Hankel operators between weighted Bergman spaces in the ball

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Summary and introduction

Let m denote the Lebesgue measure on the unit ball $B\subset\mathbf{C}^n$ and let, for $-1<\alpha<\infty$, μ_α be the measure $c_\alpha(1-|z|^2)^\alpha\,dm(z)$, where c_α is chosen such that $\mu_\alpha(B)=1$, i.e. $c_\alpha=\big(\Gamma(n+\alpha+2)\big)/\pi^n\Gamma(\alpha+1)$.

The closed subspace of all holomorphic functions in $L^2(d\mu_\alpha)$ is denoted A^α and called a weighted Bergman space. Since A^α is closed in $L^2(d\mu_\alpha)$ there is a self-adjoint projection P_α of $L^2(d\mu_\alpha)$ onto A^α, which is given by a kernel:

$$P_\alpha f(z)=\int\frac{1}{\big(1-\langle z,w\rangle\big)^{n+\alpha+1}}f(w)\,d\mu_\alpha(w).$$

The big and small Hankel operators can now be defined:

$$H_f(g)=(I-P_\alpha)(\bar f g),$$
$$\tilde H_f(g)=\bar P_\alpha(\bar f g).$$

Here, $\bar P_\alpha$ denotes the projection onto the subspace $\bar A^\alpha\subset L^2(d\mu_\alpha)$ of anti-holomorphic functions. H_f and $\tilde H_f$ will be studied as operators from A^β into $L^2(d\mu_\alpha)$ and the symbol f is assumed to be holomorphic.

For an operator T between Hilbert spaces the singular numbers are defined by $s_n(T)=\inf\{\|T-K\|\}$; rank $\{(K)\leqq n\}$, $n\geqq 0$. We denote by S_p (Schatten—von Neumann class) the ideal of operators for which $\{s_n(T)\}_{n\geqq 0}\in l^p$, $0<p<\infty$. In accordance with this definition the class of bounded operators is denoted by S_∞. In our case, operators from A^β into $L^2(d\mu_\alpha)$, the corresponding Schatten—von Neumann class is written $S_p^{\beta\alpha}$.

Finally, for $0<p\leqq\infty$ and $-\infty<s<\infty$, we define the Besov space of holomorphic functions by

$$B_p^s=\big\{f\colon (1-|z|^2)^{m-s}\mathscr{R}^m f(z)\in L^p\big((1-|z|^2)^{-1}\,dm\big)\big\},$$

where $m>s$ is a non-negative integer and $\mathcal{R}=\sum z_j \partial/\partial z_j$. Generalizing Janson's results [J] for the disk we have

Theorem 1. *Let* $\alpha, \beta>-1$, $n\geqq 2$ *and* $0<p\leqq\infty$.
(i) *If* $2n/p<1+\alpha-\beta$, *then* $H_f\in S_p^{\beta\alpha}$ *iff* $f\in B_p^{n/p+(\beta-\alpha)/2}$.
(ii) *If* $2n/p\geqq 1+\alpha-\beta$, *except in the case* $p=\infty$ *and* $\beta=\alpha+1$, *then* $H_f\in S_p^{\beta\alpha}$ *only if* $H_f=0$.

Let b_∞^s be the closure of the polynomials in B_∞^s.

Theorem 2. *Let* $\alpha, \beta>-1$ *and* $n\geqq 2$.
(i) *If* $\beta<\alpha+1$, *then* $H_f\colon A^\beta\to L^2(d\mu_\alpha)$ *is compact iff* $f\in b_\infty^{(\beta-\alpha)/2}$.
(ii) *If* $\beta\geqq\alpha+1$, *then* $H_f\colon A^\beta\to L^2(d\mu_\alpha)$ *is compact only if* $H_f=0$.

For the small operators we have

Theorem 3. *Let* $\alpha, \beta>-1$ *and* $0<p\leqq\infty$. *Then* $\tilde{H}_f\in S_p^{\beta\alpha}$ *iff* $f\in B_p^{n/p+(\beta-\alpha)/2}$.

Theorem 4. *Let* $\alpha, \beta>-1$. *Then* $\tilde{H}_f\colon A^\beta\to\bar{A}^\alpha$ *is compact iff* $f\in b_\infty^{(\beta-\alpha)/2}$.

The last two theorems are well-known (cf. Sec. 3) and included here because they are used in the proofs of Theorems 1 and 2.

1. Besov spaces in the ball

The purpose of this section is to state the facts about Besov spaces that we will use. For details and proofs, see e.g. [BB], [CR], [P] and [T].

To begin with, our definition of B_p^s is independent of the integer m. Indeed, we get an equivalent norm using the differential operators $D^k=\partial^{|k|}/\partial z_1^{k_1}\partial z_2^{k_2}\dots\partial z_n^{k_n}$, k a multi-index:

$$(1.1)\qquad B_p^s=\{f\colon (1-|z|^2)^{m-s}D^kf(z)\in L^p((1-|z|^2)^{-1}dm),\ |k|\leqq m\}.$$

It is immediate that $\mathcal{R}^u$, $u\in\mathbf{R}$, defined by $\mathcal{R}^u z^k=|k|^u z^k$, k a multi-index, and D^k define continuous mapping between Besov spaces

$$\mathcal{R}^u\colon B_p^s\to B_p^{s-u},$$
$$D^k\colon B_p^s\to B_p^{s-|k|},$$

and that the Besov spaces are decreasing in s

$$B_p^{s_1}\subseteqq B_p^{s_2},\quad s_1\geqq s_2.$$

The property of Besov spaces that will be especially useful to us is decomposition into "atoms":

Lemma 0. *Let* $0<p\leqq 1$, $-\infty<s<\infty$ *and suppose that* $N>n/p-s$. *Then there exists a sequence* $\{\xi_i\}\subset B$ *such that every* $f\in B_p^s$ *can be written*

$$f(z)=\sum_i \lambda_i(1-|\xi_i|^2)^{N+s-n/p}(1-\langle z,\xi_i\rangle)^{-N}$$

with

$$\sum_i |\lambda_i|^p\leqq C\|f\|_{B_p^s}^p.$$

Proof. For $s<0$ this is Theorem 2 in [CR]. If $0\leqq s<1$, then $T\colon B_p^s\to B_p^{s-1}$, defined by $T=Id+(N-1)^{-1}\mathscr{R}$, is continuous and injective. T is also surjective, since

$$\sum_i \lambda_i(1-|\xi_i|^2)^{N+s-1-n/p}(1-\langle z,\xi_i\rangle)^{-N}$$

is the image under T of

$$\sum_i \lambda_i(1-|\xi_i|^2)^{N-1+s-n/p}(1-\langle z,\xi_i\rangle)^{-N+1}.$$

By the Open Mapping Theorem, T is an isomorphism. The lemma follows by induction. □

The converse of the Lemma is also true, because the B_p^s-norm to the power p is a metric.

2. The cut-off

Proposition 1. *Let* $\alpha,\beta>-1$, $0<p<\infty$, $n\geqq 2$ *and let* f *be a homogeneous polynomial of degree* 1. *Then*

(i) $H_f\in S_p^{\beta\alpha}$ *iff* $2n/p<1+\alpha-\beta$.

(ii) $H_f\in S_\infty^{\beta\alpha}$ *iff* $0\leqq 1+\alpha-\beta$.

(iii) $H_f\colon A^\beta\to L^2(d\mu_\alpha)$ *is compact iff* $0<1+\alpha-\beta$.

Proof. Since the properties in question are invariant under a linear change of variables and $H_fL=LH_{L^{-1}f}$, where L is composition with such a change of variables, we may assume that $f(z)=z_1$. Let $k=(k_1,\dots,k_n)$ be a multi-index and let $\gamma_{k,\alpha}$ denote the norm of z^k in A^α. Then, for $k_1\geqq 1$,

$$P_\alpha(\bar z_1 z^k)=\frac{\langle \bar z_1 z^k, z^{(k_1-1,k_2,\dots,k_n)}\rangle}{\gamma^2_{(k_1-1,k_2,\dots,k_n),\alpha}}\,z^{(k_1-1,k_2,\dots,k_n)}$$

and

$$\|H_{z_1}(z^k)\|_\alpha^2=\|\bar z_1 z^k\|_\alpha^2-\|P_\alpha(\bar z_1 z^k)\|_\alpha^2=\begin{cases}\gamma^2_{(1,k_2,\dots,k_n),\alpha} & \text{if } k_1=0\\[4pt] \gamma^2_{(k_1+1,k_2,\dots,k_n),\alpha}-\dfrac{\gamma^4_{k,\alpha}}{\gamma^2_{(k_1-1,k_2,\dots,k_n),\alpha}} & \text{if } k_1\geqq 1.\end{cases}$$

Further $\gamma_{k,\alpha}^2 = \dfrac{k!\,\Gamma(n+\alpha+1)}{\Gamma(|k|+n+\alpha+1)}$, and a calculation yields

$$\left\| H_{z_1}\left(\frac{z^k}{\gamma_{k,\beta}}\right)\right\|_\alpha^2 \asymp \left(1-\frac{k_1}{|k|+n+\alpha}\right)(|k|+1)^{\beta-\alpha-1}. \tag{2.1}$$

It is easily seen that $\{H_{z_1}(z^k)\}$ are orthogonal, and since $\dfrac{z^k}{\gamma_{k,\beta}}$ is an ON-basis in A^β, it follows that $\left\| H_{z_1}\left(\dfrac{z^k}{\gamma_{k,\beta}}\right)\right\|_\alpha$ are the singular numbers of H_{z_1}. Hence (2.1) proves (ii) and (iii).

As to (i) we have

$$\|H_{z_1}\|_{S_p^{\beta\alpha}}^p \asymp \sum_k \left(1-\frac{k_1}{|k|+n+\alpha}\right)^{p/2} (|k|+1)^{p/2(\beta-\alpha-1)}$$

$$= \sum_{l=0}^\infty (l+1)^{p/2(\beta-\alpha-1)} \sum_{m=0}^l \binom{m+n-2}{m}\left(\frac{m+n+\alpha}{l+n+\alpha}\right)^{p/2}$$

$$\asymp \sum_{l=0}^\infty (l+1)^{p/2(\beta-\alpha-1)} \sum_{m=0}^l (m+1)^{n-2}\left(\frac{m+1}{l+1}\right)^{p/2} \asymp \sum_{l=0}^\infty (l+1)^{p/2(\beta-\alpha-1)+n-1}. \quad \square$$

Now the second parts of Theorems 1 and 2 follow as in [J].

3. The small operator

Theorems 3 and 4 are, as stated earlier, well known, at least in the case $1\leqq p\leqq\infty$ and $\alpha=\beta$, see [A] or [JPR], where the results are stated for the associated bilinear form $\tilde{H}_f(g_1, g_2)=\int \bar{f} g_1 g_2\, d\mu_2$. A proof in the general case can be obtained following e.g. [A]. When $0<p\leqq 1$ the sufficiency part is a simple consequence of Lemma 0. For the necessity part one picks an even integer $l>n/p+(\beta-\alpha)/2$ and considers the bilinear forms $(g_1, g_2)\mapsto \tilde{H}_f(D^{k_1}g_1, D^{k_2}g_2)$, $|k_i|\leqq l/2$, which are S_p-forms on $B_2^{(l/2)-(1+\beta)/2}\times B_2^{(l/2)-(1+\alpha)/2}$. Then Semmes' method [S] yields $f\in B_p^{(n/p)+(\beta-\alpha)/2}$.

4. The case $p\leqq 1$

Lemma 0 reduces the sufficiency proof to an estimate of the S_p-norms for symbols of the type $(1-\langle z, \zeta\rangle)^N$, which will be done in this section. Let $M_\zeta^s f(z)=(1-\langle z, \zeta\rangle)^{-s} f(z)$, $s\in\mathbf{R}$. We then have

Lemma 1. *If* $0<p\leqq 2$, $\alpha-\beta>2n/p$, $s<n/2[(\alpha-\beta)/n]$ *and* $(\alpha-\beta)/n\notin\mathbf{Z}$, *then*

$$\|M_\zeta^s\|_{S_p(A^\beta, A^\alpha)} \leqq C.$$

Proof. By unitary invariance we may assume that $\zeta=(t, 0, \dots, 0)$, $t>0$. If $s<1/2$, then

$$\|M_\zeta^s(z^k)\|_\alpha^2 = c\int_{B_n} |1-tz_1|^{-2s}|z^k|^2(1-|z|^2)^\alpha\, dm(z)$$

does not change if t is replaced by $e^{i\theta}t$, $e^{i\theta}\in \mathbf{T}$. Then integrate over $\mathbf{T}$ with respect to $d\theta/2\pi$ to get

$$\|M_\zeta^s(z^k)\|_\alpha^2 \leqq C\|z^k\|_\alpha^2$$

and the S_p-estimate follows by the inequality $\|T\|_{S_p}^p \leqq \sum_i \|Te_i\|^p$, $\{e_i\}$ any ON-basis.

If $1/2\leqq s<n/2$, we have

$$\|M_\zeta^s(z^k)\|_\alpha^2 = c_\alpha\int_{B_n} |1-tz_1|^{-2s}|z^k|^2(1-|z|^2)^\alpha\, dm(z)$$

$$= C\int_0^1 r_1^{2|k|+2n-1}(1-r_1^2)^\alpha\, dr_1 \int_{S_{n-1}} |1-tr_1\eta_1|^{-2s}|\eta^k|\, d\sigma(\eta)$$

$$= C\int_0^1 r_1^{2|k|+2n-1}(1-r_1^2)^\alpha\, dr_1 \int_{B_{n-1}} |1-tr_1 z_1|^{-2s}|z_1|^{2k_1}\dots|z_{n-1}|^{2k_{n-1}}(1-|z|^2)^{k_n}\, dm(z)$$

$$= C\int_0^1 r_1^{2|k|+2n-1}(1-r_1^2)^\alpha\, dr_1 \int_0^1 r_2^{2(k_1+\dots+k_{n-1})+2n-3}(1-r_2^2)^{k_n}$$

$$\int_{S_{n-2}} |1-tr_1r_2\eta_1|^{-2s}|\eta_1|^{2k_1}\dots|\eta_{n-1}|^{2k_{n-1}}\, d\sigma(\eta) = \dots$$

$$= C\int_0^1 r_1^{2|k|+2n-1}(1-r_1^2)^\alpha\, dr_1 \dots \int_0^1 r_{n-1}^{2(k_1+k_2)+2n-1-2(n-1-1)}(1-r_{n-1}^2)^{k_3}$$

$$\int_U |1-tr_1r_2\dots r_{n-1}z|^{-2s}|z|^{2k_1}(1-|z|^2)^{k_2}\, dm(z)$$

$$\leqq C\int_0^1 (1-r_1r_2\dots r_n)^{1-2s} r_n^{2k_1+1}(1-r_n^2)^{k_2} r_{n-1}^{2(k_1+k_2)+3}$$

$$(1-r_{n-1}^2)^{k_3}\dots r_1^{2|k|+2n-1}(1-r_1^2)^\alpha\, dr_1\, dr_2\dots dr_n$$

$$\asymp \int_0^1 (1-r_1^2r_2^2\dots r_n^2)^{1-2s}(r_n^2)^{k_1}(1-r_n^2)^{k_2}(r_{n-1}^2)^{k_1+k_2+1}(1-r_{n-1}^2)^{k_3}\dots(r_1^2)^{|k|+n-1}(1-r_1^2)^\alpha$$

$$r_1r_2\dots r_n\, dr_1\, dr_2\dots dr_n \asymp \int_0^1 (1-r_1r_2\dots r_n)^{1-2s} r_n^{k_1}(1-r_n)^{k_2} r_{n-1}^{k_1+k_2+1}(1-r_{n-1})^{k_3}$$

$$\dots r_1^{|k|+n-1}(1-r_1)^\alpha\, dr_1\, dr_2\dots dr_n = \sum_{j=0}^\infty \frac{\Gamma(j+2s-1)}{\Gamma(j+1)\Gamma(2s-1)}\int_0^1 r_n^{k_1+j}(1-r_n)^{k_2}\, dr_n$$

$$\times\int_0^1 r_{n-1}^{k_1+k_2+1+j}(1-r_{n-1})^{k_3}\, dr_{n-1}\dots\int_0^1 r_1^{|k|+n-1}(1-r_1)^\alpha\, dr_1$$

$$= \sum_{j=0}^\infty \frac{\Gamma(j+2s-1)}{\Gamma(j+1)\Gamma(2s-1)}\frac{\Gamma(k_1+j+1)\Gamma(k_2+1)}{\Gamma(k_1+k_2+2+j)}$$

$$\times\frac{\Gamma(k_1+k_2+2+j)\Gamma(k_3+1)}{\Gamma(k_1+k_2+k_3+3+j)}\dots\frac{\Gamma(|k|+n+j)\Gamma(\alpha+1)}{\Gamma(|k|+n+\alpha+1+j)}$$

$$= \Gamma(\alpha+1)k_2!\,k_3!\dots k_n!\sum_{j=0}^{\infty}\frac{\Gamma(j+2s-1)}{\Gamma(2s-1)\Gamma(j+1)}\frac{\Gamma(k_1+j+1)}{\Gamma(|k|+n+\alpha+1+j)}$$

$$= \frac{\Gamma(\alpha+1)k!}{\Gamma(|k|+n+\alpha+1)}\sum_{j=0}^{\infty}\frac{(2s-1)_j(k_1+1)_j}{(|k|+n+\alpha+1)_j\, j!}$$

$$= \frac{\Gamma(\alpha+1)k!}{\Gamma(|k|+n+\alpha+1)}\frac{\Gamma(|k|+n+\alpha+1)\Gamma(|k|+n+\alpha+1-k_1-1-2s+j)}{\Gamma(|k|+n+\alpha+1-2s+1)\Gamma(|k|+n+\alpha+1-k_1-1)}$$

$$\asymp \gamma_{k,\alpha}^2(|k|+1)^{2\alpha-1}(|k|-k_1+1)^{1-2s} = \gamma_{k,\alpha}^2\left(1-\frac{k_1}{|k|+1}\right)^{1-2s},$$

whence

$$\left\|M_\zeta^s\left(\frac{z^k}{\gamma_{k,\beta}}\right)\right\|_\alpha \leqq c\left(1-\frac{k_1}{|k|+1}\right)^{1/2-s}(|k|+1)^{(\beta-\alpha)/2}$$

and

$$\|M_\zeta^s\|_{S_p(A^\beta,A^\alpha)}^p \leqq c\sum_k\left(1-\frac{k_1}{|k|+1}\right)^{p(1/2-s)}(|k|+1)^{p((\beta-\alpha)/2)}$$

$$= \sum_{l=0}^{\infty}(l+1)^{p((\beta-\alpha)/2)}\sum_{m=0}^{l}\binom{m+n-2}{m}\left(\frac{m+1}{l+1}\right)^{p(1/2-s)}$$

$$\asymp \sum_{l=0}^{\infty}(l+1)^{p((\beta-\alpha)/2)}\sum_{m=0}^{l}(m+1)^{n-2}\left(\frac{m+1}{l+1}\right)^{p(1/2-s)}$$

$$\asymp \sum_{l=0}^{\infty}(l+1)^{p((\beta-\alpha)/2)+n-1}<\infty$$

if $\alpha-\beta>2n/p$. If $s<n/2[(\alpha-\beta)/n]=n/2\cdot m$ and $(\alpha-\beta)/n\notin\mathbf{Z}$, we pick p_0 such that $2n/(\alpha-\beta)<p_0\leqq\min(p,2/m)$ and put $\alpha(i)=\beta+i/m(\alpha-\beta)$. Then

$$\|M_\zeta^{s/m}\|_{S_{mp_0}(A^{\alpha(i)},A^{\alpha(i+1)})}\leqq C,\quad 0\leqq i\leqq m-1.$$

The lemma follows by the Schatten—Hölder inequality and the inclusion $S_{p_0}\subseteqq S_p$. □

Lemma 2. *If* $\alpha-\beta>n-1$ *and* $N\geqq s+1$, *then*

$$\|H_{(1-\langle z,\zeta\rangle)^N}M_\zeta^s\|_{S_2^{\beta\alpha}}\leqq C.$$

Proof. Let $b(z)=(1-\langle z,\zeta\rangle)^N\overline{(1-\langle z,\zeta\rangle)}^{-s}$. Then $H_{(1-\langle z,\zeta\rangle)^N}M_\zeta^s=H_b$. The first derivatives of b are bounded on B by a constant independent of ζ. Hence we have

$$\|H_{(1-\langle z,\zeta\rangle)^N}M_\zeta^s\|_{S_2^{\beta\alpha}} = \|H_b\|_{S_2^{\beta\alpha}}\leqq\iint\left|\frac{\bar b(z)-\bar b(w)}{(1-\langle z,w\rangle)^{n+\alpha+1}}(1-|w|^2)^{\alpha-\beta}\right|^2 d\mu_\beta(w)\,d\mu_\alpha(z)$$

$$\leqq C\iint\frac{(1-|w|^2)^{2\alpha-\beta}}{|1-\langle z,w\rangle|^{2n+2\alpha+1}}\,dm(w)\,d\mu_\alpha(z)\leqq C\int(1-|w|^2)^{\alpha-\beta-n}\,dm(w)=C.\quad\square$$

Lemma 3. *If* $0<p\leqq 1$, $\alpha-\beta>2n/p-1$ *and* $N\geqq\alpha-\beta-n/p+n+1$, *then*

$$\|H_{(1-\langle z,\zeta\rangle)^N} M_\zeta^{\alpha-\beta}\|_{S_p^{\beta\alpha}} \leqq C.$$

Proof. Define q by $1/q=1/p-1/2$. Choose γ such that $\alpha-n+1>\gamma>\beta+2n/q$ and $(\gamma-\beta)/n\notin\mathbf{Z}$. Let $s=1/2(\gamma-\beta)-n/2<n/2[(\gamma-\beta)/n]$. Then, by Lemma 1,

$$\|M_\zeta^s\|_{S_q(A^\beta,A^\gamma)} \leqq C.$$

Since $\alpha-\gamma>n-1$ and $2(\alpha-\beta-s)=2\alpha-\beta-\gamma+n$ we have by Lemma 2 also

$$\|H_{(1-\langle z,\zeta\rangle)^N} M_\zeta^{\alpha-\beta-s}\|_{S_2(A^\gamma,L^2(d\mu_\alpha))} \leqq C.$$

The lemma follows by the Schatten—Hölder inequality. □

Lemma 4. *If* $0<p\leqq 1$, $\alpha-\beta>2n/p-1$ *and* $N\geqq\alpha-\beta-n/p+n+1$, *then*

$$\|H_{(1-\langle z,\zeta\rangle)^{-N}}\|_{S_p^{\beta\alpha}} \leqq C(1-|\zeta|^2)^{((\alpha-\beta)/2)-N}.$$

Proof. Let φ_ζ be the involution that takes 0 to ζ and define

$$V_\zeta^\alpha f(z)=f\circ\varphi_\zeta(z)\left(\frac{1-|\zeta|^2}{(1-\langle z,\zeta\rangle)^2}\right)^{(n+\alpha+1)/2}.$$

Then V_ζ^α is an isometry of $L^2(d\mu_\alpha)$ onto itself which maps A^α onto itself, and we have

$$\begin{aligned}
V_\zeta^\alpha H_{(1-\langle z,\zeta\rangle)^{-N}} &= H_{(1-\langle\varphi_\zeta(z),\zeta\rangle)^{-N}} V_\zeta^\alpha = (I-P_\alpha)\overline{(1-\langle\varphi_\zeta(z),\zeta\rangle)}^{-N} V_\zeta^\alpha \\
&= (I-P_\alpha)\overline{\left(\frac{1-|\zeta|^2}{1-\langle z,\zeta\rangle}\right)}^{-N}\left(\frac{1-|\zeta|^2}{(1-\langle z,\zeta\rangle)^2}\right)^{(\beta-\alpha)/2} V_\zeta^\beta \\
&= (1-|\zeta|^2)^{((\alpha-\beta)/2)-N} H_{(1-\langle z,\zeta\rangle)^N} M_\zeta^{\alpha-\beta} V_\zeta^\beta
\end{aligned}$$

and Lemma 3 yields the required estimate. □

The sufficiency part of Theorem 1 for $p\leqq 1$ now follows by Lemmas 0 and 4.

5. The case $p=\infty$

Lemma 5. *Suppose that* $\alpha>-1$, $s<1/2$, $-1<\gamma<\alpha$ *and* $-1<\gamma+s<\alpha$. *Then*

$$\int_B \frac{|f(z)-f(w)|}{|1-\langle z,w\rangle|^{n+\alpha+1}}(1-|z|^2)^\gamma\,dm(z) \leqq C(1-|w|^2)^{\gamma-\alpha+s}\|f\|_{B_\infty^s}.$$

Note that we get cut-off $s<1/2$ when $n>1$ in contrast to $s<1$ when $n=1$. This is connected with the boundary behaviour of holomorphic Lipschitz functions ($B_\infty^s=H(B)\cap\Lambda_s$ when $0<s<1$). See Ch. 6 in [R].

Proof. If we can prove the inequality

$$|f(z)-f(w)| \leqq C\|f\|_{B^s_\infty}|1-\langle z, w\rangle|^s, \quad 0<s<\tfrac{1}{2} \tag{5.1}$$

then the lemma follows as in [J].

To prove (5.1) we assume, for simplicity, that $n=2$. By unitary invariance, we may also assume that $w=(\varrho, 0)$, $\varrho>0$. Write $z=(r_1e^{i\varphi_1}, r_2e^{i\varphi_2})$, $|\varphi_i|\leqq\pi$, and put $\mu=(r_1^2\varphi_1+r_2^2\varphi_2)/(r_1^2+r_2^2)$. We have

$$|f(z)-f(w)| \leqq \left|f(z)-f\left(\varrho\frac{z}{|z|}\right)\right|+\left|f\left(\varrho\frac{z}{|z|}\right)-f\left(\frac{\varrho}{|z|}e^{i\mu}(r_1, r_2)\right)\right|$$

$$+\left|f\left(\frac{\varrho}{|z|}e^{i\mu}(r_1, r_2)\right)-f\left(\frac{\varrho}{|z|}(r_1, r_2)\right)\right|+\left|f\left(\frac{\varrho}{|z|}(r_1, r_2)\right)-f(w)\right| = \Delta_1+\Delta_2+\Delta_3+\Delta_4.$$

Note that when z and w lie on the same complex line through the origin (5.1) is trivial, since then $|z-w|\leqq|1-\langle z, w\rangle|$. This argument takes care of Δ_1 and Δ_3.

To deal with Δ_2 and Δ_4, recall that f is Λ_{2s} along complex-tangential curves, Th. 6.4.10 in [R]. To finish the proof we need only find the appropriate curves. These are suitable portions of $t\mapsto(\varrho/|z|\,r_1e^{i(\mu+t\theta_1)}, \varrho/|z|\,r_2e^{i(\mu+t\theta_2)})$, where $\theta_1=r_2^2(\varphi_1-\varphi_2)/(r_1^2+r_2^2)$ and $\theta_2=r_1^2(\varphi_2-\varphi_1)/(r_1^2+r_2^2)$, and $t\mapsto(\varrho/|z|\cos t, \varrho/|z|\sin t)$. The lengths of these curves are clearly less than $c|z-w|$, whence

$$|\Delta_i| \leqq C\|f\|_{B^s_\infty}|z-w|^{2s} \leqq C\|f\|_{B^s_\infty}|1-\langle z, w\rangle|^s, \quad i=2, 4. \quad \square$$

Define $L^s_q=\{f \text{ measurable}: (1-|z|^2)^{-s}f(z)\in L_q\}$. Then we have

Lemma 6. *Suppose that* $-1<\alpha<\infty$ *and* $s<1/2$. *Let* $f\in B^s_\infty$ *and define*

$$K(z, w) = \frac{\overline{f(z)}-\overline{f(w)}}{(1-\langle z, w\rangle)^{n+\alpha+1}}.$$

If $0<t<\alpha+1$, $0<s+t<\alpha+1$ *and* $1\leqq q\leqq\infty$, *then the mappings*

$$u(z)\mapsto\int|K(z, w)|\,u(w)\,d\mu(w)$$

and $u(z)\mapsto\int K(z, w)u(w)d\mu_\alpha(w)$ *map* L_q^{-s-t} *into* L_q^{-t}. *In particular* H_f *then maps* B_q^{-s-t} *into* L_q^{-t}.

Proof. For $q=1$ and $q=\infty$ this follows from Lemma 5 with $\gamma=t-1$ and $\gamma=\alpha-s-t$. The case $1<q<\infty$ follows by interpolation. $\square$

Taking $q=2$, $t=(\alpha+1)/2$ and $s=(\beta-\alpha)/2$ we obtain $H_f\in S_p^{\beta\alpha}$, provided $s<1/2$ and $s+t<\alpha+1$, i.e. $\beta-\alpha<1$ and $\beta-\alpha<\alpha+1$. The restriction $\beta-\alpha<\alpha+1$ can be avoided if we, as in [J], use the integral representation of $H_fP_{\alpha+1}$, given by the kernel K of Lemma 7.

Lemma 7. *Suppose that* $\alpha>-1$ *and* $s<1/2$. *Let* $f\in B^s_\infty$ *and define*

$$K(z, w) = \frac{\overline{f(z)}-\overline{f(w)}}{(1-\langle z, w\rangle)^{n+\alpha+2}}-(n+\alpha+1)^{-1}\frac{\overline{Rf(w)}}{(1-\langle z, w\rangle)^{n+\alpha+1}}.$$

If $-1<\gamma<\alpha$ *and* $-1<\gamma+s<\alpha+1$, *then*

$$\int K(z, w)|(1-|z|^2)^\gamma dm(z) \leqq c(1-|w|^2)^{\gamma-\alpha-1+s}\|f\|_{B^s_\infty}.$$

If $-1<\gamma<\alpha+1$ *and* $0<\gamma+s<\alpha+1$, *then* $\int |K(z, w)|(1-|w|^2)^\gamma dm(w)\leqq c(1-|z|^2)^{\gamma-\alpha-1+s}\|f\|_{B^s_\infty}$.

Consequently, if $0<t<\alpha+1$, $0<s+t$ *and* $1\leqq q\leqq\infty$, *then the mappings* $u(z)\mapsto\int |K(z, w)|u(w)d\mu_{\alpha+1}(w)$ *and* $u(z)\mapsto\int K(z, w)u(w)d\mu_{\alpha+1}(w)$ *map* L_q^{-s-t} *into* L_q^{-t}. *In particular,* H_f *then maps* B_q^{-s-t} *into* L_q^{-t}.

Proof. As in [J]. □

6. The case $1<p<\infty$ and compactness

This far we have proved Theorem 1 for $p\leqq1$ and $p=\infty$. To settle the case $1<p<\infty$ we use, as in [J], interpolation.

Suppose that $\alpha, \beta>-1$ and $1<p<\infty$. If $2n<1+\alpha-\beta$, then the cut-off causes no trouble. Otherwise let $\gamma=\beta+2n/p$. Then $-1<\gamma<\alpha+1$ and $\gamma-2n<\alpha+1-2n$. Defines the fractional integration I^s, for complex s, by

$$I^s g(z) = \sum_k \hat{g}(k)(|k|+1)^{-s}z^k,$$

and define $T_z(f)$ to be $H_f I^{nz}$. Then I^s is an isomorphism of A^γ onto $B_2^{-(1+\gamma-2\operatorname{Re}s)/2}$. As in § 5 the norm in $S_1(B_2^{-(1+\gamma-2n)/2}, L^2(d\mu_\alpha))$ of H_f can be shown to be bounded by a constant times the norm of f in $B_1^{(\gamma-2n-\alpha)/2}$. It follows that $\{T_z\}$ map $B_\infty^{(\gamma-\alpha)/2}$ into $S_\infty^{\gamma\alpha}$ when $\operatorname{Re} z=0$, and $B_1^{(\gamma-\alpha)/2}$ into $S_1^{\gamma\alpha}$ when $\operatorname{Re} z=1$. By the abstract Stein interpolation theorem [CJ], $T_{1/p}$ maps $B_p^{(\gamma-\alpha)/2}$ into $S_p^{\gamma\alpha}$. Therefore $H_f=T_{1/p}I^{n/p}\in S_p^{\beta\alpha}$ if $f\in B_p^{(n/p)+(\beta-\alpha)/2}$, and the proof of Theorem 1 is complete.

The proof of Theorem 2 is the same as for $n=1$.

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