On a theorem of Marcinkiewicz and Zygmund for Taylor series

E. S. Katsoprinakis

Abstract

Let K be the class of trigonometric series of power type, i.e. Taylor series $\sum_{n=0}^{\infty} c_n z^n$ for $z=e^{ix}$, whose partial sums for all x in E, where E is a nondenumerable subset of $[0, 2\pi)$, lie on a *finite* number of circles (a priori depending on x) in the complex plane. The main result of this paper is that for every member of the class K, there exist a complex number ω, $|\omega|=1$, and two positive integers ν, $\varkappa$, $\nu<\varkappa$, such that for the coefficients c_n we have:

$$c_{\mu+\lambda(\varkappa-\nu)} = c_\mu \omega^\lambda, \quad \mu = \nu, \nu+1, \dots, \varkappa-1, \quad \lambda = 1, 2, 3, \dots .$$

Thus, every member of the class K has (with minor modifications) a representation of the form:

$$\mathsf{P}(x) \sum_{n=0}^{\infty} e^{iknx},$$

where $\mathsf{P}(x)$ is a suitable trigonometric polynomial and k a positive integer. The proof is elementary but rather long. This result is closely related to a theorem of Marcinkiewicz and Zygmund on the circular structure of the set of limit points of the sequence of partial sums of $(C, 1)$ summable Taylor series.

1. Introduction

Let

$$\sum_{n=0}^{\infty} c_n e^{inx}, \quad x \text{ in } [0, 2\pi), \tag{1.1}$$

be a trigonometric series of power type, i.e. Taylor series. The partial sums and the Cesàro means of (1.1) will be denoted by $s_n(x)$ and $\sigma_n(x)$ respectively, i.e.

$$s_n(x) = \sum_{\nu=0}^{n} c_\nu e^{i\nu x}, \quad \sigma_n(x) = \frac{1}{n+1} \sum_{\nu=0}^{n} s_\nu(x).$$

It will be convenient to introduce some more notation. By $C(z; r)$ we shall mean the circle (circumference) with centre z and radius $r \geqq 0$. By $L(x)$ we shall denote the set of limit points (including the point at infinity) of the sequence of partial sums of (1.1). $L(x)$ is a closed set which reduces to a single point if (1.1) converges (or diverges to infinity). If (1.1) is $(C, 1)$ summable to a (finite) sum $\sigma(x)$, i.e. $\lim_{n\to\infty} \sigma_n(x) = \sigma(x)$, x fixed, then we write: $m(x) = \liminf_{n\to\infty} r_n(x)$, $M(x) = \limsup_{n\to\infty} r_n(x)$, where $r_n(x) = |s_n(x) - \sigma(x)|$. The Lebesgue measure of a set E will be denoted by $m(E)$ and the cardinality of E by card E. Finally, we say that a set S in the complex plane is of "circular structure", if there is a point z_0 (centre of S), such that whenever a point z belongs to S, so does the whole circle $C(z_0; |z - z_0|)$. In other words, S is the union of a finite, denumerable or non-denumerable set of circles with common centre z_0 and radii $\geqq 0$.

A celebrated theorem of Marcinkiewicz and Zygmund states that: "If the series (1.1) is $(C, 1)$ summable to a (finite) sum $\sigma(x)$ for all x in a subset E of $[0, 2\pi)$, then for almost all x in E the set $L(x)$ is of circular structure with centre $\sigma(x)$ and extreme radii $m(x)$ and $M(x)$". (See [2] and [3] V.II, p. 178, for the proof and comments on this theorem). In connection with this result A. Zygmund asked the question: "If the series (1.1) is as in the above theorem and $m(E) > 0$, is the angular distribution, around $\sigma(x)$, of the limit points of the sequence $\{s_n(x)\}$ uniform?". The question is not exactly stated, especially when $\sigma(x) \in L(x)$, but a precise statement is one of the difficulties of the problem.

A first step in this direction and, as far as the author knows, the only one in the literature, is a recent result of J.-P. Kahane (see [1]). Roughly speaking J.-P. Kahane introduces for each complex sequence $\{z_n\}$ and each compact subset K of $\bar{\mathbf{C}}$ an "upper density" $d\{(z_n), K\}$, and following the general lines of the Marcinkiewicz and Zygmund proof, he arrives at the following theorem: "If (1.1) is $(C, 1)$ summable to a (finite) sum $\sigma(x)$ for all x in a subset E of $[0, 2\pi)$, $m(E) > 0$, then for almost all $x \in E$ we have: $d\{(s_n(x)), K_1\} = d\{(s_n(x)), K_2\}$ for all pairs of compact subsets K_1, K_2 of $\bar{\mathbf{C}}$ which are obtained from each other by rotation around $\sigma(x)$".

Marcinkiewicz and Zygmund illustrate their theorem by giving several examples for which it is easy to verify that for almost all x in $[0, 2\pi)$ the corresponding sequences $\{\arg[s_n(x) - \sigma(x)]\}$ are uniformly distributed (see § 5, Remark 4). Some of these examples lead to series of the form: $\mathsf{P}(x) \sum_{n=0}^{\infty} e^{insx}$, where $\mathsf{P}(x)$ is a suitable trigonometric polynomial and s a positive integer. In this case, except for a finite number of x's in $[0, 2\pi)$, $L(x)$ is a union of a *finite* number of concentric circles. At this point one can ask: "Is the converse true?". The answer is no. For example if we add the series $\sum e^{inx}$ to an everywhere convergent series, then for $x \not\equiv q\pi$, $q \in \mathbf{Q}$, the set $L(x)$ is one circle, but the resulting series is no longer of the above form. However, series of the above form have a special property; Not only the limit points of $\{s_n(x)\}$ are situated on a *finite* number of concentric circles, but the same is true for the sequence $\{s_n(x)\}$ itself. The main result of this paper is that the answer to the above question is "yes" if we take into account this special property. More precisely we prove:

Theorem 1. *Assume that there is a nondenumerable subset E of $[0, 2\pi)$ such that for each $x \in E$ the partial sums of* (1.1) *lie on a finite number of circles in the complex plane (the number of circles, their centres and radii a priori may depend on x). Then:*

(i) *There are two positive integers ν, $\varkappa$, $\nu < \varkappa$, and a real number ϑ, such that the series* (1.1) *has the form:*

$$P_0(x) + e^{i\nu x} P(x) \sum_{\lambda=0}^{\infty} e^{i\lambda(\varkappa-\nu)(x+\vartheta)}, \tag{1.2}$$

where

$$P_0(x) = c_0 + c_1 e^{ix} + c_2 e^{2ix} + \ldots + c_{\nu-1} e^{i(\nu-1)x} \tag{1.3}$$

and

$$P(x) = c_\nu + c_{\nu+1} e^{ix} + c_{\nu+2} e^{2ix} + \ldots + c_{\varkappa-1} e^{i(\varkappa-\nu-1)x}. \tag{1.4}$$

(ii) *There is a positive integer m, such that, for all but a finite number of x's in* $[0, 2\pi)$, *the partial sums $s_n(x)$ of* (1, 1), *with $n \geqq \nu$, lie on exactly m concentric circles.*

We observe that $(C, 1)$ summability is not needed but it follows from the theorem. We return to this point in the last paragraph of this paper where we offer also more remarks and consequences of Theorem 1.

Acknowledgements. This paper is the main part of the author's thesis made at the University of Crete (Greece) under the guidance of S. K. Pichorides, who suggested the problem of representation of $(C, 1)$ summable Taylor series by finite combinations of geometric progressions. For this reason as well as for several other useful suggestions I would like to express my gratitude to him. I am indebted to J.-P. Kahane and V. Nestoridis whose remarks on the cardinality of the set E are included in this paper (see addendum). I would also like to thank V. Nestoridis and S. Papadopoulou who read carefully the whole manuscript and made several useful remarks, which greatly improved the presentation.

2. A basic lemma and the case of one circle

As a preparation for the proof of theorem 1, which is given in § 3, we prove a basic lemma and the special case of partial sums lying on one circle. Strictly speaking this special case is not needed, but we think that it will clarify the role of the basic lemma.

We start with a remark for the partial sums of the series (1.1), which is useful for the proof.

If $n<m$ and $s_n(x)=s_m(x)$ then either $c_{n+1}=c_{n+2}=\ldots=c_m=0$, or $x\in F$, where F is a *finite* subset of $[0, 2\pi)$. Since we deal only with infinite subsets of $[0, 2\pi)$, we may suppose (excluding countably many x's if needed), that only the first case can arise here. *Thus when $s_n(x)$ and $s_m(x)$, $n<m$, differ (on an infinite set), then at least one of the coefficients $c_{n+1}, c_{n+2}, \ldots, c_m$ is not zero, and vice versa.*

Let A be a nondenumerable subset of $[0, 2\pi)$ and n, m, $n<m$, two positive integers, such that:

(i) $c_n c_m \neq 0$.

(ii) For each $x\in A$ there is a circle $C(z(x); r(x))$ (or a straight line, when $z(x)=\infty$), which passes through $s_j(x)$, $s_n(x)$, $s_m(x)$, $s_k(x)$, for some $j=j(x)<n$ and $k=k(x)>m$.

(iii) $s_k(x)\neq s_m(x)$.

Because of (i), (iii) and the preceding remark these four partial sums are distinct points of the complex plane.

Lemma 1. *Under the above assumptions we have:*

If $c_{n+1}=\ldots=c_{n+q}=0$, $c_{n+q+1}\neq 0$, $q\geqq 0$ ($q=0$ *means* $c_{n+1}\neq 0$), *then*

(a) $c_{m+1}=\ldots=c_{m+q}=0$, $c_{m+q+1}\neq 0$.

(b) *There is a nondenumerable subset B of A and nonnegative integers $N<n$ and $M>m+q$, such that $s_N(x)$, $s_n(x)$, $s_m(x)$, $s_M(x)$ lie on a circle for every x in B. Moreover, for any such N, M and for each s, where $s=1, 2, \ldots, \min(n-N, m-n-q, M-m-q)$, we have:*

$$\sum_{j=0}^{s-1} \bar{c}_{n-j} c_{n+q+s-j} = \sum_{j=0}^{s-1} \bar{c}_{m-j} c_{m+q+s-j}. \tag{2.1}$$

Proof. For every $x\in A$ consider the n circles (or straight lines) C_j, $j=0, 1, 2, \ldots, n-1$, where C_j passes through $s_n(x)$, $s_m(x)$ and $s_j(x)$. Because of (ii) there is at least one $k=k(x)$, $k>m$, such that $s_k(x)\in C_j$ for at least one $j=0, 1, 2, \ldots, n-1$. Let

$$A_{j,k} = \{x\in A\colon\ s_k(x)\in C_j\},\quad j = 0, 1, \ldots, n-1,\quad k = m+1, m+2, \ldots.$$

Since the union of all $A_{j,k}$'s is A and A is nondenumerable, the same is true for some $A_{N,M}$, i.e. there are $N<n$ and $M>m$, independent of x, such that $A_{N,M}$ is nondenumerable. We write B for such an $A_{N,M}$. We can obviously assume $c_{N+1}\neq 0$ and $c_M\neq 0$. Thus, we have four distinct partial sums $s_N(x)$, $s_n(x)$, $s_m(x)$, $s_M(x)$, of (1.1), which are situated on a circle (or a straight line) for every x in B. We conclude that the cross-ratio:

$$\frac{s_N(x)-s_n(x)}{s_m(x)-s_n(x)} : \frac{s_N(x)-s_M(x)}{s_m(x)-s_M(x)}$$

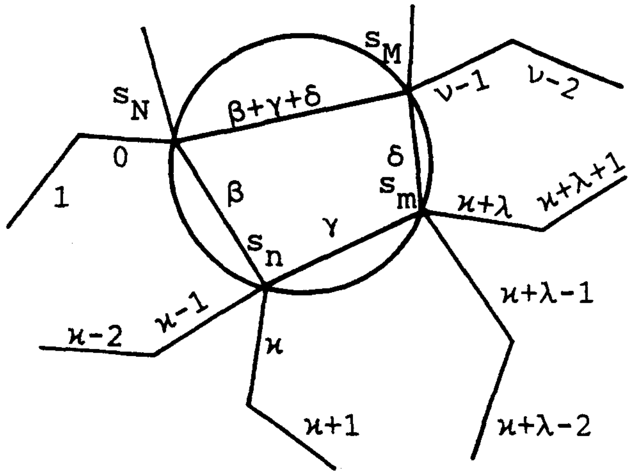

Figure 1

is real for every point x in B. (In Figure 1 for simplicity we write ϱ instead of $a_\varrho e^{i(N+\varrho+1)x}$.) We write:

$$\beta = s_n(x)-s_N(x), \quad \gamma = s_m(x)-s_n(x), \quad \delta = s_M(x)-s_m(x)$$

$$a_0 = c_{N+1},\ a_1 = c_{N+2}, \dots, a_{\varkappa-1} = c_n; \quad a_\varkappa = c_{n+1}, a_{\varkappa+1} = c_{n+2}, \dots, a_{\varkappa+\lambda-1} = c_m;$$

$$a_{\varkappa+\lambda} = c_{m+1},\ a_{\varkappa+\lambda+1} = c_{m+2}, \dots, a_{\nu-1} = c_M, \quad \text{where} \quad \nu = \varkappa+\lambda+\mu.$$

Since $c_{N+1}c_M \neq 0$ we have $a_0 a_{\nu-1} \neq 0$. Then, for every x in B, we have that $\beta\bar{\gamma}\delta(\overline{\beta+\gamma+\delta})$ is real. This means that the expression:

$$\begin{aligned}&(a_0e^{i(N+1)x}+a_1e^{i(N+2)x}+\dots+a_{\varkappa-1}e^{i(N+\varkappa)x})(\bar{a}_\varkappa e^{-i(N+\varkappa+1)x}\\&+\bar{a}_{\varkappa+1}e^{-i(N+\varkappa+2)x}+\dots+\bar{a}_{\varkappa+\lambda-1}e^{-i(N+\varkappa+\lambda)x})(a_{\varkappa+\lambda}e^{i(N+\varkappa+\lambda+1)x}\\&+a_{\varkappa+\lambda+1}e^{i(N+\varkappa+\lambda+2)x}+\dots+a_{\nu-1}e^{i(N+\nu)x})(\bar{a}_0e^{-i(N+1)x}\\&+\bar{a}_1e^{-i(N+2)x}+\dots+\bar{a}_{\nu-1}e^{-i(N+\nu)x})\end{aligned}$$

equals its conjugate, for all x in B. This in turn leads easily to:

$$\begin{aligned}&(a_0+a_1e^{ix}+\dots+a_{\varkappa-1}e^{i(\varkappa-1)x})(\bar{a}_{\varkappa+\lambda-1}+\bar{a}_{\varkappa+\lambda-2}e^{ix}+\dots+\bar{a}_\varkappa e^{i(\lambda-1)x})(a_{\varkappa+\lambda}\\&+a_{\varkappa+\lambda+1}e^{ix}+\dots+a_{\nu-1}e^{i(\mu-1)x})(\bar{a}_{\nu-1}+\bar{a}_{\nu-2}e^{ix}+\dots+\bar{a}_0e^{i(\nu-1)x})\\&=(\bar{a}_{\varkappa-1}+\bar{a}_{\varkappa-2}e^{ix}+\dots+\bar{a}_0e^{i(\varkappa-1)x})(a_\varkappa+a_{\varkappa+1}e^{ix}+\dots+a_{\varkappa+\lambda-1}e^{i(\lambda-1)x})(\bar{a}_{\nu-1}\\&+\bar{a}_{\nu-2}e^{ix}+\dots+\bar{a}_{\varkappa+\lambda}e^{i(\mu-1)x})(a_0+a_1e^{ix}+\dots+a_{\nu-1}e^{i(\nu-1)x}),\end{aligned}$$

for all x in B.

Let now $c_{n+j}=0$, $j=1, 2, 3, \ldots, q$, $c_{n+q+1}\neq 0$ or, in the new notation, $a_\varkappa= a_{\varkappa+1}=\ldots=a_{\varkappa+q-1}=0$, $a_{\varkappa+q}\neq 0$. Then we have:

$$(2.2)\quad \begin{cases}(a_0+a_1e^{ix}+\ldots+a_{\varkappa-1}e^{i(\varkappa-1)x})(\bar a_{\varkappa+\lambda-1}+\bar a_{\varkappa+\lambda-2}e^{ix}+\ldots+\bar a_{\varkappa+q}e^{i(\lambda-q-1)x})\\ (a_{\varkappa+\lambda}+a_{\varkappa+\lambda+1}e^{ix}+\ldots+a_{\varkappa+\lambda+q-1}e^{i(q-1)x}+a_{\varkappa+\lambda+q}e^{iqx}+\ldots+a_{\nu-1}e^{i(\mu-1)x})\\ (\bar a_{\nu-1}+\bar a_{\nu-2}e^{ix}+\ldots+\bar a_0e^{i(\nu-1)x})=\\ (\bar a_{\varkappa-1}+\bar a_{\varkappa-2}e^{ix}+\ldots+\bar a_0e^{i(\varkappa-1)x})(a_{\varkappa+q}+a_{\varkappa+q+1}e^{ix}+\ldots+a_{\varkappa+\lambda-1}e^{i(\lambda-1)x})\\ (\bar a_{\nu-1}+\bar a_{\nu-2}e^{ix}+\ldots+\bar a_{\varkappa+\lambda}e^{i(\mu-1)x})(a_0+a_1e^{ix}+\ldots+a_{\nu-1}e^{i(\nu-1)x})e^{iqx},\end{cases}$$

for all x in B.

The second member of (2.2) is a trigonometric polynomial whose term of lowest degree is $\bar a_{\varkappa-1}a_{\varkappa+q}\bar a_{\nu-1}a_0e^{iqx}$. Since the coefficient of this term is not zero and $a_0a_{\varkappa+\lambda-1}a_{\nu-1}\neq 0$, (2.2) gives $a_{\varkappa+\lambda}=c_{m+1}=0$. In the same way we obtain $a_{\varkappa+\lambda+1}= c_{m+2}=0, \ldots, a_{\varkappa+\lambda+q-1}=c_{m+q}=0$ and finally $a_0\bar a_{\varkappa+\lambda-1}a_{\varkappa+\lambda+q}\bar a_{\nu-1}=\bar a_{\varkappa-1}a_{\varkappa+q}\bar a_{\nu-1}a_0$, $a_0a_{\nu-1}\neq 0$. Hence:

$$(2.3)\qquad \bar a_{\varkappa+\lambda-1}a_{\varkappa+\lambda+q}=\bar a_{\varkappa-1}a_{\varkappa+q} \quad\text{or}\quad \bar c_nc_{n+q+1}=\bar c_mc_{m+q+1},$$

which gives $c_{m+q+1}\neq 0$ and completes the proof of (a).

In order to prove (b), we set first in (2.2) $a_{\varkappa+\lambda}=a_{\varkappa+\lambda+1}=\ldots=a_{\varkappa+\lambda+q-1}=0$. Then, if $\varrho=\min(\varkappa, \lambda-q, \mu-q)$, (2.2) takes the form:

$$(2.4)\quad (P_1+Q_1)(P_2+Q_2)(P_3+Q_3)(P_4+Q_4)=(P_2^*+Q_2^*)(P_3^*+Q_3^*)(P_4+Q_4^*)(P_1+Q_1^*)$$

where,

$$\begin{aligned}
P_1&=P_1(x)=a_0+a_1e^{ix}+\ldots+a_{\varrho-1}e^{i(\varrho-1)x};\\
Q_1&=Q_1(x)=a_\varrho e^{i\varrho x}+a_{\varrho+1}e^{i(\varrho+1)x}+\ldots+a_{\varkappa-1}e^{i(\varkappa-1)x};\\
P_2&=P_2(x)=\bar a_{\varkappa+\lambda-1}+\bar a_{\varkappa+\lambda-2}e^{ix}+\ldots+\bar a_{\varkappa+\lambda-\varrho}e^{i(\varrho-1)x};\\
Q_2&=Q_2(x)=\bar a_{\varkappa+\lambda-\varrho-1}e^{i\varrho x}+\bar a_{\varkappa+\lambda-\varrho-2}e^{i(\varrho+1)x}+\ldots+\bar a_{\varkappa+q}e^{i(\lambda-q-1)x};\\
P_3&=P_3(x)=a_{\varkappa+\lambda+q}+a_{\varkappa+\lambda+q+1}e^{ix}+\ldots+a_{\varkappa+\lambda+q+\varrho-1}e^{i(\varrho-1)x};\\
Q_3&=Q_3(x)=a_{\varkappa+\lambda+q+\varrho}e^{i\varrho x}+a_{\varkappa+\lambda+q+\varrho+1}e^{i(\varrho+1)x}+\ldots+a_{\nu-1}e^{i(\mu-q-1)x};\\
P_4&=P_4(x)=\bar a_{\nu-1}+\bar a_{\nu-2}e^{ix}+\ldots+\bar a_{\nu-\varrho}e^{i(\varrho-1)x};\\
Q_4&=Q_4(x)=\bar a_{\nu-\varrho-1}e^{i\varrho x}+\ldots+\bar a_0e^{i(\nu-1)x};\\
P_2^*&=P_2^*(x)=\bar a_{\varkappa-1}+\bar a_{\varkappa-2}e^{ix}+\ldots+\bar a_{\varkappa-\varrho}e^{i(\varrho-1)x};\\
Q_2^*&=Q_2^*(x)=\bar a_{\varkappa-\varrho-1}e^{i\varrho x}+\ldots+\bar a_0e^{i(\varkappa-1)x};\\
P_3^*&=P_3^*(x)=a_{\varkappa+q}+a_{\varkappa+q+1}e^{ix}+\ldots+a_{\varkappa+q+\varrho-1}e^{i(\varrho-1)x};\\
Q_3^*&=Q_3^*(x)=a_{\varkappa+q+\varrho}e^{i\varrho x}+a_{\varkappa+q+\varrho+1}e^{i(\varrho+1)x}+\ldots+a_{\varkappa+\lambda-1}e^{i(\lambda-q-1)x};\\
Q_4^*&=Q_4^*(x)=\bar a_{\nu-\varrho-1}e^{i\varrho x}+\bar a_{\nu-\varrho-2}e^{i(\varrho+1)x}+\ldots+\bar a_{\varkappa+\lambda+q}e^{i(\mu-q-1)x};\\
Q_1^*&=Q_1^*(x)=a_\varrho e^{i\varrho x}+a_{\varrho+1}e^{i(\varrho+1)x}+\ldots+a_{\nu-1}e^{i(\nu-1)x}.
\end{aligned}$$

Of course,

(i) if $\varrho = \varkappa$, then $Q_1 = Q_2^* = 0$,

(ii) if $\varrho = \lambda - q$, then $Q_2 = Q_3^* = 0$,

(iii) if $\varrho = \mu - q$, then $Q_3 = Q_4^* = 0$.

For every x in B (2.4) gives:

$$P_1 P_2 P_3 P_4 + P_1 P_2 P_3 Q_4 + P_1 P_2 Q_3 (P_4 + Q_4) + P_1 Q_2 (P_3 + Q_3)(P_4 + Q_4) + Q_1 (P_2 + Q_2)(P_3 + Q_3)(P_4 + Q_4) = P_1 P_4 P_2^* P_3^* + P_1 P_4 P_2^* Q_3^* + P_1 P_4 Q_2^* (P_3^* + Q_3^*) + P_1 Q_4^* (P_2^* + Q_2^*)(P_3^* + Q_3^*) + Q_1^* (P_2^* + Q_2^*)(P_3^* + Q_3^*)(P_4 + Q_4^*),$$

where all the summands, except the first in both sides, do not contain terms of degree less than ϱ. Since B is nondenumerable, we conclude that the coefficients of the terms of degree less than ϱ of the trigonometric polynomial

$$P_1(x) P_4(x) [P_2^*(x) P_3^*(x) - P_2(x) P_3(x)], \tag{2.5}$$

are zero. We also have:

$$P_2^*(x) P_3^*(x) = (\bar{a}_{\varkappa-1} + \bar{a}_{\varkappa-2} e^{ix} + \ldots + \bar{a}_{\varkappa-\varrho} e^{i(\varrho-1)x})(a_{\varkappa+q} + \ldots + a_{\varkappa+q+\varrho-1} e^{i(\varrho-1)x}) = b_0^* + b_1^* e^{ix} + \ldots + b_{\varrho-1}^* e^{i(\varrho-1)x} + b_\varrho^* e^{i\varrho x} + \ldots + b_{2\varrho-2}^* e^{i(2\varrho-2)x},$$

where

$$b_{s-1}^* = \sum_{j=1}^{s} \bar{a}_{\varkappa-j} a_{\varkappa+q-j+s}, \quad s = 1, 2, 3, \ldots, \varrho. \tag{2.6}$$

Similarly, the corresponding coefficients b_{s-1} of $P_2(x) P_3(x)$ are given by:

$$b_{s-1} = \sum_{j=1}^{s} \bar{a}_{\varkappa+\lambda-j} a_{\varkappa+\lambda+q-j+s}, \quad s = 1, 2, 3, \ldots, \varrho. \tag{2.7}$$

So, (2.5) becomes:

$$(a_0 + \ldots + a_{\varrho-1} e^{i(\varrho-1)x})(\bar{a}_{\nu-1} + \ldots + \bar{a}_{\nu-\varrho} e^{i(\varrho-1)x})[(b_0^* - b_0) + (b_1^* - b_1) e^{ix} + \ldots + (b_{\varrho-1}^* - b_{\varrho-1}) e^{i(\varrho-1)x} + \ldots + (b_{2\varrho-2}^* - b_{2\varrho-2}) e^{i(2\varrho-2)x}]. \tag{2.8}$$

Since, $a_0 a_{\nu-1} \neq 0$ and $a_0 \bar{a}_{\nu-1}(b_0^* - b_0) = 0$, we have $b_0^* = b_0$. Substituting in (2.8) and dividing by e^{ix} we conclude as before that $b_1^* = b_1$ and in the same manner we obtain $b_2^* = b_2, \ldots, b_{\varrho-1}^* = b_{\varrho-1}$. Thus, we have:

$$b_{s-1}^* = b_{s-1}, \quad s = 1, 2, 3, \ldots, \varrho,$$

which (in the old notation) are the relations (2.1). This completes the proof of lemma 1. □

We pass now to the proof of the special case of partial sums lying on exactly one circle. More precisely we prove the following theorem:

Theorem 2. *Assume that there is a nonnegative integer* n_0, *such that for* $n \geqq n_0$ *the partial sums of* (1.1) *are situated on some circle* $C(\tau(x); r(x))$ *for all* x *in* E, *where* E *is a nondenumerable subset of* $[0, 2\pi)$. *Then, there are, a positive integer* k, *a real number* ϑ *and two complex numbers* a, b, *with* $\bar{a}b$ *real, such that* (1.1) *has the form:*

$$\mathsf{P}_0(x)+e^{i(n_0+1)x}(a+be^{ik(x+\vartheta)})\sum_{n=0}^{\infty} e^{2ikn(x+\vartheta)} \tag{2.9}$$

where

$$\mathsf{P}_0(x) = c_0+c_1e^{ix}+\ldots+c_{n_0}e^{in_0x}.$$

Conversely, every series of the form (2.9) *has its partial sums* $s_n(x)$ *for* $n \geqq n_0$ *and for all, except a finite number,* $x \in [0, 2\pi)$, *on one circle if and only if* $\bar{a}b$ *is real.*

Proof. We may assume that $c_{n_0+1} \neq 0$ and $c_{n_0+2} = \ldots = c_{n_0+k} = 0$, $c_{n_0+k+1} \neq 0$ ($k=1$, means $c_{n_0+2} \neq 0$). If $c_{n_0+k+2} = \ldots = c_{n_0+k+s} = 0$, $c_{n_0+k+s+1} \neq 0$, lemma 1 gives

$$s = k \quad \text{and} \quad \bar{c}_{n_0+1}c_{n_0+k+1} = \bar{c}_{n_0+k+1}c_{n_0+2k+1}. \tag{2.10}$$

Hence, $c_{n_0+2k+1} \neq 0$. In the same manner we conclude that the only non zero coefficients of (1.1) with indices greater than n_0 are the c_{n_0+1+jk}, $j=0, 1, 2, 3, \ldots$. It will be convenient to use the notation:

$$a_j = c_{n_0+1+jk}, \quad j = 0, 1, 2, 3, \ldots.$$

In the new notation (2.10) becomes $\bar{a}_0a_1 = \bar{a}_1a_2$ and similarly:

$$\bar{a}_ja_{j+1} = \bar{a}_{j+1}a_{j+2}, \quad j = 0, 1, 2, 3, \ldots. \tag{2.11}$$

For every j (2.11) gives $|a_j| = |a_{j+2}|$ and if we set $a_2 = a_0\omega$, then $|\omega| = 1$ i.e. $\bar{\omega} = \dfrac{1}{\omega}$. Since $a_0 \neq 0$, $\bar{a}_0a_1 = \bar{a}_2a_3$ implies $a_3 = a_1\omega$ and using induction we obtain:

$$a_{2j} = a_0\omega^j, \quad a_{2j+1} = a_1\omega^j, \quad j = 0, 1, 2, 3, \ldots. \tag{2.12}$$

Thus, (1.1) has the form:

$$\begin{aligned} &c_0+c_1e^{ix}+\ldots+c_{n_0}e^{in_0x}+e^{i(n_0+1)x}(a_0+a_1e^{ikx}+a_0\omega e^{2ikx}+a_1\omega e^{3ikx}+\ldots) \\ &\quad = \mathsf{P}_0(x)+e^{i(n_0+1)x}(a_0+a_1e^{ikx})\sum_{n=0}^{\infty} e^{2ikn(x+\vartheta)} \end{aligned} \tag{2.13}$$

where ϑ is a real number such that $e^{2ik\vartheta} = \omega$. Setting $a_0 = a$, $a_1e^{-ik\vartheta} = b$ we obtain (2.9) from (2.13). From $\bar{a}_0a_1 = \bar{a}_1a_2$ and $a_2 = a_0e^{2ik\vartheta}$ we have: $\bar{a}b = \bar{a}_0a_1e^{-ik\vartheta} = \bar{a}_1a_0e^{2ik\vartheta}e^{-ik\vartheta} = a\bar{b}$, i.e. $\bar{a}b$ is real.

Conversely, it is obvious that it suffices to consider the series:

$$(a+be^{ikx})\sum_{n=0}^{\infty} e^{2iknx}.$$

For this series we have:

$$s_{2kn}(x) = \frac{a+be^{ikx}}{1-e^{2ikx}} - \frac{b+ae^{ikx}}{1-e^{2ikx}} e^{ik(2n+1)x},$$

$$s_{(2n+1)k}(x) = \frac{a+be^{ikx}}{1-e^{2ikx}} - \frac{a+be^{ikx}}{1-e^{2ikx}} e^{ik(2n+2)x},$$

for every but a finite number $x \in [0, 2\pi)$. Thus, for these x's, the $s_{2kn}(x)$'s are situated on the circle with centre $\tau(x) = (a+be^{ikx})/(1-e^{2ikx})$ and radius $r_1(x) = |(b+ae^{ikx})/(1-e^{2ikx})|$. Similarly, the $s_{(2n+1)k}(x)$'s lie on the circle with the same centre and radius $r_2(x) = |(a+be^{ikx})/(1-e^{2ikx})|$. These two concentric circles coincide if and only if $\bar{a}b$ is real, as one can easily verify. This completes the proof of theorem 2. □

3. The general case

We divide the proof of theorem 1 into five steps.

Step 1. There exist, a nondenumerable subset E^* of E and two positive integers m, n_0, with the following property:

For each x in E^*, there are m complex numbers $z_j(x)$ and m *positive* real numbers $r_j(x)$, $j=1, 2, \ldots, m$, such that:

(i) $n \geqq n_0$ implies that $s_n(x) \in \bigcup_{j=1}^{m} C(z_j(x); r_j(x))$ and for each x in E^* and each j in $\{1, 2, \ldots, m\}$, card $\{n: s_n(x) \in C(z_j(x); r_j(x))\} = \infty$.

(ii) If $x \in E^*$ and $j \in \{1, 2, \ldots, m\}$, then there is $n = n(x, j) < n_0$, such that $s_n(x)$ lies on $C(z_j(x); r_j(x))$ (this will permit us to apply lemma 1 with $n \geqq n_0$).

Proof. We know that for each x in E there are $M(x)$ complex numbers and $M(x)$ positive real numbers, which we enumerate in an arbitrary manner as $z_j(x)$, $r_j(x)$, $j=1, 2, \ldots, M(x)$, such that, for all $n=0, 1, 2, \ldots$, the partial sums $s_n(x)$ of (1.1) lie on the union of the circles $C(z_j(x); r_j(x))$. We fix x in E and consider the set:

$$A(x) = \{j \in \{1, 2, \ldots, M(x)\}: \text{card}\{n: s_n(x) \in C(z_j(x);\ r_j(x))\} = \infty\}.$$

If $m(x) = \text{card}\, A(x)$, then $A(x)$ can be writen as: $A(x) = \{i_1, i_2, \ldots, i_{m(x)}\}$. We set: $C_1(x) = C(z_{i_1}(x); r_{i_1}(x)), \ldots, C_{m(x)}(x) = C(z_{i_{m(x)}}(x); r_{i_{m(x)}}(x))$. As one can easily verify there is a (minimum) positive integer $k(x)$, such that:

(a) $n \geqq k(x)$ implies $s_n(x) \in C_1(x) \cup \ldots \cup C_{m(x)}(x)$.

(b) For each $j=1, 2, \ldots, m(x)$, card $\{n: s_n(x) \in C_j(x)\} = \infty$.

(c) For each $j=1, 2, \ldots, m(x)$, there is $n = n(x, j) < k(x)$: $s_n(x) \in C_j(x)$.

As in the proof of lemma 1 we see that there is a nondenumerable subset E^* of E and two positive integers m, n_0, such that, for all x in E^*, $m(x) = m$ and $k(x) =$

n_0. It is evident from (a), (b), (c) that m, n_0 and E^* satisfy the properties (i) and (ii). This completes the proof of step 1. □

From now on we shall assume, as we may, that E itself satisfies (i) and (ii) of step 1.

Step 2. The following assertion is an essential part of the proof.

Assertion. There are, a nondenumerable subset E_1 of E and positive integers ν, $k_1, k_2, \ldots, k_{2m}$, with $n_0 \leqq \nu < k_1 < \ldots < k_{2m}$, such that:

(i) $c_\nu \neq 0$.

(ii) *For all x in E_1 the partial sums $s_\nu(x), s_{k_1}(x), \ldots, s_{k_{2m}}(x)$ lie on one of the m circles mentioned in step 1. We denote this circle by $C(\tau(x); r(x))$.*

(iii) *There are $2m$ real numbers ϑ_ϱ, $\varrho = 1, 2, \ldots, 2m$, such that for all x in E_1,*

$$s_{q-j}(x) - \tau(x) = (s_{\nu-j}(x) - \tau(x))\omega_\varrho, \quad \omega_\varrho = e^{i(q-\nu)(x+\vartheta_\varrho)}, \tag{3.1}$$
$$q = k_\varrho, \quad \varrho = 1, 2, \ldots, 2m, \quad j = 0, 1.$$

The proof, which depends heavily on lemma 1, will be given in paragraph 4.

Step 3. Let r, p, s positive integers, $r \geqq n_0$, and φ a real number. If for all x in E_1

(a) $s_r(x)$, $s_{r+p}(x)$ lie on one of the m circles, and this circle is concentric to $C(\tau(x); r(x))$ mentioned in step 2;

(b) $s_{r+p-j}(x) - \tau(x) = (s_{r-j}(x) - \tau(x))e^{ip(x+\varphi)}$, $j = 0, 1$;

(c) $c_r \neq 0$;

(d) $c_{r+1} = c_{r+2} = \ldots = c_{r+s-1} = 0$, $c_{r+s} \neq 0$,

then, for all x in E_1;

(e) $s_{r+p+s}(x) - \tau(x) = (s_{r+s}(x) - \tau(x))e^{ip(x+\varphi)}$, and

(f) $c_{r+p+1} = c_{r+p+2} = \ldots = c_{r+p+s-1} = 0$, $c_{r+p+s} \neq 0$.

Proof. (a) allows us to apply lemma 1, which gives (f) and

$$\bar{c}_r c_{r+s} = \bar{c}_{r+p} c_{r+p+s}. \tag{3.2}$$

Subtracting the two relations $(j = 0, 1)$ of (b) we obtain:

$$c_{r+p} e^{i(r+p)x} = c_r e^{irx} e^{ip(x+\varphi)}, \quad \text{i.e.} \quad c_{r+p} = c_r e^{ip\varphi}. \tag{3.3}$$

(3.2), (3.3) and (c) imply:

$$c_{r+p+s} = c_{r+s} e^{ip\varphi}. \tag{3.4}$$

Now (e) is an immediate consequence of (b), (3.3) and (3.4). □

Step 4. In this step we prove that the series (1.1) has the desired form (1.2).

Proof. It suffices to show that

$$c_{\mu+\varkappa-\nu} = c_\mu e^{i(\varkappa-\nu)\vartheta}, \quad \mu = \nu, \nu+1, \nu+2, \ldots \tag{3.5}$$

with $\varkappa = k_1$, $\vartheta = \vartheta_1$ (k_1, ϑ_1, as in step 2). Indeed, if (3.5) holds, then,

$$c_{\mu+2(\varkappa-\nu)} = c_\mu \omega^2, \ldots, c_{\mu+\lambda(\varkappa-\nu)} = c_\mu \omega^\lambda, \ldots, \tag{3.6}$$

where $\omega = e^{i(\varkappa-\nu)\vartheta}$ and $\mu = \nu, \nu+1, \nu+2, \ldots, \varkappa-1$.

Thus (1.1) takes the form:

$$c_0 + c_1 e^{ix} + \ldots + c_{\nu-1} e^{i(\nu-1)x} + c_\nu e^{i\nu x} + \ldots + c_{\varkappa-1} e^{i(\varkappa-1)x}$$
$$+ c_\nu \omega e^{i\varkappa x} + \ldots + c_{\varkappa-1} \omega e^{i(2\varkappa-\nu-1)x} + c_\nu \omega^2 e^{i(2\varkappa-\nu)x} + \ldots$$
$$= \mathrm{P}_0(x) + e^{i\nu x} \mathrm{P}(x) \sum_{\lambda=0}^{\infty} (\omega e^{i(\varkappa-\nu)x})^\lambda,$$

which is (1.2).

In order to show (3.5) we recall first that $\varkappa = k_1 > \nu$. If now $c_{\nu+1} = \ldots = c_{\nu+s-1} = 0$, $c_{\nu+s} \neq 0$, then, by step 2, the hypotheses of step 3 are fulfilled with $r = \nu$, $p = q - \nu$ and $\varphi = \vartheta_\varrho$, where $q = k_\varrho$, $\varrho = 1, 2, \ldots, 2m$.

Hence, for all x in E_1

$$s_{q+s}(x) - \tau(x) = \big(s_{\nu+s}(x) - \tau(x)\big) e^{i(q-\nu)(x+\vartheta_\varrho)}, \quad q = k_\varrho, \quad \varrho = 1, 2, \ldots, 2m, \tag{3.7}$$

and $c_{q+1} = c_{q+2} = \ldots = c_{q+s-1} = 0$, $c_{q+s} \neq 0$. Obviously (3.1) and (3.7) imply:

(3.8)

$$s_{q+j}(x) - \tau(x) = \big(s_{\nu+j}(x) - \tau(x)\big)\omega_\varrho, \quad q = k_\varrho, \quad \varrho = 1, 2, \ldots, 2m; \quad j = -1, 0, \ldots, s.$$

(3.7) means that for all x in E_1 the partial sums $s_{q+s}(x)$, $q = \nu, k_1, \ldots, k_{2m}$ lie on a circle C_s centered at $\tau(x)$, which a priori may not be one of the m circles mentioned in step 1. But evidently at least three among the above $(2m+1)$ partial sums lie on one of these m circles, which implies that C_s is indeed one of them. So, we can apply the result of step 3 again, with $r = \nu + s$, $p = q - \nu$ and $\varphi = \vartheta_\varrho$, $q = k_\varrho$, $\varrho = 1, 2, \ldots, 2m$. It follows that if $c_{\nu+s+1} = \ldots = c_{\nu+s+t-1} = 0$, $c_{\nu+s+t} \neq 0$, then, $c_{q+s+1} = \ldots = c_{q+s+t-1} = 0$, $c_{q+s+t} \neq 0$, $q = k_\varrho$, $\varrho = 1, 2, \ldots, 2m$ and for all x in E_1

$$s_{q+j}(x) - \tau(x) = \big(s_{\nu+j}(x) - \tau(x)\big)\omega_\varrho, \quad q = k_\varrho, \quad \varrho = 1, 2, \ldots, 2m; \tag{3.9}$$
$$j = -1, 0, \ldots, s+t.$$

Thus, for all x in E_1, the partial sums $s_{q+s+t}(x)$, $q = \nu, k_1, \ldots, k_{2m}$, lie on one circle centered at $\tau(x)$, which by the same argument as before must be one of the m circles. Continuing in the same way we see that (3.9) holds for all $j = -1, 0, 1, 2, \ldots$.

To prove (3.5), we consider (3.9) for $q = k_1 = \varkappa$, and $\vartheta_1 = \vartheta$ (i.e. $\omega_1 = e^{i(\varkappa-\nu)(x+\vartheta)}$). Subtracting now the corresponding equalities for $j = \mu - \nu - 1$ and $j = \mu - \nu$, we obtain.

$$c_{\mu+\varkappa-\nu}e^{i(\mu+\varkappa-\nu)x} = c_\mu e^{i\mu x} e^{i(\varkappa-\nu)(x+\vartheta)},$$

which is equivalent to (3.5). □

Step 5. Here we complete the proof of theorem 1, by showing that there is a positive integer m, such that, for all but a finite number of x's in $[0, 2\pi)$, the partial sums $s_n(x)$ of (1.1), with $n \geqq \nu$, lie on exactly m concentric circles.

Proof. So far we saw that the series (1.1) has the form (1.2) from which we shall conclude the desired results. It is obvious that it suffices to consider a series of the form:

$$\text{(3.10)} \qquad \mathsf{P}_1(x)\sum_{n=0}^{\infty} e^{iknx},$$

where $\mathsf{P}_1(x) = a_0 + a_1 e^{ix} + \dots + a_{k-1}e^{i(k-1)x}$, for some positive integer k.
If we set:

$$\text{(3.11)} \qquad \begin{aligned} P_1(x) &= \mathsf{P}_1(x), \\ P_2(x) &= a_{k-1} + a_0 e^{ix} + \dots + a_{k-2}e^{i(k-1)x}, \\ P_3(x) &= a_{k-2} + a_{k-1}e^{ix} + \dots + a_{k-3}e^{i(k-1)x}, \\ &\vdots \\ P_k(x) &= a_1 + a_2 e^{ix} + \dots + a_0 e^{i(k-1)x}, \end{aligned}$$

then, for $x \neq 0, 2\pi/k, \dots, 2(k-1)\pi/k$ and $n = 0, 1, 2, 3, \dots,$ the partial sums of (3.10) are given by the formulae:

$$\text{(3.12)} \qquad \begin{aligned} s_{kn+k-1}(x) &= \frac{P_1(x)}{1-e^{ikx}} - \frac{P_1(x)}{1-e^{ikx}} e^{i(kn+k)x}, \\ s_{kn+k-2}(x) &= \frac{P_1(x)}{1-e^{ikx}} - \frac{P_2(x)}{1-e^{ikx}} e^{i(kn+k-1)x}, \\ &\vdots \\ s_{kn}(x) &= \frac{P_1(x)}{1-e^{ikx}} - \frac{P_k(x)}{1-e^{ikx}} e^{i(kn+1)x}. \end{aligned}$$

(3.12) means that for these x's the partial sums of (3.10) lie on a finite number of *concentric* circles with common centre

$$\text{(3.13)} \qquad \tau(x) = \frac{P_1(x)}{1-e^{ikx}}$$

and radii:

$$\text{(3.14)} \qquad r_j(x) = |P_j(x)/(1-e^{ikx})|, \quad j = 1, 2, \dots, k.$$

It remains to show that for all but a finite number of x's in $[0, 2\pi)$, the corresponding number of circles is constant.

If $i\neq j$, $i, j\in\{1, 2, ..., k\}$, then, $r_i(x)=r_j(x)$ if and only if the trigonometric polynomial $P_i(x)\bar{P}_i(x)-P_j(x)\bar{P}_j(x)$ equals 0. It follows that, either the above equation has a finite number of solutions in $[0, 2\pi)$, or it holds for all x. Let $r_{j_1}(x), ..., r_{j_m}(x)$, be representatives of the equivalence classes of the relation:

$$\text{“}r_i(x) = r_j(x) \quad \text{for all } x\text{”}.$$

Then, to all x in $[0, 2\pi)-\{0, 2\pi/k, ..., 2(k-1)\pi/k\}$, except the finitely many solutions of the equations $r_{j_t}(x)=r_{j_s}(x)$, $t\neq s$, $t, s\in\{1, 2, ..., m\}$, correspond exactly m circles. This completes the proof of theorem 1. □

4. Proof of the assertion of § 3

In step 1 of paragraph 3, we saw that there exist a nondenumerable subset E of $[0, 2\pi)$ and two positive integers m, n_0, such that for each $x\in E$ there are m circles with the property: On each of them lie infinitely many partial sums $s_n(x)$ of the series (1.1), with $n\geqq n_0$, and at least one with $n=n(x)<n_0$. So, we can apply lemma 1 with $n\geqq n_0$.

We remark that the above m circles are not supposed to be concentric (this is true, but the proof given in paragraph 3 used the assertion we are going to prove). We also recall that for each $x\in E$ the partial sums $s_n(x)$ are "essentially" distinct complex numbers. More precisely:

$$s_n(x) = s_{n+k}(x) \quad \text{if and only if} \quad c_{n+1} = c_{n+2} = ... = c_{n+k} = 0.$$

For each x in E we enumerate, in an arbitrary manner, the m circles corresponding to it: $C_1(x), C_2(x), ..., C_m(x)$.

We may obviously assume that $c_{n_0}\neq 0$. The existence of zero coefficients in the series (1.1) causes some technical difficulties. To avoid them we shall work with partial sums $s_n(x)$ with $c_n\neq 0$. So, we let n_1 be the smallest integer greater than n_0 such that $c_{n_1}\neq 0$, n_2 the smallest integer greater than n_1 such that $c_{n_2}\neq 0$ and so on.

It will simplify the exposition of the proof if we introduce a function F defined on the set E, as follows:

$$F(x) = (t_0, t_1, ..., t_N),$$

where t_j, $j=0, 1, ..., N$, is the smallest integer in $\{1, 2, ..., m\}$, such that $s_{n_j}(x)$ belongs to the circle $C_{t_j}(x)$, and $N=4(2m+1)^2m^{2m+1}$. The reason behind this choice of N will be clarified in a moment.

Since F takes at most m^{N+1} values and E is nondenumerable, there is a nondenumerable subset E_0 of E such that:

For all x in E_0, F has a *constant value,* say

$$F(x) = (v_0, v_1, \ldots, v_{p-1}),$$

where $v_q = (t_{q(2m+1)}, t_{q(2m+1)+1}, \ldots, t_{q(2m+1)+2m})$, $q=0, 1, \ldots, p-1$ and

$$p = 4(2m+1)m^{2m+1}.$$

We observe now that the set of possible v_q's is independent of q and has m^{2m+1} elements. This implies that at least $p/m^{2m+1} = 4(2m+1)$ v_q's are identical. This means that there is an increasing sequence of non negative integers $q_0, q_1, \ldots, q_{4(2m+1)-1}$, such that:

$$v_{q_\mu} = (t_0^*, t_1^*, \ldots, t_{2m}^*), \quad \mu = 0, 1, \ldots, 4(2m+1)-1,$$

where

(4.1) $$t_j^* = t_{q_0(2m+1)+j}, \quad j = 0, 1, \ldots, 2m.$$

(4.2). Since t_j^* takes m values, $1, 2, \ldots, m$, and j $2m+1$ values, $0, 1, \ldots, 2m$, there are i, j in $\{0, 1, \ldots, 2m\}$, with $s=j-i \geqq 2$, such that $t_i^* = t_j^*$.

We pause for a moment to say a few words about the significance of our results for the partial sums of (1.1). If we write

$$q_\mu(2m+1)+i = v_\mu, \quad \mu = 0, 1, \ldots, 4(2m+1)-1,$$

then the partial sums of (1.1) with indices n_k, $v_\mu \leqq k \leqq v_\mu + s$, follow a succession of circles, which is independent of μ. More precisely, using also (4.1) and (4.2), for all x in E_0 and λ in $\{0, 1, \ldots, s\}$, the partial sums with indices $n_{v_\mu+\lambda}$, $\mu=0, 1, \ldots, 4(2m+1)-1$, lie on the circle $C_{t_{v_0+\lambda}}(x)$ and for each x in E_0 if $\lambda=0$ or s these circles coincide (see remark 5 for the geometric interpretation).

To conclude the proof we shall choose $(2m+1)$ numbers, among the $4(2m+1)$ $n_{v_\mu+1}$'s with $0 \leqq \mu < 4(2m+1)$, say $v, k_1, k_2, \ldots, k_{2m}$, so that (iii) of the assertion holds. We note that the way we defined $n_{v_\mu+1}$ guarantees automatically the validity of (i) and (ii) of the assertion.

The crucial step to achieve this choice is the following lemma:

Lemma 2. *If*

(i) $c_{r_j} c_{p_j} \neq 0$, $j=0, 1, \ldots, s+1$,

(ii) $c_{r_j+1} = \ldots = c_{r_j+l_j} = 0$, $l_j = r_{j+1} - r_j - 1$,
$c_{p_j+1} = \ldots = c_{p_j+l'_j} = 0$, $l'_j = p_{j+1} - p_j - 1$, $j=0, 1, \ldots, s$,

(iii) *for all x in E_0, $s_{r_j}(x)$, $s_{p_j}(x)$, $j=0, 1, \ldots, s$, lie on one of the m circles,*

(iv) *for all x in E_0, $s_{r_0}(x)$, $s_{r_s}(x)$, $s_{p_0}(x)$, $s_{p_s}(x)$, lie on the same circle whose centre we denote by $\tau(x)$,* *where* $r_0<r_1<\ldots<r_s$, $p_0<p_1<\ldots<p_s$, $s\geqq 2$, *and* $n_0\leqq r_s\leqq p_0$, *then,*

(v) $l_j=l'_j$, $j=0, 1, \ldots, s$,

(vi) *if* $c_{p_0}=c_{r_0}\omega$, *then* $|\omega|$ *can take only the values* 1, $\left|\frac{c_{r_1}c_{r_3}}{c_{r_0}c_{r_2}}\right|^{1/2}$,

(vii) *if* $|\omega|=1$ *and we write* $\omega=e^{i(p_0-r_0)\vartheta}$,
$s_{p_0}(x)-\tau(x)=(s_{r_0}(x)-\tau(x))e^{i\varphi}$, ϑ, φ *real numbers, then, either*
$\varphi=(p_0-r_0)(x+\vartheta)(\text{mod } 2\pi)$, *or*
$\varphi=\{(p_0-r_0)(x+\vartheta)-2\,\text{Arg}\,[(s_{r_0}(x)-\tau(x))(\overline{s_{r_s}(x)-s_{r_0}(x)})]\}\,(\text{mod } 2\pi)$.

Proof. To simplify the notation we write:

$$c_{r_j}=a_j,\quad c_{p_j}=b_j,\quad j=0, 1, \ldots, s+1;\quad a=a(x)=s_{r_s}(x)-s_{r_0}(x),$$

$$b=b(x)=s_{p_s}(x)-s_{p_0}(x),\quad A=A(x)=s_{r_0}(x)-\tau(x),\quad B=B(x)=s_{p_0}(x)-\tau(x),$$

$$\gamma=(p_0-r_0)(x+\vartheta)\quad\text{and}\quad \delta=\text{Arg}\,[(s_{r_0}(x)-\tau(x))(\overline{s_{r_s}(x)-s_{r_0}(x)})]=\text{Arg}\,(A\bar{a}).$$

The hypotheses (i), (ii), (iii) and lemma 1 give (v) and

$$\bar{b}_j b_{j+1}=\bar{a}_j a_{j+1},\quad j=0, 1, \ldots, s. \tag{4.3}$$

Since, in the new notation, $b_0=a_0\omega$, (4.3), with $j=0$, implies: $b_1=\frac{a_1}{\bar{\omega}}$. Substituting b_1 in (4.3), with $j=1$, we obtain: $b_2=a_2\omega$. Continuing in the same way we have:

$$b_{2j}=a_{2j}\omega,\quad b_{2i+1}=\frac{a_{2i+1}}{\bar{\omega}},\quad \max(2j, 2i+1)\leqq s+1. \tag{4.4}$$

Case 1. "$l_0\neq l_2$". Let $l_0<l_2$ (in the opposite case the proof is essentially the same). Applying (2.1) of lemma 1 we obtain:

$$\bar{c}_{r_0}\cdot c_{r_2}+\underbrace{\bar{0}0+\ldots+\bar{0}0}_{l_0\text{ terms}}+\bar{c}_{r_1}\cdot 0=\bar{c}_{p_0}\cdot c_{p_2}+\underbrace{\bar{0}0+\ldots+\bar{0}0}_{l_0\text{ terms}}+\bar{c}_{p_1}\cdot 0,\quad\text{or,}\quad \bar{a}_0 a_2=\bar{b}_0 b_2,$$

which, by (4.4), gives $\bar{a}_0 a_2=\bar{a}_0 a_2\omega\bar{\omega}$, i.e. $|\omega|=1$.

Case 2. "$l_0=l_2$". Applying again (2.1) of lemma 1 we obtain as before:

$$\bar{a}_0 a_2+\bar{a}_1 a_3=\bar{b}_0 b_2+\bar{b}_1 b_3.$$

Hence,

$$\bar{a}_0 a_2+\bar{a}_1 a_3=\bar{a}_0 a_2|\omega|^2+(\bar{a}_1 a_3)/|\omega|^2,$$

or

$$(1-|\omega|^2)\left(\bar{a}_0 a_2-\frac{\bar{a}_1 a_3}{|\omega|^2}\right)=0,$$

from which (vi) follows immediately.

If now $|\omega|=1$, i.e. $\bar{\omega}=1/\omega$, then (4.4) implies $b_j=a_j\omega$, or

$$c_{p_j}=c_{r_j}\omega,\quad j=0, 1, \ldots, s+1 \tag{4.5}$$

(4.5) with (v) and $\omega=e^{i(p_0-r_0)\vartheta}$ give

$$b = a_1\omega e^{ip_0x}+\ldots+a_s\omega e^{ip_sx} = a\omega e^{i(p_0-r_0)x} = ae^{i\gamma}.$$

From (iv) we have $|A+a|=|B+b|$, which together with the obvious relations $|A|=|B|$, $|a|=|b|$, implies $\operatorname{Re}(A\bar a)=\operatorname{Re}(B\bar b)$, or, $\operatorname{Re}(e^{i\delta})=\operatorname{Re}(e^{i[\delta+(\varphi-\gamma)]})$. Hence, $\varphi=\gamma(\operatorname{mod} 2\pi)$, or $\varphi=(\gamma-2\delta)(\operatorname{mod} 2\pi)$, which are the desired relations of (vii). □

We return to the proof of the assertion.

Let

$$I = \{0, 1, 2, \ldots, 4(2m+1)-1\},$$
$$J = \{0, 1, 2, \ldots, s\}.$$

We saw that if $\lambda\in J$, then for all x in E_0 and for all μ in I the partial sums of (1.1), with indices $n_{\nu_\mu+\lambda}$, lie on one of the m circles and the circles corresponding to $\lambda=0$ and $\lambda=s$ coincide. We denote the centre of this circle by $\tau(x)$. We have also $c_{n_{\nu_\mu+\lambda}}\neq 0$. Thus, lemma 2 applies and gives a partition of I in two sets I_1, I_2, defined as follows: If $c_{n_{\nu_\mu}}=c_{n_{\nu_0}}\omega_\mu$, then, $I_1=\{\mu\in I\colon |\omega_\mu|=1\}$, and $I_2=I-I_1$. By (vi) of lemma 2, $I_2\subset\{\mu\in I\colon |\omega_\mu|=\Omega\}$, $\Omega=\left|\dfrac{c_{n_{\nu_0+1}}c_{n_{\nu_0+3}}}{c_{n_{\nu_0}}c_{n_{\nu_0+2}}}\right|^{1/2}$. Since $\operatorname{card} I=4(2m+1)$, one of these two sets has cardinality at least $2(2m+1)$. Let $\mu_0<\mu_1<\ldots<\mu_{2(2m+1)-1}$ be the first $2(2m+1)$ elements of this set and denote by I^* the set $\{\mu_0, \mu_1, \ldots, \mu_{2(2m+1)-1}\}$. It is trivial, if $I^*\subset I_1$, and very easy, if $I^*\subset I_2$, to see that

$$|\omega_j^*| = |c_{\beta_j}/c_{\beta_0}| = 1, \quad \beta_j = n_{\nu_{\mu_j}}, \quad j=0, 1, \ldots, 2(2m+1)-1. \tag{4.6}$$

Hence: (a) There are $2(2m+1)$ real numbers ϑ_j, such that

$$\omega_j^* = e^{i(\beta_j-\beta_0)\vartheta_j}, \quad j=0, 1, \ldots, 2(2m+1)-1. \tag{4.7}$$

(b) If we write

$$s_{\beta_j}(x)-\tau(x) = \big(s_{\beta_0}(x)-\tau(x)\big)e^{i\varphi_j}, \quad j=0, 1, \ldots, 2(2m+1)-1, \tag{4.8}$$

φ_j real numbers, then, (vii) of lemma 2 gives that for fixed x in E_0, the set I^* is partitioned in two sets $I_1^*(x)$, $I_2^*(x)$, as follows:

$$I_1^*(x) = \{\mu_j\in I^*\colon \varphi_j = (\beta_j-\beta_0)(x+\vartheta_j)(\operatorname{mod} 2\pi)\},$$
$$I_2^*(x) = I^*-I_1^*(x).$$

Obviously, one of these two sets has cardinality at least $2m+1$. Let $i_0<i_1<\ldots<i_{2m}$ be the first $2m+1$ elements of this set (which depends on x) and $k_\varrho(x)=n_{\nu_{i_\varrho}+1}$, $\varrho=0, 1, \ldots, 2m$. Since the number of possibilities for

$$\big(k_0(x), k_1(x), \ldots, k_{2m}(x)\big)$$

is finite and E_0 is nondenumerable, there are, a nondenumerable subset E_0^* of E_0 and $2m+1$ positive integers $k_0<k_1<\ldots<k_{2m}$, such that, for all x in E_0^*,

$$k_0(x)=k_0,\ k_1(x)=k_1,\ \ldots,\ k_{2m}(x)=k_{2m}.$$

Since, for each x in E_0^*, $k_\varrho\in I_1^*(x)$ or $k_\varrho\in I_2^*(x)$, for all $\varrho=0,1,\ldots,2m$, and E_0^* is nondenumerable, it follows that one of the sets:

$$E_1^*=\{x\in E_0^*:\ k_\varrho\in I_1^*(x),\ \varrho=0,1,2,\ldots,2m\},$$
$$E_1^{**}=E_0^*-E_1^*=\{x\in E_0^*:\ k_\varrho\in I_2^*(x),\ \varrho=0,1,\ldots,2m\}$$

is nondenumerable. We call this set E_1 and we have:

Case 1. Let $E_1=E_1^*$. We observe that, for all x in E_0, $\mu_0\in I_1^*(x)$. Thus, since $k_0\in I_1^*(x)$, $s_{k_0-1}(x)=s_{\beta_0}(x)$. Using the same notation as in the definition of $I_1^*(x)$ and the above observation, we have:

$$(4.9)\qquad s_{k_\varrho-1}(x)-\tau(x)=\big(s_{k_0-1}(x)-\tau(x)\big)e^{i\varphi_\varrho},\quad \varphi_\varrho=(k_\varrho-k_0)(x+\vartheta_\varrho'),$$

$\varrho=1,2,\ldots,2m$, where ϑ_ϱ' is defined as follows: "For each ϱ the definition of k_ϱ implies the existence of an index j, $j=j(\varrho)$, with $i_\varrho=\mu_j$; we put $\vartheta_\varrho'=\vartheta_j$, where ϑ_j is given by (4.6), (4.7)".

The definition of ϑ_ϱ' and (4.5) imply

$$(4.10)\qquad c_{k_\varrho}=c_{k_0}e^{i(k_\varrho-k_0)\vartheta_\varrho'},\quad \varrho=1,2,\ldots,2m.$$

Case 2. Let $E_1=E_1^{**}$, then we have:

$$(4.11)\quad s_{k_\varrho-1}(x)-\tau(x)=\big(s_{\beta_0}(x)-\tau(x)\big)e^{i\varphi_\varrho'},\quad \varphi_\varrho'=(k_\varrho-\beta_0-1)(x+\vartheta_\varrho')-2\delta,$$

$\varrho=1,2,\ldots,2m$, ϑ_ϱ' is defined as before and δ is a real number (independent of ϱ; see (vii) of lemma 2).

(4.11) implies

$$\big(s_{k_\varrho-1}(x)-\tau(x)\big)/\big(s_{k_0-1}(x)-\tau(x)\big)=e^{i(\varphi_\varrho'-\varphi_0')},$$

or,

$$s_{k_\varrho-1}(x)-\tau(x)=\big(s_{k_0-1}(x)-\tau(x)\big)e^{i\varphi_\varrho},$$

where

$$\varphi_\varrho=\varphi_\varrho'-\varphi_0'=(k_\varrho-k_0)x+k_\varrho\vartheta_\varrho'-k_0\vartheta_0'-(\beta_0+1)(\vartheta_\varrho'-\vartheta_0')=(k_\varrho-k_0)(x+\vartheta_\varrho''),$$
$$\varrho=1,2,\ldots,2m\quad\text{and}\quad \vartheta_\varrho''=[k_\varrho\vartheta_\varrho'-k_0\vartheta_0'-(\beta_0+1)(\vartheta_\varrho'-\vartheta_0')](k_\varrho-k_0)^{-1}.$$

Also, (4.5), (4.6), (4.7) imply

$$c_{k_\varrho}=c_{\beta_0+1}e^{i(k_\varrho-\beta_0)\vartheta_\varrho'},\quad \varrho=0,1,\ldots,2m,$$

which gives

$$c_{k_\varrho}/c_{k_0}=e^{i[k_\varrho\vartheta_\varrho'-k_0\vartheta_0'-(\beta_0+1)(\vartheta_\varrho'-\vartheta_0')]}=e^{i(k_\varrho-k_0)\vartheta_\varrho''},\quad \varrho=1,2,\ldots,2m.$$

Thus, we have again (4.9), (4.10), with ϑ''_ϱ instead of ϑ'_ϱ. It follows that in either case, if we write ϑ_ϱ for ϑ'_ϱ, ϑ''_ϱ, we have:

(4.12) $\quad s_{k_\varrho-1}(x)-\tau(x)=\big(s_{k_0-1}(x)-\tau(x)\big)e^{i(k_\varrho-k_0)(x+\vartheta_\varrho)}, \quad \varrho=1,2,\ldots,2m.$

(4.12) and (4.10) give

$$\begin{aligned} s_{k_\varrho}(x)-\tau(x) &= s_{k_\varrho-1}(x)-\tau(x)+c_{k_\varrho}e^{ik_\varrho x} \\ &= \big(s_{k_0-1}(x)-\tau(x)\big)e^{i(k_\varrho-k_0)(x+\vartheta_\varrho)}+c_{k_0}e^{i(k_\varrho-k_0)\vartheta_\varrho}e^{ik_0x} \\ &= \big(s_{k_0}(x)-\tau(x)\big)e^{i(k_\varrho-k_0)(x+\vartheta_\varrho)}. \end{aligned}$$

This relation combined with (4.12) gives that for all x in E_1,

$$s_{q-j}(x)-\tau(x)=\big(s_{v-j}(x)-\tau(x)\big)\omega_\varrho, \quad \omega_\varrho=e^{i(q-v)(x+\vartheta_\varrho)},$$

where $q=k_\varrho$, $\varrho=1,2,\ldots,2m$, $v=k_0$, $j=0,1$, i.e. (3.1). This completes (iii) of the assertion and the proof is finished. □

5. Remarks

1. The theorem of Marcinkiewicz and Zygmund, given in § 1, suggests that it would be natural to assume in theorem 1 that the circles are concentric. This hypothesis would have made the proof somewhat simpler, but as we have already seen it is not needed.

2. From (1.2) we see that if $|z|<1$, then the function

$$F(z)=\sum_{n=0}^{\infty}c_nz^n$$

is a rational function of a special form. More precisely we have:

$$F(z)=\frac{Q(z)}{1-(\zeta z)^{\varkappa-v}},$$

where $Q(z)$ is a polynomial of degree less than $\varkappa$ and $\zeta=e^{i\vartheta}$. Thus, (1.1) becomes:

$$Q(e^{ix})\sum_{\lambda=0}^{\infty}e^{i\lambda(\varkappa-v)(x+\vartheta)}.$$

Conversely, it is easy to see that every rational function of the form:

$$F(z)=P(z)\sum_{\lambda=0}^{\infty}z^{\lambda\mu},$$

where $P(z)$ is a polynomial, not necessarily of degree less than μ, can be written, for $z=e^{ix}$, in the form (1.2).

It is also easy to see (using arguments analogous to those in step 5 of § 3) that if we multiply in the obvious way $P(e^{ix})$ with $\sum e^{i\lambda\mu x}$, then the resulting series is $(C, 1)$ summable to $F(e^{ix})$, which in turn coincides with the common centre $\tau(x)$ of the circles referred to in theorem 1.

The general case of a rational function F presents interesting problems, which will not be considered here. We note only that, as expected from theorem 1, it is easy to find examples of rational functions whose partial sums do not lie on a finite number of circles (see [3] V.II, p. 180, example (iii)).

3. There are interesting problems concerning degeneracy related to the number of circles in theorem 1. More precisely, considering the simpler form (3.10), under what conditions on $P_1(x)$ this series leads to $1, 2, \ldots, k$ circles, i.e., taking into account (3.14), under what conditions some relations of the form $|P_i(x)| \equiv |P_j(x)|$, $i \neq j$, $i, j = 1, 2, \ldots, k$, hold. We shall not examine here the general case, but we give some partial results in this direction:

(i) If $|P_j(x)| \equiv |P_{j+1}(x)| \equiv |P_{j+2}(x)|$, for some $j = 1, 2, \ldots, k$ (if $j = k-1$, then $P_{j+2}(x) = P_1(x)$ and if $j = k$, then $P_{j+1}(x) = P_1(x)$ and $P_{j+2}(x) = P_2(x)$), then all the circles coincide ($a_n \neq 0$, see (3.11)).

(ii) Theorem 2 gives the necessary and sufficient conditions in order that the number of circles reduces to 1. We can also give the corresponding theorem in order to have exactly two circles.

"If the series (1.1) has one of the following forms (with the usual notation):

$$(\alpha)\ (a + b e^{ik(x+\vartheta)}) \sum_{n=0}^{\infty} e^{in(k+m)(x+\vartheta)},$$

$$(\beta)\ (a + b e^{ik(x+\vartheta)} + c e^{i(k+m)(x+\vartheta)}) \sum_{n=0}^{\infty} e^{in(k+m+q)(x+\vartheta)},$$

$$(\gamma)\ (a + b e^{ik(x+\vartheta)} + c e^{2ik(x+\vartheta)} + d e^{ik(m+2)(x+\vartheta)}) \sum_{n=0}^{\infty} e^{2ikn(m+1)(x+\vartheta)},$$

$$(\delta)\ (a + b e^{ik(x+\vartheta)} + c e^{ik(m+1)(x+\vartheta)} + d e^{ik(m+2)(x+\vartheta)} + f e^{2ik(m+1)(x+\vartheta)}) \sum_{n=0}^{\infty} e^{ikn(3m+2)(x+\vartheta)},$$

then, the partial sums of (1.1) lie on exactly two circles, if and only if:

$m \neq k$ or $\bar{a}b$ is not real, in case (α),

($q = m$ and $\bar{a}c = b\bar{c}$) or ($q = k$ and $\bar{a}c = a\bar{b}$) *if* $k \neq m$,

($q = k$ and $\bar{a}b = \bar{b}c \neq \bar{c}a$ or $\bar{b}c = \bar{c}a \neq \bar{a}b$ or $\bar{a}c = a\bar{b} \neq b\bar{c}$) or ($q \neq 0, k$ and $\bar{a}b = \bar{b}c$) if $k = m$, or ($q = 0$ and $(a+c)\bar{b}$ is not real), in case (β),

($\bar{a}b = \bar{b}c$ and $\bar{b}d = b\bar{d}$) if $m > 1$,

($\bar{a}b = \bar{b}c \neq \bar{c}d$ and $\bar{b}d = b\bar{d}$) if $m = 1$, in case (γ),

($b = a\bar{\omega}^2$, $c = a\omega$, $d = a\bar{\omega}$, $f = a\omega^2$, where $\omega \neq 0, 1$ satisfies the equation $\omega^4 + \bar{\omega} = 2\omega\bar{\omega}^2$) if $m = 1$,

($b=a\omega^3/2$, $c=a\omega$, $d=a\omega^4/4$, $f=a\omega^2$, where ω satisfies the equation $\omega^8=16$) if $m>1$, in case (δ).

Conversely, if the partial sums of (1.1) lie on exactly two circles, then (1.1) has (possibly with trivial modifications) one of the above forms".

4. It is easy to see that when the series (1.1) has the form (1.2), or when it is the sum of two or more series of the this form, or it differs from such a series by a convergent series, then its partial sums have uniform "angular distribution". By this we mean that given an angle with vertex $\tau(x)$ and opening α, $0<\alpha<2\pi$, then for all but a denumerable number of x's,

$$\lim_{N\to\infty} \frac{1}{N}\varphi(N, x) = \frac{1}{2\pi}\alpha,$$

where $\varphi(N, x)$ is the number of partial sums $s_n(x)$ with $n\leqq N$, which lie on this angle. This is obviously equivalent to the uniform distribution of the sequence $\{\arg[s_n(x)-\tau(x)]\}$.

5. It will be helpful to interpret geometrically the proof of the "assertion" given in paragraph 4. We take for definiteness $m=6$ and represent each of the 6 circles by a straight line (Figure 2).

The main idea of the proof consisted in finding a nondenumerable subset E_0 of E, such that for all x in E_0 the polygonal line with vertices $s_{n_0}(x), s_{n_1}(x), \ldots$, follows the same succession of circles and contains "sufficiently many loops". By loop we mean a connected part of this polygonal line whose endpoints lie in the same "straight line". This succession corresponds to the constant value

$$(v_0, v_1, \ldots, v_{p-1}), \quad p = 4\cdot 13\cdot 6^{13} = 679\,156\,088\,832,$$

of the function F on E_0.

In figure 2 v_{q_0} is v_1, v_{q_1} is $v_3, \ldots$, where v_{q_μ}, $\mu=0, 1, \ldots, 4\cdot 13-1$, are as in § 4 and the boldface parts of the graph represent the $52=4(2m+1)$ "loops" found in (4.2).

In the rest of the proof we chose first $2(2m+1)=26$ loops whose corresponding sides have the same length and finally, reducing E_0 to E_1, we chose $2m+1=13$ loops which are completely identical, i.e. one follows from the other by rotation about $\tau(x)$. In figure 2 these are the shadowed loops. The numbers $\nu, k_1, k_2, \ldots, k_{2m}$, of § 4, are the indices of the partial sums corresponding to the endpoints of the first sides of the above loops (we could as well have chosen any other vertex of these loops).

We finally remark that, by geometric arguments, we could simplify the proof a little by avoiding the reduction from E_0 to E_1.

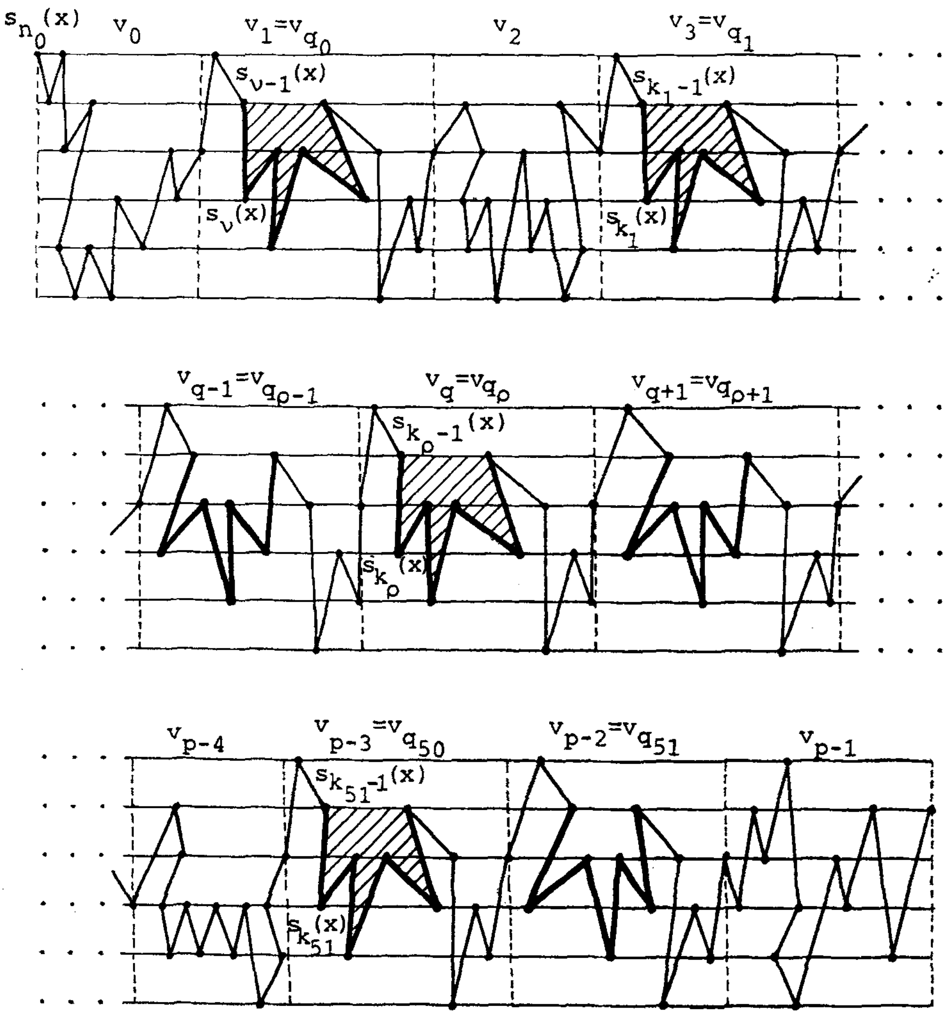

Figure 2

Added to the proof. After this paper had been written some questions and remarks of J.-P. Kahane and V. Nestoridis clarified further the role of the cardinality of the set E in theorem 1. More precisely, it can be shown that in theorem 1 the hypothesis "E nondenumerable" can be replaced by "E infinite", if we assume that the number of circles has an upper bound independent of x (it is easy to see that the hypothesis "E finite" is not sufficient). The example (communicated to the author by V. Nestoridis) $\sum_{n=0}^{\infty} e^{in\varphi} e^{2^n ix}$, x in $[0, 2\pi)$, for a convenient choice of the real number φ, shows that "E nondenumerable" is essential in theorem 1.

References

1. KAHANE, J.-P., Sur la structure circulaire des ensembles de points limites des sommes partielles d'une série de Taylor, *Acta Sci. Math. (Szeged)* **45** (1983), 247—251.
2. MARCINKIEWICZ, J. and ZYGMUND, A., On the behavior of trigonometric series and power series, *Trans. Amer. Math. Soc.* **50** (1941), 407—453.
3. ZYGMUND, A., *Trigonometric series*, 2nd ed. reprinted, I, II. Cambridge University Press, 1979.

Received Jan. 12, 1988

Department of Mathematics
University of Crete
P.O. Box 1470
Iraklion, Greece