On John and Nirenberg's theorem

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Introduction

A well-known theorem by John and Nirenberg states that for a function f in $BMO(\mathbf{R}^n)$ with $\|f\|_{BMO}=K$ we have for every cube Q with sides parallel to the axes:

$$\left|\{x\in Q;\ |f(x)-a_Q|>\sigma\}\right| \leqq c_1 e^{-c_2\sigma K^{-1}}|Q|. \tag{1}$$

The constant c_2 which is obtained normally is of the form 2^{-cn}. In the paper [2] John and Nirenberg claim that the constant c_2 can be improved to be of the order $\log n/n$. (c_1 is an absolute constant e.g. 2.)

In this paper we introduce the more general notion of *a false cube* and an associated BMO-norm, $\|f\|'_{BMO}$. We will show that (1) is true with this norm for all false cubes Q with a constant c_2 which then is *independent of* n (Theorem 1). We also will show (Theorem 2) that the quotient of $\|f\|'_{BMO}$ and $\|f\|_{BMO}$ is at most of the order $\sqrt{n}$, which means that we can improve c_2 in (1) to the order of $n^{-1/2}$ and at the same time allow Q to be any false cube.

Definitions and notations

A *cube* will always mean a cube in $\mathbf{R}^n$ with sides parallel to the axes.

A *false cube* is an n-dimensional rectangle in $\mathbf{R}^n$ whose sides are parallel to the axes and for some s have side lengths either s or $2s$, i.e. its proportions are $2\times 2\times\ldots\times 2\times 1\times 1\times\ldots\times 1$.

The *Lebesgue measure* of a set E is denoted by $|E|$. If f is a real-valued function in L^1_{loc} we define the *sharp function,* $f^\#$, by

$$f^\#(t)=\sup_{Q\ni t}\frac{1}{|Q|^2}\int_Q\int_Q |f(x)-f(y)|\,dx\,dy, \tag{2}$$

where the supremum is taken over all cubes Q containing t.

$f^{\#'}$ is also defined by (2) but with the less severe restriction that Q varies over all false cubes containing t. Obviously

$$f^{\#'}(t) \geqq f^{\#}(t).$$

We use the following norms:

$$\|f\|_{BMO} = \|f^{\#}\|_{\infty} \quad \text{and} \quad \|f\|'_{BMO} = \|f^{\#'}\|_{\infty}.$$

In [3] it is shown that

$$\|f\|^*_{BMO} \leqq \|f\|_{BMO} \leqq 2\|f\|^*_{BMO},$$

where $\|f\|^*_{BMO}$ denotes the usual BMO-norm as defined in [2]. We will let a_Q denote a *median value* of f in a (false) cube Q, i.e. a real number with the property that

$$|(x\in Q;\ f(x) \geqq a_Q)| \geqq \tfrac{1}{2}|Q|, \quad |(x\in Q;\ f(x) \leqq a_Q)| \geqq \tfrac{1}{2}|Q|.$$

B_m consists of the vertices of the unit cube in $\mathbf{R}^m$. If $v=(v_1, v_2, \ldots, v_m)$ and $\mu=(\mu_1, \mu_2, \ldots, \mu_m)$ are two elements of B_m the *Hamming distance* between v and μ is:

$$d(v, \mu) = \sum_{i=1}^m |v_i - \mu_i|.$$

Theorems and proofs

Theorem A. *Let a be a real function on B_m, such that*

$$|a(v) - a(\mu)| \leqq d(v, \mu).$$

Then

$$\sum_{v\in B_m} \sum_{\mu\in B_m} |a(v)-a(\mu)| \leqq \sum_{k=0}^m \sum_{p=0}^m \binom{m}{p}\binom{m}{k}|k-p| = m\binom{2m}{m} \leqq 4^m\sqrt{\frac{m}{\pi}}.$$

An elegant graph theoretic proof of this theorem is now available [1].

Lemma 1. *Let E be a subset of $\mathbf{R}^n$ with finite Lebesgue measure and ϱ a real number in $(0, 1)$. Suppose that E is contained in a false cube Q (or just $\mathbf{R}^n$) and that $|E| \leqq \varrho|Q|$. Then there exists a sequence $\{Q_v\}_1^\infty$ of dyadic false cubes, dyadic with respect to Q, and contained in Q such that they have disjoint interiors and*

1) $\frac{1}{2}\varrho|Q_v| < |Q_v \cap E| \leqq \varrho|Q_v|$

2) $\bigcup_1^\infty Q_v \supset E'$,

where E' is the set of density points of E.

Proof. Let x be a point in E'. Then there exists a dyadic cube, $Q_{x,1}$, containing x, such that

$$|Q_{x,1} \cap E| > \varrho|Q_{x,1}|.$$

We double the volume of the cube $Q_{x,1}$ and obtain a false dyadic cube $Q_{x,2}$ by doubling the length of the side parallel to the x_1-axis and proceed with the other dimensions in order. After the n:th doubling we have got a new (true) cube and start again with the first dimension. After finitely many, say $(p-1)$, doublings we reach a first false cube with an "E-density" of at most ϱ, i.e.

$$|Q_{x,p}\cap E| \leqq \varrho|Q_{x,p}|.$$

Since

$$|Q_{x,p-1}\cap E| > \varrho|Q_{x,p-1}|$$

we have

$$|Q_{x,p}\cap E| \geqq |Q_{x,p-1}\cap E| > \varrho|Q_{x,p-1}| = \frac{\varrho}{2}|Q_{x,p}|$$

i.e.

$$\frac{\varrho}{2}|Q_{x,p}| < |Q_{x,p}\cap E| \leqq \varrho|Q_{x,p}|.$$

This procedure can be carried out for every x in E'. We obtain a family of false cubes $\{Q_{x,p}\}_{x\in E'}$. (p will depend on x.) We will now show that two such false cubes Q_{x_1,p_1} and Q_{x_2,p_2} have disjoint interior unless one of them is contained in the other.

If the two false cubes do not have disjoint interiors their intersection must contain at least one dyadic cube Q_0. It is evident that both Q_{x_1,p_1} and Q_{x_2,p_2} are obtained by successively (within Q) doubling Q_0 dyadically in the dimensions $1, 2, 3, \dots$. Therefore either the two cubes coincide or one of them is a subset of the other.

We delete from the family those false cubes that are contained in others and the remaining cubes, which then have disjoint interiors, can be numbered by decreasing volume. Thus we have obtained the sequence claimed in the lemma.

Note. We note that if Q is a (true) cube we may prescribe not only that the false cubes should be disjoint but also that their side lengths do not increase when taken in a certain common order, which in the proof above is chosen to be the order given by the $x_1, x_2, \dots, x_n$ axes.

Theorem 1. *Suppose that f is a function in $BMO(\mathbf{R}^n)$ and*

$$\|f\|'_{BMO} = K.$$

Then there exists a constant α ($\alpha=16$ will do) such that for every false cube Q

$$|S_\sigma| \leqq 2\cdot 2^{-\sigma(\alpha K)^{-1}}|Q|, \tag{3}$$

where

$$S_\sigma = \{x\in Q;\ |f(x)-a_Q| \geqq \sigma\}.$$

This can also be expressed as

$$\int_Q e^{c|f(x)-a_Q|}\,dx \leqq \frac{2\alpha}{\ln 2(\alpha-cK)}|Q| \quad \textit{for} \quad c<\frac{\alpha}{K}.$$

Proof. We may assume that $\|f\|'_{BMO}=1$. Let Q be an arbitrary false cube and E_0 a subset of Q with measure $|Q|/2$ such that

$$f(x) \leqq a_Q \quad \text{on} \quad E_0, \quad \text{and} \quad f(x) \geqq a_Q \quad \text{on} \quad Q \setminus E_0.$$

Put, for $k \geqq 0$

$$E_k = \{x \in Q;\ f(x)-a_Q \geqq k\}.$$

We now use Lemma 1 to cover E_k', $k \geqq 0$, with a sequence $\{Q_{k,\nu}\}_{\nu=1}^{\infty}$ of disjoint false cubes, dyadic with respect to Q, such that

$$\frac{1}{4}|Q_{k,\nu}| < |E_k \cap Q_{k,\nu}| \leqq \frac{1}{2}|Q_{k,\nu}|. \tag{4}$$

Therefore

$$|\complement(E_k) \cap Q_{k,\nu}| \geqq \frac{1}{2}|Q_{k,\nu}|.$$

and we have for t in $Q_{k,\nu}$ and $l>0$

$$f^{\#\prime}(t) \geqq \frac{2}{|Q_{k,\nu}|^2}\int_{E_{k+l}\cap Q_{k,\nu}} dy \int_{\complement(E_k)\cap Q_{k,\nu}} (f(y)-f(x))\,dx \geqq l\,\frac{|E_{k+l}\cap Q_{k,\nu}|}{|Q_{k,\nu}|},$$

and, since $\|f^{\#\prime}\|_\infty = 1$,

$$|E_{k+l}\cap Q_{k,\nu}| \leqq \frac{|Q_{k,\nu}|}{l}.$$

We use (4) and make a summation over ν. This gives

$$|E_{k+l}| \leqq \frac{1}{l}\sum |Q_{k,\nu}| < \frac{4}{l}\sum |E_k \cap Q_{k,\nu}| = \frac{4}{l}|E_k|.$$

We can obtain the same estimate for the sets

$$\{x \in Q;\ f(x)-a_Q \leqq k\}, \quad k<0.$$

The estimates combine to

$$|S_{k+l}| \leqq \frac{8}{l}|S_k|.$$

We take $l=16$ and find

$$|S_{k+16}| \leqq \tfrac{1}{2}|S_k|,$$

which, for any positive integer p implies:

$$|S_{16p}| \leqq 2^{-p}|S_0| \leqq 2^{-p}|Q|,$$

from which (3) follows and the theorem is proved.

We will now prove a couple of lemmas which will make it possible for us to compare $\|f\|_{BMO}$ and $\|f\|'_{BMO}$. Theorem 1 then will provide a good constant in John and Nirenberg's theorem.

Lemma 2. *Suppose that f is a function in $BMO(\mathbf{R}^n)$, Q an arbitrary false cube and a_Q a median value of f in Q. Then*

(5)
$$\frac{1}{|Q|}\int_Q |f(x)-a_Q|\,dx \leqq \frac{1}{|Q|^2}\int_Q\int_Q |f(x)-f(y)|\,dx\,dy \leqq \frac{2}{|Q|}\int_Q |f(x)-a_Q|\,dx.$$

Proof. Let E_1 and E_2 be disjoint subsets of Q such that

$$f(x) \geqq a_Q \quad \text{for} \quad x\in E_1, \quad f(x) \leqq a_Q \quad \text{for} \quad x\in E_2 \quad \text{and} \quad |E_1| = |E_2| = \tfrac{1}{2}|Q|.$$

Then

$$\begin{aligned}\int_Q\int_Q |f(x)-f(y)|\,dx\,dy &= \int_{E_1}\int_{E_1} |f(x)-f(y)|\,dx\,dy \\ &+\int_{E_2}\int_{E_2} |f(x)-f(y)|\,dx\,dy+2\int_{E_1}\int_{E_2} |f(x)-f(y)|\,dx\,dy.\end{aligned} \tag{6}$$

The right inequality of (5) is an immediale consequence of the triangle inequality. Using only the third member of the right-hand side of (6) we can also derive:

$$\begin{aligned}\int_Q\int_Q |f(x)-f(y)|\,dx\,dy &\geqq 2\int_{E_1}\int_{E_2} [(f(x)-a_Q)+(a_Q-f(y))]\,dx\,dy \\ &= |Q|\int_{E_1} |f(x)-a_Q|\,dx+|Q|\int_{E_2} |a_Q-f(y)|\,dy = |Q|\int_Q |f(x)-a_Q|\,dx,\end{aligned}$$

which proves the left inequality of (5).

Corollary. *Suppose f is a function in $BMO(\mathbf{R}^n)$. Then, for every cube Q, we have*

$$\frac{1}{|Q|}\int_Q |f(x)-a_Q|\,dx \leqq \|f\|_{BMO}.$$

The same inequality with $\|f\|_{BMO}$ replaced by $\|f\|'_{BMO}$ holds for every false cube Q. The corollary is an immediate consequence of (5) and the definition of $\|f\|_{BMO}$ and $\|f\|'_{BMO}$ as suprema.

Lemma 3. *Suppose f is a function in $BMO(\mathbf{R}^n)$ and Q_1 and Q_2 two adjacent cubes with the same side length. Then*

$$|a_{Q_1}-a_{Q_2}| \leqq 6\|f\|_{BMO},$$

where a_{Q_1} and a_{Q_2} are median values of f in Q_1 and Q_2 respectively.

Proof. We may assume that $\|f\|_{BMO}=1$. Let Q be the cube which consists of half of Q_1 and half of Q_2 and a_Q the median value of f in Q. Then for some $t\leqq 1$ we use the corollary above to see that:

$$\int_{Q\cap Q_1}|f(x)-a_Q|\,dx = t|Q|, \quad \int_{Q\cap Q_1}|f(x)-a_{Q_1}|\,dx \leqq |Q|$$

$$\int_{Q\cap Q_2}|f(x)-a_Q|\,dx \leqq (1-t)|Q|, \quad \int_{Q\cap Q_2}|f(x)-a_{Q_2}|\,dx \leqq |Q|.$$

By the triangle inequality

$$\int_{Q\cap Q_1}|a_Q-a_{Q_1}|\,dx \leqq (1+t)|Q|, \quad \text{i.e.} \quad |a_Q-a_{Q_1}| \leqq 2(1+t)$$

and

$$|a_Q-a_{Q_2}| \leqq 2(2-t).$$

Thus

$$|a_{Q_1}-a_{Q_2}| \leqq 2(1+t)+2(2-t) = 6,$$

which proves the lemma.

Theorem 2. *Let f be a function in $BMO(\mathbf{R}^n)$. Then*

$$\|f\|'_{BMO} \leqq \left(2+6\sqrt{\frac{n}{\pi}}\right)\|f\|_{BMO}.$$

Proof. Without loss of generality we may assume that $\|f\|_{BMO}=1$. Let Q be an arbitrary false cube consisting of 2^m, $1\leqq m<n$ cubes. We may also assume that Q has side lengths $2s$ in the first m dimensions and side lengths s in the remaining $n-m$ dimensions. Put

$$b_k = \min_{x\in Q} x_k, \quad k = 1, 2, \ldots, n.$$

The midpoints of the subcubes of Q then have the coordinates

$$\left(b_1+\left(\frac{1}{2}+\varepsilon_1\right)s,\ b_2+\left(\frac{1}{2}+\varepsilon_2\right)s,\ \ldots,\ b_m+\left(\frac{1}{2}+\varepsilon_m\right)s,\ b_{m+1}+\frac{s}{2},\ \ldots,\ b_n+\frac{s}{2}\right),$$

where the ε_k:s $k=1, 2, \ldots, m$ are either zero or one. We number the subcubes by the m-digit numbers $v=\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m$ which also will denote vertices of B_m.

a_ν is a median value of f in Q_ν. We have:

$$\int_Q \int_Q |f(x)-f(y)|\,dx\,dy = \sum_{\nu,\mu=1}^{2^m} \int_{Q_\nu} dx \int_{Q_\mu} |f(x)-f(y)|\,dy$$

$$\leqq \sum_{\nu,\mu=1}^{2^m} \int_{Q_\nu} dx \int_{Q_\mu} \big(|f(x)-a_\nu|+|a_\nu-a_\mu|+|f(y)-a_\mu|\big)\,dy$$

$$= 2|Q| \sum_{\nu=1}^{2^m} \int_{Q_\nu} |f(x)-a_\nu|\,dx + \frac{|Q|^2}{2^{2m}} \sum_{\nu,\mu=1}^{2^m} |a_\nu - a_\mu|.$$

Since $\|f\|_{BMO}=1$ we use the corollary of Lemma 2 to see that

$$2|Q| \sum_{\nu=1}^{2^m} \int_{Q_\nu} |f(x)-a_\nu|\,dx \leqq 2|Q| \sum_{\nu=1}^{2^m} |Q_\nu| = 2|Q|^2.$$

By Lemma 3

$$|a_\nu - a_\mu| \leqq 6d(\nu, \mu),$$

where $d(\nu, \mu)$ is the Hamming distance between ν and μ regarded as elements of B_m. This implies, using Theorem A, that

$$\|f\|'_{BMO} \leqq \left(2+6\sqrt{\frac{n}{\pi}}\right) < 6\sqrt{n}$$

and the theorem is proved.

As a corollary to this theorem and Theorem 1 we obtain the following version of John and Nirenberg's theorem.

Corollary. *Suppose that f is a function in $BMO(\mathbf{R}^n)$ and*

$$\|f\|_{BMO} = K.$$

Then there exists a constant α $\left(\alpha=16\left(2+6\sqrt{n/\pi}\right)\right.$ will do$\left.\right)$ such that for every false cube Q

$$|S_\sigma| \leqq 2\cdot 2^{-\sigma(\alpha K)^{-1}}|Q|,$$

where

$$S_\sigma = \{x\in Q;\ |f(x)-a_Q| \geqq \sigma\}.$$

References

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Received Feb. 2, 1989

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