A new proof of a sharp inequality concerning the Dirichlet integral

Donald E. Marshall*

If f is an analytic function defined in the unit disk, Δ, let

$$\mathscr{D}(f) = \left(\int_\Delta |f'(z)|^2 \frac{dx\,dy}{\pi}\right)^{1/2} \quad (z = x+iy)$$

be the Dirichlet integral of f. In this note, we will give a new proof of the following theorem.

Theorem 1 [4]. *There is a constant $C<\infty$ such that if f is analytic on Δ, $f(0)=0$, and $\mathscr{D}(f)\leqq 1$ then*

$$\int_0^{2\pi} e^{|f(e^{i\theta})|^2}\,d\theta \leqq C.$$

See [4] for motivation. Following a suggestion of L. Carleson, we give a proof of this theorem based on an unpublished result of A. Beurling. In Section 1, we give a proof of a version of Beurling's theorem. In Section 2, we use Beurling's theorem to reduce our problem to an integral inequality due to J. Moser [7]. Finally, in Section 3, we give a proof of Moser's inequality. We would like to thank P. Jones for helpful discussions.

1. Beurling's theorem

If f is analytic on Δ, let γ_t be the level curves of f defined by $\gamma_t=\{z\in\Delta\colon\ |f(z)|=t\}$ and let $|f(\gamma_t)|$ denote the length of the image of these curves under the map f. In other words, $|f(\gamma_t)|=\int_{\gamma_t}|f'(z)|\,|dz|$ where $|dz|$ denotes arc length. If E is a subset of the complex plane, let cap (E) denote the logarithmic capacity of E.

* Supported in part by the National Science Foundation and the American—Scandinavian Foundation.

Theorem 2 (Beurling). *Suppose f is analytic in a neighborhood of the closed unit disk and suppose that $|f(z)| \leqq M$ whenever $|z| \leqq r$. Then*

$$\operatorname{cap}\{e^{i\theta}\colon |f(e^{i\theta})| > x\} \leqq r^{-1/2} \exp\left\{-\pi \int_M^x \frac{dt}{|f(\gamma_t)|}\right\}.$$

We remark that if f is also univalent and $f(0)=0$, we may take M to be the radius of the largest disk centered at 0 contained in the image $f(\Delta)$. In this case, two applications of Koebe's 1/4-theorem show $r^{-1/2} \leqq 4$. However, the proof below can be modified, in this case, to replace the constant $r^{-1/2}$ by $\left(\frac{|f'(0)|}{M}\right)^{1/2}$, which is at most 2. This latter result is sharp, for if f is the conformal map to a slit disk of radius x, the inequality is actually an equality. This case, with constant 4 can also be found in Haliste [6, Theorem 3.2], and is related to a classical distortion estimate of Ahlfors. We will use Beurling's estimate to obtain an integral inequality that is delicate both in its growth dependence on x and on the initial value M, which we must take larger than the distance from 0 to the complement of $f(\Delta)$ when f is not univalent. Tsuji [8, p. 112] has proved a similar result, where the upper limit x is replaced by τx, $\tau < 1$, and the coefficient $r^{-1/2}$ is replaced by a coefficient that becomes unbounded as $\tau \to 1$. To prove Theorem 1, we need $\tau = 1$.

Proof. We adopt the notations and definitions of [2]. If Ω is a domain in the plane and if $E_1, E_2 \subset \partial\Omega$, let $d_\Omega(E_1, E_2)$ denote the extremal distance between E_1 and E_2 with respect to the region Ω. It will be clear from the context which regions Ω we consider below, so we will drop the subscript Ω. For example, if C_a denotes the circle centered at 0 of radius a, then $d(C_t, C_s) = (1/2\pi) \log s/t$, for $s > t$. By Theorems 2.4 and 4.9 of [2]

$$-\frac{1}{\pi} \log(\operatorname{cap} E) = \lim_{t \to 0} \{d(C_t, E) - d(C_t, C_1)\}$$

for subsets E of $\partial\Delta$. Since the disk of radius r is contained in $\{z\colon |f(z)| < M\}$ by assumption, we obtain

$$-\frac{1}{\pi} \log\left(\operatorname{cap}\{e^{i\theta}\colon |f(e^{i\theta})| > x\}\right) \geqq \lim_{t \to 0} d(C_t, \gamma_x) + \frac{1}{2\pi} \log t$$

$$\geqq \lim_{t \to 0} d(C_t, C_r) + d(C_r, \gamma_M) + d(\gamma_M, \gamma_x) + \frac{1}{2\pi} \log t$$

$$\geqq \frac{1}{2\pi} \log r + d(\gamma_M, \gamma_x).$$

In the case where f is univalent and $f(0)=0$,

$$\lim_{t\to 0} d(C_t, \gamma_x)+\frac{1}{2\pi}\log t = \lim_{t\to 0} d(\gamma_{t|f'(0)|}, \gamma_x)+\frac{1}{2\pi}\log t$$

$$\geqq \lim_{t\to 0} d(\gamma_{t|f'(0)|}, \gamma_M)+d(\gamma_M, \gamma_x)+\frac{1}{2\pi}\log t.$$

By the conformal invariance of the extremal distance, this latter quantity equals

$$d(\gamma_M, \gamma_x)+\frac{1}{2\pi}\log\frac{M}{|f'(0)|}.$$

To prove our theorem, we must finally show

$$d(\gamma_M, \gamma_x) \geqq \int_M^x \frac{dt}{|f(\gamma_t)|}.$$

We define a metric on Δ by

$$\varrho(z)=\frac{|f'(z)|}{|f(\gamma_t)|} \quad \text{if} \quad z\in\gamma_t.$$

Let $\Omega_{M,x}=\{z\colon M<|f(z)|<x\}$. By converting to polar coordinates on the Riemann surface $f(\Delta)$, we obtain

$$\int_{\Omega_{M,x}} \varrho^2(z)\,dx\,dy = \int_M^x \frac{\int_{\gamma_t}|f'(z)|\,|dz|\,dt}{|f(\gamma_t)|^2} = \int_M^x \frac{1}{|f(\gamma_t)|}\,dt.$$

If γ is a curve in $\Omega_{M,x}$ such that $\gamma(0)\in\gamma_M$ and $\gamma(1)\in\gamma_x$, then

$$\int_\gamma \varrho|dz| = \int_\gamma \frac{|f'(z)|\,|dz|}{|f(\gamma_t)|} = \int_{f(\gamma)}\frac{|dw|}{|f(\gamma_t)|} \geqq \int_M^x \frac{dt}{|f(\gamma_t)|}.$$

Thus $d(\gamma_M, \gamma_x)\geqq\int_M^x \frac{dt}{|f(\gamma_t)|}$, proving the theorem.

2. Proof of Theorem 1

We need the following elementary lemma.

Lemma. *There is a universal constant $r>0$ such that: for each f analytic on Δ with $f(0)=0$ and $\mathscr{D}(f)\leqq 1$, there is an M (depending upon f) with*

(i) $0<M\leqq 1$

(ii) $\{z\colon |z|<r\}$ *is contained in* $\{z\colon |f(z)|<M\}$

(iii) $\int_0^M |f(\gamma_t)|\,dt\geqq\pi M^2/3$.

Assuming the lemma for the moment, we proceed as follows.

We first note that we may suppose f is analytic in a neighborhood of the closed unit disk. One way to see this is to observe that if $f_r(z)=f(rz)$, then $\int e^{|f_r|^2}d\theta=\sum_{n=0}^{\infty} 1/n!\int |f_r|^{2n}d\theta$. Since the L^{2n} norms of f_r increase with r to the L^{2n} norm of f, we see that $\lim_{r\to 1}\int e^{|f_r|^2}d\theta=\int e^{|f|^2}d\theta$.

By Theorem 2.7 of [2], the lemma above and Beurling's theorem

$$\sin\frac{|\{\theta\colon |f(e^{i\theta})|>x\}|}{4}\leqq r^{-1/2}\exp\left\{-\pi\int_M^x\frac{dt}{|f(\gamma_t)|}\right\}$$

where $|E|$ denotes the linear measure of a subset E of $\partial\Delta$. Thus

$$\int_0^{2\pi} e^{|f(e^{i\theta})|^2}d\theta = 2\pi+2\int_0^{\infty}|\{\theta\colon |f(e^{i\theta})|>x\}|e^{x^2}x\,dx$$

$$\leqq 2\pi+4\pi\int_0^M e^{x^2}x\,dx+\pi r^{-1/2}\int_M^{\|f\|_\infty}\exp\left\{x^2-\pi\int_M^x\frac{dt}{|f(\gamma_t)|}\right\}x\,dx.$$

Let

$$\alpha(x)=\begin{cases}\dfrac{\pi Mx}{\int_0^M |f(\gamma_t)|\,dt} & \text{if } 0\leqq x\leqq M\\[2ex] \pi\displaystyle\int_M^x\frac{dt}{|f(\gamma_t)|}+\frac{\pi M^2}{\int_0^M|f(\gamma_t)|\,dt} & \text{if } M\leqq x<\|f\|_\infty.\end{cases}$$

By the lemma $\alpha(M)\leqq 3$, and $M\leqq 1$, so it suffices to bound

(1) $$\int_0^{\|f\|_\infty} e^{x^2-\alpha(x)}x\,dx.$$

Note that $\alpha'(x)>0$ and $\alpha(0)=0$. Let

$$\varphi(y)=\begin{cases}x & \text{if } y=\alpha(x)\\ \|f\|_\infty & \text{if } y>\|\alpha\|_\infty.\end{cases}$$

Then $\varphi(0)=0$ and $\int_0^\infty (\varphi'(y))^2\,dy=\int_0^{\|f\|_\infty}\frac{|f(\gamma_t)|}{\pi}\,dt=\mathscr{D}^2(f)\leqq 1$. Using this change of variables, (1) becomes

$$\int_0^\infty e^{\varphi^2(y)-y}\varphi(y)\varphi'(y)\,dy.$$

By an approximation we may suppose φ' is continuous and has compact support in $(0,\infty)$. Integrating by parts, we find that it suffices to show that there is a constant $C<\infty$ such that if φ is absolutely continuous, $\varphi(0)=0$, and $\int_0^\infty(\varphi'(y))^2\,dy\leqq 1$ then

$$\int_0^\infty e^{\varphi^2(y)-y}\,dy\leqq C.$$

This was done by Moser [7]. If the assumption that $\varphi(0)=0$ is dropped, this latter estimate fails. That is why we need control on the ratio $\pi M^2/\int_0^M |f(\gamma_t)|\,dt$ as provided by the lemma, and consequently need the correct lower limit in Beurling's integral estimate.

Proof of the lemma. If $f(z)=\sum_{n=1}^{\infty} a_n z^n$ then $\mathcal{D}^2(f)=\sum_{n=1}^{\infty} n|a_n|^2$, so by the Cauchy—Schwarz inequality

$$|f(z)| \leqq \mathcal{D}(f)\left(\log\frac{1}{1-|z|^2}\right)^{1/2} \leqq \left(\log\frac{1}{1-|z|^2}\right)^{1/2}.$$

Hence $|f(z)|<M$ if $|z|<(1-e^{-M^2})^{1/2}$. If the analog of Bloch's theorem for disks centered at 0 were true here, i.e. if f covered a disk of some fixed radius M_0, *centered at* 0, then the lemma would follow with $r=(1-e^{-M_0^2})^{1/2}$. Unfortunately, this is not the case, so we proceed as follows: Note

$$\int_0^R |f(\gamma_t)|\,dt \geqq \pi R^2$$

for small R. If $\int_0^R |f(\gamma_t)|\,dt>\dfrac{\pi R^2}{2}$ for all R, $0<R<1$, let $M=1$ and $r=(1-e^{-1})^{1/2}$. Otherwise we may assume $\pi M^2/3\leqq\int_0^M |f(\gamma_t)|\,dt\leqq\pi M^2/2$ for some M, $0<M\leqq 1$. Then $f(\Delta)$ omits at least two points a, b with $M/2\leqq|a|\leqq|b|\leqq M$ and $|b-a|\geqq M/2$. If λ denotes the covering map of Δ onto $\mathbf{C}\setminus\{0,1\}$, then by the monodromy theorem $g(z)=\lambda^{-1}\left(\dfrac{f(z)-a}{b-a}\right)$ can be defined to be analytic and bounded by 1 in Δ. Note that $\dfrac{1}{4}\leqq\left|\dfrac{-a}{b-a}\right|\leqq 2$ and $\left|\dfrac{-a}{b-a}-1\right|\geqq\dfrac{1}{4}$, so that $|g(0)|<c<1$ for some universal constant c. By Schwarz's lemma applied to g, $\dfrac{f(z)-a}{b-a}$ must lie in a disk of radius at most 1/8, if $|z|<r$ with r sufficiently small and independent of f and M. Hence $|f(z)|<M$ on $|z|<r$.

In the course of discovering the lemma, the following problem arose, which we were unable to solve:

Does there exist a universal constant $r>0$ such that if f is analytic on $\Delta, f(0)=0$, and for some M, $\int_0^M |f(\gamma_t)|\,dt<\pi M^2$, then $\{z\colon |z|<r\}$ is contained in $\{z\colon |f(z)|<M\}$? Here, as before, $\gamma_t=\{z\in\Delta\colon |f(z)|=t\}$ and $|f(\gamma_t)|=\int_{\gamma_t}|f'(z)|\,|dz|$. If f is univalent, $r=1/16$ will work by two applications of Koebe's 1/4-theorem.

3. Moser's theorem

In this section, we give a shorter proof of Moser's theorem.

Theorem 3 (Moser [7]). *There is a constant* $C<\infty$ *such that if* $\int_0^\infty \psi^2(y)\,dy\leqq 1$ *then* $\int_0^\infty e^{-F(t)}dt\leqq C$ *where* $F(t)=t-(\int_0^t \psi(y)\,dy)^2$.

The key lemma in Adams [1] is a generalization of this theorem. Adams attributes the technique of the proof of that lemma to a private communication of A. Garsia to J. Moser. Extracting the technique in Adams' lemma and applying it to Moser's theorem, as stated, the following remarkably simple proof emerges.

Proof. Note that $(\int_0^t \psi(y)\,dy)^2\leqq t\int_0^\infty \psi^2(y)\,dy\leqq t$, so that $F(t)\geqq 0$. Let $E_\lambda=\{t\geqq 0\colon F(t)\leqq\lambda\}$. It suffices to show that if $t_1, t_2\in E_\lambda$ with $2\lambda\leqq t_1<t_2$, then $t_2-t_1\leqq 20\lambda$. For then $|E_\lambda|\leqq 22\lambda$ and $\int_0^\infty e^{-F(t)}dt=\int_0^\infty |E_\lambda|e^{-\lambda}d\lambda\leqq 22$. Note that if $t\in E_\lambda$,

$$t-\lambda\leqq\Big(\int_0^t \psi(y)\,dy\Big)^2\leqq t\int_0^t \psi^2(y)\,dy\leqq t-t\int_t^\infty \psi^2(y)\,dy,$$

and so $\int_t^\infty \psi^2(y)\,dy\leqq\lambda/t$. Thus

$$t_2-\lambda\leqq\Big[\int_0^{t_1}\psi(y)\,dy+\int_{t_1}^{t_2}\psi(y)\,dy\Big]^2\leqq\Big[t_1^{1/2}+(t_2-t_1)^{1/2}\Big(\int_{t_1}^\infty \psi^2(y)\,dy\Big)^{1/2}\Big]^2$$

$$\leqq[t_1^{1/2}+(t_2-t_1)^{1/2}(\lambda/t_1)^{1/2}]^2\leqq t_1+2(t_2-t_1)^{1/2}\lambda^{1/2}+(t_2-t_1)\tfrac{1}{2}.$$

We conclude $\frac{3}{2}(t_2-t_1)\leqq\lambda+2(t_2-t_1)^{1/2}\lambda^{1/2}+t_2-t_1=(\lambda^{1/2}+(t_2-t_1)^{1/2})^2$ and hence

$$t_2-t_1\leqq\frac{\lambda}{((\frac{3}{2})^{1/2}-1)^2}<20\lambda.$$

A similar proof shows that if $\int_0^\infty \psi^p(y)\,dy\leqq 1$ then $\int_0^\infty e^{-F(t)}dt\leqq C_p$ where $F(t)=t-(\int_0^t \psi(y)\,dy)^q$, $\frac{1}{p}+\frac{1}{q}=1$, $1<p<\infty$. As Adams remarks, this technique does not seem delicate enough to find the best possible constant C. See Carleson and Chang [3] for a numerical estimate of the best C. After this paper was written, a more general version of Theorem 1 was discovered by Essén [5], using somewhat different methods.

References

1. Adams, D., A sharp inequality of J. Moser for higher order derivatives, *Ann. of Math.* **128** (1988), 385—398.
2. Ahlfors, L., *Conformal Invariants*, McGraw-Hill, New York, 1973.
3. Carleson, L. and Chang, S.-Y., On the existence of an extremal function for an inequality of J. Moser, *Bull. Sc. Math.* **110** (1986), 113—127.

4. CHANG, S.-Y. and MARSHALL, D. E., A sharp inequality concerning the Dirichlet integral, *Amer. J. Math.* **107** (1985), 1015—1033.
5. ESSÉN, M., Sharp estimates of uniform harmonic majorants in the plane, *Ark. Mat.* **25** (1987), 15—28.
6. HALISTE, K., Estimates of harmonic measures, *Ark. Mat.* **6** (1967), 1—31.
7. MOSER, J., A sharp form of an inequality by N. Trudinger, *Indiana Univ. Math. J.* **20** (1971), 1077—1092.
8. TSUJI, M., *Potential Theory*, Chelsea, New York, 1975.

Received Sept. 27, 1984, in revised form (Sec. 3), July 12, 1988

Department of Mathematics
University of Washington
Seattle, WA 98195
U.S.A.

Editors' note: The Arkiv för matematik apologizes for the very delayed publication of this paper. The delay was caused by an unfortunate misunderstanding on the part of the editors.