sum_{a,b\in L,\,[a]+[b]=m}2\big(|v_{a(X)}|^2+(\varepsilon')^{50}\big)|p_{a(X)}(X)|^2\cdot\varepsilon^{\max\{(m^2/4-1/2),1/4\}}$$
$$+\sum_{m>l(r)}2(10m+10)^6(\varepsilon')^{-10}\cdot\varepsilon^{m^2/4}<|v_{a(X)}|^2|p_{a(X)}(X)|^2\cdot\varepsilon^{1/5}+(\varepsilon')^{20}.$$

Now 3.5 and 3.6 imply:

$$|h(X)|^2>|v_{a(X)}|^2|p_{a(X)}(X)|^2(3/4-\varepsilon^{1/5})-(\varepsilon')^{10}$$
$$>\varepsilon^2(1-|f(X)|^2)/(|1-\varphi(X)|^4+(\varepsilon')^{10})-(\varepsilon')^4.$$

(C) is now proved.

(A), (B), and (C) in this section are similar to (A), (B), (C) in the proof of

Lemma 1. The following (D) will be the same as (D) in the proof of Lemma 1 and will be proved in the same way.

(D) For all $\delta>0$, $\mu>0$ there exist $r_0>0$, $r_0=r_0(\delta,\mu)$ so that if $0<r<r_0$ in the definition of h then for every $X\in U'$ such that $d(X,V)>\delta$ (distance in the coordinates) we have $|h(X)|<\mu$.

Proof of (D): Fix $X=(x,y,z,w)\in U'$ so that $d(X,U)>\delta$. Define $M(X)$ as in 1.29. Then:

$$|h(X)|^2 = \left|\sum\nolimits_{a,b\in L}(v_a, v_b)p_a(X)\bar{p}_b(X)\right|$$
$$= \sum\nolimits_{M(X)\leqq m}\sum\nolimits_{a,b\in L,\,[a]+[b]=m}(\varepsilon')^{-10}|p_a(X)\bar{p}_b(X)|$$
$$\leqq \sum\nolimits_{M(X)\leqq m}(\varepsilon')^{-10}\cdot(m+2)^6\varepsilon^{m^2/4}\varepsilon^{2w/r}.$$

Like the sum in 1.30 this sum can be made arbitrarily small uniformly on $X\in U'$, $d(X,V)>\delta$, as we shrink $r>0$.

Let us now choose $\delta>0$ as in 1.31 and $\mu=(\varepsilon')^{10}$ and $r>0$ in the definition of h will be chosen so that $|h|<(\varepsilon')^{10}$ outside of V^δ. The proof of the claim in 1.34 makes it clear that if $\varepsilon'>0$ is smaller than some constant (and we assume it is so), then there exists a C^∞ map $\hat{g}\colon \bar{B}^2\to\mathbf{C}^3$, holomorphic in B^2 such that:

3.7. for all $z\in W'$,

$$|\hat{g}(z)-h(z)|<(\varepsilon')^8 \quad\text{and for}\quad z\in\bar{B}^2\setminus W',\quad |\hat{g}(z)|<(\varepsilon')^8.$$

This implies (among other things) that:

3.8. for all $z\in K$,

$$|\hat{g}(z)|<(\varepsilon')^3.$$

3.9. Let us define for $z\in\bar{B}^2$:

$$g(z)=\big(1-\varphi(z)\big)^2\hat{g}(z).$$

Then (A) and 3.7 and 3.9 imply that for all $z\in bB^2$:

3.10. $$\big|\big(f(z),g(z)\big)\big|\leqq(\varepsilon')^3|1-\varphi(z)|^2.$$

Now (B) and 3.7 and 3.9 imply that for all $z\in bB^2$:

3.11. $$|g(z)|^2\leqq\big(\varepsilon^{1/2}\big(1-|f(z)|^2\big)/\big(|1-\varphi(z)|^4+(\varepsilon')^{10}\big)+(\varepsilon')^3\big)|1-\varphi(z)|^4$$
$$\leqq\varepsilon^{1/2}\big(1-|f(z)|^2\big)+(\varepsilon')^3|1-\varphi(z)|^4.$$

Thus 3.10 and 3.11 imply that for $z\in bB^2$:

3.12. $$1-|f(z)+g(z)|^2\geqq 1-|f(z)|^2-|g(z)|^2-2\big|\big(f(z),g(z)\big)\big|$$
$$\geqq\big(1-|f(z)|^2\big)(1-\varepsilon^{1/2})-(\varepsilon')^3\big(|1-\varphi(z)|^4+2|1-\varphi(z)|^2\big)$$
$$\geqq\big(1-|f(z)|^2\big)/2-10(\varepsilon')^3|1-\varphi(z)|^2$$

(using assumptions (2) and (3) in the statement of Lemma 2, for the first time)

$$\geqq (C/2)|1-\varphi(z)|^2-10(\varepsilon')^3|1-\varphi(z)|^2 \geqq (C/4)|1-\varphi(z)|^2.$$

So (a) (in the statement of Lemma 2) is now proved. We could, of course, obtain a better (larger) constant then $C/4$, but it has no importance at all (as long as we obtain a positive constant).

We have (by 3.10) that for all $z\in bB^2$:

3.13. $$|f(z)+g(z)|^2-|f(z)|^2 > -2|(f(z), g(z))| > -\varepsilon'$$

thus (c) is proved.

From 3.8 (and 3.9) (d) follows, and 3.11 implies (e). Now (f) follows from the Definition 3.9, so it remains to prove (b).

Let $z\in W\cap bB^2$, (C) and 3.7 imply that:

3.14.

$$|g(z)|^2 \geqq \varepsilon^2(1-|f(z)|^2)(|1-\varphi(z)|^4+(\varepsilon')^{10})^{-1}|1-\varphi(z)|^4-(\varepsilon')^3|1-\varphi(z)|^4.$$

So by 3.10 we have:

3.15. $$|f(z)+g(z)|^2-|f(z)|^2 \geqq |g(z)|^2-2|(f(z), g(z))|$$

$$\geqq \varepsilon^2(1-|f(z)|^2)(|1-\varphi(z)|^4+(\varepsilon')^{10})^{-1}\cdot|1-\varphi(z)|^4-10(\varepsilon')^3|1-\varphi(z)|^2.$$

If $1-|f(z)|^2>\varepsilon^{100}$, then assumption (4) implies that $|1-\varphi(z)|>\varepsilon'$ and by 3.15:

3.16.

$$|f(z)+g(z)|^2-|f(z)|^2 \geqq \varepsilon^2(1-|f(z)|^2)(1+|1-\varphi(z)|^{-4}(\varepsilon')^{10})^{-1}-10(\varepsilon')^3$$

$$> \varepsilon^2(1-|f(z)|^2)(1-\varepsilon')-\varepsilon'.$$

It is clear that (b) holds in this case. If $1-|f(z)|^2\leqq\varepsilon^{100}$, then by 3.13 we have $|f(z)+g(z)|^2-|f(z)|^2>-\varepsilon'>\varepsilon^{100}-\varepsilon^{50}>\varepsilon^2(1-|f(z)|^2)-\varepsilon^{50}$ and (b) holds also in this case. Lemma 2 is proved.

Proof of Theorem 3.

By Lemma 1 of Globevnik [12] there exists a continuous extension of $\hat{f}$, f: $\bar{B}^N\to\bar{B}^{N+1}$, which is holomorphic in B^N such that for all $z\in bB^N$:

4.1. $$1 \geqq |f(z)| > 1/2.$$

Choose an integer $n_0>1$ so that for all $z\in K$:

4.2. $$(|1+\varphi(z)|/2)^{n_0} < \varepsilon/2.$$

Recall that we fixed the the function φ before the statement of Lemma 2, φ is a peak function on E and $\mathrm{Re}(\varphi)\geqq 0$.

Define for $z\in\bar{B}^N$

4.3. $$f_1(z)=((1+\varphi(z))/2)^{n_0}f(z).$$

Thus when $z\in bB^N$:

4.4. $$|f_1(z)|=(|1+\varphi(z)|/2)^{n_0}|f(z)|>2^{-(n_0+1)}.$$

Define $R_0=2^{-(n_0+1)}$. Note that for all $z\in K$:

4.5. $$|f_1(z)|<\varepsilon/2.$$

Before we proceed we need the following simple fact:

If $\lambda\in\mathbf{C}$, $|\lambda|\leqq 1$, then

4.6. $$1-|(1+\lambda)/2|^2\geqq(1/4)|1-\lambda|^2.$$

Proof. This is equivalent to:

$$4-(1+\lambda)(1+\bar{\lambda})\geqq(1-\lambda)(1-\bar{\lambda})\Leftrightarrow 3-|\lambda|^2-(\lambda+\bar{\lambda})\geqq 1+|\lambda|^2-(\lambda+\bar{\lambda})\Leftrightarrow 2\geqq 2|\lambda|^2.$$

So 4.6 is true.

Using 4.6 we have for all $z\in\bar{B}^2$:

4.7. $$\begin{aligned}1-|f_1(z)|^2&=1-\big|(1+\varphi(z))/2\big|^2\big|(1+\varphi(z))/2\big|^{2(n_0-1)}|f(z)|^2\\&\geqq 1-\big|(1+\varphi(z))/2\big|^2\geqq(1/4)\big|(1-\varphi(z))\big|^2.\end{aligned}$$

So f_1 fulfils condition (2) in Lemma 2 with $C=1/4$.

We shall follow now a similar process, and in most parts identical, to the one in the proof of Theorem 1 of constructing $\{f_n\colon 1\leqq n<\infty\}$. We will not repeat identical details.

4.8. Let $W_1,\dots,W_M$ be relatively open subsets of $\bar{B}^N$ which have the properties of W in Lemma 2, and so that $bB^N\subseteqq\bigcup\{W_i\colon 1\leqq i\leqq M\}$. Let K_n be defined by 2.2 (where $D=B^N$) and $\bar{n}$ for $n>1$ be defined by 2.3 and A, ε_n (for $n\geqq 1$) by 2.4.

4.9. Let $n\geqq 1$, and assume inductively that the maps $g_1,\dots,g_{n-1},f_1,\dots,f_n$ are defined, and that for $1\leqq i\leqq n$, $f_i\colon\bar{B}^N\to\bar{B}^{N+1}$ is continuous and holomorphic in B^N, $f_i\equiv f$ on E, and that there exists, $C_i>0$, so that $1-|f_i(z)|^2\geqq C_i|1-\varphi(z)|^2$ for all $z\in\bar{B}^N$. Assume also that for $1\leqq i\leqq n-1$ $g_i\colon\bar{B}^N\to\mathbf{C}^{N+1}$ are continuous and holomorphic in B^N, and $f_n=f_1+g_1+\dots+g_{n-1}$.

4.10. Let $\varepsilon_n'>0$ be so that

$$((\varepsilon_n)^{1000}C_n)/(C_n+1)>\varepsilon_n'$$

and $\{z\in\bar{B}^N\colon |1-\varphi(z)|<\varepsilon_n'\}\subseteqq\{z\in\bar{B}^N\colon 1-|f_n(z)|^2<(\varepsilon_n)^{100}\}$.

By Lemma 2 there exists $g_n\colon\bar{B}^N\to\bar{B}^{N+1}$, continuous and holomorphic in B^N so that the following (a)—(f) holds (with $C_{n+1}=C_n/4$):

4.11. (a) for all $z\in bB^N$,

$$1-|f_n(z)+g_n(z)|^2\geqq C_{n+1}|1-\varphi(z)|^2\geqq 0,$$

(b) for all $z \in W_{\bar{n}} \cap bB^N$,

$$|f_n(z)+g_n(z)|^2-|f_n(z)|^2 > (1-|f_n(z)|^2)(\varepsilon_n)^2-(\varepsilon_n)^{50},$$

(c) for all $z \in bB^N$,

$$|f_n(z)+g_n(z)|^2-|f_n(z)|^2 > -\varepsilon_n',$$

(d) for all $z \in K_n$,

$$|g_n(z)| < \varepsilon_n',$$

(e) for all $z \in bB^N$,

$$|g_n(z)|^2 < (\varepsilon_n)^{1/2}(1-|f_n(z)|^2)+\varepsilon_n',$$

(f) for all $z \in E$,

$$g_n(z) = 0.$$

4.12. We will define $f_{n+1}=f_n+g_n$. It is easy to see that our induction hypothesis holds for $n+1$. Property (d) implies that f_n converges on compacta. We will call its limit F. Exactly the same proof (in fact, we do not need to change even one word) as in 2.11—2.21 can be applied here to show that there exists $C>0$ so that for $n>1$ and $z \in bB^N$:

4.13. $$1-|f_n(z)|^2 \leqq C/n^{10}.$$

It follows then from (e) that f_n converges uniformly on $\bar{B}^N$, so F can be extended continuously to the boundary. By 4.13 $|F(z)|=1$ when $z \in bB^N$, by (f) $F(z)=\hat{f}(z)$ for $z \in E$, and by 4.5, 2.4 (ii), 4.11 (d), $|f(z)|<\varepsilon$ when $z \in K$. We choose $\varepsilon_0>0$ with respect to $R_0/2$ and by having $0<\varepsilon_n'$ small enough, we can assume that $\sum_{1\leqq n} \varepsilon_n' < (R_0)^2/2$. So (c) and 4.4 now imply that for $n>1$, $z \in bB^N$, $|f_n(z)|>R_0/2$. The proof of Theorem 3 is now completed.

Theorem 4. *Let $\varepsilon>0$. There exists a continuous map $F\colon \bar{B}^N \to \bar{B}^{N+1}$ which is holomorphic in B^N such that $F(bB^N)=bB^{N+1}$ and $F((1-\varepsilon)B^N) \subseteqq \varepsilon B^{N+1}$. The map F can be the extension of any continuous map $f\colon \mathbf{R}^N \cap bB^N \to bB^{N+1}$.*

This gives a positive answer to an open question by Globevnik [12].

Proof. We will follow the Globevnik [12] proof. Let $\hat{A} \subset b\Delta$ ($\Delta=\{z \in \mathbf{C}\colon |z|<1\}$) be a Cantor subset. Assume $1, -1 \notin \hat{A}$. The set $A=\hat{A}\times\{(0, 0, \ldots, 0)\}$ is an interpolation set of $A(B^N)$ (see Rudin [18], 10.1.5). Let $B=\mathbf{R}^N \cap bB^N$, so B is also an interpolation set of $A(B^N)$ and so is $A \cup B$ (see [18, 10]). There exists $g\colon A \to bB^{N+1}$, a continuous map so that $g(A)=bB^{N+1}$ (see [15] p. 166). If $f\colon B \to bB^{N+1}$ is continuous, then since $A \cap B=\emptyset$, we can define for $z \in A \cup B$:

$$\hat{f}(z) = g(z) \quad \text{for} \quad z \in A \quad \text{and} \quad \hat{f}(z) = f(z) \quad \text{for} \quad z \in B.$$

So $\hat{f}\colon A \cup B \to bB^{N+1}$ is continuous and by Theorem 3 there exists $F\colon \bar{B}^N \to \bar{B}^{N+1}$, continuous and holomorphic in B^N such that $F=\hat{f}$ on $A \cup B$, $F(bB^N) \subseteqq bB^{N+1}$ and

$F((1-\varepsilon)B^N)\subseteq \varepsilon B^{N+1}$. Since $\hat{f}(A)=bB^{N+1}$ then $F(bB^N)=bB^{N+1}$. Theorem 4 is thus proved.

It is interesting that f can be any real analytic function, for example $f(z)=(z, 0)$ for $z\in B$. So while F can not be C^∞ in an open subset of bB^N (it would imply (by [3]) that F is rational) it can be real analytic in a smaller set like $\mathbf{R}^N\cap bB^N$.

Section 3

In this section the results obtained in Section 1, 2 are generalized to dimension $N\geqq 2$. The following Lemma 3 will provide us with the basic tool that is needed to generalize Lemma 1 and Lemma 2 to dimension $N\geqq 2$. We fix $N\geqq 2$ until the end of the section. For an integer n we define here $\bar{n}$ to be the unique integer in $\{1, 2, ..., N\}$ so that $n-\bar{n}$ is an integer product of N. For $1\leqq i\leqq N$ we define:

5.1. $$S_i = \{a = (a_1, ..., a_{N-1})\in \mathbf{Z}^{N-1}\colon \overline{a_1+...+a_{N-1}} = i\}$$

where $\mathbf{Z}$ is the set of all integers. The following standard notation will be used: when $x\in\mathbf{R}^{N-1}$, $r>0$ then $B(x, r)=\{y\in\mathbf{R}^{N-1}\colon |y-x|<r\}$.

Lemma 3. *Let* $0<\alpha<1/4$ *then there exist* $v_1, ..., v_N\in\mathbf{R}^{N-1}$, $|v_i|<\alpha$ $(1\leqq i\leqq N)$, *and* $\beta>0$ *so that if we define*

5.2. $$\check{S}_i = \{a+v_i | a\in S_i\}$$

then for every $x\in\mathbf{R}^{N-1}$ *there exists* $1\leqq i\leqq N$ *so that if* $d=d(x, \check{S}_i)$ (the distance of x from $\check{S}_i$) *then there exists only one element in* $B(x, d+\beta)\cap\check{S}_i$. *The constant* $\beta>0$ *depends only on the dimension* N *and on* α *and it does not depend on* x.

In other words, after shifting each of the sets S_i, we obtain sets $\{\check{S}_i\}$ such that for every point x in $\mathbf{R}^{N-1}$ there is $\check{S}_i$, and a point $y\in\check{S}_i$ which is closer to x, by a difference of a constant $\beta>0$, than any other point in $\check{S}_i$.

Proof. When $N=2$ then Lemma 3 is trivial with $\beta=1$ and $v_i=0$ $(i=1, 2)$ we will therefore assume throughout the proof of Lemma 3 that $N>2$. Choose $0<y_{ij}<\alpha/N$, for all $1\leqq i\leqq N$, $1\leqq j\leqq N-1$, so that the set (of $N(N-1)+1$ real numbers): $\{y_{ij}|1\leqq i\leqq N,\ 1\leqq j\leqq N-1\}\cup\{1\}$ is linearly independent over $\mathbf{Q}$. Where $\mathbf{R}$ is viewed as a vector space over $\mathbf{Q}$.

Define:

5.3. $$v_i = (y_{i1}, ..., y_{i,N-1}) \quad 1 \leqq i \leqq N.$$

We will prove that Lemma 3 holds with these $\{v_i\}$, for some $\beta>0$, where (as in 5.2) $\check{S}_i=S_i+v_i$ for all $1\leqq i\leqq N$. We define for $a\in S_i$ and $1\leqq i\leqq N$:

5.4. $$\check{a} = a+v_i.$$

The following notation will be used, when $v, w \in \mathbf{R}^{N-1}$ then:

5.5. $$[v, w] = (v+w)/2+(v-w)^{\perp}$$

(we use here the standard notation, for $v \in \mathbf{R}^{N-1}$ $v^{\perp}=\{u \in \mathbf{R}^{N-1}: (u, v)=0\}$). Note the simple fact that for $x, v, w \in \mathbf{R}^{N-1}$:

$$|x-w|^2 = |x-v|^2 \Leftrightarrow |x|^2+|w|^2-2(x, w) = |x|^2+|v|^2-2(x, v)$$
$$\Leftrightarrow 2(x, v-w) = (v+w, v-w) \Leftrightarrow (x-(v+w)/2, v-w) = 0 \Leftrightarrow x \in [v, w].$$

So $[v, w]$ is the set of points in $\mathbf{R}^{N-1}$ which have the same distance from v and w.

The following claim is of central importance in the proof of Lemma 3.

5.6. *Claim.* If $a_i, b_i \in S_i$, $1 \leqq i \leqq N$ and $a_i \neq b_i$ then

$$\bigcap \{[\check{a}_i, \check{b}_i]:\ 1 \leqq i \leqq N\} = \emptyset.$$

Proof. Let us assume (to get a contradiction) that there exists

$$p = (p_1, \dots, p_{N-1}) \in \mathbf{R}^{N-1}$$

so that $p \in [\check{a}_i, \check{b}_i]$ for all $1 \leqq i \leqq N$. Then we have for all $1 \leqq i \leqq N$:

$$(p-((a_i+b_i)/2+v_i),\ a_i-b_i) = 0;$$

which is equivalent to:

5.7. $$(p, a_i-b_i) = (|a_i|^2-|b_i|^2)/2+(v_i, a_i-b_i) \quad \text{for all} \quad 1 \leqq i \leqq N.$$

Let us look again at $\mathbf{R}$ as a vector space over $\mathbf{Q}$ and define:

$$W = \mathrm{sp}_{\mathbf{Q}}(p_1, \dots, p_{N-1})$$

and

$$\gamma_i = (|a_i|^2-|b_i|^2)/2+(v_i, a_i-b_i) \quad (1 \leqq i \leqq N).$$

Then 5.7 implies that $\gamma_i \in W$ for all $1 \leqq i \leqq N$. Since the dimension of W over $\mathbf{Q}$ is no more than $N-1$ we must conclude that $\gamma_1, \dots, \gamma_N$ are linearly dependent. But this contradicts our assumption that $\{y_{ij} | 1 \leqq i \leqq N,\ 1 \leqq j \leqq N-1\} \cup \{1\}$ are linearly independent over $\mathbf{Q}$. So claim 5.6 is proved.

In general a claim like 5.6 can not be proved if the number of families $\{S_i\}$ is $K<N$. In this case we would have in claim 5.6 $K<N$ hyperplanes in $\mathbf{R}^{N-1}$, we can write them as $V_i = u_i+(w_i)^{\perp}$ where $u_i, v_i \in \mathbf{R}^{N-1}$ $1 \leqq i \leqq K$. Then $\bigcap\{V_i:\ 1 \leqq i \leqq K\} = \emptyset$ is equivalent to saying that there is no $p \in \mathbf{R}^{N-1}$ so that

$$(p-u_i, w_i) = 0 (\Leftrightarrow (p, w_i) = (u_i, w_i))$$

for all $1 \leqq i \leqq K$. But when $\{w_i:\ 1 \leqq i \leqq K\}$ are linearly independent then of course there is such $p \in \mathbf{R}^{N-1}$.

The following is derived from claim 5.6.

5.8. *Claim.* There exists $\beta'>0$ such that if $a_i, b_i \in S_i$, $a_i \neq b_i$ $(1 \leqq i \leqq N)$ and there is $a \in S_N$ such that for all $1 \leqq i \leqq N$, $a_i, b_i \in B(a, 2N)$ then

$$\cap \{[\check{a}_i, \check{b}_i]+B(0,\beta'):\ 1 \leqq i \leqq N\} = \emptyset.$$

($\check{a}_i$, $\check{b}_i$ are defined by 5.4).

Proof. Let us define $a_i'=a_i-a$, $b_i'=b_i-a$ $(1 \leqq i \leqq N)$ then $[\check{a}_i, \check{b}_i]=a+[\check{a}_i', \check{b}_i']$ so it is apparent that it is enough to prove 5.8 with the assumption that $a=0$, and then with the same $\beta'>0$ it will be true for any $a \in S_N$.

Define $T_i=\{[\check{a}_i, \check{b}_i]: a_i, b_i \in S_i \cap B(0, 2N), a_i \neq b_i\}$ for $1 \leqq i \leqq N$ and $T=T_1 \times \ldots \times T_N$ and let $V=(V_1, \ldots, V_N) \in T$. Claim 5.6 implies that $\cap \{V_i: 1 \leqq i \leqq N\}=\emptyset$, and since $V_1, \ldots, V_N$ are affine hyperplanes of $\mathbf{R}^{N-1}$ there exists $\beta'(V)>0$ so that $\cap \{V_i+B(0, \beta'(V)): 1 \leqq i \leqq N\}=\emptyset$ (see Claim 5.11 after the end of this proof). Take $\beta'=\min \{\beta'(V): V \in T\}$ ($\beta'>0$ is well defined as the minimum is taken over a finite set). Then for every $V=(V_1, \ldots, V_N) \in T$, $\cap \{V_i+B(0, \beta'): 1 \leqq i \leqq N\}=\emptyset$, and 5.8 is now proved.

Define $\beta=\beta'/N$. Now take $x \in \mathbf{R}^{N-1}$, and choose for every $1 \leqq i \leqq N$ $a_i \in S_i$ so that $d(x, _{D})=d(x, \check{S}_i)$ and $b_i \in S_i \setminus \{a_i\}$ so that $d(x, \check{b}_i)=d(x, \check{S}_i \setminus \{\check{a}_i\})$ ($\check{a}_i$ is the closest element to x in $\check{S}_i$ and $\check{b}_i$ is the next closest). It is clear that there exists an $a \in S_N$ so that $a_i, b_i \in B(a, 2N)$ for all $1 \leqq i \leqq N$. Therefore 5.8 implies that $\cap \{[\check{a}_i, \check{b}_i]+B(0, \beta'): 1 \leqq i \leqq N\}=\emptyset$. Thus there exist $1 \leqq i_0 \leqq N$ so that $x \notin [\check{a}_{i_0}, \check{b}_{i_0}]+B(0, \beta')$. We calculate that:

$$\begin{aligned} 5.9.\quad 0 &\leqq |x-\check{b}_{i_0}|-|x-\check{a}_{i_0}| = (|x-\check{b}_{i_0}|^2-|x-\check{a}_{i_0}|^2)/(|x-\check{b}_{i_0}|+|x-\check{a}_{i_0}|) \\ &\geqq (|x-\check{b}_{i_0}|^2-|x-\check{a}_{i_0}|^2)/2N = \big(2(x, \check{a}_{i_0}-\check{b}_{i_0})+|\check{b}_{i_0}|^2-|\check{a}_{i_0}|^2\big)/2N \\ &= \big(x-(\check{a}_{i_0}+\check{b}_{i_0})/2,\ \check{a}_{i_0}-\check{b}_{i_0}\big)/N. \end{aligned}$$

Let us define $t=\big(x-(\check{a}_{i_0}+\check{b}_{i_0})/2,\ \check{a}_{i_0}-\check{b}_{i_0}\big)$ then:

$$5.10.\qquad \big((x-t\cdot(\check{a}_{i_0}-\check{b}_{i_0})/|\check{a}_{i_0}-\check{b}_{i_0}|^2)-(\check{a}_{i_0}+\check{b}_{i_0})/2,\ \check{a}_{i_0}-\check{b}_{i_0}\big) = 0.$$

Since $a_{i_0}-b_{i_0}=\check{a}_{i_0}-\check{b}_{i_0}$ then by 5.10:

$$x-t(a_{i_0}-b_{i_0})/|a_{i_0}-b_{i_0}|^2 \in [\check{a}_{i_0}, \check{b}_{i_0}].$$

As $|a_{i_0}-b_{i_0}|>1$ we obtain that $x \in [\check{a}_{i_0}, \check{b}_{i_0}]+B(0, t)$. But by our assumption (before 5.9) $x \notin [\check{a}_{i_0}, \check{b}_{i_0}]+B(0, \beta')$ therefore $t>\beta'$. Thus by 5.9 $|x-\check{b}_{i_0}|-|x-\check{a}_{i_0}| \geqq t/N > \beta'/N=\beta$. Lemma 3 is now proved.

Note that the proof of Lemma 3 implies that we would obtain the same result (using the same proof with trivial modifications) with some different definitions of the initial sets S_i. We could define for example (for $1 \leqq i \leqq N$):

$$S_i = \{a = (a_1, \ldots, a_{N-1}) \in \mathbf{Z}^{N-1}: \bar{a}_1 = i\},$$

or (for $1 \leqq i \leqq N$): $S_i = \{a = (a_1, \ldots, a_{N-1}) \in \mathbf{Z}^{N-1}: \bar{P}(a) = i\}$ where P is a polynomial of $N-1$ variables with integer coefficients, and $\bar{P}(\mathbf{Z}^{N-1}) = \{1, \ldots, N\}$. The main point in the proof of Lemma 3 is the way the sets S_i are being shifted to $\check{S}_i$.

We are using the following elementary fact in the proof of Lemma 3.

5.11. *Claim.* Let $1 \leqq n$ and let $V_1, \ldots, V_m$ $(1 \leqq m)$ be affine hyperplanes of $\mathbf{R}^n$ such that $\bigcap \{V_i: 1 \leqq i \leqq m\} = \emptyset$, then there exists $\varepsilon > 0$ such that

$$\bigcap \{V_i + B(0, \varepsilon): 1 \leqq i \leqq m\} = \emptyset$$

(where $B(0, \varepsilon) = \{x \in \mathbf{R}^n: |x| < \varepsilon\}$).

Proof. Let $n \geqq 2$, we will assume inductively that the claim is true for $\mathbf{R}^{n-1}$ (it is trivially true for $\mathbf{R}^1$). Let $V_1, \ldots, V_m$ be affine hyperplanes of $\mathbf{R}^n$ and $u_1, w_1, \ldots, u_m, w_m \in \mathbf{R}^m$ be such that $V_i = u_i + (w_i)^\perp$ (for all $1 \leqq i \leqq m$). Assume that $\bigcap \{V_i: 1 \leqq i \leqq m\} = \emptyset$. This assumption is equivalent to the following:

(1) there is no $p \in \mathbf{R}^n$ such that $(p - u_i, w_i) = 0$ for all $1 \leqq i \leqq m$.

Let us assume (to get a contradiction) that there is no $\varepsilon > 0$ so that

$$\bigcap \{V_i + B(0, \varepsilon): 1 \leqq i \leqq m\} = \emptyset.$$

It follows that for every $1 \leqq k < \infty$, $1 \leqq i \leqq m$ there exist $p_k, v_{k,i} \in \mathbf{R}^n$, $|v_{k,i}| < 1/k$ so that for all $1 \leqq k < \infty$, $1 \leqq i \leqq m$:

$$(2) \qquad (p_k + v_{k,i} - u_i, w_i) = 0.$$

If $\{p_k\}_{1 \leqq k < \infty}$ has a bounded subsequence then it has a converging subsequence, we call its limit p'. It is obvious from (2) that $(p' - u_i, w_i) = 0$ for all $1 \leqq i \leqq m$ and this contradicts (1), therefore

$$(3) \qquad \lim_{k \to \infty} |p_k| = \infty.$$

We can assume that the sequence $\{p_k/|p_k|\}_{1 \leqq k < \infty}$ converges, let $q = \lim_{k \to \infty} p_k/|p_k|$. It follows from (2) that for all $1 \leqq k < \infty$, $1 \leqq i \leqq m$ $(p_k/|p_k| + (v_{k,i} - u\,)/|p_k|, w_i) = 0$. If we let $k \to \infty$ we obtain

$$(4) \qquad (q, w_i) = 0 \quad \text{for all} \quad 1 \leqq i \leqq m.$$

After an orthonormal change of coordinates we can assume that $q = (0, \ldots, 0, 1)$. Define, for $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$, $\hat{y} = (y_1, \ldots, y_{n-1}, 0)$. Since (from (4)) $w_1, \ldots, w_m \in \mathbf{R}^{n-1} \times \{0\}$ then (1) implies that:

(5) there is no $p \in \mathbf{R}^{n-1} \times \{0\}$ so that $(p - \hat{u}_i, w_i) = 0$ for all $1 \leqq i \leqq m$.

It follows from (2) that for all $1\leqq k<\infty$, $1\leqq i\leqq m$ $(\hat{p}_k+\hat{v}_{k,i}-\hat{u}_i, w_i)=0$, but this (with (5)) contradicts our induction assumption that the claim holds for $\mathbf{R}^{n-1}$.

Until the end of this section we will fix $\alpha=10^{-5}$ and $\beta>0$ will be the constant associated to $\alpha=10^{-5}$ by Lemma 3 (N is fixed).

Proof of Lemma 1 for dimension N $(N\geqq 2)$

6.1. Let $X=(x, y, z, w)$ be the coordinates system in the neighborhood of Z_0 described in 0.1—0.3 and U, U', V, V', W, W' are defined by 0.2—0.4 where $d_1=\beta\cdot 10^{-(10N)!}$ and $d=(d_1)^2/2$ (the meaning of this choice is explained by 6.11 and later by 6.21—6.27). The functions $u_{\bar{X}}$ (for $\bar{X}\in V'$) will be defined by 0.5. Sublemma 1 implies that $u_{\bar{X}}$ (where $\bar{X}=(\bar{x}_1, \dots, \bar{x}_{N-1}, \bar{y}_1, \dots, \bar{y}_{N-1}, \bar{z}, 0)\in V'$) has the following properties:

6.2. (i) $\mathrm{Re}(u_{\bar{X}})>0$ on $U'\setminus\{\bar{X}\}$ and $u_{\bar{X}}(\bar{X})=0$

(ii) If $X=(x_1, \dots, x_{N-1}, y_1, \dots, y_{N-1}, z, w)\in U'$ then:

$$u_{\bar{X}}(X) = w+\sum_{1\leqq j\leqq N-1}(x_j)^2(y_j-\bar{y}_j+z-\bar{z})^2+x_j(x_j-\bar{x}_j)^4$$

$$+\Big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\Big)(z-\bar{z})^2+\Big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\Big)^{-3/2}\Big(\sum_{1\leqq j\leqq N-1}x_j(\bar{x}_j-x_j)\Big)^4$$

$$-i\sum_{1\leqq j\leqq N-1}(x_j)^2(1-w)^2(y_j-\bar{y}_j)-i(1-w)^2(z-\bar{z})+R_{\bar{X}}(Z).$$

Where the remainder term $R_{\bar{X}}$ is bounded in the following way:

$$|R_{\bar{X}}(X)| \leqq 10^5\Big(w+|z-\bar{z}|^2+\sum_{1\leqq j\leqq N-1}|y_j-\bar{y}_j+z-\bar{z}|^2+|x_j-\bar{x}_j|^4\Big)$$

$$\times\Big(|z-\bar{z}|^{1/2}+\sum_{1\leqq j\leqq N-1}|y_j-\bar{y}_j+z-\bar{z}|^{1/2}+w+\sum_{1\leqq j\leqq N-1}|\bar{x}_j-x_j|\Big)$$

and:

$$\big|\mathrm{Im}\,(R_{\bar{X}}(X))\big| \leqq 10^4\Big(w+|z-\bar{z}|^2+\sum_{1\leqq j\leqq N-1}|y_j-\bar{y}_j+z-\bar{z}|^2+|x_j-\bar{x}_j|^4\Big)$$

$$\times\Big(|z-\bar{z}|^{1/2}+\sum_{1\leqq j\leqq N}|y_j-\bar{y}_j+z-\bar{z}|^{1/2}\Big).$$

The significant point is that $R_{\bar{X}}(X)=o\big(\mathrm{Re}\,(u_{\bar{X}}(X))\big)$. It was also proved by Sublemma 1 that:

6.3. $$\mathrm{Re}\big(u_{\bar{X}}(X)\big) \geqq 99/100\Big(w+\sum_{1\leqq j\leqq N-1}\big((x_j)^2(y_j-\bar{y}_j+z-\bar{z})^2+x_j(x_j-\bar{x}_j)^4\big)$$

$$+\Big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\Big)(z-\bar{z})^2+\Big(\Big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\Big)^{-3/2}\Big)\Big(\sum_{1\leqq j\leqq N-1}(x_j)(\bar{x}_j-x_j)\Big)^4\Big).$$

We will state an assumption parallel to the one in 1.5:

6.4. Let $\varepsilon^{1/(\beta^2\varepsilon)}>r>0$ be so that when $X, X'\in V'$ and $|X-X'|<(\log(1/r))^{-1}$ then:

$$|f(X)-f(X')| < \varepsilon^{100}$$

and so that $(\log\varepsilon)/(r^{1/2}\cdot N^5\cdot 2\pi)$ is an integer.

We will present now a long (and tiresome) list of definitions and notations that describes the locations of the peak points of our peak functions. This is a crucial part of the proof of Lemma 1. The motivation for these definitions will be clear from their use in 6.21—6.27.

The following convention will always be used: when we have

$$X = (X_1, \ldots, X_{2N-1})\in\mathbf{R}^{2N-1},$$

we call $(X_1, \ldots, X_{N-1})$ the x-coordinates and $(X_N, \ldots, X_{2N-2})$ the y-coordinates and we call X_{2N-1} the z-coordinate, x, y and z each having a different role in the proof of Lemma 1.

6.5. Define $c_1=c_2=\ldots=c_{N-1}=(d_1)^{-1/4}r^{1/4}$ and $c_N=c_{N+1}=\ldots=c_{2N-2}=(d_1)^{-1}r^{1/2}$ and $c_{2N-1}=r^{1/2}$.

6.6. Define

$$L = \{a = (a_1, \ldots, a_{2N-1})\in\mathbf{Z}^{2N-1}\colon\ d((a_1c_1, \ldots, a_{2N-1}c_{2N-1}, 0), V) < r^{0.2}\}$$

and

$$L' = \{a\in\mathbf{Z}^{2N-1}\colon\ (a_1c_1, \ldots, a_{2N-1}c_{2N-1}, 0)\in V'\}$$

(here $d(\,,)$ is the usual distance in $\mathbf{C}^{2N}$; V, V' are defined in 0.2).

6.7. When $Z=(Z_1, \ldots, Z_{2N-1})\in\mathbf{C}^{2N-1}$, we define $Z'=(Z_N, \ldots, Z_{2N-2})$.

Take $a\in L'$. There is (one) $1\leqq i\leqq N$ so that $a'\in S_i$ (S_i is defined by 5.1). Let $v_i\in\mathbf{R}^{N-1}$ be defined as in 5.3, and define:

$$\hat{a} = (\hat{a}_1, \ldots, \hat{a}_{2N-1}) = a+(0, \ldots_{(N-1\text{ times})}\ldots, 0, v_i, 0)$$

(we move the y-coordinates by v_i). Define also:

$$X_a = (c_1\hat{a}_1, \ldots, c_{2N-1}\hat{a}_{2N-1}, 0)$$

$$= (c_1a_1, \ldots, c_{2N-1}a_{2N-1}, 0)+(0, \ldots_{(N-1\text{ times})}\ldots, 0, c_Nv_i, 0, 0).$$

Note that the "shrinking" constants $c_1, c_2, \ldots, c_{2N-1}$ act uniformly on each of the coordinates x, y, z.

6.8. Next we define for $a\in L$, $u_a=u_{X_a}$ and $R_a=R_{X_a}$ (see 6.2).

6.9. Now let $L_i'=\{a\in L'\colon\ a'\in S_i\}$ and $L_i=\{a\in L\colon\ a'\in S_i\}$ for $1\leqq i\leqq N$.

Take $a\in L$ and $X=(X_1, \ldots, X_{2N})=(x_1, \ldots, x_{N-1}, y_1, \ldots, y_{N-1}, z, w)\in U'$ and define $t_j=X_j/c_j$ $(1\leqq j\leqq 2N-1)$, using 6.2 we have:

$$6.10.\quad u_a(X) = u_{X_a}(X) = w+\sum_{1\leqq j\leqq N-1}\big((x_j)^2(c_{N+j-1}(t_{N+j-1}-\hat{a}_{N+j-1})$$
$$+c_{2N-1}(t_{2N-1}-a_{2N-1}))^2+x_j(c_j(t_j-a_j))^4\big)+\big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\big)(c_{2N-1}(t_{2N-1}-a_{2N-1}))^2$$
$$+\big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\big)^{-3/2}\big(\sum_{1\leqq j\leqq N-1}x_jc_j(t_j-a_j)\big)^4$$
$$-i\sum_{1\leqq j\leqq N-1}(x_j)^2(1-w)^2c_{N+j-1}(t_{N+j-1}-\hat{a}_{N+j-1})$$
$$-ic_{2N-1}(1-w)^2(t_{2N-1}-a_{2N-1})+R_a(X)$$
$$= w+\sum_{1\leqq j\leqq N-1}r\big((x_j/d_1)^2((t_{N+j-1}-\hat{a}_{N+j-1})+d_1(t_{2N-1}-a_{2N-1}))^2+(x_j/d_1)(t_j-a_j)^4\big)$$
$$+r\big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\big)(t_{2N-1}-a_{2N-1})^2$$
$$+r\big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\big)^{-3/2}\big(\sum_{1\leqq j\leqq N-1}x_j(d_1)^{-1/4}(t_j-a_j)\big)^4$$
$$-i(d_1)^{-1}r^{1/2}(1-w)^2\sum_{1\leqq j\leqq N-1}(x_j)^2(t_{N+j-1}-\hat{a}_{N+j-1})$$
$$-ir^{1/2}(1-w)^2(t_{2N-1}-a_{2N-1})+R_a(X).$$

Define now for $a\in L$, $X\in U'$:

$$6.11.\qquad p_a(X) = \exp\big(u_a(X)\cdot(\log\varepsilon)/(rN^5)\big),$$

and for $a\in L'\setminus L$ we (formally) define $p_a\equiv 0$.

By 6.2 when $a\in L$ $1>|p_a|>0$ on $U'\setminus\{X_a\}$, $p_a(X_a)=1$, and $|p_a(X)|$ decreases rapidly as $X\in U'$ moves away from X_a. We need a good estimate of $|p_a(X)|=\varepsilon^{\mathrm{Re}(u_a(X)/(rN^5))}$. It follows from 6.10 that:

$$\mathrm{Re}\big((u_a(X)-R_a(X))/(rN^5)\big) = N^{-5}\big(w/r+\sum_{1\leqq j\leqq N-1}((x_j/d_1)^2((t_{N+j-1}-\hat{a}_{N+j-1})$$
$$+d_1(t_{2N-1}-a_{2N-1}))^2+(x_j/d_1)(t_j-a_j)^4)+\big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\big)(t_{2N-1}-a_{2N-1})^2$$
$$+\big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\big)^{-3/2}\big(\sum_{1\leqq j\leqq N-1}x_j(d_1)^{-1/4}(t_j-a_j)\big)^4\big).$$

This term will will play a central role in the main step of the proof which is in 6.21—6.27. Note that by 6.1 $1-d_1\leqq x_j/d_1\leqq 1+d_1$ (for $1\leqq j\leqq N-1$) and $d_1=\beta\cdot 10^{-(10N)!}$.

We have from 6.10 and 6.11 that for

$$X=(x,y,z,0)=(X_1,\ldots,X_{2N-1},0)\in V'\quad (t_j=X_j/c_j\ (1\leqq j\leqq 2N-1)),\quad (\mathscr{E}\overset{\text{def}}{=}\varepsilon):$$
$$p_a(X)=\mathscr{E}^{\mathrm{Re}(u_a(X)/(rN^5))}\exp\big(-i(\log\varepsilon)r^{-1/2}N^{-5}(d_1)^{-1}\sum_{1\leqq j\leqq N-1}(x_j)^2(t_{N+j-1}-\hat{a}_{N+j-1})\big)$$
$$\times\exp\big(-i(\log\varepsilon)r^{-1/2}N^{-5}(t_{2N-1}-a_{2N-1})\big)\cdot\theta_a(X)$$
$$=\mathscr{E}^{\mathrm{Re}(u_a(X)/(rN^5))}\exp\big(-i(\log\varepsilon)r^{-1/2}N^{-5}(d_1)^{-1}\sum_{1\leqq j\leqq N-1}(x_j)^2(t_{N+j-1}-\hat{a}_{N+j-1})\big)$$
$$\times\exp\big(-i(\log\varepsilon)r^{-1/2}N^{-5}t_{2N-1}\big)\cdot\theta_a(X)$$

where

$$\theta_a(X) = \exp\big(i(\log \varepsilon)\cdot r^{-1}\cdot N^{-5}\,\mathrm{Im}\,(R_a(X))\big).$$

The last equality holds since a_{2N-1} is an integer and by 6.4 $(\log \varepsilon)/(r^{1/2}\cdot N^5\cdot 2\pi)$ is an integer.

We obtain that for all $a, b\in L$:

6.12.

$$p_a(X)\cdot \bar{p}_b(X) = \varepsilon^{\mathrm{Re}(u_a(X)/(rN^5))+\mathrm{Re}(u_b(X)/(rN^5))}$$

$$\times\exp\big(-i(\log\varepsilon)r^{-1/2}N^{-5}(d_1)^{-1}\textstyle\sum_{1\le j\le N-1}(x_j)^2(\hat{b}_{N+j-1}-\hat{a}_{N+j-1})\big)\cdot\theta_a(X)\cdot\bar{\theta}_b(X).$$

It follows that if $a'=b'$ then:

$$p_a(X)\cdot\bar{p}_b(X) = |p_a(X)\cdot\bar{p}_b(X)|\cdot\theta_a(X)\cdot\bar{\theta}_b(X).$$

Note that $a'=b'$ iff y-coordinates of X_a is equal to the y-coordinates of X_b.

6.13. The fact that $\theta_a(X)$, $\theta_b(X)$ are very close to 1 when X_a, X_b are close enough to X is critical and will be used as in Section 1. We can evaluate $\theta_a(X)$ in the way that it was done in Section 1 (1.12) and obtain that if $\mathrm{Re}\,(u_a(X)/(rN^5))<10^5$ then:

$$|\theta_a(X)-1| < 10^{-10}.$$

Note also that $|\theta_a|\equiv 1$.

The following definition is equivalent to the one in 1.14.

6.14. Let $v_1, v_2, \dots, v_N$: $V'\to\mathbf{C}^{N+1}$ be continuous functions such that for every $X\in V'$, $\{f(X), v_1(X), \dots, v_N(X)\}$ are mutually orthogonal and $|v_i(X)|=1$ (for all $1\le i\le N$). By shrinking r further (in 6.4, 6.5) we can assume that when $X, X'\in V'$ are such that $|X-X'|<(-\log r)^{-1}$ then we have for $i=1, 2, \dots, N$:

$$|v_i(X)-v_i(X')| < \varepsilon^{60}.$$

Let $a\in L'$. There exists unique $i\in\{1, \dots, N\}$ such that $a'\in S_i$. Define:

$$v_a = \big(2\varepsilon(1-|f(X_a)|^2)\big)^{1/2}v_i(X_a).$$

The set $\{v_a\colon a\in L'\}$ has the following properties (compare with 1.14):

(i) $$(v_a, f(X_a)) = 0,$$

(ii) $$|v_a|^2 = 2\varepsilon\big(1-|f(X_a)|^2\big).$$

Let $a, b\in L'$, $|a-b|<-\log r$, where $a'\in S_i$, $b'\in S_j$. Then:

(iii) if $i\neq j$,

$$|(v_a, v_b)| < \varepsilon^{60},$$

(iv) if $i=j$,

$$\big|(v_a, v_b)-|v_a|^2\big| < \varepsilon^{60}.$$

The proof that (i)—(iv) holds is essentially the same as the proof in 1.14 and we will not repeat it.

The map h is defined as in Sections 1, 2.

6.15. For $Z\in\bar{D}$:

$$h(Z)=\sum_{a\in L}p_a(Z)v_a.$$

As before the distinction between Z and $X(Z)$ is suppressed. Define for $X=(x_1,\dots,x_{N-1},y_1,\dots,y_{N-1},z,w)\in U'$ and $n\geqq 0$ (see 6.11):

6.16. $L'(X,n)=\{a\in L'\colon n^2<\operatorname{Re}\big((u_a(X)-(w+R_a(X)))/(rN^5)\big)\leqq(n+1)^2\}$.

It is essentially the same definition as 1.16 with a slight technical difference. The process that will follows is parallel to the one in Section 1. Propositions (A), (B), (C), (D) are as there and the proofs of (A), (B), (D) are essentially the same as there. The proof of (C) is different and Lemma 3 is the basis of it.

Let $X\in V'$ then:

6.17. (A) $|(f(X),h(X))|<\varepsilon^{100}$,

(B) $|h(X)|^2<\varepsilon^{1/2}(1-|f(X)|^2)+\varepsilon^{50}$,

(C) when $X\in V$ then $|h(X)|^2>\varepsilon^2(1-|f(X)|^2)-\varepsilon^{50}$.

Note that obtaining a holomorphic map that satisfies only (A) and (B) is trivial (take $h\equiv 0$) and the same is true for a holomorphic map that satisfies only (C) (take $h\equiv 1$). Obviously the difficulty is to construct a holomorphic map for which (A), (B) and (C) holds. For the map h that we constructed above, the proofs of (A) and (B) are rather simple and do not use arguments that depend on the co-dimension. On the other hand the proof of (C) requires careful consideration of the co-dimension and the distribution of X_a, with use of Lemma 3.

6.18. We will use the facts that $\operatorname{car}(L(X,n))<(N^{10}(n+2))^{2N}$ (not a sharp estimate) and when $a\in L(X,n)$ then $|p_a(X)|<\varepsilon^{n^2/2}$. Simple implications of these facts will also be used without mention.

6.19. As in Sections 1, 2, when $X\in U'$ is fixed we define for $a\in L$ $[a]=n$ where n is the only integer so that $a\in L(X,n)$. Since the proofs of (A) and (B) here are basically the same as proofs of (A) and (B) in Section 1, we will just go briefly through them.

Proof of (A):

$$|(f(X),h(X))|=\Big|\sum_{a\in L}(f(X),v_a)\bar{p}_a(X)\Big|$$

$$=\Big|\sum_{0\leqq n\leqq 100}\sum_{a\in L(X,n)}(f(X)-f(X_a),v_a)\bar{p}_a(X)\Big|+\Big|\sum_{100<n}\sum_{a\in L(X,n)}(f(X),v_a)\bar{p}_a(X)\Big|$$

$$\leqq\sum_{0\leqq n\leqq 100}\sum_{a\in L(X,n)}|f(X)-f(X_a)|\cdot|v_a|+\sum_{100<n}\sum_{a\in L(X,n)}|p_a(X)|$$

$$<4\sum_{0\leqq n\leqq 100}\varepsilon^{100}\cdot\varepsilon^{1/2}(N^{10}(n+2))^{2N}+\sum_{100<n}\varepsilon^{n^2/2}\cdot(N^{10}(n+2))^{2N}<\varepsilon^{100}.$$

Proof of (B): Fix $X\in V'$:

$$|h(X)|^2 = \left|\sum_{a,b\in L}(v_a, v_b)p_a(X)\bar{p}_b(X)\right|$$
$$\leqq \sum_{0\leqq m\leqq 100}\sum_{a,b\in L,\,[a]+[b]=m}|(v_a, v_b)p_a(X)\bar{p}_b(X)|$$
$$+\sum_{100<m}\sum_{a,b\in L,\,[a]+[b]=m}|p_a(X)\bar{p}_b(X)|$$
$$< \sum_{0\leqq m\leqq 100}\left(2\varepsilon\left(1-|f(X)|^2\right)+\varepsilon^{100}\right)\left(N^{10}(m+2)\right)^{4N}\cdot\varepsilon^{m^2/4}+\sum_{100<m}\left(N^{10}(m+2)\right)^{4N}\cdot\varepsilon^{m^2/4}$$
$$< (4N)^{40N}\cdot\varepsilon\cdot\left(1-|f(X)|^2\right)+\varepsilon^{60}.$$

Proof of (C):

6.20. Define for $Z\in\bar{B}^N$ and $1\leqq i\leqq N$:

$$h_i(Z) = \sum_{a\in L_i} p_a(Z)\cdot v_a.$$

We have $h=h_1+h_2+\ldots+h_N$.

The following definitions and propositions (6.21—6.27) which are based on Lemma 3 are the basis of the proof of (C) and therefore of Lemma 1. While (A) and (B) use mainly 6.14 (i), (ii) and the fact that p_a are located in some "regulated" (lattice type) way. In the proof of (C) we need to look closely at the properties of the peak functions and their locations.

6.21. Let $X\in V$, $X=(x, y, z, 0)=(X_1, \ldots, X_{2N-1}, 0)$ $(x, y\in\mathbf{R}^{N-1}, z\in\mathbf{R})$ and let $t_j=X_j/c_j$ $(j=1, \ldots, 2N-1)$ and $t=(t_1, \ldots, t_{2N-1})$, X (and t) will be fixed until the end of 6.27. We will make use now of the definitions in 6.7—6.9.

$(*)$ By Lemma 3 there exists $i\in\{1, \ldots, N\}$ and $a\in L_i$ so that $|t'-\hat{a}'|+\beta<|t'-\hat{b}'|$ for every $b\in L_i$ such that $a'\neq b'$ (note that only the y-coordinates are involved here). Let us choose one such i and call it $i(X)$.

6.22. Choose $a(X)\in L_{i(X)}$ so that the following holds:

$$\mathrm{Re}\left(\left(u_{a(X)}(X)-R_{a(X)}(X)\right)/(rN^5)\right) = \min\left\{\mathrm{Re}\left(\left(u_a(X)-R_a(X)\right)/(rN^5)\right):\ a\in L_{i(X)}\right\}$$

(note that the choice for the minimum may not be unique). Looking at the explicit term for $\mathrm{Re}\left(\left(u_a(X)-R_a(X)\right)/(rN^5)\right)$ in 6.11 and the definition of L_i (and 6.1) we can see that:

6.23. (1) $\mathrm{Re}\left(\left(u_{a(X)}(X)-R_{a(X)}(X)\right)/(rN^5)\right)<1/3$

and

(2) if $a\in L$ and $|a-t|\geqq N^{20}$ ($|\ |$ is the standard Euclidean norm) then $\mathrm{Re}\left(u_a(X)-R_a(X)\right)/(rN^5)>2$ (not the best estimates but they suffice).

When $a\in L_{i(X)}$, it follows directly from 6.11 and the definition of $a(X)$ that:

6.24. $\mathrm{Re}\left(\left(u_a(X)-R_a(X)\right)/(rN^5)\right)-\mathrm{Re}\left(\left(u_{a(X)}(X)-R_{a(X)}(X)\right)/(rN^5)\right)$

$$\geqq N^{-5}\Big(\Big(\sum_{1\leqq j\leqq N-1}(x_j/d_1)^2\left((t_{N+j-1}-\hat{a}_{N+j-1})+d_1(t_{2N-1}-a_{2N-1})\right)^2$$
$$-\sum_{1\leqq j\leqq N-1}(x_j/d_1)^2\left((t_{N+j-1}-\hat{a}_{N+j-1}(X))+d_1\left(t_{2N-1}-a_{2N-1}(X)\right)\right)^2\Big)$$
$$+\Big(1-\sum_{1\leqq j\leqq N-1}(x_j)^2\Big)\left((t_{2N-1}-a_{2N-1})^2-\left(t_{2N-1}-a_{2N-1}(X)\right)^2\right)\Big).$$

An equality occurs in 6.24 when $a_j=a_j(X)$ for all $1\leqq j\leqq N-1$ (i.e. a is equal to $a(X)$ in the x-coordinates).

When we consider the facts that $d_1=\beta\cdot 10^{-(10N)!}$, and $1-d_1\leqq x_j/d_1\leqq 1+d_1$ and the definition of $a(X)$ we obtain (from 6.24 and the remark below it) the following.

6.25. (1) When $a\in L_{i(X)}$, $|a-t|<N^{20}$ and $a_j=a_j(X)$ for all $j\in\{1, \dots, N-1\}\cup\{2N-1\}$ (thus a is equal to $a(X)$ in the x and z coordinates) then:

$$0\leqq \operatorname{Re}\big((u_a(X)-R_a(X))/(rN^5)\big)-\operatorname{Re}\big((u_{a(X)}(X)-R_{a(X)}(X))/(rN^5)\big)$$
$$< N^{-5}\big(|t'-\hat{a}'|^2-|t'-\hat{a}'(X)|^2+d_2\big)$$

where $d_2\overset{\text{def}}{=\!=}\beta\cdot 10^{-(10N)!/2}$.

Thus

$$(\square)\qquad |t'-\hat{a}'|^2>|t'-\hat{a}'(X)|^2-d_2.$$

Note that $(\square)$ holds for all $a\in L_{i(X)}$. First we can always assume that a is equal to $a(X)$ in the x and z coordinates, without effecting $(\square)$, then if $|a-t|\geqq N^{20}$, we have $|t'-\hat{a}'|^2>N^{20}$, and $(\square)$ is trivial.

(2) When $a\in L_{i(X)}$, $|a-t|<N^{20}$ and $a_j=a_j(X)$ for all $j\in\{1, \dots, 2N-2\}$ (a is equal to $a(X)$ in the x, y-coordinates), then:

$$0\leqq \operatorname{Re}\big((u_a(X)-R_a(X))/(rN^5)\big)-\operatorname{Re}\big((u_{a(X)}(X)-R_{a(X)}(X))/(rN^5)\big)$$
$$< N^{-5}\big((t_{2N-1}-a_{2N-1})^2-(t_{2N-1}-a_{2N-1}(X))^2+d_2\big).$$

Thus: $(t_{2N-1}-a_{2N-1})^2>(t_{2N-1}-a_{2N-1}(X))^2-d_2$.

(3) When $a\in L_{i(X)}$ and $|a-t|<N^{20}$ then:

$$\operatorname{Re}\big((u_a(X)-R_a(X))/(rN^5)\big)-\operatorname{Re}\big((u_{a(X)}(X)-R_{a(X)}(X))/(rN^5)\big)$$
$$> N^{-5}\big(|t'-\hat{a}'|^2-|t'-\hat{a}'(X)|^2+(t_{2N-1}-a_{2N-1})^2-(t_{2N-1}-a_{2N-1}(X))^2-d_2\big).$$

From $(\square)$ (in 6.25 (1)) and the choice of $i(X)$ in 6.21 (recall that Lemma 3 was used there) it follows that for all $a\in L_{i(X)}$

$$-d_2<|t'-\hat{a}'|^2-|t'-\hat{a}'(X)|^2=\big(|t'-\hat{a}'|-|t'-\hat{a}'(X)|\big)\big(|t'-\hat{a}'|+|t'-\hat{a}'(X)|\big),$$

thus since $|t'-\hat{a}'|+|t'-\hat{a}'(X)|\geqq|\hat{a}'-\hat{a}'(X)|=|a'-a'(X)|$ then: $-d_2<|t'-\hat{a}'|-|t'-\hat{a}'(X)|$. Therefore in view of 6.21 $(*)$:

6.26. $|t'-\hat{a}'(X)|=\min\{|t'-\hat{a}'|\colon a\in L_{i(X)}\}$, which is equivalent to (using 6.21 $(*)$): when $a\in L_{i(X)}$ and $a'\neq a'(X)$:

$$|t'-\hat{a}'|^2-|t'-\hat{a}'(X)|^2=\big(|t'-\hat{a}'|-|t'-\hat{a}'(X)|\big)\big(|t'-\hat{a}'|+|t'-\hat{a}'(X)|\big)>\beta.$$

We conclude from 6.26, 6.25 (2), (3) (and the marginality of $R_a(X)$, $R_{a(X)}(X)$ which is described by 6.1 (see also 1.12)) that if $a\in L_{i(X)}$, $a'\neq a'(X)$ and

$|a-t|<N^{20}$ then

$$\mathrm{Re}\,(u_a(X)/(rN^5))-\mathrm{Re}\,(u_{a(X)}(X)/(rN^5)) > \beta N^{-10}.$$

6.23 (and the marginality of the remainder term) implies that this is true also when $|a-t|\geqq N^{20}$.

Thus we conclude the following (see 6.11):

6.27. If $a\in L_{i(X)}$ and $a'\neq a'(X)$ then:

$$|p_a(X)/p_{a(X)}(X)| < \varepsilon^{\beta N^{-10}}.$$

Let us fix $X\in V$ until the end of proof of (C). The following is the first step in the proof of (C).

6.28. When $1\leqq i,j\leqq N$ and $i\neq j$, then (see Def. 6.20):

$$\big|(h_i(X),\, h_j(X))\big| < \varepsilon^{55}.$$

Proof (using 6.14 (iii)):

$$\big|(h_i(X),\, h_j(X))\big| = \Big|\sum_{0\leqq m\leqq 100}\sum_{a\in L_i,\, b\in L_j,\, [a]+[b]=m}(v_a,\, v_b)\,p_a(X)\bar p_b(X)$$

$$+\sum_{100<m}\sum_{a\in L_i,\, b\in L_j,\, [a]+[b]=m}(v_a,\, v_b)\,p_a(X)\bar p_b(X)\Big|$$

$$< \sum_{0\leqq m\leqq 100}\varepsilon^{60}\cdot\varepsilon^{m^2/4}\big(N^{10}(m+2)\big)^{4N}+\sum_{100<m}\varepsilon^{m^2/4}\cdot\big(N^{10}(m+2)\big)^{4N} < \varepsilon^{55}.$$

6.29. *Claim.* $|h_{i(X)}(X)|^2 > \varepsilon^2(1-|f(X)|^2)-\varepsilon^{55}$.

The proof of (C) is concluded once we prove this claim since

$$|h(X)|^2 = |h_1(X)+\ldots+h_N(X)|^2$$

$$= |h_1(X)|^2+\ldots+|h_N(X)|^2+2\,\mathrm{Re}\big(\sum_{1\leqq i<j\leqq N}(h_i(X),\, h_j(X))\big)$$

and (C) follows from 6.28 and 6.29.

6.30. Let

$$A(X) = \{a\in L_{i(X)}\colon [a]\leqq 100,\ a' = a'(X)\}$$

and

$$B(X) = \{a\in L_{i(X)}\colon a'\neq a'(X)\ \text{or}\ [a]>100\}.$$

Since

$$h_{i(X)}(X) = \sum_{a\in L_{i(X)}} p_a(X)v_a = \sum_{a\in A(X)} p_a(X)v_a+\sum_{a\in B(X)} p_a(X)v_a$$

then:

6.31. $|h_{i(X)}(X)|^2 \geqq \big|\sum_{a\in A(X)} p_a(X)v_a\big|^2-2\big|\sum_{a\in A(X),\, b\in B(X)}(v_a,\, v_b)p_a(X)\bar p_b(X)\big|$.

Let us look at the first term:

$$\Big|\sum_{a\in A(X)} p_a(X)v_a\Big|^2 \geqq \mathrm{Re}\big(\sum_{a,b\in A(X)}(v_a,\, v_b)p_a(X)\bar p_b(X)\big).$$

It follows from 6.12, 6.13 that that when $a, b\in A(X)$, then since $a'=b'$ and $[a],[b]\leqq 100$

$$\mathrm{Re}\big(p_a(X)\bar p_b(X)\big) > 0$$

and since $a, b \in L_{i(X)}$ then 6.14 (iv), 6.4 imply that

$$\left|(v_a, v_b) - |v_{a(X)}|^2\right| < \varepsilon^{59}.$$

Therefore $\mathrm{Re}\,((v_a, v_b) p_a(X) \bar{p}_b(X)) > -\varepsilon^{58}$.

Since $a(X) \in A(X)$ and $\mathrm{card}\,(A(X)) \leqq \sum_{0 \leqq n \leqq 100} (N^{10}(n+2))^{2N}$ we obtain that:

6.32.
$$\left|\sum_{a \in A(X)} p_a(X) v_a\right|^2 > |v_{a(X)}|^2 |p_{a(X)}(X)|^2 - \varepsilon^{57}.$$

Let $b \in B(X)$, then if $b' \neq a'(X)$ 6.27 implies that:

6.33.
$$|p_b(X)/p_{a(X)}(X)| < \varepsilon^{\beta N - 10}$$

and if $b' = a'(X)$ then $[b] > 100$ and by 6.3, 6.16 the claim of 6.33 clearly holds.

Using the fact that

$$\mathrm{car}\,\{b \in B(X) \colon [b] \leqq 100\} \leqq \sum_{0 \leqq n \leqq 100} (N^{10}(n+2))^{2N} \overset{\mathrm{def}}{=} M_0$$

we have:

6.34.
$$\begin{aligned} &\left|\sum_{a \in A(X),\, b \in B(X)} (v_a, v_b) p_a(X) \bar{p}_b(X)\right| \\ &\leqq \left|\sum_{a \in A(X),\, b \in B(X),\, [b] \leqq 100} (v_a, v_b) p_a(X) \bar{p}_b(X)\right| \\ &\quad + \left|\sum_{a \in A(X),\, b \in B(X),\, [b] > 100} (v_a, v_b) p_a(X) \bar{p}_b(X)\right| \\ &< \sum_{a \in A(X),\, b \in B(X),\, [b] \leqq 100} (|v_{a(X)}|^2 + \varepsilon^{59}) |p_{a(X)}(X)|^2 \varepsilon^{\beta N - 10} \\ &\quad + \sum_{a \in A(X),\, b \in B(X),\, [b] > 100} |p_a(X) \bar{p}_b(X)| < (M_0)^2 (|v_{a(X)}|^2 + \varepsilon^{59}) |p_{a(X)}(X)|^2 \varepsilon^{\beta N - 10} \\ &\quad + \sum_{n \geqq 100} \Big(\sum_{0 \leqq j \leqq 100} (N^{10}(j+2))^{2N}\Big) \big((N^{10}(n+2))^{2N}\big) \varepsilon^{n^2/2} \\ &< |p_{a(X)}(X)|^2 |v_{a(X)}|^2 \varepsilon^{(\beta N - 20)} + \varepsilon^{59}. \end{aligned}$$

Combining the estimates 6.31, 6.32, 6.34 and the fact (6.11 and 6.23) that $|p_{a(X)}(X)|^2 > \varepsilon$ we obtain:

6.35.
$$\begin{aligned} |h_{i(X)}(X)|^2 &> |v_{a(X)}|^2 |p_{a(X)}(X)|^2 (1 - \varepsilon^{\beta N - 20}) - \varepsilon^{56} \\ &> \varepsilon \cdot |p_{a(X)}(X)|^2 \cdot (1 - |f(X)|^2) - \varepsilon^{55} > \varepsilon^2 (1 - |f(X)|^2) - \varepsilon^{55}. \end{aligned}$$

Claim 6.29 is now proved and thus (C) is proved.

(D) will be exactly the same as in Section 1 and the proof is essentially identical and will be omitted.

The globalization process is identical to the one in Section 1 and we will not repeat it. Since 1.33 and 1.34 hold, then the proof of Lemma 1 is now completed. This proof obviously holds when the target ball is B^M, for any $M \geqq N+1$.

Proof of Lemma 2 for dimension $N \geqq 2$:

It is quite clear at this stage that the proof that will follow is a composition of the proof of Lemma 1 (for dimension $N \geqq 2$) with the proof of Lemma 2 (for dimension 2).

The definitions and statements in the proof of Lemma 1 until (and including) 6.13 will be adopted here with one change, the constant $r>0$ in the definition of the (first stage) correction function will need to be smaller (as in 3.1).

7.1. Let r be so that $(\varepsilon')^{1/\varepsilon'}>r>0$ and when $X, Y \in V'$, $|X-Y|<(-\log r)^{-1}$ then:

(i) $|f(X)-f(Y)| < (\varepsilon')^{100}$,

(ii) $\left|1/(|1-\varphi(X)|^4+(\varepsilon')^{10})-1/(|1-\varphi(Y)|^4+(\varepsilon')^{10})\right| < (\varepsilon')^{100}$,

(iii) $\sum_{m \geqq l(r)} (1/2)^{m^2/4} < (\varepsilon')^{100}$ where $l(r) \overset{\text{def}}{=} -\log r$,

(iv) $(\log \varepsilon)/(r^{1/2} \cdot N^5 \cdot 2\pi)$ is an integer.

The definition of v_a, $a \in L'$ is slightly different than in 6.14 (see 3.3). The process done in 6.14 (and 1.14) implies (after changing the constants that are involved in the process and shrinking r) that for every $a \in L'$ we can assign $v_a \in \mathbf{C}^{N+1}$ so that the following will hold (the projection into the y-coordinates $a \to a'$ is defined in 6.7, S_i is defined in 5.1).

7.2. For $a, b \in L'$

(i) $$(v_a, f(X_a)) = 0,$$

(ii) $$|v_a|^2 = 2\varepsilon\left(1-|f(X_a)|^2\right)/\left(|1-\varphi(X_a)|^4+(\varepsilon')^{10}\right).$$

Let $a, b \in L'$, $|a-b|<2l(r)$, (where $l(r) \overset{\text{def}}{=} -\log r$) so that $a' \in S_i$, $b' \in S_j$ then:

(iii) if $i \neq j$ $$|(v_a, v_b)| < (\varepsilon')^{50},$$

(iv) if $i = j$ $$\left|(v_a, v_b)-|v_a|^2\right| < (\varepsilon')^{50}.$$

7.3. We will adopt here 6.15, 6.16 without a change. We will prove now (A), (B), (C) as in the case $N=2$ (Section 3). The proofs consist mostly of the changes done (by the use of Lemma 3, as in the proof of Lemma 1 for dimension $N \geqq 2$) to adapt to the higher dimension.

7.4. For $X \in V'$:

(A) $\left|(f(X), h(X))\right| < (\varepsilon')^4$,

(B) $|h(X)|^2 < \varepsilon^{1/2}\left(1-|f(X)|^2\right)/\left(|1-\varphi(X)|^4+(\varepsilon')^{10}\right)+(\varepsilon')^4$,

(C) when $X \in V$ then $|h(X)|^2 > \varepsilon^2\left(1-|f(X)|^2\right)/\left(|1-\varphi(X)|^4+(\varepsilon')^{10}\right)-(\varepsilon')^4$.

The remark following 3.4 will be used.

Proof of (A) $(l(r) \stackrel{\text{def}}{=} -\log r)$:

$$|(f(X), h(X))| = \left|\sum_{a\in L}(f(X), v_a)\bar{p}_a(X)\right|$$
$$= \left|\sum_{0\le n\le l(r)}\sum_{a\in L(X,n)}(f(X)-f(X_a), v_a)\bar{p}_a(X)\right| + \left|\sum_{l(r)<n}\sum_{a\in L(X,n)}(f(X), v_a)\bar{p}_a(X)\right|$$
$$\le \sum_{0\le n\le l(r)}\sum_{a\in L(X,n)}|f(X)-f(X_a)|\,|v_a|\,|p_a(X)| + \sum_{l(r)<n}\sum_{a\in L(X,n)}|p_a(X)|\,|v_a|$$
$$< \sum_{0\le n\le l(r)}\left(N^{10}(n+2)\right)^{2N}(\varepsilon')^{100}\varepsilon^{n^2/2}(\varepsilon')^{-5} + 2\sum_{l(r)<n}\left(N^{10}(n+2)\right)^{2N}\varepsilon^{n^2/2}(\varepsilon')^{-5} < (\varepsilon')^4.$$

Proof of (B): Fix $X\in V'$:

$$|h(X)|^2 = \left|\sum_{a,b\in L}(v_a, v_b)p_a(X)\bar{p}_b(X)\right|$$
$$\le \sum_{0\le m\le l(r)}\sum_{a,b\in L,\,[a]+[b]=m}|(v_a, v_b)p_a(X)\bar{p}_b(X)|$$
$$+\sum_{l(r)<m}\sum_{a,b\in L,\,[a]+[b]=m}|p_a(X)\bar{p}_b(X)|\,(\varepsilon')^{-10}$$
$$< \sum_{0\le m\le l(r)}\left(|v_{a(X)}|^2+(\varepsilon')^{40}\right)\left(N^{10}(m+2)\right)^{4N}\varepsilon^{m^2/4} + \sum_{l(r)<m}(\varepsilon')^{-10}\left(N^{10}(m+2)\right)^{4N}\varepsilon^{m^2/4}$$
$$< (4N)^{40N}\varepsilon\left(1-|f(X)|^2\right)/\left(|1-\varphi(X)|^4+(\varepsilon')^{10}\right)+(\varepsilon')^4.$$

Proof of (C): We will adopt definitions and propositions 6.20—6.27. The following is equivalent to 6.28:

7.5. When $1\le i,j\le N$ and $i\ne j$ then:

$$|(h_i(X), h_j(X))| < (\varepsilon')^{10}.$$

Proof.

$$|(h_i(X), h_j(X))| \le \left|\sum_{0\le m\le l(r)}\sum_{a\in L_i,\, b\in L_j,\,[a]+[b]=m}(v_a, v_b)p_a(X)\bar{p}_b(X)\right.$$
$$\left.+\sum_{l(r)<m}\sum_{a\in L_i,\, b\in L_j,\,[a]+[b]=m}(v_a, v_b)p_a(X)\bar{p}_b(X)\right|$$
$$< \sum_{0\le m\le l(r)}(\varepsilon')^{50}\varepsilon^{m^2/4}\left(N^{10}(m+2)\right)^{4N} + \sum_{l(r)<m}(\varepsilon')^{-10}\varepsilon^{m^2/4}\left(N^{10}(m+2)\right)^{4N} < (\varepsilon')^{10}.$$

7.6. *Claim.*

$$|h_{i(X)}(X)|^2 > \varepsilon^2\left(1-|f(X)|^2\right)/\left(|1-\varphi(X)|^4+(\varepsilon')^{10}\right)-(\varepsilon')^{10}.$$

Proof. Let $A(X)$, $B(X)$ be as in 6.30 then

$$h_{i(X)}(X) = \sum_{a\in L_{i(X)}} p_a(X)v_a = \sum_{a\in A(X)} p_a(X)v_a + \sum_{a\in B(X)} p_a(X)v_a$$

and:

7.7. $|h_{i(X)}(X)|^2 \ge \left|\sum_{a\in A(X)} p_a(X)v_a\right|^2 - 2\left|\sum_{a\in A(X),\, b\in B(X)}(v_a, v_b)p_a(X)\bar{p}_b(X)\right|.$

It follows from 6.13 that when $a, b\in A(X)$ then by using the fact that $a'=b'$ we have

$$\mathrm{Re}\left(p_a(X)\bar{p}_b(X)\right) > 0$$

and when $a, b \in L_{i(X)}$ and $[a], [b] < 2l(r)$ then 7.1, 7.2 imply that

7.8. $$\left|(v_a, v_b) - |v_{a(X)}|^2\right| < (\varepsilon')^{40}.$$

Therefore for $a, b \in A(X)$.

7.9. $\operatorname{Re}\left((v_a, v_b) p_a(X) \bar{p}_b(X)\right) > -(\varepsilon')^{30}$.

7.10. Looking at the first term we have:

$$\left|\sum_{a \in A(X)} p_a(X) v_a\right|^2 \geqq \operatorname{Re}\left(\sum_{a,b \in A(X)} (v_a, v_b) p_a(X) \bar{p}_b(X)\right).$$

Using 7.9 and the fact that $\operatorname{card}(A(X)) \leqq \sum_{0 \leqq n \leqq 100} (N^{10}(n+2))^{2N} \overset{\text{def}}{=} M_0$ we obtain that:

$$\left|\sum_{a \in A(X)} p_a(X) v_a\right|^2 > |v_{a(X)}|^2 |p_{a(X)}(X)|^2 - (\varepsilon')^{25}.$$

When we look at the second term in the right side of 7.7 and apply 6.27, 6.18 we have:

7.11. $$\begin{aligned} &\left|\sum_{a \in A(X),\, b \in B(X)} (v_a, v_b) p_a(X) \bar{p}_b(X)\right| \\ &\leqq \left|\sum_{a \in A(X),\, b \in B(X),\, [b] \leqq 100} (v_a, v_b) p_a(X) \bar{p}_b(X)\right| \\ &+ \left|\sum_{a \in A(X),\, b \in B(X),\, 100 < [b] \leqq l(r)} (v_a, v_b) p_a(X) \bar{p}_b(X)\right| \\ &+ \left|\sum_{a \in A(X),\, b \in B(X),\, l(r) < [b]} (v_a, v_b) p_a(X) \bar{p}_b(X)\right| \\ &< \sum_{a \in A(X),\, b \in B(X),\, [b] \leqq 100} \left(|v_{a(X)}|^2 + (\varepsilon')^{40}\right) |p_{a(X)}(X)|^2 \varepsilon^{\beta N - 10} \\ &+ \sum_{a \in A(X),\, b \in B(X),\, 100 < [b] \leqq l(r)} \left(|v_{a(X)}|^2 + (\varepsilon')^{40}\right) |p_{a(X)}(X)|^2 \varepsilon^{([b]^2 - 1)/2} \\ &+ \sum_{a \in A(X),\, b \in B(X),\, l(r) < [b]} (\varepsilon')^{-10} |p_a(X) \bar{p}_b(X)| < (M_0)^2 \left(|v_{a(X)}|^2 + (\varepsilon')^{40}\right) |p_{a(X)}(X)|^2 \varepsilon^{\beta N - 10} \\ &+ \left(\sum_{n \geqq 100} (M_0 (N^{10}(n+2))^{2N}) \varepsilon^{(n^2 - 1)/2}\right) \left(|v_{a(X)}|^2 + (\varepsilon')^{40}\right) |p_{a(X)}(X)|^2 + (\varepsilon')^{80} \\ &< |p_{a(X)}|^2 |v_{a(X)}|^2 \varepsilon^{\beta N - 20} + (\varepsilon')^{25}. \end{aligned}$$

Combining the above estimates we obtain:

$$\begin{aligned} |h_i(X)|^2 &> |v_{a(X)}|^2 |p_{a(X)}|^2 (1 - \varepsilon^{\beta N - 20}) - (\varepsilon')^{20} \\ &> \varepsilon^2 (1 - |f(X)|^2) / \left(|1 - \varphi(X)|^4 + (\varepsilon')^{10}\right) - (\varepsilon')^4. \end{aligned}$$

So now (C) is proved. From this point the proof continues exactly like the proof of Lemma 2 in the case $N=2$ (see (D) there) as the dimension is not used there (from (D) on) at all. Like the proof of Lemma 1, this proof holds (without any change) in the case the target ball is B^M where $M > N$.

Remark. After this work was completed I learned that Monique Hakim has obtained similar results. Our work was done entirely independently.

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