Compactification of varieties

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Introduction

In [22] Morgan and Shalen construct a compactification of the space $X(\mathbf{C})$ of $\mathbf{C}$-rational points of an algebraic variety X defined over a subfield k of the field $\mathbf{C}$ of complex numbers. The motivation for this construction goes back to the work of Thurston on Teichmüller spaces ([33]). A special feature of the construction of Morgan and Shalen is the use of the valuation theory of the function field $k(X)$ of the k-variety X. Brumfiel ([5]; [6]) noted that, if k is contained in the field $\mathbf{R}$ of real numbers and if one wants to compactify the space $X(\mathbf{R})$ of $\mathbf{R}$-rational points, the real spectrum ([1]; [2]; [8]; [18]; [21]) of the coordinate ring $A=k[X]$ is closely related to the construction in [22]. The space of closed points in the real spectrum Sper (A) of A is a compactification of $X(\mathbf{R})$. Brumfiel realized that in the real setting this compactification can be mapped onto the one of Morgan and Shalen. Taking into account that Morgan and Shalen describe the additional points of their compactification in terms of valuations to me it seems that the connection with the real spectrum compactification stems from the fact that in ordered fields valuation rings appear naturally as convex subrings ([23]; [25]).

This observation is the starting point for the investigations in this paper. In chapter I a functor "valuation spectrum" from the category of rings to the category of spectral spaces ([14]) is defined. The valuation spectrum Spev (A) of a ring A is closely connected with the valuations of the residue fields of A at its prime ideals. The notion of valuations used here comprises not only Krull valuations but also archimedean valuations or absolute values ([13], Chapter 6; [34], Chapter 1). This notion of a valuation makes it possible to embed $X(\mathbf{R})$ into Spev (A) if A is the coordinate ring of an affine $\mathbf{R}$-variety X. In fact, Spev (A) contains an isomorphic copy of the real spectrum Sper (A) of A, and therefore $X(\mathbf{R})$ may be considered as a subspace of Spev (A). So Spev (A) contains the real spectrum compactification of $X(\mathbf{R})$. There is another compactification of $X(\mathbf{R})$ inside Spev (A). Its additional points are associated with certain Krull valuations of residue fields Quot (A/P)

of A into $\mathbf{R}^{>}$, the multiplicative group of positive real numbers. There is a natural map from the real spectrum compactification onto this one. Brumfiel's map from the real spectrum compactification to the Morgan—Shalen compactification factors through this new compactification.

The approach using the real spectrum of the coordinate ring of the variety X is, of course, limited to the real setting. To compactify the space of $\mathbf{C}$-rational points some additional ideas are required. Brumfiel suggests the use of some kind of complex spectrum of a ring ([5]; see also [26]). The basic idea is that $\mathbf{C}^n$ may be identified with the real affine space $\mathbf{R}^{2n}$. Under this identification the $\mathbf{C}$-rational points $X(\mathbf{C})\subset\mathbf{C}^n$ of an affine $\mathbf{C}$-variety are the $\mathbf{R}$-rational points of an affine $\mathbf{R}$-variety. This real algebraic set can be treated by methods of real algebraic geometry. Huber ([15]) has defined and systematically investigated the notion of a complex spectrum of a ring (independently of Brumfiel's suggestions). In chapter II of the present paper another notion of a complex spectrum is defined. The complex spectrum is a functor from the category of rings to the category of spectral spaces ([14]). The complex spectrum $\operatorname{Spec}\mathrm{x}(A)$ of the ring A is a subspace of $\operatorname{Spev}(A)$ by definition. The complex spectrum has many properties in common with the real spectrum. For example, the specializations of a point form a chain with respect to specialization. Consequently the space of closed points in a complex spectrum is compact. Returning to the original setting this can be used to compactify varieties: Let $k\subset\mathbf{C}$ be a subfield containing i. The inclusion defines a canonical absolute value $\varkappa\in\operatorname{Spec}\mathrm{x}(k)$. If A is the coordinate ring of the affine k-variety X then $\operatorname{Spec}\mathrm{x}_k(A)$ denotes the fibre of the functional map $\operatorname{Spec}\mathrm{x}(A)\to\operatorname{Spec}\mathrm{x}(k)$ over $\varkappa$. For $x\in X(\mathbf{C})$ the evaluation $x^*\colon A\to\mathbf{C}\colon a\to a(x)$ defines a point $e(x)\in\operatorname{Spec}\mathrm{x}_k(A)$. The map $e\colon X(\mathbf{C})\to\operatorname{Spec}\mathrm{x}_k(A)$ defined in this way maps $X(\mathbf{C})$ homeomorphically onto a dense open subspace of the space of closed points of $\operatorname{Spec}\mathrm{x}_k(A)$. In particular, the space of closed points of $\operatorname{Spec}\mathrm{x}_k(A)$ compactifies $X(\mathbf{C})$. This compactification, which is called the complex spectrum compactification, is completely analogous to the real spectrum compactification in the real setting.

In chapter III the complex spectrum is used to define yet another compactification of $X(\mathbf{C})$ inside $\operatorname{Spev}(A)$ which is very close to the Morgan—Shalen compactification. This new compactification is the image of the complex spectrum compactification (in a natural way) and can be mapped onto the Morgan—Shalen compactification. In fact, in some cases the map onto the Morgan—Shalen compactification is a homeomorphism. So in these cases the valuation spectrum allows an alternative construction of the Morgan—Shalen compactification. Therefore the valuation spectrum may help understanding the constructions of Morgan and Shalen.

In this paper several closely related compactifications of $X(\mathbf{C})$ are exhibited. Going back to the original motivation for all these compactifications (Thurston's work on Teichmüller spaces, see [33]) one should of course ask if these new constructions do in any way contribute to a better understanding of the compactifications of Teichmüller spaces. A discussion of these questions is deferred to future investigations.

Comparing the valuation spectrum and the complex spectrum with the real spectrum one may ask if these new spectra have geometric applications similar to the real spectrum (cf. [1]; [2]; [8]; [13]; [29]; [30]). This is another question which is not investigated in this paper. Conceivably an investigation of the geometry of these spectra may lead to new aspects causing slight modifications in the notions of the valuation spectrum and the complex spectrum. Huber's seminar notes ([16]) contain some hints in this direction.

I. The valuation spectrum

For an affine algebraic variety X defined over some subfield $k \subset \mathbf{C}$, Morgan and Shalen constructed a compactification of $X(\mathbf{C})$, the space of $\mathbf{C}$-valued points of X ([22]). In the compactification there are two essentially different kinds of points:

In the first place there are the $\mathbf{C}$-rational points. These may be considered as absolute values $K \to \mathbf{C}$ with K a residue field of the affine coordinate ring $A = k[X]$ of X at some prime ideal $p \subset A$.

The points used to compactify $X(\mathbf{C})$ can be obtained from Krull valuations $K^* \to \mathbf{R}^{>}$ (multiplicative group of positive real numbers) with K a residue field of A again.

These two kinds of points are put together in a rather complicated way ([22], p. 413—418) to produce the compactification.

In number theory there is a notion of valuations $K \to \mathbf{R}^{\geqq}$ (K a field, $\mathbf{R}^{\geqq}$ the nonnegative real numbers) comprising both absolute values and Krull valuations ([13], chapter 6; [34], chapter 1). The idea behind our construction of a compactification is that a common generalization of absolute values and Krull valuations should allow the definition of one space containing both $\mathbf{C}$-rational points and points coming from Krull valuations, avoiding the problem of putting two different spaces together. To construct this desired space, a spectrum will be associated with the ring $A = k[X]$ such that the compactification is contained in the space of closed points of the spectrum. The first step in the construction of this spectrum is the definition of the valuation spectrum of a ring.

For all facts about general valuation theory we refer to [3]; [11]; [27].

1. The spectral space Spev (A). One way to associate a valuation spectrum with a ring A is to consider all homomorphisms $A \to K$, where K is a field equipped with a valuation (in the sense of [13], chapter 6 or [34], chapter 1). Quasi-compactness of spectral spaces ([14]) obtained in this say can be proved sometimes by use of the compactness theorem of model theory ([7], p. 67; [24], p. 68). But the use of model theory makes it impossible to restrict our attention to the archimedean value group $\mathbf{R}^{>}$. For the property of being an archimedean totally ordered group cannot be axiomatized in first order logic ([7], p. 67). So we will have to work with not necessarily archimedean value groups. To have a definition which contains both Krull valuations and absolute values we use the multiplicative groups of positive elements in arbitrary real closed fields as value groups.

Let $\mathfrak{L}(\leqq)$ be the language of ordered fields, $Th(rc)$ the theory of real closed fields. If A is a ring then $\mathfrak{L}_A(\leqq)$ is obtained from $\mathfrak{L}(\leqq)$ by adding a new constant c_a for every $a \in A$. The extension $Th_A(rc)$ of $Th(rc)$ is given by the following axioms:

(a) For every $a \in A$: $c_a \geqq 0$

(b) $c_0 = 0$; $c_1 = c_{-1} = 1$; $(c_2 = 2) \vee (c_2 \leqq 1)$

(c) For all $a, b \in A$: $c_{ab} = c_a c_b$; $c_{a+b} \leqq c_a + c_b$; $c_a = 0 \Rightarrow c_b = c_{a+b}$

(d) For all $a, b \in A$: $(c_2 \leqq 1) \Rightarrow [(c_{a+b} \leqq c_a) \vee (c_{a+b} \leqq c_b)]$.

By $\operatorname{Spev}^\circ(A)$ we denote the class of all models of $Th_A(rc)$. We think of the elements of $\operatorname{Spev}^\circ(A)$ as mappings $f\colon A \to R^{\geqq}$ (with $R^{>}$ the multiplicative group of positive elements in the real closed field R, $R^{\geqq} = R^{>} \cup \{0\}$) having the following properties:

(b′) $f(0) = 0$; $f(1) = 1 = f(-1)$; $f(2) = 2 \vee f(2) \leqq 1$

(c′) $f(ab) = f(a)f(b)$; $f(a+b) \leqq f(a) + f(b)$; $f(a) = 0 \Rightarrow f(a+b) = f(b)$

(d′) $f(2) \leqq 1 \Rightarrow (f(a+b) \leqq f(a) \vee f(a+b) \leqq f(b))$.

To get a spectral space out of the class $\operatorname{Spev}^\circ(A)$ of models of $Th_A(rc)$ some of the models have to be identified via an equivalence relation. Then the equivalence classes are the points of the space. There are different identifications possible, and for different purposes different identifications may be appropriate. The following is one possible approach for the purposes of this paper. (From the seminar notes ([16]) of Huber I know that there are several other ways which would also work.)

Let Z be the set $\{0, 1, \infty\}$. With every $(f\colon A \to R^{\geqq}) \in \operatorname{Spev}^\circ(A)$ we associate a map $\alpha_f\colon A \times A \leftarrow Z$ by defining:

$$\alpha_f(a, b) = \begin{cases} 0 & \text{if } \ f(a) = 0 \quad \text{or} \quad f(b) = 0 \\ 1 & \text{if } \ 0 < f(a) \leqq f(b) \\ \infty & \text{if } \ 0 < f(b) < f(a). \end{cases}$$

The set

$$\operatorname{Spev}(A) = \{\alpha_f \in Z^{A \times A} \mid f \in \operatorname{Spev}^\circ(A)\}$$

is the *valuation spectrum* of A.

The meaning of the different elements of Z and of this definition will be explained shortly. But first it is useful to note the following:

Proposition 1. *If* $\alpha \in \mathrm{Spev}(A)$ *then the set* $\{a \in A \mid \alpha(a, 1) = 0\}$ *is a prime ideal of* A, *called the* support *of* α, *denoted by* $\mathrm{supp}(\alpha)$.

Proof. Let $f\colon A \to R^{\geqq}$ be a model of $Th_A(rc)$ with $\alpha = \alpha_f$. Then $\mathrm{supp}(\alpha) = f^{-1}(0)$. From the definition of $Th_A(rc)$ it is immediately clear that this is a prime ideal of A. □

Example 2. Let A be a ring, $P \subset A$ a prime ideal, $K = \mathrm{Quot}(A/P)$ the quotient field of A/P. Let $v\colon K^* \to \Gamma$ be a Krull valuation. If $\tilde{\Gamma}$ is the divisible hull of Γ then $\tilde{\Gamma}$ can be embedded into $R^{>}$, $R = \mathbf{R}((\tilde{\Gamma}))$, the power series field with coefficients from $\mathbf{R}$ and exponents from $\tilde{\Gamma}$. This is a real closed field ([25], p. 55, Satz 13). Thus v may be considered as a valuation $K \to R^{\geqq}$ (mapping 0 to 0). Altogether this gives a map

$$f\colon A \xrightarrow{\pi} K \xrightarrow{v} R^{\geqq}$$

which is a model of $Th_A(rc)$ as one checks. With f we associate $\alpha_f \in Z^{A \times A}$ according to the definition. Then we have

$\alpha_f(a, b) = 0$ if and only if $a \in P$ or $b \in P$,
$\alpha_f(a, b) = 1$ if and only if $a, b \notin P$ and $v(\pi(a)) \leqq v(\pi(b))$,
$\alpha_f(a, b) = \infty$ if and only if $a, b \notin P$, $v(\pi(a)) > v(\pi(b))$.

This example also helps explain the term "valuation spectrum". The connections with valuations are explained more fully in section 2.

So far the valuation spectrum is nothing but a set. This set is endowed with some structure via the following structures on Z:

(i) *Total order on* Z: $0 < 1 < \infty$.
(ii) *Constructible topology on* Z = *discrete topology on* Z. *Notation:* Z_c.
(iii) *Weak topology on* Z: ϕ, $\{\infty\}$, $\{1, \infty\}$, Z. *Notation:* Z_w.

If I is any set and $M \subset Z^I$ is a subset then the topologies induced on M by Z_c^I and Z_w^I are also called the constructible topology and the weak topology. In particular, $\mathrm{Spev}(A)$ has a constructible and a weak topology.

A subset $K \subset \mathrm{Spev}(A)$ is said to be *constructible* if K belongs to the Boolean algebra of subsets of $\mathrm{Spev}(A)$ generated by all sets $\{\alpha \in \mathrm{Spev}(A) \mid \alpha(a, b) = z\}$ with $a, b \in A$, $z \in Z$.

A key result about the constructible topology of $\mathrm{Spev}(A)$ is

Theorem 3. *If* $K \subset \mathrm{Spev}(A)$ *is constructible and* $K = \bigcup_{i \in I} K_i$ *is a cover by constructible subsets then there is a finite subcover.*

Proof. Every constructible subset $C \subset \mathrm{Spev}(A)$ can be described by a formula of the language $\mathfrak{L}_A(\leqq)$. That means, there is a formula ϕ of the language $\mathfrak{L}_A(\leqq)$ such that

$$\{(f\colon A \to R^{\leqq}) \in \mathrm{Spev}^{\circ}(A) \mid (f\colon A \to R^{\leqq}) \models \phi\}$$

is the class C^0 of those models of $Th_A(rc)$ for which $\alpha_f \in C$.

To prove this it suffices to consider constructible sets generating the Boolean algebra of constructible sets. For example, let $C = \{\alpha \in \mathrm{Spev}(A) | \alpha(a, b) = 1\}$ and let C^0 be the class of $(f\colon A \to R^{\leqq}) \in \mathrm{Spev}^{\circ}(A)$ with $\alpha_f \in C$. If ϕ is the following formula:

$$(c_a \neq 0) \wedge (c_b \neq 0) \wedge (c_a \leqq c_b),$$

then C^0 is the class of models of $Th_A(rc) \cup \{\phi\}$.

Now suppose that K and K_i, $i \in I$, are constructible subsets of $\mathrm{Spev}(A)$, $K = \bigcup_{i \in I} K_i$. We choose formulas ϕ, ϕ_i, $i \in I$, defining these constructible subsets. The constructible set $K \setminus K_i$ is defined by the formula $\phi \wedge \neg \phi_i$. Since $\bigcap_{i \in I} (K \setminus K_i) = \phi$, the set $\{\phi \wedge \neg \phi_i | i \in I\}$ of formulas does not have a model, i.e., it is inconsistent (completeness theorem of model theory — see [7], chapter 2.1; [24], chapter 1.5). By the compactness theorem of model theory ([7], loc. cit.; [24], loc. cit.), there is a finite subset JCI such that $\{\wedge \phi - \phi_i|\} i \in J\}$ is inconsistent. By the completeness Theorem this means that there is no model for the $(r_c) \bigcap \{\phi\Phi - \phi_i|_{i \ni J}\}$, i. e., $\bigcap(_{i \ni J}(K/K_i) = \phi$. Thus $K = \bigcap_{i \ni J} K_i$ as claimed. □

This result has the following immediate consequences:

Corollary 4. Spev (A) *with the constructible topology is a Boolean space.*

Corollary 5. *A subset* $C \subset \mathrm{Spev}(A)$ *is constructible if and only if C is open and closed in the constructible topology.*

Corollary 6. *Let* Z *carry an arbitrary topology and consider* Spev (A) *as a subspace of* $Z^{A \times A}$ *with the product topology. Then a subset* $K \subset \mathrm{Spev}(B)$ *is open and constructible if and only if it is open and quasi-compact. The open quasi-compact subsets are closed under finite intersections. They form a basis of the topology of* Spev (B).

Using [14], proposition 7 we conclude:

Corollary 7. *Let* Z *carry an arbitrary* T_0*-topology (e.g., the weak topology). Then* Spev (A) *with the restriction of the product topology of* $Z^{A \times A}$ *is a spectral space in the sense of Hochster* ([14]).

From now on we will consider Spev (A) only with the constructible topology or the weak topology. If nothing is said about the topology then we always mean the weak topology.

There are many different T_0-topologies on the set Z. It may happen that in some context one of these topologies should be used and that for other purposes another topology is more appropriate. It turns out that the weak topology is well suited for the purposes of this paper. However, this choice of topology is not only justified by its success in this paper but also by our geometric intuition. This will be discussed in some more detail at the end of section 2.

Concerning the choice of topology, there is a similar situation in semi-algebraic geometry. In the abstract setting the real spectrum of a ring ([8]; [2], Chapter VII) is used as the basic topological space ([29]; [30]). The real spectrum of the ring A can be considered as a subspace of $\{0, 1\}^A$. On $\{0, 1\}$ there are three T_0-topologies, namely the discrete topology, the *weak topology* $\{\phi, \{1\}, \{0, 1\}\}$ and the *inverse topology* $\{\phi, \{0\}, \{0, 1\}\}$. The weak topology is very well suited for working with locally semi-algebraic spaces ([29]; [30]; [10]). Recently it turned out that the inverse topology should be used in the abstract discussion of weakly semi-algebraic spaces ([19]; [31]; [32]).

So far the spectral space $\mathrm{Spev}(A)$ has been associated with the ring A. This construction can be extended to give a functor from the category of rings to the category of spectral spaces:

Proposition 8. *Let $\varphi: A \to B$ be a homomorphism of rings. φ induces a map $Z^{\varphi \times \varphi}: Z^{B \times B} \to Z^{A \times A}$ which restricts to $\varphi^*: \mathrm{Spev}(B) \to \mathrm{Spev}(A)$. φ^* is a morphism of spectral spaces.*

Proof. The definition of $Z^{\varphi \times \varphi}$ is obvious. Let $(f: B \to R^{\geqq})$ be a model of $Th_B(rc)$. Then $(f\varphi: A \to R^{\geqq}) \in \mathrm{Spev}^{\circ}(A)$ and $Z^{\varphi \times \varphi}(\alpha_f) = \alpha_{f\varphi}$. This defines the map ϕ^*: Spev (B) — Spev (A). Since $Z^{\varphi \cdot \varphi}$ is clearly continuous in both the constructible topology and the weak topology the same is true for φ^*. □

2. *Connections with valuations.* We saw in example 2 that at least some points of the valuation spectrum $\mathrm{Spev}(A)$ of the ring A arise from Krull valuations of the residue fields of A. The next example shows that absolute values also define points of the valuation spectrum:

Example 9. Let R be a real closed field with algebraic closure C. Then there is the absolute value

$$|.|: C \to R^{\geqq}: a+ib \to \sqrt{a^2+b^2}.$$

If $\varphi: A \to C$ is a ring homomorphism then

$$f: A \to R^{\geqq}: a \to |\varphi(a)|$$

belongs to $\mathrm{Spev}^{\circ}(A)$ and, hence, defines a point of the valuation spectrum. In this case we have $f(2)=2$.

We will see that the examples exhibited in example 2 and example 9 show fairly typical points of valuation spectra. To start with we show that every point of Spev (A) is of the form $\varphi^*(\alpha)$ with $\varphi\colon A\to K$ a homomorphism into a field K, $\alpha\in\operatorname{Spev}(K)$.

Proposition 10. *Let A be a ring, $P\subset A$ a prime ideal, $i\colon A\to A_P$ the canonical homomorphism. Then $i^*\colon \operatorname{Spev}(A_P)\to\operatorname{Spev}(A)$ is a homeomorphism onto*

$$\{\alpha\in\operatorname{Spev}(A)|\operatorname{supp}(\alpha)\subset P\}.$$

Proof. If $\beta\in\operatorname{Spev}(A_P)$ then

$$\operatorname{supp}(i^*(\beta))=i^{-1}(\operatorname{supp}(\beta))\subset i^{-1}(P_P)=P.$$

Pick $\alpha\in\operatorname{Spev}(A)$ with $\operatorname{supp}(\alpha)\subset P$ and let $(f\colon A\to R^{\geqq})\in\operatorname{Spev}^\circ(A)$ be such that $\alpha=\alpha_f$. We define $g\colon A_P\to R^{\geqq}$ by $g\left(\frac{a}{b}\right)=\frac{f(a)}{f(b)}$. This is well-defined since $\operatorname{supp}(\alpha)\subset P$. An easy computation shows that $g\in\operatorname{Spev}^\circ(A_P)$. Moreover, $\alpha=i^*(\alpha_g)$, and the image of i^* is as claimed.

i^* is injective: Let $i^*(\beta)=\alpha=i^*(\gamma)$ and pick representatives $g,h\in\operatorname{Spev}^\circ(A_P)$ of β,γ. Then $\beta\left(\frac{a}{b},\frac{c}{d}\right)=1$ is equivalent to $0<g\left(\frac{a}{b}\right)\leqq g\left(\frac{c}{d}\right)$. Because of $\frac{b}{1},\frac{d}{1}\in A_P^*$ this is the same as $0<g\left(\frac{ad}{1}\right)=gi(ad)\leqq g\left(\frac{bc}{1}\right)=gi(bc)$. Since $\alpha=\alpha_{gi}$ this is equivalent to $\alpha(ad,bc)=1$. Similarly, this is also equivalent to $\gamma\left(\frac{a}{b},\frac{c}{d}\right)=1$. The same computations with 0 and ∞ in place of 1 show that $\beta=\gamma$.

Since i^* is a morphism of spectral spaces it remains to prove that $i^*\colon\operatorname{Spev}(A_P)\to im(i^*)$ is open. If suffices to show that, given $\frac{a}{b},\frac{c}{d}\in A_P$, $U\subset Z_w$ open, the image of $V=\left\{\beta\in\operatorname{Spev}(A_p)\middle|\beta\left(\frac{a}{b},\frac{c}{d}\right)\in U\right\}$ is open in $im(i^*)$. It is clear that $\beta\left(\frac{a}{b},\frac{c}{d}\right)\in U$ if and only if $i^*(\beta)(ad,bc)=\beta\left(\frac{ad}{1},\frac{bc}{1}\right)\in U$. Thus,

$$i^*(V)=im(i^*)\cap\{\alpha\in\operatorname{Spev}(A)\,|\,\alpha(ad,bc)\in U\}$$

is open. □

Proposition 11. *Let $P\subset A$ be a prime ideal, $\pi\colon A\to A/P$ the canonical homomorphism. Then $\pi^*\colon\operatorname{Spev}(A/P)\to\operatorname{Spev}(A)$ is a homeomorphism onto the set $\{\alpha\in\operatorname{Spev}(A)|P\subset\operatorname{supp}(\alpha)\}$.*

Proof. It is obvious that $im(\pi^*)\subset\{\alpha\in\operatorname{Spev}(A)|P\subset\operatorname{supp}(\alpha)\}$. On the other hand, pick α in this set and let $(f\colon A\to R^{\geqq})\in\operatorname{Spev}^\circ(A)$ be a representative of α. Then $P\subset\operatorname{supp}(\alpha)=f^{-1}(0)$, and the definition of $\operatorname{Spev}^\circ(A)$ shows that f factors through $\pi\colon A\to A/P$ to give $g\colon A/P\to R^{\geqq}$. One readily checks that $g\in\operatorname{Spev}^\circ(A/P)$. Since $\alpha=\pi^*(\alpha_g)$ the image is as claimed. Surjectivity of π implies injectivity of π^*. So, π^* is a bijection onto the image.

Since π^* is continuous it remains to prove that $\pi^*\colon\operatorname{Spev}(A/P)\to im(\pi^*)$ is open. So, let $\pi(a),\pi(b)\in A/P$ and pick some open subset $U\subset Z_w$. Define $V=\{\beta\in\operatorname{Spev}(A/P)|\beta(\pi(a),\pi(b))\in U\}$. Since $\pi^*(\beta)(a,b)=\beta(\pi(a),\pi(b))$ it follows that

$$\pi^*(V) = im(\pi^*)\cap\{\alpha\in\operatorname{Spev}(A)\,|\,\alpha(a,b)\in U\},$$

proving that $\pi^*(V)$ is open in $im(\pi^*)$. □

As a consequence of proposition 10 and proposition 11 we record

Corollary 12. *Let* $P\subset A$ *be a prime ideal,* $A(P)$ *the residue field of* A *at* P, $\varphi\colon A\to A(P)$ *the canonical homomorphism. Then* φ^* *maps* $\operatorname{Spev}(A(P))$ *homeomorphically onto* $\{\alpha\in\operatorname{Spev}(A)|\operatorname{supp}(\alpha)=P\}$.

As consequence of these results we may restrict our attention to fields when studying individual elements of valuation spectra.

Let K be a field, $\alpha\in\operatorname{Spev}(K)$ with representative $(f\colon K\to R^{\geqq})\in\operatorname{Spev}^\circ(K)$. Then $f(K^*)\subset R^{>}$ is a totally ordered subgroup. We will see now that this totally ordered group depends solely on α, not on the representative chosen: To see this let

$$U = \{x\in K^*\,|\,\alpha(x,1) = 1 \;\;\&\;\; \alpha(1,x) = 1\}.$$

It is clear that $U=f^{-1}(1)$. Therefore $f(K^*)$ and K^*/U are isomorphic as groups. Now we define $U<xU$ in K^*/U if and only if $\alpha(x,1)=\infty$. With this order K^*/U is a totally ordered group and the above isomorphism is an isomorphism of totally ordered groups. Since the totally ordered group has been defined solely by referring to α we see that $f(K^*)$, as a totally ordered group, depends only on α and not on the representative f. This group is denoted by Γ_α. We write $\overline{\Gamma_\alpha}$ for $f(K^*)\cup\{0\}$. The map $v_\alpha\colon K\to\overline{\Gamma_\alpha}$ is the same as f with the range restricted to $f(K)$.

Theorem 13. *Let K be a field,* $\alpha\in\operatorname{Spev}(K)$, $v_\alpha\colon K\to\overline{\Gamma_\alpha}$ *as above. Then the following statements are equivalent:*

(a) v_α *is a Krull valuation.*
(b) $\alpha(2,1)\leqq 1$.
(c) $V_\alpha=\{x\in K|\alpha(x,1)\leqq 1\}$ *is a valuation ring.*

Proof. (a)⇒(c): Note that $\alpha(x, 1) \leqq 1$ if and only if $v_\alpha(x) \leqq 1$. — (c)⇒(b): Since $2 \in V_\alpha$, the definition of V_α shows that $\alpha(2, 1) \leqq 1$. — (b)⇒(a): Let $(f\colon K \to R^{\geqq}) \in \operatorname{Spev}^\circ(A)$ be a representative of α. Then $f(2) \leqq 1$ and the definition of $\operatorname{Spev}^\circ(K)$ shows that $f(x+y) \leqq f(x)$ or $f(x+y) \leqq f(y)$ for all $x, y \in K$. The same is true for v_α, and we see that v_α is a Krull valuation. □

As a consequence of corollary 12 and theorem 13 we note:

Corollary 14. *Let A be a ring, $\alpha \in \operatorname{Spev}(A)$, $P = \operatorname{supp}(\alpha)$, $\varphi\colon A \to A(P)$ the canonical homomorphism into the residue field of A at P. If $\alpha(2, 1) \leqq 1$ then α is induced by a unique Krull valuation of $A(P)$.*

Concerning $\alpha \in \operatorname{Spev}(A)$ with $\alpha(2, 1) = \infty$ we have the following result:

Theorem 15. *Let K be a field, $\alpha \in \operatorname{Spev}(K)$ with $\alpha(2, 1) = \infty$. If Γ_α is archimedean then $v_\alpha\colon K \to \overline{\Gamma_\alpha}$ is an archimedean valuation of K (in the sense of* [21], *p. 9).*

Proof. Since Γ_α is archimedean Γ_α may be considered as subgroup of $\mathbf{R}^{>}$ with $v_\alpha(2) = 2$ (Hölder's theorem — see [12], p. 73, Satz 1 or [25], p. 8, Satz 4). From the definition of the valuation spectrum (section 1) it follows immediately that v_α is a valuation in the sense of [34], p. 1 (with $C = 2$). It is clear that this is an archimedean valuation. □

It remains an open problem if every $\alpha \in \operatorname{Spev}(A)$ with $\alpha(2, 1) = \infty$ is induced by a homomorphism $A \to C$ into an algebraically closed field and an absolute value of C (as in example 9). In any event, such an element of the valuation spectrum can always be used to define a Krull valuation:

Proposition 16. *Let K be a field, $\alpha \in \operatorname{Spev}(K)$ with $\alpha(2, 1) = \infty$. Let $\Gamma'_\alpha \subset \Gamma_\alpha$ be the convex subgroup generated by $v_\alpha(2)$. Then $K \xrightarrow{v_\alpha} \overline{\Gamma_\alpha} \to \overline{\Gamma_\alpha / \Gamma'_\alpha}$ is a Krull valuation.*

The easy proof is omitted.

Based on the results of this section we will call the elements of Spev (A) *valuations* of A. Valuations α are called *Krull valuations* or *absolute values* according as $\alpha(2, 1) \leqq 1$ or $\alpha(2, 1) = \infty$. The correspondin map $A \to A(\operatorname{supp}(\alpha)) \xrightarrow{v_\alpha} \overline{\Gamma_\alpha}$ is also denoted by v_α and is called a *valuation.* Γ_α is the value group.

We continue to consider a field K. The set $\{\alpha \in \operatorname{Spev}(K) | \alpha(2, 1) \leqq 1\}$ is canonically bijective to the set of equivalence classes of Krull valuations and also to the set of valuation rings of K. The abstract Riemann surface of K is the set of valuation rings with the topology having all sets

$$\{V | a_1, \ldots, a_n \in V\}, \quad a_1, \ldots, a_n \in K$$

as a basis of open subsets ([35], chapter VI, § 17). With a valuation ring $V \subset K$ we associate the following element $\alpha_V \in \mathrm{Spev}(K)$:

$$\alpha_V(a, b) = \begin{cases} 0 & \text{if} \quad a = 0 \quad \text{or} \quad b = 0 \\ 1 & \text{if} \quad ab \neq 0, \quad \dfrac{a}{b} \in V \\ \infty & \text{if} \quad ab \neq 0, \quad \dfrac{a}{b} \notin V. \end{cases}$$

In this way we consider the abstract Riemann surface of K as a subset of $\mathrm{Spev}(K)$ and of $Z^{K \times K}$. To get the above topology of the abstract Riemann surface one can use the following topology of Z: $\{\phi, \{1\}, \{1, \infty\}, Z\}$. So the weak topology we are using does not restrict to the usual topology on the abstract Riemann surface.

Both in Huber's seminar notes ([16]) and in de la Puente Muñoz' dissertation ([26]) the Krull valuations of a ring are considered (by de la Puente Muñoz under the name abstract Riemann surface of a ring). They also consider topologies different from the weak topology.

The reason for using the weak topology in this paper is the following: Let A be a ring and let $K \subset \mathrm{Spev}(A)$ be the set of Krull valuations. Choosing a topology on Z one makes a decision about which of the following subsets of K are open:

$$\{\alpha \in K \mid v_\alpha(a) \leqq 1\},$$
$$\{\alpha \in K \mid v_\alpha(a) \geqq 1\},$$
$$\{\alpha \in K \mid v_\alpha(a) > 1\},$$

where $a \in A$. To start with, there is no obvious reason to decide that certain of these sets should be open and others should be closed. So the choice of topology seems to be somewhat arbitrary. However, in our setting $\mathrm{Spev}(A)$ also contains absolute values. And for these it is easy to decide which subsets ought to be open. For example, let A be the affine coordinate ring of an affine $\mathbf{C}$-variety X. Then every $\mathbf{C}$-rational point x defines an absolute value by composing the evaluation homomorphism x^*: $A \to \mathbf{C}$: $a \to a(x)$ with the canonical absolute value $\mathbf{C} \to \mathbf{R}^{\geqq}$. In this way $X(\mathbf{C})$ may be considered as a subset of $\mathrm{Spev}(A)$. We even want to consider $V(\mathbf{C})$ as a subspace of $\mathrm{Spev}(A)$. To get this it is necessary to consider the sets

$$\{\alpha \in \mathrm{Spev}(A) \mid \alpha(a, 1) = \infty\},$$
$$\{\alpha \in \mathrm{Spev}(A) \mid \alpha(a, 1) \neq 0\}$$

as open subsets of $\mathrm{Spev}(A)$. For, their restrictions to $V(\mathbf{C})$ are the sets

$$\{x \in V(\mathbf{C}) \,\big|\, |a(x)| > 1\},$$
$$\{x \in V(\mathbf{C}) \,\big|\, |a(x)| > 0\},$$

and these are open subsets of $V(\mathbf{C})$. So, if one wants to consider $V(\mathbf{C})$ (with the strong topology) as a subset of Spev (A) then the weak topology has to be used.

3. Connections with other spectra. If A is a ring we have seen that the Krull valuations of the residue fields of A at its prime ideals yield elements of Spev (A). The set of Krull valuations of A,

$$\{\alpha \in \operatorname{Spev}(A) \mid \alpha(2, 1) \leqq 1\},$$

is a closed constructible subspace of Spev (A). This subspace is also a spectral space, called the Krull valuation spectrum.

In proposition 1 we defined the support of a valuation of a ring.

Proposition 17. *The support map* supp: Spev $(A) \to$ Spec (A) *is a morphism of spectral spaces.*

Proof. The sets $D(a) = \{P \in \operatorname{Spec}(A) | a \notin P\}$, $a \in A$, form a basis of the Zariski topology of Spec (A) and at the same time generate the Boolean algebra of constructible subsets of Spec (A). Therefore it suffices to show that $\operatorname{supp}^{-1}(D(a))$ is open and constructible for all $a \in A$. Because of

$$\operatorname{supp}^{-1}(D(a)) = \{\alpha \in \operatorname{Spev}(B) \mid \alpha(a, 1) \neq 0\}$$

this is clear. □

We may also consider Spec (A) is a closed subspace of Spev (A):

Proposition 18. *If* $P \in \operatorname{Spec}(A)$ *then* $\tau(P) \in Z^{A \times A}$ *is defined by*

$$\tau(P)(a, b) = \begin{cases} 0 & \text{if } a \in P \text{ or } b \in P \\ 1 & \text{if } ab \notin P. \end{cases}$$

This defines a map τ: Spec $(A) \to$ Spev (A). τ *is an isomorphism onto the closed subspace*

$$\bigcap_{a, b \in A} \{\alpha \mid \alpha(a, b) \leqq 1\} \subset \operatorname{Spev}(A).$$

Proof. For $P \in \operatorname{Spec}(A)$ define $f(P)\colon A \to \mathbf{R}^{\geqq}$ by $f(P)(a) = 0$ if $a \in P$, $f(P)(a) = 1$ if $a \notin P$. Then $f(P) \in \operatorname{Spev}^{\circ}(A)$ and $\tau(P) = \alpha_{f(P)}$. Thus τ is a map Spec $(A) \to$ Spev (A). It is clear that the image of τ is as claimed. Because of $\operatorname{supp} \circ \tau = \mathrm{id}$ it follows that τ is injective.

It remains to show that τ is both continuous and open. First pick $a, b \in A$ and set

$$U_\infty = \{\alpha \in \operatorname{Spev}(A) \mid \alpha(a, b) = \infty\},$$

$$U_1 = \{\alpha \in \operatorname{Spev}(A) \mid \alpha(a, b) \neq 0\}.$$

Then $\tau^{-1}(U_\infty) = \phi$, $\tau^{-1}(U_1) = \{P \in \operatorname{Spec}(A) | ab \notin P\}$. This proves continuity. Now we determine the image of $D(a)=\{P\in \mathrm{Spec}(A) | a\notin P\}$ under τ:

$$\tau(D(a)) = im(\tau)\cap\{\alpha\in \mathrm{Spev}(A) \mid \alpha(a, 1) \neq 0\},$$

and τ is open. □

The image of τ is the set of *trivial valuations* of A, i.e., the valuations α with $\Gamma_\alpha=\{1\}$.

Let $\mathrm{Sper}(A)$ be the real spectrum of A (see [1]; [2]; [8]; [18]; [21]). For $\mu\in \mathrm{Sper}(A)$ let ϱ_μ: $A\to\varrho(\mu)$ be the canonical homomorphism into the real closure of the residue field $A(\mathrm{supp}(\mu))$ of A at $\mathrm{supp}(\mu)=\mu\cap -\mu$ with respect to the total order defined on $A(\mathrm{supp}(\mu))$ by μ. On $\varrho(\mu)$ there is the canonical absolute value $|.|$: $\varrho(\mu)\to\varrho(\mu)^{\geqq}$: $x\to\max\{x, -x\}$. Composing this with ϱ_μ we obtain

$$f(\mu)\colon A \xrightarrow{\varrho_\mu} \varrho(\mu) \xrightarrow{|\cdot|} \varrho(\mu)^{\geqq}$$

which belongs to $\mathrm{Spev}^\circ(A)$. Thus $\omega(\mu)=\alpha_{f(\mu)}$ defines a map ω: $\mathrm{Sper}(A)\to \mathrm{Spev}(A)$.

Proposition 19. *ω maps* $\mathrm{Sper}(A)$ *isomorphically onto a pro-constructible subspace of* $\mathrm{Spev}(A)$.

Proof. Injectivity: Suppose that $\lambda\neq\mu$, $\lambda, \mu\in \mathrm{Sper}(A)$. *Case 1.* $\mathrm{supp}(\lambda)\neq \mathrm{supp}(\mu)$. We may assume that there is some $a\in \mathrm{supp}(\lambda)\setminus \mathrm{supp}(\mu)$. Then $\omega(\lambda)(a, 1)=0$, $\omega(\mu)(a, 1)\neq 0$, and this proves $\omega(\lambda)\neq\omega(\mu)$. *Case 2.* $\mathrm{supp}(\lambda)= \mathrm{supp}(\mu)$. Let $K=A(\mathrm{supp}(\lambda))$. With K_λ and K_μ we denote K with the total order induced by λ, resp. μ. Then there is some $x\in K$ such that $0<x$ in K_λ, $x<0$ in K_μ. Replacing x by x^{-1} if necessary we may assume that $0<x<1$ in K_λ. Let $x=\dfrac{\bar{y}}{\bar{z}}$ with $y, z\in A$, $\bar{y}, \bar{z}\in A/\mathrm{supp}(\lambda)$ their canonical images. Assume by way of contradiction that $\omega(\lambda)=\omega(\mu)$. From $0<\bar{y}<\bar{z}$ we see that $\omega(\alpha)(y, z)=1=\omega(\beta)(y, z)$. This implies $-1<x<0$ in K_μ. Then $1<1+x$ in K_λ and $0<1+x<1$ in K_μ, and $\omega(\alpha)(y+z, z)=\infty$, $\omega(\beta)(y+z, z)=1$, a contradiction. This proves that ω is injective.

To prove that ω is a morphism of spectral spaces, we pick $a, b\in A$ and determine the inverse images of

$$U_\infty = \{\alpha\in \mathrm{Spev}(A) \mid \alpha(a, b) = \infty\}$$

$$U_1 = \{\alpha\in \mathrm{Spev}(A) \mid \alpha(a, b) \neq 0\}$$

under ω:

$$\omega^{-1}(U_\infty) = \{\mu\in \mathrm{Sper}(A) \mid 0 < \varrho_\mu(b)^2 < \varrho_\mu(a)^2\},$$

$$\omega^{-1}(U_1) = \{\mu\in \mathrm{Sper}(A) \mid \varrho_\mu(a)\varrho_\mu(b) \neq 0\}.$$

These are both open and constructible, showing that ω is a morphism of spectral spaces. In particular, $im(\omega)\subset \mathrm{Spev}(A)$ is a pro-constructible subspace.

To show that ω is open onto the image we note that, for $a\in A$,

$$\omega(\{\mu\in \mathrm{Sper}(A)\,|\,\varrho_\mu(a)>0\}) = im(\omega)\cap V$$

with

$$V=\{\alpha\in \mathrm{Spev}(A)\,|\,\alpha(a+1,a)=\infty \;\;\&\;\; \alpha(a+a^2,a)=\infty\}. \quad \square$$

4. Specializations in the valuation spectrum. If $\alpha,\beta\in \mathrm{Spev}(A)$ we say that β is a *specialization* of α and α is a *generalization* of β if $\beta\in\overline{\{\alpha\}}$. In real spectra the specializations of a point have the remarkable property of forming a chain with respect to specialization, i.e., if $\beta,\gamma\in\overline{\{\alpha\}}$ then $\beta\in\overline{\{\gamma\}}$ of $\gamma\in\overline{\{\beta\}}$ (cf. [8], proposition 2.1). This statement is false in the Zariski spectrum of A. Since $\mathrm{Spec}(A)$ may be considered as subspace of $\mathrm{Spev}(A)$ (proposition 18) it is also false in $\mathrm{Spev}(A)$. To exclude this trivial case we will discuss mostly $\mathrm{Spev}(A)\setminus\mathrm{Spec}(A)$ in this section. In this space the specializations of a point have some rather special properties, although they do not necessarily form a chain. First we record

Lemma 20. *Let* $\alpha,\beta\in \mathrm{Spev}(A)$, $\beta\in\overline{\{\alpha\}}$. *Then* $\mathrm{supp}(\beta)\supset\mathrm{supp}(\alpha)$.

Proof. supp is continuous (proposition 17). $\square$

Before continuing with the investigation of specializations we note that, if β is a specialization of α and if we are only interested in the valuations α and β, proposition 10 in connection with lemma 20 allows us to assume that $A=A_{\mathrm{supp}(\beta)}$. Whenever this leads to a simplification of arguments this assumption will be made without much comment.

To gain a good understanding of specializations in $\mathrm{Spev}(A)$ it is important to establish connections between the value groups Γ_α and Γ_β if $\beta\in\overline{\{\alpha\}}$: By lemma 20, $\mathrm{supp}(\alpha)\subset\mathrm{supp}(\beta)$. Suppose that $A=A_{\mathrm{supp}(\beta)}$ and let $i\colon A\to A_{\mathrm{supp}(\alpha)}$ be the canonical homomorphism. Let $\Gamma_{\beta\alpha}\subset\Gamma_\alpha$ be the image of the group homomorphism

$$v_\alpha i\colon\; A^*\to A^*_{\mathrm{supp}(\alpha)}\to\Gamma_\alpha .$$

Lemma 21. $\ker(v_\alpha i)\subset\ker(v_\beta)$.

Proof. Pick $a\in A^*$ such that $v_\alpha i(a)=1$. This means that $v_\alpha(a)=1=v_\alpha(1)$, i.e., $\alpha(a,1)=1$ and $\alpha(1,a)=1$. Since β is a specialization of α either $\beta(a,1)=0$ or: $\beta(a,1)=1$ and $\beta(1,a)=1$. Since $\beta(a,1)=0$ contradicts $a\in A^*$ we have $\beta(a,1)=1$ and $\beta(1,a)=1$, i.e., $v_\beta(a)=1$. $\square$

As a consequence of lemma 21 there is a canonical homomorphism $\pi_{\beta\alpha}\colon\Gamma_{\beta\alpha}\to\Gamma_\beta$. Both groups are totally ordered ($\Gamma_{\beta\alpha}$ as a subgroup of Γ_α).

Lemma 22. $\pi_{\beta\alpha}$ *is order compatible.*

Proof. Pick $a \in A^*$ and suppose that $v_\alpha i(a) \geqq 1$ in $\Gamma_{\beta\alpha}$. Then either $v_\alpha(a) > 1$ or $v_\alpha(a) = 1$. This means either $\alpha(a, 1) = \infty$ or: $\alpha(a, 1) = 1$ and $\alpha(1, a) = 1$. Assume that $v_\beta(a) < 1$. Then $\beta(1, a) = \infty$ and, since $\beta \in \overline{\{\alpha\}}$, $\alpha(1, a) = \infty$ follows, a contradiction.

Corollary 23. *Let* $\alpha \in \operatorname{Spev}(A)$, $P \subset A$ *a prime ideal with* $\operatorname{supp}(\alpha) \subset P$. *Then* $\overline{\{\alpha\}} \cap \{\beta \in \operatorname{Spev}(A) | \operatorname{supp}(\beta) = P\}$ *is a chain with respect to specialization.*

Proof. We may assume that $A = A_P$ (proposition 10, lemma 20). Pick $\beta, \gamma \in \overline{\{\alpha\}}$ with $\operatorname{supp}(\beta) = P = \operatorname{supp}(\gamma)$. Then the subgroups $\Gamma_{\beta\alpha}$, $\Gamma_{\gamma\alpha}$ of Γ_α agree. Let $\Delta_{\beta\alpha} = \ker(\pi_{\beta\alpha})$, $\Delta_{\gamma\alpha} = \ker(\pi_{\gamma\alpha})$. These are convex subgroups of $\Gamma_{\beta\alpha} = \Gamma_{\gamma\alpha}$ (lemma 22). Hence $\Delta_{\beta\alpha} \subset \Delta_{\gamma\alpha}$ or $\Delta_{\gamma\alpha} \subset \Delta_{\beta\alpha}$. In the first case $\gamma \in \overline{\{\beta\}}$, in the second case $\beta \in \overline{\{\gamma\}}$. □

In particular we have

Corollary 24. *If K is a field and* $\alpha \in \operatorname{Spev}(K)$ *then* $\overline{\{\alpha\}}$ *is a chain.*

We continue to discuss $\alpha, \beta \in \operatorname{Spev}(A)$, β a specialization of α. We assume that $A = A_{\operatorname{supp}(\beta)}$. Moreover we choose representatives $(f\colon A \to R^{\geqq}), (g\colon A \to S^{\geqq}) \in \operatorname{Spev}^\circ(A)$ of α and β.

Lemma 25. *If* $a, b \in A$ *are such that* $f(a) \leqq f(b)$ *and* $g(b) \neq 0$ *then* $g(a) \leqq g(b)$.

Proof. Assuming that $g(a) > g(b)$ we have $\beta(a, b) = \infty$. This implies $\alpha(a, b) = \infty$ since $\beta \in \overline{\{\alpha\}}$ and hence $f(a) > f(b)$, a contradiction. □

Proposition 26. *If* Γ_α, Γ_β *are both nontrivial then* $f(a) \leqq f(b)$ *implies* $g(a) \leqq g(b)$ *for all* $a, b \in A$.

Proof. If $g(b) \neq 0$ then the claim follows from lemma 25. We assume by way of contradiction that $g(b) = 0$ and $g(a) > g(b)$.

First suppose that $f(2) \leqq 1$, i.e., $\alpha(2, 1) \leqq 1$. This implies $\beta(2, 1) \leqq 1$, i.e., $g(2) \leqq 1$. Since Γ_β is nontrivial there is some $c \in A$ such that $0 < g(c) < g(a)$ (observe that $A = A_{\operatorname{supp}(\beta)}$). This implies $0 < f(c) < f(a) \leqq f(b)$ (since $\beta \in \overline{\{\alpha\}}$). Both v_α and v_β are induced by Krull valuations of the residue fields of A at $\operatorname{supp}(\alpha)$ and $\operatorname{supp}(\beta)$ (corollary 12, theorem 13). Therefore $f(a) \leqq f(b) = f(b+c)$ and $0 < g(c) = g(b+c)$. Lemma 25 implies that $g(a) \leqq g(b+c) = g(c)$, a contradiction.

From now on we assume that $f(2) = 2$, i.e., $\alpha(2, 1) = \infty$. We will need

Lemma 27. *Let* $f \in \operatorname{Spev}^\circ(A)$ *be such that* $f(2) = 2$. *Then* $f(n) = n$ *for all* $n \in \mathbf{N}$.

Proof. First note that $f(2^r) = f(2)^r$ for all $r \in \mathbf{N}$ (since f is multiplicative). Trivially we have $f(n) \leqq n$ for all $n \in \mathbf{N}$. Choose $r \in \mathbf{N}$ such that $n < 2^r$ and write $2^r = n + p$. Then $2^r = f(2^r) \leqq f(n) + f(p) \leqq n + p = 2^r$. This is possible only if $f(n) = n$. □

Proof of Proposition 26 continued. For f we have the following inequalities:

$$f(a) < 2f(a) = f(2a) \leqq f(2b) < f(2b) + (f(2b) - f(a)) =$$
$$= 2f(2b) - f(a) \leqq f(4b) - f(a) \leqq f(4b - a).$$

Because of $g(4b-a)=g(-a)=g(a)>0$ lemma 25 shows that $g(2a)\leqq g(4b-a)=g(a)$, i.e., we have $g(2)\leqq 1$. By corollary 12 and theorem 13, v_β is induced by a Krull valuation of the residue field of A at $\operatorname{supp}(\beta)$. Since $\beta\in\overline{\{\alpha\}}$, $\beta(1, 2)=0$ or $\beta(1, 2)=1$.

First assume that $\beta(1, 2)=1$. Since Γ_β is nontrivial there is some $c\in A=A_{\operatorname{supp}(\beta)}$ such that $g(c)>1$. Then also $f(c)>1$ and $g(a)<g(ca)$. From the inequalities

$$f(a) < f(ca) \leqq f(cb) < f(cb) + (f(cb) - f(a)) =$$
$$= f(2cb) - f(a) \leqq f(2cb - a)$$

in connection with $0<g(a)=g(-a)=g(2cb-a)$ (by $g(b)=0$) we conclude by use of lemma 25 that $g(ca)\leqq g(2cb-a)=g(a)$, a contradiction.

Finally we assume that $\beta(1, 2)=0$ i.e., $g(2)=0$. Then $g(3)=1$. From lemma 27 we learn that $3f(a)=f(3a)\leqq f(3b)=3f(b)$. Again pick $c\in A$ such that $g(c)>1$. The same computations as above show that $f(ca)\leqq f(3cb-a)$, $g(3cb-a)=g(a)\neq 0$. Now lemma 25 yields $g(ca)\leqq g(3cb-a)=g(a)$, a contradiction. □

The statement of proposition 26 can be reformulated by saying that there is a natural homomorphism $f(A)\to g(A)$ of ordered sets.

Next we study the space of closed points in certain subsets of the valuation spectrum.

Theorem 28. *Let A be a ring, $K'\subset\operatorname{Spev}(A)$ a closed subset. Pick $a_1, \ldots, a_n\in A$ and set $a_0=2$. Then*

$$K = \{\alpha\in K' \mid \alpha(a_0, 1) = \infty \text{ or} \ldots \text{or } \alpha(a_n, 1) = \infty\}$$

is a constructible subset of K'. Every $\alpha\in K$ has a unique closed (in K) specialization in K.

Proof. K is obviously a pro-constructible subset of $\operatorname{Spev}(A)$, hence is a spectral space. Therefore every point of K has a closed specialization. It remains to prove uniqueness. So let $\alpha\in K$ and pick a representative $(f\colon A\to R^{\geqq})\in\operatorname{Spev}^\circ(A)$ of α. We set $U_\alpha=\{x\in R^{\geqq} \mid \exists a\in A\colon x\leqq f(a)\}$. By definition of K, U_α contains a nontrivial convex subgroup of $R^{>}$. Let V_α be the largest such subgroup and set $W_\alpha=U_\alpha\setminus V_\alpha$. We define $P=\{a\in A \mid f(a)\in W_\alpha\}$. It is immediately clear (from the definition of $\operatorname{Spev}^\circ(A)$ — see section 1) that $P\subset A$ is a prime ideal. We pick a closed specialization $\beta\in\overline{\{\alpha\}}$ in K and a representative $(g\colon A\to S^{\geqq})\in\operatorname{Sper}^\circ(A)$ of β. If we can show that $\operatorname{supp}(\beta)=P$ then an application of corollary 23 finishes the proof. So it remains to prove that $\operatorname{supp}(\beta)=P$.

First assume that there is some $a\in\operatorname{supp}(\beta)\setminus P$. Then there exists some $b\in A$ such that $1\leqq f(ab)$. Proposition 26 implies that $1\leqq g(ab)$. But this contradicts $g(ab)=g(a)g(b)=0$. We see that $\operatorname{supp}(\beta)\subset P$. Conversely assume that $\operatorname{supp}(\beta)\subsetneqq P$. Using g instead of f, we define the subsets $U_\beta, V_\beta, W_\beta\subset S^{\geqq}$ in the same way as we defined $U_\alpha, V_\alpha, W_\alpha\subset R^{\geqq}$. Then we also have $P=\{a\in A \mid g(a)\in W_\beta\}$ and $W_\beta\neq\{0\}$ (since $\operatorname{supp}(\beta)\subsetneqq P$).

Case 1. v_β is a Krull valuation. In this case define

$$h\colon A\to S^{\geqq}\colon a\to\begin{cases}0 & \text{if } a\in P\\ g(a) & \text{if } a\notin P.\end{cases}$$

Noting that $x\in P, a\notin P$ implies $g(a)=g(a+x)$ one sees that $(h\colon A\to S^{\geqq})\in\operatorname{Spev}^\circ(A)$ and α_h is a proper specialization of β. Moreover, $g(a_i)>1$ for some i implies $h(a_i)>1$ for the same i. We see that $\alpha_h\in K$, and β is not closed in K, a contradiction.

Case 2. v_β is an absolute value. In this case $g(2)=2$. Let $C\subset S$ be the convex subring generated by $g(A)$, let $M\subset C$ be the maximal ideal. Then $M=W_\beta\cup -W_\beta$. It is easy to check that

$$h\colon A\xrightarrow{g}C\to C/M$$

is in $\operatorname{Spev}^\circ(A)$. α_h is a proper specialization of β since $h^{-1}(0)=P$. Moreover $h(2)=2$ so that $\alpha_h\in K$. Again this contradicts the choice of β as a closed point of K. □

Corollary 29. *With the notation of theorem* 28, $K^{\max}$, *the space of closed points of K, is a compact space.*

Proof. Since K is quasi-compact the same is true for $K^{\max}$. To prove the Hausdorff property let $\alpha, \beta\in K$ be closed points. By theorem 28, α and β do not have common generalizations. Thus α and β have disjoint neighborhoods in K, hence also in $K^{\max}$. □

Still sticking to the same K, a retraction $r\colon K\to K^{\max}$ is defined by letting $r(\alpha)$ be the closed point in $\overline{\{\alpha\}}$ for all $\alpha\in K$.

Proposition 30. *r is continuous.*

Proof. Pick $\alpha\in K$ and set $\beta=r(\alpha)$. If $U\subset K^{\max}$ is a neighborhood of β, then $C=K^{\max}\setminus U$ is closed in $K^{\max}$, hence compact. For $\gamma\in C$, β, γ do not have common generalizations. There are open constructible neighborhoods U_γ of γ, V_γ of β in K with $U_\gamma\cap V_\gamma=\phi$. $C\subset\bigcup_{\gamma\in C}U_\gamma$ is an open cover. By compactness there is a finite subcover $C\subset\bigcup_{i=1}^{n} U_{\gamma_i}=W$. If $V=\bigcap_{i=1}^{n} V_{\gamma_i}$ then V is an open neighborhood of β in K. V is also a neighborhood of α in K. By construction, $V\cap W=\phi$ and W is an open neighborhood of C in K. This implies $r(V)\subset U$, and continuity of r has been proved. □

The closed points of the space K of theorem 28 have the following characterization:

Theorem 31. *If* $\alpha\in K$, *K as in theorem* 28, *then α is a closed point if and only if Γ_α is nontrivial archimedean.*

Proof. If $\alpha\in K$ and $(f\colon A\to R^{\geqq})\in \mathrm{Spev}^\circ(A)$ is a representative of α then $\Gamma_\alpha\subset R^{>}$ is the subgroup generated by $f(A\setminus\mathrm{supp}(\alpha))$. Since $f(a_i)>1$ for some $i=0, \ldots, n$, Γ_α is nontrivial. Now suppose that α is a closed point. The subsets $U_\alpha, V_\alpha, W_\alpha\subset R^{\geqq}$ are defined as in the proof of theorem 28. It was shown there that $f^{-1}(W_\alpha)$. is the support of the closed point in $\{\bar{\alpha}\}$. Since α is closed, $\mathrm{supp}(\alpha)=f^{-1}(W_\alpha)$. Therefore V_α is the convex subgroup of $R^{>}$ generated by Γ_α. If $i\in\{0, \ldots, n\}$ is chosen to be minimal such that $f(a_i)>1$ and V_α is the convex subgroup of V_α generated by $f(a_i)$ then Γ_α is the convex subgroup of Γ_α generated by $v_\alpha(a_i)$. There are two cases:

Case 1. $i=0$. In this case α is an absolute value and $f(2)=2$. If $C\subset R$ is the convex subring generated by V_α then C is generated by $f(2)=2$, i.e., C is the convex hull of $\mathbf{Q}$. Let $M\subset C$ be the maximal ideal. Then with

$$g\colon A\xrightarrow{f} C\to C/M,$$

α_g is a specialization of α and belongs to K since $g(2)=2$. Since α is a closed point of K, $\alpha=\alpha_g$. Since C/M is an archimedean real closed field, Γ_α is archimedean.

Case 2. $i>0$. It is easy to check that, if $\Gamma'_\alpha\subset\Gamma_\alpha$ is the largest convex subgroup not containing $f(a_i)$, the map

$$v_\beta\colon A\to\overline{\Gamma_\alpha}\to\overline{\Gamma_\alpha/\Gamma'_\alpha}$$

defines a Krull valuation β of A which specializes α. Since $v_\beta(f(a_i))>1$ by construction, β belongs to K. Since α is a closed point of K, $\beta=\alpha$, i.e., $\Gamma_\alpha=\Gamma_\alpha/\Gamma'_\alpha$ is archimedean.

Conversely suppose that Γ_α is archimedean. If β is a proper specialization of α, then we have the order compatible epimorphism $\pi_{\beta\alpha}\colon\Gamma_{\beta\alpha}\to\Gamma_\beta$ (lemma 22) with nontrivial kernel. Since Γ_β is nontrivial (see the beginning of the proof) $\Gamma_{\beta\alpha}$ is nonarchimedean. Since $\Gamma_{\beta\alpha}\subset\Gamma_\alpha$ is a subgroup, Γ_α is nonarchimedean, a contradiction. □

II. The complex spectrum

The valuation spectrum of the ring A contains the real spectrum of A (proposition I 19). The connections between these two spectra can be used to give a rather satisfactory construction of the real version of the compactification of Morgan and Shalen ([22]): If $k \subset \mathbf{R}$ is a subfield and $A = k[a_1, \ldots, a_n]$ then $\mathrm{Sper}_k(A)$ denotes those points of $\mathrm{Sper}(A)$ which induce the given total order on k. With $a_0 = 2$ and

$$K = \{\alpha \in \mathrm{Spev}(A) \mid \alpha(a_0, 1) = \infty \text{ or} \ldots \text{or } \alpha(a_n, 1) = \infty\}$$

the set $L = K \cap \overline{\omega(\mathrm{Sper}_k(A))}$ is closed in K. $L^{\max}$ is a compact space (corollary I 29) and contains the image of

$$e\colon\ X(\mathbf{R}) \to \mathrm{Spev}(A)$$
$$x \to \left(\begin{matrix} A \to \mathbf{R} \\ a \to |a(x)| \end{matrix}\right)$$

(theorem I 31). From proposition I 19 and since $X(\mathbf{R})$ with the strong topology can be considered as a subspace of $\mathrm{Sper}_k(A)$ ([8], section 3) it follows that $L^{\max}$ is a compactification of $V(\mathbf{R})$. The compactification of Morgan and Shalen is a quotient of this compactification.

This is a reformulation of Brumfiel's construction of the compactification of Morgan and Shalen ([5]; [6]). The basic idea for dealing with the normal case is to define the notion of a complex spectrum of a ring. This complex spectrum should play a similar role in the nonreal case as the real spectrum does in the real case. Such an approach has also been suggested by Brumfiel ([5]). The idea of a complex spectrum has been realized independently by Huber ([15]) and de la Puente Muñoz ([26]). However, Huber's complex spectrum does not fit into the valuation spectrum. We will define another notion of a complex spectrum which is a subspace of the valuation spectrum. This complex spectrum has many properties in common with the real spectrum. Using this spectrum the construction of the real compactification of Morgan and Shalen pointed out above carries over to the nonreal situation. In preparation for the construction of compactifications in chapter III this chapter contains a discussion of the complex spectrum.

1. Compatible valuations. In real algebraic geometry for some purposes usual commutative algebra is not the adequate algebraic tool. It is only natural that, if one is dealing with genuinely real questions, partial or total orders must enter in some way into the algebra used. Currently such a brand of commutative algebra is developed under the name "real algebra" ([4]; [20]; [21]). Similarly, a complex spectrum requires some algebraic methods of its own. Here a few valuation theoretic notions and results are assembled which are necessary for the discussion of the complex spectrum.

Definition 1. A *complex field* is a triple consisting of an algebraically closed field C, a distinguished square root i of -1 and a real closed subfield R such that $C=R(i)$.

The notion of a morphism of complex fields is obvious. The complex fields form a category.

The language of the theory of complex fields consists of the usual language of the theory of fields, a constant i, a one-place relation R and a two-place relation $\leqq$. This language is denoted by $\mathfrak{L}(cx)$. The theory of complex fields is the theory of algebraically closed fields supplemented in the following way:

The constant i distinguishes a particular square root of -1.

The one-place relation R singles out a real closed subfield over which the algebraically closed field is algebraic.

The two-place relation $\leqq$ restricts to the unique total order on the real closed subfield. On the algebraically closed field it is the componentwise lattice order with respect to the basis $\{1, i\}$ over the real closed subfield.

The theory of complex fields is denoted by $Th(cx)$. Its models are usually denoted by (C, R, i) or just by C if the meaning is clear from the context.

On a complex field $C=(C, R, i)$ there is a natural absolute value

$$|.|: C \to \mathbf{R}^{\geqq}: x = x_1+ix_2 \to |x| = \sqrt{x_1^2+x_2^2}.$$

This absolute value defines an element of the valuation spectrum of C. The absolute value will be used to connect the complex spectrum of a ring with its valuation spectrum.

The order compatible valuation rings of the real closed field R are exactly the convex subrings of R ([23]; [25]). They are Henselian valuation rings with real closed residue field and divisible value group (loc. cit.). Any such valuation ring $V\subset R$ extends uniquely to a valuation ring $W\subset C$ ([11]; [27]). The residue field of W is the algebraic closure of the residue field of V, the value groups of V and W agree (loc. cit.).

Definition 2. A valuation ring (resp. valuation) of the complex field $C=(C, R, i)$ is *compatible* with the complex structure if the restriction to R is an order compatible valuation ring (resp. valuation). The smallest compatible valuation ring (resp. finest compatible valuation) is the *natural* valuation ring (resp. valuation) of C.

By this definition a compatible valuation ring of C may be considered as a couple (W, V) of valuation rings $W\subset C$, $V=W\cap R\subset R$ such that V is convex in R. Let $M_W\subset W$, $M_V\subset V$ be the maximal ideals. Then W/M_W canonically contains the real closed field V/M_V and is its algebraic closure.

Definition 3. If $C=(C, R, i)$ is a complex field and (W, V) is a compatible valuation ring, then $(W/M_W, V/M_V, i)$ is the *induced complex structure* on W/M_W, where $i \in W/M_W$ is the canonical image of the distinguished square $i \in W \subset C$ of -1.

Definition 4. The complex field (C, R, i) is *archimedean* if R is archimedean. (C, R, i) is *archimedean over the subring* $A \subset C$ if $\{|x| \; x \in A\}$ is cofinal in R.

The following is a consequence of Hölder's theorem ([12], p. 74, Satz 1; [25], p. 8, Satz 4):

Proposition 5. *If* (C, R, i) *is an archimedean complex field then there is a unique embedding* $(C, R, i) \to (\mathbf{C}, \mathbf{R}, i)$.

If (C, R, i) is a complex field and (W, V) is the natural valuation ring then the complex field $(W/M_w, V/M_v, i)$ is archimedean.

As a consequence of the real Hahn embedding theorem ([25], p. 62, Satz 21) we note

Proposition 6. *If* (C, R, i) *is a complex field with natural valuation* $w\colon C^* \to \Gamma$ *then there is an embedding* $(C, R, i) \to (\mathbf{C}((\Gamma)), \mathbf{R}((\Gamma)), i)$ *of complex fields.*

2. The spectral space $\operatorname{Spec} x(A)$. If A is a ring containing a square root of -1 the complex spectrum of A shall consist of homomorphisms $A \to (C, R, i)$ into complex fields under an appropriate identification. We proceed as follows:

Let A be a ring containing a squareroot of -1. We distinguish one of these and denote it by $i(A)$. The language $\mathfrak{L}(cx)$ of complex fields is extended by constants c_a for every $a \in A$. The resulting language is denoted by $\mathfrak{L}_A(cx)$. The theory $Th_A(cx)$ is obtained from the theory $Th(cx)$ of complex fields by adding the following axioms:

$$c_0 = 0, \quad c_1 = 1, \quad c_{i(A)} = i, \quad c_a + c_b = c_{a+b}, \quad c_{ab} = c_a c_b.$$

The class of models of $Th_A(cx)$ is denoted by $\operatorname{Spec} x^0(A)$. Usually we think of the elements of $\operatorname{Spec} x^0(A)$ as ring homomorphisms $f\colon A \to C$, where C is a field carrying a complex structure, such that $f(i(A)) = i$.

With $(f\colon A \to (C, R, i)) \in \operatorname{Spec} x^0(A)$ we associate the map

$$\alpha_f\colon A \times A \to Z\colon (a, b) \to \begin{cases} 0 & \text{if } f(a)f(b) = 0 \\ 1 & \text{if } 0 < f(a) \leq f(b) \\ \infty & \text{if } 0 < f(b) < f(a). \end{cases}$$

The subset

$$\operatorname{Spec} x(A) = \{\alpha_f \in Z^{A \times A} \mid f \in \operatorname{Spec} x^0(A)\}$$

is the *complex spectrum* of A. An immediate consequence of the definition is that the complex spectrum of A is a subset of the valuation spectrum of A. In fact, every $\alpha_f \in \operatorname{Spec} x(A)$ is an absolute value of A. As a subset of $\operatorname{Spev}(A)$, $\operatorname{Spec} x(A)$ has a constructible topology and a weak topology. Without proof we note that the same methods as in chapter I, section 1 apply to prove

Theorem 7. Specx(A) *with the weak topology is a spectral space.*

An immediate consequence of this in

Corollary 8. Specx(A) *is a pro-constructible subspace of* Spev(A).

We have the following sufficient condition for elements of Specx$^0(A)$ to give the same point in the complex spectrum of A:

Proposition 9. *Let* $(f\colon A\to(C_1, R_1, i_1))$, $(g\colon A\to(C_2, R_2, i_2))\in \text{Specx}^0(A)$ *and suppose that* $h_1\colon (C_1, R_1, i_1)\to(C, R, i)$, $h_2\colon (C_2, R_2, i_2)\to(C, R, i)$ *are morphisms of complex fields such that* $h_1 f = h_2 g$. *Then* $\alpha_f = \alpha_g$ *in* Specx(A).

Proof. It suffices to show that $\alpha_f = \alpha_{h_1 f}$. Let $a, b\in A$, $f(a)=a_1+i_1 a_2$, $f(b)=b_1+i_1 b_2$ (with $a_1, a_2, b_1, b_2\in R_1$). Then $h_1 f(a)=h_1(a_1)+ih_1(a_2)$, $h_1 f(b)=h_1(b_1)+ih_1(b_2)$ and it follows that $|f(a)|<|f(b)|$ if and only if $|h_1 f(a)|<|h_1 f(b)|$.

Since Specx(A) is contained in Spev(A) every element of Specx(A) has a support. The support map is a morphism of spectral spaces Specx$(A)\to$Spec(A).

The proof of proposition I.8 can be copied to prove

Proposition 10. *If* $\varphi\colon A\to B$ *is a ring homomorphism then* $Z^{\varphi\times\varphi}\colon Z^{B\times B}\to Z^{A\times A}$ *restricts to a map* $\varphi^*\colon$ Specx$(B)\to$Specx(A). φ^* *is a morphism of spectral spaces.*

An immediate consequence of proposition 10 is that Specx is a functor from the category of rings to the category of spectral spaces.

The results of proposition I 10, proposition I 11, corollary I 12 also carry over to the complex spectrum:

Proposition 11. *Let* $P\subset A$ *be a prime ideal.*

(a) *If* $i\colon A\to A_p$ *is canonical then* $i^*\colon$ Specx$(A_p)\to$Specx(A) *is an isomorphism onto the pro-constructible subspace* $\{\alpha\in \text{Specx}(A) | \text{supp}(\alpha)\subset P\}$ *of* Specx(A).

(b) *If* $\pi\colon A\to A/P$ *is canonical then* $\pi^*\colon$ Specx$(A/P)\to$Specx(A) *is an isomorphism onto the pro-constructible subspace* $\{\alpha\in \text{Specx}(A) | P\subset \text{supp}(\alpha)\}$ *of* Specx(A).

(c) *If* $\varphi\colon A\to A(P)=\text{Quot}(A/P)$ *is canonical then* $\varphi^*\colon$ Specx$(A(P))\to$ Specx(A) *is an isomorphism onto the pro-constructible subspace*

$$\{\alpha\in \text{Specx}(A) | \text{supp}(\alpha) = P\}$$

of Specx(A).

Other approaches to the complex spectrum of a ring may be found in [15] and in [26]. The basic idea is always the same: the points of the complex spectrum shall arise from homomorphisms into complex fields. The difference between these approaches is the way different homomorphisms into complex fields are identified. proposition 9 shows that the identification used here is coarser than the one used in the references.

3. Specializations in the complex spectrum. The complex spectrum is a subspace of the valuation spectrum. Thus, for $\alpha, \beta \in \operatorname{Spec x}(A)$, β specializes α in $\operatorname{Spec x}(A)$ if and only if β specializes α in $\operatorname{Spev}(A)$. All the results about specializations in $\operatorname{Spev}(A)$ also apply to $\operatorname{Spec x}(A)$. In particular we mention that $\beta \in \overline{\{\alpha\}}$ implies $\operatorname{supp}(\beta) \supset \operatorname{supp}(\alpha)$ and, if $(f\colon A \to (C_1, R_1, i_1))$, $(g\colon A \to (C_2, R_2, i_2)) \in \operatorname{Spec x}^0(A)$ are representatives of α and β, then $|f(a)| \leqq |f(b)|$ implies $|g(a)| \leqq |g(b)|$ for all $a, b \in A$.

Since the elements of $\operatorname{Spec x}(A)$ are rather special valuations of A we can prove some results about specializations in $\operatorname{Spec x}(A)$ which go beyond those of chapter I, section 4.

Proposition 12. *Let* $\beta \in \overline{\{\alpha\}}$ *in* $\operatorname{Spec x}(A)$ *and let* $(f\colon A \to (C_\alpha, R_\alpha, i_\alpha))$, $(g\colon A \to (C_\beta, R_\beta, i_\beta)) \in \operatorname{Spec x}^0(A)$ *be representative of* α *and* β. *If* (W, V) *is the smallest compatible valuation ring of* C_α *containing* $f(A)$ *then* $\operatorname{supp}(\beta) = f^{-1}(P)$, *where* $P \subset W$ *is some prime ideal*

Proof. The set

$$P = \{x \in W \mid \exists n \in \mathbf{N}\ \exists a \in \operatorname{supp}(\beta)\colon\ |x|^n \leqq |f(a)|\}$$

is a prime ideal of W. By definition, $f(\operatorname{supp}(\beta)) \supset P$. On the other hand, suppose that $f(a) \in P$, i.e., there are $n \in \mathbf{N}$ and $b \in \operatorname{supp}(\beta)$ such that $|f(a)|^n = |f(a^n)| \leqq |f(b)|$. By proposition I 26 this implies $|g(a^n)| \leqq |g(b)| = 0$, i.e., $g(a) = 0$. □

Theorem 13. *In* $\operatorname{Spec x}(A)$ *the specializations of a point form a chain.*

Proof. Let $(f_\alpha\colon A \to C_\alpha)$, $f_\beta\colon A \to C_\beta)$, $(f_\gamma\colon A \to C_\gamma)$ be representatives of α, β, γ in $\operatorname{Spec x}^0(A)$ and suppose that $\beta, \gamma \in \overline{\{\alpha\}}$. By proposition 12 $\operatorname{supp}(\beta)$ and $\operatorname{supp}(\gamma)$ are inverse images of prime ideals of W, where $(W, V) \subset (C_\alpha, R_\alpha, i_\alpha)$ is the natural valuation ring. Thus $\operatorname{supp}(\beta)$ and $\operatorname{supp}(\gamma)$ are comparable. If $\operatorname{supp}(\beta) = \operatorname{supp}(\gamma)$ then corollary I 23 shows that β, γ are comparable.

Assume now that $\operatorname{supp}(\beta) \subsetneqq \operatorname{supp}(\gamma)$. It will be shown that $\gamma \in \overline{\{\beta\}}$. With the notation used in chapter I, section 4 we have the subgroups $\Gamma_{\gamma\alpha} \subset \Gamma_{\beta\alpha} \subset \Gamma_\alpha$ and the epimorphisms $\pi_{\gamma\alpha}\colon \Gamma_{\gamma\alpha} \to \Gamma_\gamma$, $\pi_{\beta\alpha}\colon \Gamma_{\beta\alpha} \to \Gamma_\beta$ of ordered groups. Let $\Delta_{\gamma\alpha} \subset \Gamma_{\gamma\alpha}$, $\Delta_{\beta\alpha} \subset \Gamma_{\beta\alpha}$ be the kernels of $\pi_{\gamma\alpha}$ and $\pi_{\beta\alpha}$. The convex subgroups of Γ_α generated by $\Delta_{\gamma\alpha}$ and $\Delta_{\beta\alpha}$ are denoted by $[\Delta_{\gamma\alpha}]$, $[\Delta_{\beta\alpha}]$. From the definition it is clear that $[\Delta_{\gamma\alpha}] \cap \Gamma_{\gamma\alpha} = \Delta_{\gamma\alpha}$ and $[\Delta_{\beta\alpha}] \cap \Gamma_{\beta\alpha} = \Delta_{\beta\alpha}$. The claim follows immediately if we can show that $[\Delta_{\beta\alpha}] \subsetneqq [\Delta_{\gamma\alpha}]$.

Since the convex subgroups $[\Delta_{\beta\alpha}], [\Delta_{\gamma\alpha}] \subsetneqq \Gamma_\alpha$ are comparable it suffices to show that $[\Delta_{\gamma\alpha}] \setminus [\Delta_{\beta\alpha}] \neq \phi$.

Pick some $a \in \operatorname{supp}(\gamma) \setminus \operatorname{supp}(\beta)$. Then $f_\gamma(1+a) = f_\gamma(1) = 1$ and $|f_\alpha(1+a)| \in \Delta_{\gamma\alpha}$. Assume by way of contradiction that $[\Delta_{\gamma\alpha}] \subset [\Delta_{\beta\alpha}]$. Then also $\Delta_{\gamma\alpha} \subset \Delta_{\beta\alpha}$ and we have $|f_\alpha(1+a)| \in \Delta_{\beta\alpha}$. In the same way, $2a \in \operatorname{supp}(\gamma) \setminus \operatorname{supp}(\beta)$ and $|f_\alpha(1+2a)| \in \Delta_{\gamma\alpha} \subset \Delta_{\beta\alpha}$. Writing $f_\beta(a) = a_1 + i_\beta a_2$ with $a_1, a_2 \in R_\beta$ we obtain

$$1 = |f_\beta(1+a)|^2 = (1+a_1)^2 + a_2^2,$$
$$1 = |f_\beta(1+2a)|^2 = (1+2a_1)^2 + 4a_2^2.$$

An easy computation shows that this implies $a_1 = 0 = a_2$, i.e., $f_\beta(a) = 0$, a contradiction. □

As immediate consequences of theorem 13 we record:

Corollary 14. *If* $\alpha, \beta \in \operatorname{Spec} \mathrm{x}(A)$ *are not comparable with respect to specialization then they have disjoint neighborhoods.*

Corollary 15. *For any pro-constructible subset* $K \subset \operatorname{Spec} \mathrm{x}(A)$ *the space* $K^{\max}$ *of closed points is compact.*

Theorem 13 can be used to define a retraction r: $\operatorname{Spec} \mathrm{x}(A) \to \operatorname{Spec} \mathrm{x}^{\max}(A)$: $r(\alpha)$ is the closed point of $\overline{\{\alpha\}}$. It is easy to show that r is continuous (see the proof of proposition I. 30). We will see now how $r(\alpha)$ can be determined: Let

$$(f\colon A \to C_\alpha) \in \operatorname{Spec} \mathrm{x}^0(A)$$

be a representative of α. The smallest compatible valuation ring in C_α containing $f(A)$ is denoted by (W, V). Then

$$g\colon A \xrightarrow{f} W \to W/M_W$$

belongs to $\operatorname{Spec} \mathrm{x}^0(A)$ (where W/M_w carries the induced complex structure).

Proposition 16. α_g *is the unique closed point in* $\overline{\{\alpha\}}$.

Proof. Obviously $\beta = \alpha_g \in \overline{\{\alpha\}}$ holds. Suppose that $\gamma \in \overline{\{\beta\}}$. Since $\operatorname{supp}(\gamma) \supset \operatorname{supp}(\beta)$ holds trivially, proposition 12 and the definition of g imply that $\operatorname{supp}(\gamma) = \operatorname{supp}(\beta)$. By proposition 11 (a) we may assume that A is local with maximal ideal $\operatorname{supp}(\beta)$. Then we have $\Gamma_{\beta\alpha} = \Gamma_{\gamma\alpha} \subset \Gamma_\alpha$ and $\Delta_{\beta\alpha} \subset \Delta_{\gamma\alpha}$ where $\Delta_{\beta\alpha}$, $\Delta_{\gamma\alpha}$ are the kernels of $\pi_{\beta\alpha}$ and $\pi_{\gamma\alpha}$. It suffices to prove that $\Delta_{\beta\alpha} = \Delta_{\gamma\alpha}$ (since then $\beta = \gamma$).

Assume by way of contradiction that there is some $a \in A^*$ with $|f(a)| \in \Delta_{\gamma\alpha} \setminus \Delta_{\beta\alpha}$. Then $|f(a)|$ is not a 1-unit in V. Without loss of generality we may assume that $|f(a)| > 1$. Then the interval

$$(1 - |f(a)|, |f(a)| - 1) \subset V$$

is not contained in M_V. Since $\{|f(b)|\,|\, b\in A^*\}$ is cofinal in V there is some $b\in A^*$ such that $|f(2b)|\in(1-|(f(a))|, |f(a)|-1)$, i.e., $1<1+|f(2b)|<|f(a)|$. Write $f(b)$ as $b_1+i_\alpha b_2$ with $b_1, b_2\in R_\alpha$. Replacing b by $-b$ if necessary we assume that $b_1\geqq 0$. Then

$$1 < |f(1+2b)| \leqq 1+|f(2b)| < |f(a)|.$$

By choice of a and proposition I. 26,

$$1 = |h(1)| \leqq |h(1+2b)| \leqq |h(a)| = 1$$

(where $(h\colon A\to C_\gamma)\in \operatorname{Spec} \mathrm{x}^0(A)$ is a representative of γ). This implies $|h(1+2b)|=1$. In the same way it can be shown that $|h(1+b)|=1$. An easy computation yields $h(b)=0$. This is a contradiction since b was chosen in A^*. □

4. An ultrafilter theorem for the complex spectrum. If k is a field containing a square root of -1 and A is a k-algebra and if $(f\colon A\to(C, R, i))\in \operatorname{Spec} \mathrm{x}^0(A)$ has the property that C is archimedean over $f(k)$ then it is an immediate consequence of proposition 16 that α_f is a closed point of $\operatorname{Spec} \mathrm{x}(A)$.

In particular, let (C, R, i) be a complex field, $A=C[a_1, \dots, a_n]$ a finitely generated C-algebra. We have the structural map $\varphi\colon C\to A$ and the functorial map $\varphi^*\colon \operatorname{Spec} \mathrm{x}(A)\to \operatorname{Spec} \mathrm{x}(C)$. Let $\varkappa\in \operatorname{Spec} \mathrm{x}(C)$ be the canonical absolute value. The we denote $\varphi^{*-1}(\varkappa)$ by $\operatorname{Spec} \mathrm{x}_C(A)$. Let X be the scheme $\operatorname{Spec}(A)$ over C, $X(C)$ the set of C-valued points of X. Given $x\in X(C)$ we have the evaluation map $x^*\colon A\to C\colon a\to a(x)$, which belongs to $\operatorname{Spec} \mathrm{x}^0(A)$. By $e(x)$ we denote the corresponding point of $\operatorname{Spec} \mathrm{x}(A)$. Since $x^*\circ\varphi\colon C\to A\to C$ is the identity we even see that $e(x)\in \operatorname{Spec} \mathrm{x}_C(A)$. By the above remark, $e(x)$ in a closed point of $\operatorname{Spec} \mathrm{x}(A)$, hence also of $\operatorname{Spec} \mathrm{x}_C(A)$. So a map $e\colon X(C)\to \operatorname{Spec} \mathrm{x}_C^{\max}(A)$ has been defined.

Since $X(C)$ is contained in $C^n=R^{2n}$ the interval topology of R defines a topology on $X(C)$ which we call the strong topology.

Theorem 17. *e maps $X(C)$ with the strong topology homeomorphically onto $im(e)\subset \operatorname{Spec} \mathrm{x}_C(A)$.*

Proof. First we prove that e is injective: For $x\in X(C)$ let $M_x\subset A$ be the maximal ideal belonging to x. Then $\operatorname{supp}(e(x))=M_x$. If $x\neq y$ then $M_x\neq M_y$ and hence the supports of $e(x)$ and $e(y)$ are different.

e is continuous: Let $U\subset \operatorname{Spec} \mathrm{x}_C(A)$ be an open neighborhood of $e(x)$. We may assume that $U=U_1\cap\dots\cap U_k$, where each U_i is an open set of one of the following two forms:

$$V = \{\alpha\in \operatorname{Spec} \mathrm{x}_C(A) \mid \alpha(a, b) = \infty\},$$

$$W = \{\alpha\in \operatorname{Spec} \mathrm{x}_C(A) \mid \alpha(a, b) \neq 0\}$$

with $a, b \in A$. Since

$$e^{-1}(V) = \{x \in X(C) \,|\, |a(x)| > |b(x)| > 0\},$$

$$e^{-1}(W) = \{x \in X(C) \,|\, a(x)b(x) \neq 0\}$$

are both open it follows that e is continuous.

Finally, e is open onto the image: $x=(x_1, \ldots, x_n) \in X(C)$ has a neighborhood basis consisting of open polycylinders

$$P(x, \varepsilon) = \{y \in X(C) \,|\, |y_1 - x_1| < \varepsilon \;\&\; \ldots \;\&\; |y_n - x_n| < \varepsilon\}.$$

On the other hand

$$U = \left\{\alpha \in \operatorname{Spec} \mathrm{x}_C(A) \,|\, \forall_i \colon \alpha(\varepsilon, a_1 - x_1) = \infty \quad \text{or} \quad \alpha\left(\frac{\varepsilon}{2}, a_1 - x_1 - \frac{\varepsilon}{4}\right) = \infty\right\}$$

is open in $\operatorname{Spec} \mathrm{x}_C(A)$. Since $e(P(x, \varepsilon)) = im(e) \cap U$ the proof is finished. □

In real algebraic geometry there is a natural correspondence between semi-algebraic subsets of an affine variety defined over a real closed field and constructible subsets of the real spectrum of the coordinate ring ([8], section 5). Similar results can be established in the complex setting:

Definition 18. Let $e\colon X(C) \to \operatorname{Spec} \mathrm{x}_C(A)$ be the embedding introduced above. A subset $M \subset X(C)$ is *complex* if it is of the form $e^{-1}(K)$ with $K \subset \operatorname{Spec} \mathrm{x}_C(A)$ constructible.

The main result about the connections between constructible subsets of $\operatorname{Spec} \mathrm{x}_C(A)$ and complex subsets of $X(C)$ is

Theorem 19. *The map $K \to e^{-1}(K)$ from the set of constructible subsets of* $\operatorname{Spec} \mathrm{x}_C(A)$ *to the set of complex subsets of $X(C)$ is a bijection.*

Proof. It suffices to show that $e^{-1}(K)$ is nonempty if K is nonempty. So, pick $K \subset \operatorname{Spec} \mathrm{x}_C(A)$ constructible, $K \neq \phi$. Let $\varphi\colon C[X_1, \ldots, X_n] \to A = C[a_1, \ldots, a_n]$ be an epimorphism over C. Then $\varphi^*(K) \subset \operatorname{Spec} \mathrm{x}_C C[X_1, \ldots, X_n]$ is constructible. If e also denotes the embedding $C^n \to \operatorname{Spec} \mathrm{x}_C C[X_1, \ldots, X_n]$ it suffices to prove that $\varphi^*(K) \cap e(C^n) \neq \phi$. So we may assume that $A = C[X_1, \ldots, X_n]$. With A we associate the real polynomial ring $B = R[Y_1, \ldots, Y_n, Z_1, \ldots, Z_n]$ by separating the real and imaginary parts of the elements of A (cf. [9]; [15]): Replacing X_k by $Y_k + iZ_k$ for every $a \in A$ we obtain $a(Y_1 + iZ_1, \ldots, Y_n + iZ_n) = b + ic$ with $b, c \in B$. We call b the real part of a, c the imaginary part of a. To study the connections between A and B

more closely we look at the diagram

$$\begin{array}{ccccc} R & \longrightarrow & C & & \\ \downarrow & & \downarrow & \searrow & \\ B & \xrightarrow{\sigma} & B\otimes_R C & \xleftarrow{\tau} & A \\ & & Y_k+iZ_k & \longleftarrow & X_k \end{array}$$

Since the left-hand square is a push out every homomorphism $B \xrightarrow{f} R_1$ into a real closed field yields a unique homomorphism $f\otimes C$: $B\otimes_R C\to R_1\otimes_R C$ over C into the complex field $(R_1\otimes_R C, R_1, 1\otimes_i)$. It follows from proposition 9 and the amalgamation property of real closed fields ([24, Satz 3.22, Satz 4.7; [28], section 17) that, if g: $B\to R_2$ defines the same element of Sper(B) as does f, then $g\otimes C$: $B\otimes_R C\to R_2\otimes_R C$ defines the same element of Spec $\mathrm{x}_C(B\otimes_R C)$ as does $f\otimes C$. So, we have defined a map Sper$(B)\xrightarrow{\sigma_*}$Spec $\mathrm{x}_C(B\otimes_R C)$. We show that σ_* is a morphism of spectral spaces:

Pick $a=a_1+ia_2$, $b=b_1+ib_2\in B\otimes_R C$ with $a_1, a_2, b_1, b_2\in R$ and consider the open constructible subsets

$$V=\{\alpha\in \operatorname{Spec} \mathrm{x}_C(B\otimes_R C)\,|\,\alpha(a,b)=\infty\},$$

$$W=\{\alpha\in \operatorname{Spec} \mathrm{x}_C(B\otimes_R C)\,|\,\alpha(a,b)\neq 0\}$$

of Spec $\mathrm{x}_C(B\otimes_R C)$. If f: $B\to R_1$ is a homomorphism into a real closed field then $\alpha_{f\otimes C}\in V$ if and only if $f(a_1^2+a_2^2)>f(b_1^2+b_2^2)>0$. So, $\sigma_*^{-1}(V)$ is open and constructible in Sper(B). Similarly, $\alpha_{f\otimes C}\in W$ if and only if $f(a_1^2+a_2^2)f(b_1^2+b_2^2)\neq 0$. This shows that $\sigma_*^{-1}(W)$ is also open and constructible in Sper(B).

Composing the morphisms σ_* and τ^*: Spec $\mathrm{x}_C(B\otimes_R C)\to$Spec $\mathrm{x}_C(A)$ of spectral spaces we obtain a morphism $\tau^*\sigma_*$: Sper$(B)\to$Spec $\mathrm{x}_C(A)$. Considering R^{2n} as the subspace of R-valued points in Sper(B) (cf. [8], Section 5) we restrict $\tau^*\sigma_*$ to R^{2n}. It is immediately clear from the definition that the image of $(y_1, \dots, y_n, z_1, \dots, z_n)\in R^{2n}$ under $\tau^*\sigma_*$ is $(y_1+iz_1, \dots, y_n+iz_n)\in C^n$. So $\tau^*\sigma_*$ restricts to the usual identification of $C^n=R^{2n}$.

Now we return to the constructible subset $K\subset$Spec $\mathrm{x}_C(A)$. We have shown that $(\tau^*\sigma_*)^{-1}(K)\supset$Sper$(B)$ is constructible. By the connections between constructible subsets of Sper(B) and semi-algebraic subsets of R^{2n} (cf. [8], section 5) we know that $R^{2n}\cap(\tau^*\sigma_*)^{-1}(K)\neq\phi$. But then $C^n\cap K\subset\tau^*\sigma_*(R^{2n}\cap(\tau^*\sigma_*)^{-1}(K))\neq\phi$. □

Corollary 20. *If V is an affine C-variety with coordinate ring A then there is a bijection between* Spec $\mathrm{x}_C(A)$ *and the set of ultrafilters of complex subsets of $V(C)$.*

III. Valuation spectrum and compactification

After the preparations in the first two chapters now complex varieties will be compactified by using the valuation spectrum and the complex spectrum. In section 1 we describe the construction of the compactification. In section 2 the connections with the compactification of Morgan and Shalen ([22]) are established.

In this entire chapter $k \subset \mathbf{C}$ is a fixed subfield containing i, where i denotes a fixed square root of -1. We consider a finitely generated integral k-algebra $A=k[a_1, \ldots, a_n]$ and set $B=A\otimes_K \mathbf{C}$. The schemes $\operatorname{Spec}(A)$ and $\operatorname{Spec}(B)$ are denoted by X and Y, their $\mathbf{C}$-valued points (over k and $\mathbf{C}$) by $X(\mathbf{C})$ and $Y(\mathbf{C})$. The canonical absolute values $k \to \mathbf{R}^{\geqq}$ and $\mathbf{C} \to \mathbf{R}^{\geqq}$ given by $x \to |x|$ determine points of $\operatorname{Spec} \mathrm{x}(k)$ and $\operatorname{Spec} \mathrm{x}(\mathbf{C})$ which are denoted by $\varkappa_k$ and $\varkappa_{\mathbf{C}}$. The canonical homomorphisms $k \to A$, $\mathbf{C} \to B$ are denoted by j_k, $j_{\mathbf{C}}$. The fibres of f_k^*: $\operatorname{Spec} \mathrm{x}(A) \to \operatorname{Spec} \mathrm{x}(k)$ and $j_{\mathbf{C}}^*$: $\operatorname{Spec}(B) \to \operatorname{Spec} \mathrm{x}(\mathbf{C})$ over $\varkappa_k$ and $\varkappa_{\mathbf{C}}$ are denoted by $\operatorname{Spec} \mathrm{x}_k(A)$ and $\operatorname{Spec} \mathrm{x}_{\mathbf{C}}(B)$ (as in chapter II). The canonical maps $X(\mathbf{C}) \to \operatorname{Spec} \mathrm{x}_k(A)$, $Y(\mathbf{C}) \to \operatorname{Spec} \mathrm{x}_{\mathbf{C}}(B)$ defined by evaluation at the points of $X(\mathbf{C})$ and $Y(\mathbf{C})$ are denoted by e. τ_k, $\tau_{\mathbf{C}}$ are the trivial valuations in $\operatorname{Spev}(k)$, $\operatorname{Spev}(\mathbf{C})$.

1. Compactification through the valuation spectrum

Let φ: $k \to \mathbf{C}$ be the inclusion and ψ: $k \to \mathbf{C}$ another embedding with $\psi(i)=i$. Suppose that ψ^*: $\operatorname{Spec} \mathrm{x}(\mathbf{C}) \to \operatorname{Spec} \mathrm{x}(k)$ maps $\varkappa_{\mathbf{C}}$ to $\varkappa_k$. Then for all $a \in k$ and all $r \in \mathbf{Q}^{\geqq}$ we have:

$$\varkappa_{\mathbf{C}}(\varphi(a), r) = \varkappa_k(a, r) = \varkappa_{\mathbf{C}}(\psi(a), r),$$

$$\varkappa_{\mathbf{C}}(\varphi(a)-1, r) = \varkappa_k(a-1, r) = \varkappa_{\mathbf{C}}(\psi(a)-1, r),$$

$$\varkappa_{\mathbf{C}}(\varphi(a)-i,\ r) = \varkappa_k(a-i,\ r) = \varkappa_{\mathbf{C}}(\psi(a)-i,\ r).$$

This shows that $\varphi(a)$ and $\psi(a)$ lie at the same distance from 0, 1 and i. But then they must be the same. This proves

Proposition 1. *If ψ: $k \to \mathbf{C}$ is an embedding with $\psi(i)=i$ and $\psi^*(\varkappa_{\mathbf{C}})=\varkappa_k$ then ψ is the inclusion.*

By the same method one can prove

Lemma 2. *The map e: $X(\mathbf{C}) \to \operatorname{Spec} \mathrm{x}_k(A)$ is injective.*

Proof. If $x, y \in X(\mathbf{C})$, $x \neq y$ then there is some $a \in A$ with $a(x) \neq a(y)$. Since $a(x)$ and $a(y)$ cannot lie at the same distance from all of $0, 1$ and i it follows that $e(x)(a, r) \neq e(y)(a, r)$ or $e(x)(a-1, r) \neq e(y)(a-1, r)$ or $e(x)(a-i, r) \neq e(y)(a-i, r)$ for some $0 \leqq r \in \mathbf{Q}$. Thus, $e(x) \neq e(y)$. $\square$

In the valuation spectra $\operatorname{Spev}(A)$ and $\operatorname{Spev}(B)$ we consider the constructible subsets

$$K_A = \{a \in \operatorname{Spev}(A) \mid \alpha(a_0, 1) = \infty \text{ or} \dots \text{or } \alpha(a_n, 1) = \infty\},$$

$$K_B = \{\alpha \in \operatorname{Spev}(B) \mid \alpha(a_0, 1) = \infty \text{ or} \dots \text{or } \alpha(a_n, 1) = \infty\}$$

with $a_0 = 2$ (cf. chapter I, section 4). We set

$$L_A = \overline{\operatorname{Spec} \mathrm{x}_k(A)} \cap K_A, \quad L_B = \overline{\operatorname{Spec} \mathrm{x}_{\mathbf{C}}(B)} \cap K_B.$$

By proposition I 30 there are continuous retractions $r_A\colon L_A \to L_A^{\max}$, $r_B\colon L_B \to L_B^{\max}$. Since $\operatorname{Spec} \mathrm{x}_k(A) \subset \operatorname{Spev}(A)$, $\operatorname{Spec} \mathrm{x}_{\mathbf{C}}(B) \subset \operatorname{Spev}(B)$ are pro-constructible subsets their closures consist of the specializations of their points. Therefore r_A, r_B restrict to surjective maps

$$r_A\colon \operatorname{Spec} \mathrm{x}_k^{\max}(A) \to L_A^{\max}, \quad r_B\colon \operatorname{Spec} \mathrm{x}_{\mathbf{C}}^{\max}(B) \to L_B^{\max}$$

of compact spaces.

We study the points of $L_A^{\max}$:

Proposition 3. *Pick* $\alpha \in \operatorname{Spec} \mathrm{x}_k(A)$, $(f\colon A \to (C, R, i)) \in \operatorname{Spec} \mathrm{x}^0(A)$ *a representative. Let* (W, V) *be the smallest compatible valuation ring of* (C, R, i) *containing* $f(A)$. *Let* $\lambda\colon (W, V) \to (W/M_w, V/M_v, i)$ *be canonical.*

(a) *If* $(W/M_w, V/M_v, i)$ *is archimedean then* $r_A(\alpha)$ *is given by*

$$(\lambda f\colon A \to (W/M_w, V/M_v, i)).$$

(b) *If* $(W/M_w, V/M_v, i)$ *is nonarchimedean then there is a largest nontrivial compatible valuation ring* $(W, V') \subset (W/M_w, V/M_v, i)$. *Let* $w'\colon W/M_w \to \bar{\Gamma}$ *be the corresponding Krull valuation. Then* $r_A(\alpha)$ *has* $(w'\lambda f\colon A \to \bar{\Gamma})$ *as its representative.*

Proof. From the definitions it is clear that $(\lambda f\colon A \to (W/M_w, V/M_v, i)) \in \operatorname{Spec} \mathrm{x}^0(A)$ is the representative of a specialization of α. Moreover, $\alpha_{\lambda f}$ restricts to $\varkappa_k$ on k. For, $\alpha_{\lambda f}$ defines a specialization of $\varkappa_k$ on k with nontrivial value group. Since the value group of $\varkappa_k$ is archimedean we see that $\alpha_{\lambda f}$ induces $\varkappa_k$ on k, i.e., $\alpha_{\lambda f} \in \operatorname{Spec} \mathrm{x}_k(A)$. We may assume now that (C, R, i) is archimedean over $f(A)$.

(a) If (C, R, i) is archimedean then α is closed in L_A (theorem I. 31).

(b) Suppose that (C, R, i) is nonarchimedean. There is some $i \in \{1, \dots, n\}$ such that $\{|f(a_i)|^l \mid l \in \mathbf{N}\}$ is cofinal in R. Let $V' \subset R$ be the largest convex subring not containing $|f(a_i)|$. Then $(W' = V'[i], V')$ is the largest nontrivial compatible valuation ring. It has nontrivial archimedean value group Γ. Let $w'\colon C \to \bar{\Gamma}$ be the corresponding Krull valuation. Since $w'(|f(a_i)|) > 1$ in Γ it is clear that the valuation $(w'f\colon A \to C \to \bar{\Gamma})$ has nontrivial archimedean value group. So $\alpha_{w'f}$ is closed in K_A (theorem I. 31). By construction it is clear that $\alpha_{w',f}\in\overline{\{\alpha\}}$. This proves that $\alpha_{w',f}=r_A(\alpha)$. □

Another description of $L_A^{\max}$ is contained in

Proposition 4. *$L_A^{\max}$ is the set of closed points of $j_k^{*-1}(\varkappa_k, \tau_k)\cap K_A$.*

Proof. First let $\alpha\in L_A^{\max}$. Then α is a closed point of K_A and (by continuity of j_k^*) belongs to $j_k^{*-1}(\varkappa_k, \tau_k)$. Conversely, let α be a closed point of $j_k^{*-1}(\varkappa_k, \tau_k)\cap K_A$. First suppose that $j_k^*(\alpha)=\varkappa_k$, i.e., $\alpha(2, 1)=\infty$. Since α is closed in K_A the value group of α is nontrivial archimedean. So we may consider the associated valuation as a valuation v_α: $A\to A(\operatorname{supp}(\alpha))\to\mathbf{R}^{\geqq}$. This may be normalized so that $v_\alpha(2)=2$ (because of $\alpha(2, 1)=\infty$). This shows that $A(\operatorname{supp}(\alpha))\to\mathbf{R}^{\geqq}$ is an archimedean valuation in the sense of [34], section I-3. By the result of [34], sections I-7, I-8 there is an embedding $A(\operatorname{supp}(\alpha))\to\mathbf{C}$ such that the canonical absolute value of $\mathbf{C}$ restricts to the archimedean valuation of $A(\operatorname{supp}(\alpha))$. This shows that α is determined by the homomorphism $A\to A(\operatorname{supp}(\alpha))\to(\mathbf{C}, \mathbf{R}, i)$ into the complex field $\mathbf{C}$. By composition with j_k we obtain an embedding ψ: $k\to\mathbf{C}$ such that $\psi^*(\varkappa_{\mathbf{C}})=\varkappa_k$. So $\alpha\in\operatorname{Spec}\mathrm{x}_k(A)$, and this proves that $\alpha\in L_A^{\max}$.

Now suppose that $j_k(\alpha)=\tau_k$, i.e., $\alpha(2, 1)=1$. Let w_α: $A(\operatorname{supp}(\alpha))\to\Gamma_\alpha$ be the Krull valuation determined by α. By [17], section 4, there is a valuation preserving embedding $A(\operatorname{supp}(\alpha))\to C((\tilde{\Gamma}_\alpha))$ into a field of formal power series with its natural valuation (where C is an algebraically closed field of cardinality at least max $\{|A|, |\mathbf{C}|\}$ and $\tilde{\Gamma}_\alpha$ is the divisible hull of Γ_α) such that the image of $k\to A\to A(\operatorname{supp}(\alpha))\to C((\tilde{\Gamma}_\alpha))$ is contained in the subfield C. Let g: $\mathbf{C}\to C$ be an extension of $k\to C$ (this exists by the cardinality assumption about C). If $R\subset C$ is a maximal real closed subfield containing $g(\mathbf{R})$ then (C, R, i) (with $i=g(i)$) is a complex field. Now $(C((\tilde{\Gamma}_\alpha)), R((\tilde{\Gamma}_\alpha)), i)$ is a complex field as well. By construction $(f$: $A\to A(\operatorname{supp}(\alpha))\to(C((\tilde{\Gamma}_\alpha)), R((\tilde{\Gamma}_\alpha)), i))\in\operatorname{Spec}\mathrm{x}^0(A)$ defines $\alpha_f\in\operatorname{Spec}\mathrm{x}_k(A)$ and $\alpha\in\overline{\{\alpha_f\}}$ (proposition 3). □

If $x\in X(\mathbf{C})$ then the value group of $e(x)\in K_A$ is archimedean. Theorem I 31 shows that $e(x)$ is closed in K_A. Therefore e maps $X(\mathbf{C})$ into $L_A^{\max}$.

Corollary 5. *$j_k^{*-1}(\varkappa_k)\cap L_A^{\max}=im(e)=\{\alpha\in L_A^{\max}|\alpha(2, 1)=\infty\}$ and $j_k^{*-1}(\tau_k)\cap L_A^{\max}=j^{*-1}(\tau_k)\cap K_A^{\max}=\{\alpha\in L_A^{\max}|\alpha(2, 1)=1\}$.*

Proof. The inclusions $im(e)\subset j_k^{*-1}(\varkappa_k)\cap L_A^{\max}\subset\{\alpha\in L_A^{\max}|\alpha(2, 1)=\infty\}$ are trivial. The proof of proposition 4 shows that $\alpha\in L_A^{\max}$, $\alpha(2, 1)=\infty$ implies $\alpha\in im(e)$. The other statement is clear from proposition 4. □

All the results proved so far for $L_A^{\max}$ are also true for $L_B^{\max}$. Just set $k=\mathbf{C}$.

Theorem 6. *e maps $X(\mathbf{C})$ homeomorphically onto the dense open subspace $\{\alpha|\alpha(2, 1)=\infty\}\subset L_A^{\max}$.*

Proof. e is injective by lemma 2. Continuity of e is proved exactly as in the proof of theorem II 17. To show that $e: X(\mathbf{C}) \to im(e)$ is open we note that $x \in X(\mathbf{C})$ has a neighborhood basis consisting of open polycylinders

$$P(x, \varepsilon) = \{y \in X(\mathbf{C}) \mid |y_1 - x_1| < \varepsilon \ \& \dots \& \ |y_n - x_n| < \varepsilon\}$$

with $0 < \varepsilon \in \mathbf{Q}$. The same argument as in the proof of theorem II 17 shows that $e(P(x, \varepsilon))$ is open in $im(e)$. The image of e has been determined in corollary 5.

It remains to be shown that $im(e)$ is dense in $L_A^{\max}$. First note that $e(Y(\mathbf{C}))$ is dense in $L_B^{\max}$. This follows from continuity of r_B (proposition I 30) and from theorem II 19. If $j: A \to B$ denotes the canonical homomorphism then j induces a morphism $j^*: K_B \to K_A$ which restricts to $\operatorname{Spec} x_{\mathbf{C}}(B) \to \operatorname{Spec} x_k(A)$, hence also to $L_B \to L_A$ and (by theorem I 31) to $L_B^{\max} \to L_A^{\max}$. This map is also denoted by j^*. The canonical morphism $Y \to X$ of schemes yields a map $Y(\mathbf{C}) \to X(\mathbf{C})$ which is also denoted by j^*. This is continuous and the diagram

$$\begin{tikzcd} Y(\mathbf{C}) \arrow[r, "j^*"] \arrow[d, "e"'] & X(\mathbf{C}) \arrow[d, "e"] \\ L_B^{\max} \arrow[r, "j^*"] & L_A^{\max} \end{tikzcd}$$

commutes. To prove that $e(X(\mathbf{C}))$ is dense in $L_A^{\max}$ it suffices to show that $j^*: L_B^{\max} \to L_A^{\max}$ is surjective. So pick $\alpha \in L_A^{\max}$.

First suppose that $\alpha(2, 1) = \infty$, i.e., $\alpha = e(x)$ for some $x \in X(\mathbf{C})$. With the evaluation map $x^*: A \to \mathbf{C}: a \to a(x)$ we have the commutative square

$$\begin{tikzcd} k \arrow[r, "j_k"] \arrow[d, "\cap"'] & A \arrow[d, "x^*"] \\ \mathbf{C} \arrow[r, "="] & \mathbf{C} \end{tikzcd}$$

(cf. proposition 1). This yields a homomorphism $f: B \to \mathbf{C}$ such that $x^* = fj$ and $\mathrm{id}_{\mathbf{C}} = fj_{\mathbf{C}}$. Therefore $(f(a_1), \dots, f(a_n)) = \in Y(\mathbf{C})$ and $j^*(f(a_1), \dots, f(a_n)) = x$.

Now suppose that $\alpha(2, 1) = 1$, i.e., $\alpha \in K_A^{\max}$ and $j_k^*(\alpha) = \tau_k$. α is defined by a Krull valuation $v: A(\operatorname{supp}(\alpha))^* \to \Gamma$. As in the proof of proposition 4, the valued field $(A(\operatorname{supp}(\alpha)), v)$ is embedded into a formal power series field $C((\tilde{\Gamma}))$ with its natural valuation ([17], section 4). Again, $\tilde{\Gamma}$ is the divisible hull of Γ and C is algebraically closed, $|C| \geqq |\mathbf{C}|$. We may assume that the image of $k \to A \to A(\operatorname{supp}(\alpha)) \to C((\tilde{\Gamma}))$ is contained in C. By the cardinality hypothesis, $k \to C$ can be extended to $\mathbf{C} \to C$. The commutative diagram

$$\begin{tikzcd} k \arrow[r] \arrow[d] & A \arrow[d] \\ \mathbf{C} \arrow[r] & C \subset C((\tilde{\Gamma})) \end{tikzcd}$$

yields a homomorphism $(f\colon B\to C((\tilde{\Gamma})))\in \mathrm{Spev}^{\circ}(B)$ such that

$$\begin{tikzcd} A \arrow[rr, "j"] \arrow[dr] & & B \arrow[dl, "f"] \\ & C((\tilde{\Gamma})) & \end{tikzcd}$$

commutes. By construction we have $\alpha_f\in K_B^{\max}$ (cf. theorem I 31) and $j_{\mathbf{C}}^*(\alpha_f)=\tau_{\mathbf{C}}$. corollary 5 shows that $\alpha_f\in L_B^{\max}$. Since $j^*(\alpha_f)=\alpha$ by construction, the claim is proved. □

It is a consequence of theorem 6 that $L_A^{\max}$ is a compactification of $X(\mathbf{C})$. This compactification will be related to the Morgan—Shalen compactification in the next section.

2. *The Morgan—Shalen compactification.* First the construction of Morgan and Shalen ([22]) is briefly recalled:

The space $X(\mathbf{C})$ is locally compact. So $X(\mathbf{C})$ has a one point compactification $X(\mathbf{C})^+=X(\mathbf{C})\cup\{+\}$. Let $j\colon X(\mathbf{C})\to X(\mathbf{C})^+$ be the inclusion.

Pick a subset $\mathfrak{F}\subset A$ generating A over k. A map $\theta'\colon X(\mathbf{C})\to(\mathbf{R}^{\geqq})^{\mathfrak{F}}$ is defined by $x\to(\log(|f(x)|+2))_{f\in\mathfrak{F}}$. The group $\mathbf{R}^{>}$ operates on $(\mathbf{R}^{>})^{\mathfrak{F}}\setminus\{0\}$ by $s(r(f)_{f\in\mathfrak{F}})=(sr(f))_{f\in\mathfrak{F}}$. The set of equivalence classes is denoted by $\mathfrak{P}(\mathfrak{F})$, the canonical map $(\mathbf{R}^{\geqq})^{\mathfrak{F}}\setminus\{0\}\to\mathfrak{P}(\mathfrak{F})$ by $p(\mathfrak{F})$. $(\mathbf{R}^{\geqq})^{\mathfrak{F}}$ is equipped with the product topology, $\mathfrak{P}(\mathfrak{F})$ with the quotient topology. The map $\theta\colon X(\mathbf{C})\to\mathfrak{P}(\mathfrak{F})$ is defined to be the composition $p(\mathfrak{F})\theta'$. This map is continuous and has compact image ([22], proposition I 3.1).

The two maps $j\colon X(\mathbf{C})\to X(\mathbf{C})^+$ and $\theta\colon X(\mathbf{C})\to\mathfrak{P}(\mathfrak{F})$ together define the continuous map $(j,\theta)\colon X(\mathbf{C})\to X(\mathbf{C})^+\times\mathfrak{P}(\mathfrak{F})$. If the range of (j,θ) is restricted to the closure X of $im(j,\theta)$ then the resulting map $X(\mathbf{C})\to\hat{X}$ is denoted by η. $\hat{X}$ is compact and η maps X homeomorphically onto $\eta(X)$ ([22], p. 415), so $\hat{X}$ is a compactification of X. If $p\colon\hat{X}\to X(\mathbf{C})^+\times\mathfrak{P}(\mathfrak{F})\to X(\mathbf{C})^+$ is the projection then $\hat{X}=\eta(X)\cup B$ with $B=p^{-1}(+)$. B can be considered as a subspace of $\mathfrak{P}(\mathfrak{F})$. To study these additional points in the compactification the valuation theory of the quotient field K of A is used:

The abstract Riemann surface of K over k is denoted by $S=S(K/k)$ ([35], Chapter VI, § 17). Let $S_0\subset S$ be the set of valuation rings V maximal with the property that $A\not\subset V$. If $v\colon K^*\to\Lambda_v$ is the valuation corresponding to $V\in S_0$ then there is a smallest nontrivial convex subgroup $\Lambda_v'\subset\Lambda_v$ and for all $a\in A$, $a\neq 0$ one has $v(a)\leqq 1$ or $v(a)\in\Lambda_v'$. There is some $f\in\mathfrak{F}$ with $v(f)\geqq 1$. By Hölder's theorem ([12], p. 74, Satz 1; [25], p. 8, Satz 4) the archimedean group Λ_v' can and will be considered as a subgroup of $\mathbf{R}^{>}$. This is unique only up to a positive exponent. A map $U'\colon S_0\to(\mathbf{R}^{\geqq})^{\mathfrak{F}}\setminus\{0\}$ is defined by $v\to(\log(\max\{1,v(f)\}))_{f\in\mathfrak{F}}$. Composition with $p(\mathfrak{F})$ gives $U: S_0 \to \mathfrak{P}(\mathfrak{F})$ ([22], p. 416). U is continuous ([22], theorem I 3.4) and maps S_0 onto B ([22], theorem I 3.6).

This last result means that the Morgan—Shalen compactification is obtained by putting two spaces together, namely the space $X(\mathbf{C})$ and the space S_0 of Krull valuations. In [16] it is a nontrivial problem to put these spaces together. This is done by using certain sequences in $X(\mathbf{C})$, called valuating and pre-valuating sequences. In the present paper both $X(\mathbf{C})$ and S_0 belong to the valuation spectrum of A. It will be shown now that the valuation spectrum can be used to construct the compactification of Morgan and Shalen. In this way at least the problem of putting the spaces $X(\mathbf{C})$ and S_0 together vanishes completely.

The homomorphism $k \to K$ induces a morphism $\operatorname{Spev}(K) \to \operatorname{Spev}(k)$ of valuation spectra. By $\operatorname{Spev}(K/k)$ we denote the fibre of this morphism over the trivial valuation of k. As a set this agrees with the abstract Riemann surface $S(K/k)$ ([22], Chapter VI, § 17). However the topologies are different. Let $\mathfrak{T}_1$ be the topology induced on S_0 by $\operatorname{Spev}(K/k)$, $\mathfrak{T}_2$ the topology induced by $S(K/k)$. It is shown in [22], lemma I 3.3 that $(S_0, \mathfrak{T}_2) \to (S_0, \mathfrak{T}_1)$ is continuous.

$$S_0' = \bigcup_{i=1}^{n} \{\alpha \in \operatorname{Spev}(K/k) | \alpha(a_i, 1) = \infty\}$$

is open and constructible in $\operatorname{Spev}(K/k)$, S_0 is the set of closed points of S_0'. Corollary I 29 shows that $(S_0, \mathfrak{T}_1)$ is a compact space.

Let j: $A \to K$ be the canonical injection. Then by functoriality there is the map $\operatorname{Spev}(K) \xrightarrow{j^*} \operatorname{Spev}(A)$ which is an isomorphism onto $\{\alpha \in \operatorname{Spev}(A) | \operatorname{supp}(\alpha) = (0)\}$ (corollary I 12). Therefore $j^*(S_0) \subset \operatorname{Spev}(A)$ is a compact subspace which we identify with S_0. By definition of S_0, $S_0 \subset K_A$. If r: $K_A \to K_A^{\max}$ is the retraction of proposition I 30 then $r(S_0) \subset K_A^{\max}$ is denoted by S_1. If $\alpha \in S_1$ then $j_k^*(\alpha) = \tau_k$ (since the Krull valuations in S_0 are trivial on k). So we see that $S_1 \subset j_k^{*-1}(\tau_k) \cap L_A^{\max}$ (corollary 5). Conversely, if $\alpha \in j_k^{*-1}(\tau_k) \cap L_A^{\max}$ then α is determined by a homomorphism f: $A \to F$, where F is a field with Krull valuation v: $F^* \to \Gamma$ over k, Γ an archimedean ordered group. Moreover, $vf(a_i) > 1$ for some $i \in \{1, \ldots, n\}$. The valued field F can be embedded into a formal power series field $C((\tilde{\Gamma}))$ with its natural valuation (C an algebraically closed field, $\tilde{\Gamma}$ the divisible hull of Γ). The homomorphism $A \xrightarrow{f} F \subset C((\tilde{\Gamma}))$ extends to a place $K \xrightarrow{\lambda} C((\tilde{\Gamma})) \cup \{\infty\}$ ([13], chap. 6, § 2 no. 4, proposition 3). Let $V_1 \subset K$ be the corresponding valuation ring, $V_0 \subset V_1$ the inverse image of the natural valuation ring of $C((\tilde{\Gamma}))$. Since $\tilde{\Gamma}$ is archimedean there are no valuation rings between V_0 and V_1. By construction we have $A \not\subset V_0$, $A \subset V_1$. This proves that $V_0 \in S_0$. Let α_0 be the corresponding point of K_A. Then it is clear that $r(\alpha_0) = \alpha$. We have proved

Proposition 7. *If we identify* $X(\mathbf{C})$ *with* $\{\alpha | \alpha(2, 1) = \infty\} \subset L_A^{\max}$ *(theorem 6) then* $X(\mathbf{C}) = j_k^{*-1}(\varkappa_k) \cap L_A^{\max}$, $S_1 = j_k^{*-1}(\tau_k) \cap L_A^{\max}$ *and* $L_A^{\max} = X(\mathbf{C}) \cup S_1$.

As before we choose subset $\mathfrak{F}\subset A$ generating A over k. We will now define a continuous map $L_A^{\max}\to X(\mathbf{C})^+\times\mathfrak{P}(\mathfrak{F})$ whose image is exactly the compactification of Morgan and Shalen. The definition is done componentwise:

First λ_1: $L_A^{\max}\to X(\mathbf{C})^+$ is defined by $\lambda_1|X(\mathbf{C})=\mathrm{id}$, $\lambda_1(\alpha)=+$ for $\alpha\notin X(\mathbf{C})$. This is clearly continuous.

If $\alpha\in L_A^{\max}$ let v_α: $A\to\mathbf{R}^{\geqq}$ be the corresponding valuation (note that the value group is archimedean). v_α is unique only up to a positive exponent. If $\alpha\in X(\mathbf{C})$ then we can normalize v_α such that $v_\alpha(2)=2$. But if $\alpha\in S_1$ then there is no natural normalization for v_α. So for the time being we do not normalize at all. We define

$$\lambda_2'\colon L_A^{\max}\to(\mathbf{R}^{\geqq})^{\mathfrak{F}}\setminus\{0\}$$
$$\alpha\to\begin{cases}(\log(v_\alpha(f)+2))_{f\in\mathfrak{F}} & \text{if } \alpha\in X(\mathbf{C})\\ (\log(\max\{1,v_\alpha(f)\}))_{f\in\mathfrak{F}} & \text{if } \alpha\in S_1.\end{cases}$$

λ_2: $L_A^{\max}\to\mathfrak{P}(\mathfrak{F})$ is defined to be $p(\mathfrak{F})\lambda_2'$. At this point it becomes obvious why the space $\mathfrak{P}(\mathfrak{F})$ has to be used: lacking a canonical way to normalize the Krull valuations v_α $(\alpha\in S_1)$ we have chosen them arbitrary. So λ_2' cannot be expected to be continuous. Composition of λ_2' with $p(\mathfrak{F})$ makes the construction canonical. For, different choices v_α and v_α^s $(s\in\mathbf{R}^{>})$ for the Krull valuation associated with α yield the same point

$$p(\mathfrak{F})(\log(\max\{1,v_\alpha(f)\})_{f\in\mathfrak{F}})=p(\mathfrak{F})(s(\log(\max\{1,v_\alpha(f)\}))_{f\in\mathfrak{F}})$$

in $\mathfrak{P}(\mathfrak{F})$.

Together λ_1 and λ_2 yield the map

$$\lambda=(\lambda_1,\lambda_2)\colon L_A^{\max}\to X(\mathbf{C})^+\times\mathfrak{B}(\mathfrak{F}).$$

The restriction $\lambda|X(\mathbf{C})$ agrees with the map i: $X(\mathbf{C})\to X(\mathbf{C})^+\times\mathfrak{P}(\mathfrak{F})$ of Morgan and Shalen ([22], p. 415). The composition $S_0\xrightarrow{r}S_1\subset L_A^{\max}\xrightarrow{\lambda}X(\mathbf{C})^+\times\mathfrak{P}(\mathfrak{F})$ agrees with the map U: $S_0\to\{+\}\times\mathfrak{P}(\mathfrak{F})\subset X(\mathbf{C})^+\times\mathfrak{P}(\mathfrak{F})$ of Morgan and Shalen ([22], p. 416). So it has already been shown that λ maps $L_A^{\max}$ onto the compactification of Morgan and Shalen. Of course λ can play a reasonable role only if it is continuous. This will be proved now:

We noted above that λ_1 is continuous. It remains to consider λ_2. If $\mathfrak{E}\subset A$ is another subset containing $\mathfrak{F}$ then v_2': $L_A^{\max}\to(\mathbf{R}^{\geqq})^{\mathfrak{E}}$ and $v_2=p(\mathfrak{E})v_2'$ are defined exactly as λ_2' and λ_2. The commutative diagram

$$\begin{tikzcd}
 & (\mathbf{R}^{\geqq})^{\mathfrak{E}} \arrow[r, "p(\mathfrak{E})"] \arrow[d, "pr"] & \mathfrak{B}(\mathfrak{E}) \arrow[d, "pr"] \\
L_A^{\max} \arrow[ur, "v_2'"] \arrow[dr, "\lambda_2'"'] & (\mathbf{R}^{\geqq})^{\mathfrak{F}} \arrow[r, "p(\mathfrak{F})"] & \mathfrak{P}(\mathfrak{F}) \\
\end{tikzcd}$$

(with pr the canonical projections) shows that it suffices to prove that ν_2 is continuous. Therefore we assume now that $\mathfrak{F}=A$ and prove that λ is continuous in this case.

For every $i=0, \ldots, n$ the subsets

$$U_i = \{(r(f)) \mid r(a_i^2) > r(a_i)\} \subset (\mathbf{R}^{\geqq})^{\mathfrak{F}},$$
$$V_i = p(\mathfrak{F})(U_i) \subset \mathfrak{P}(\mathfrak{F})$$

are open. By definition of λ_2' we have

$$W_i = \lambda_2^{-1}(V_i) = \lambda_2'^{-1}(U_i) = \{\alpha \in L_A^{\max} \mid \alpha(a_i, 1) = \infty\}$$

which is an open subset of $L_A^{\max}$. We must prove that the restriction $\lambda_{2,i}$: $W_i \to V_i$ is continuous for all $i=0, \ldots, n$. Since $W_0 = X(\mathbf{C})$ it is immediately clear from the definition that $\lambda_{2,0}$ is continuous. So suppose that $1 \leqq i \leqq n$. For $\alpha \in W_i$ now we normalize v_α such that $\log(v_\alpha(a_i)+2)=1$ if $\alpha \in X(\mathbf{C})$ and $\log(v_\alpha(a_i))=1$ if $\alpha \in S_1$. As explained above, $\lambda_{2,i}$ is not affected by this normalization at all. However

$$\lambda_{2,i}'\colon W_i \to U_i\colon \alpha \to \begin{cases} (\log(v_\alpha(a)+2))_{a\in\mathfrak{F}} & \text{if} \quad \alpha \in X(\mathbf{C}) \\ (\log(\max\{1, v_\alpha(a)\}))_{a\in\mathfrak{F}} & \text{if} \quad \alpha \in S_1 \end{cases}$$

is now continuous: To prove this we fix some $a \in \mathfrak{F}$ and show that

$$\lambda_{2,i,a}'\colon W_i \to \mathbf{R}^{\geqq}\colon \alpha \to \begin{cases} \log(v_\alpha(a)+2) & \text{if} \quad \alpha \in X(\mathbf{C}) \\ \log(\max\{1, v_\alpha(a)\}) & \text{if} \quad \alpha \in S_1 \end{cases}$$

is continuous.

If $\alpha \in X(\mathbf{C})$ then continuity at α is clear since $v_\alpha(a)$ is $|a(\alpha)|$ up to a positive exponent. So suppose that $\alpha \in W_i \cap S_1$. If $\lambda_{2,i,a}(\alpha) < r$ for some $r \in \mathbf{R}^{>}$ then there is a neighborhood $W \subset W_i$ of α such that $\beta \in W$ implies $\lambda_{2,i,a}(\beta) < r$. To see this pick $s \in (\lambda_{2,i,a}'(\alpha), r) \cap \mathbf{Q}$ and write $s = \frac{t}{u}$ with $t, u \in \mathbf{N}$. If we define

$$W_{k,i} = \{\beta \in W_i \mid \beta(a_i^t, (2a)^u) = \infty\}$$

then

$$u\lambda_{2,i,a}'(\alpha) = \log(\max\{1, v_\alpha(a^u)\}) = \log(\max\{1, v_\alpha(2a)^l\}) < \log(v_\alpha(a_i^t)) = t$$

implies that $\alpha \in W_{t,u}$. Clearly, $W_{t,u}$ is open. If $\beta \in W_{t,u} \cap S_1$ then an easy computation shows that $\lambda_{2,i,a}(\beta) < s$. Finally we have to deal with $\alpha \in W_{t,u} \cap X(\mathbf{C})$. We set

$$W_{t,u}' = \{\beta \in W_{t,u} \mid \beta(a_i^t, 4^u) = \infty\}.$$

Since $\alpha \in S_1$ we know that v_α is a Krull valuation over k and hence $W_{t,u}$ is an open neighborhood of α. If $\beta \in W_{t,u}' \cap X(\mathbf{C})$ is such that $|a(\beta)| \geqq 2$ then

$$(|a(\beta)|+2)^u \leqq |2a(\beta)|^u < |a_i^t(\beta)| < (|a_i(\beta)|+2)^t.$$

If $\beta \in W'_{t,u} \cap X(\mathbf{C})$ is such that $|a(\beta)|<2$ then

$$(|a(\beta)|+2)^u < 4^u \leqq |a_i^t(\beta)| \leqq (|a_i(\beta)|+2)^t.$$

In any event we have

$$u\lambda'_{2,i,a}(\beta) = \log((|a(\beta)|+2)^u)$$
$$< \log((|a_i(\beta)|+2)^t) = t,$$

and $\lambda'_{2,i,a}(\beta)<s$ for all $\beta \in W'_{t,u}$.

Similar computations show that there is an open neighborhood $W \subset W_i$ of α with $\lambda'_{2,i,a}(W) \subset (r, \infty)$ if $\lambda'_{2,i,a}(\alpha)>r$ for $r \in \mathbf{R}^>$.

We see that, for every $a \in \mathfrak{F}$, $\lambda'_{2,i,a}$ is continuous. This implies continuity of $\lambda'_{2,i}$. Finally λ_2 is continuous. We have proved

Theorem 8. *The Morgan—Shalen compactification of $X(\mathbf{C})$ is a continuous image of the compactification $L_A^{\max}$ of $X(\mathbf{C})$.*

In general the Morgan—Shalen compactification is a proper image of $L_A^{\max}$. However there is one case in which $L_A^{\max}$ is actually equal to the Morgan—Shalen compactification:

Theorem 9. *If $\mathfrak{F}=A$ then the continuous map λ: $L_A^{\max}X(\mathbf{C})^+ \times \mathfrak{P}(\mathfrak{F})$ is a homeomorphism onto the Morgan—Shalen compactification.*

Proof. We only have to prove that λ is injective. Pick $\alpha, \beta \in L_A^{\max}$ with $\alpha \neq \beta$. If $\alpha, \beta \in X(\mathbf{C})$ then $\lambda(\alpha)=(\alpha, \lambda_2(\alpha)) \neq (\beta, \lambda_2(\beta))=\lambda(\beta)$. If $\alpha \in X(\mathbf{C})$, $\beta \in S_1$ then $\lambda(\alpha)=(\alpha, \lambda_2(\alpha)) \neq (+, \lambda_2(\beta))=\lambda(\beta)$.

Finally suppose that $\alpha, \beta \in S_1$. Assume by way of contradiction that $\lambda(\alpha)=\lambda(\beta)$. Then $v_\alpha(a)=0$ if and only if $v_\beta(a)=0$, i.e., $\mathrm{supp}(\alpha)=\mathrm{supp}(\beta)$. Suppose that $\alpha \in W_i$ (notation as in the proof of theorem 8). Then $\lambda_2(\alpha)=\lambda_2(\beta)$ implies

$$(\log(v_\alpha(a_i)) = \log(\max\{1, v_\alpha(a_i)\}) \colon \log(v_\alpha(1)) = 0)$$
$$= (\log(\max\{1, v_\beta(a_i)\} \colon \log(v_\beta(1)) = 0).$$

Since $\log(v_\alpha(a_i))>0$ we see that $\log(\max\{1, v_\beta(a_i)\}>0)$. This is possible only if $v_\beta(a_i)>1$, i.e., $\beta(a_i, 1)=\infty$.

As in the proof of theorem 8 we now assume that v_α and v_β are normalized such that $\log(v_\alpha(a_i))=1=\log(v_\beta(a_i))$. Now pick $a \in A \setminus \mathrm{supp}(\alpha)$. If $v_\alpha(a)>1$ then

$$(\log(\max\{1, v_\alpha(a)\}) \colon \log(v_\alpha(1)) = 0) = (\log(\max\{1, v_\beta(a)\}) \colon \log(v_\beta(1)) = 0)$$

implies that $\log(\max\{1, v_\beta(a)\})>0$, which is possible only if $v_\beta(a)>1$. Since α and β have been normalized the equality $\lambda_2(\alpha)=\lambda_2(\beta)$ implies $v_\alpha(a)=v_\beta(a)$. Now assume that $v_\alpha(a) \leqq 1$. Then (with $p, q \in \mathbf{N}$) $\log(v_\alpha(a^p a_i^q))>0$ if and only if $p\log(v_\alpha(a))>-q$, i.e., if and only if $\log(v_\alpha(a))>-\frac{q}{p}$. Since $\mathfrak{F}=A$, $\lambda_\alpha(\alpha)$ determines the set

$$D_\alpha(a) = \{(p, q) \in \mathbf{N}\times\mathbf{N} \mid \log(v_\alpha(a^p a_1^q)) > 0\}.$$

So $\lambda_\alpha(\alpha)$ also determines

$$\log(v_\alpha(a)) = \sup\left\{-\frac{q}{p}\,\middle|\,(p, q)\in D(a)\right\}.$$

The same also holds with β instead of α. Because of $\lambda_2(\alpha)=\lambda_2(\beta)$ we see that $D_\alpha(a)= D_\beta(a)$. This proves that $v_\alpha(a)=v_\beta(a)$.

Finally, if $a, b\in A\setminus\operatorname{supp}(\alpha)$ then it has been shown that $v_\alpha(a)=v_\beta(a)$ and $v_\alpha(b)=v_\beta(b)$. This implies that $\alpha(a, b)=\beta(a, b)$. □

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