Removable singularities of CR-functions

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0. Introduction, statement and discussion of the results. There are well-known theorems concerning removable singularities of analytic or harmonic functions from various classes in planar domains, see for example [11]. There are also generalizations to higher-dimensional domains and general elliptic differential operators instead of the $\bar{\partial}$ or Laplace operator [4]. The description of removable singularities depends on the class of functions and is usually given in terms of capacity or Hausdorff measure. For operators appearing in the theory of several complex variables such as the Cauchy—Riemann system or the $\bar{\partial}_b$ operator (the boundary Cauchy—Riemann operator for smooth domains in $\mathbf{C}^n$) we have to expect new phenomena so that the complete description cannot be given in the terms mentioned. This is suggested, for example, by the well-known Hartogs theorem ([5], Theorem 2.3.2, [8], 16.3.6): every function f analytic in the connected set $\Omega\setminus E$, Ω being a domain in $\mathbf{C}^n$ $(n>1)$ and E a compact subset of Ω, is the restriction to $\Omega\setminus E$ of a function analytic in the whole of Ω. So for the Cauchy—Riemann system a "removable set of singularities" E is not necessarily small in measure or capacity, it can even have a nonempty interior.

Suppose now that a closed set E is situated in $\operatorname{Clos}\Omega$ (not necessarily in Ω!). What the Hartogs theorem suggests in this case is that the removability of E (with respect to the class of all functions analytic in $\Omega\setminus E$) depends only on the behaviour of E near the boundary $\operatorname{Fr}\Omega$ or maybe depends only on $E\cap\operatorname{Fr}\Omega$ ($\operatorname{Clos} A$ means the closure of the set A, $\operatorname{Fr} A$ its boundary). This fact must imply Hartogs type theorems for the boundary Cauchy—Riemann operator. That means that we have to expect the existence of sets $A\subset\operatorname{Fr}\Omega$ which are removable singularities for the boundary Cauchy—Riemann operator and are "large" in some sense. So they are not necessarily removable for arbitrary differential operators of first order.

Now we shall give precise statements of the mentioned results. In the statement and proof of the results we restrict ourselves to the case of the unit ball $B(=B^n)$ in $\mathbf{C}^n$ although it is not hard to see that the main results are true for strictly pseudoconvex domains Ω with sufficiently smooth boundary (as a rule of class C^2). (For weakly pseudoconvex domains the complete description becomes more complicated, see, for example, [17] and the references there and also [12].) We assume that $E \cap \mathrm{Fr}\, B$ is contained in some open manifold M, $M \subset \mathrm{Fr}\, B$, (or that the singularities of solutions of the boundary Cauchy—Riemann operator are contained in M, respectively) and give the answer to the questions raised above in terms of M.

We shall even get more general results. We consider a domain of the form $\Omega \cap V$, Ω being strictly pseudoconvex with smooth boundary and V being a neighborhood of M. The functions are supposed to be analytic in $(\Omega \cap V) \setminus E$ with $E \cap (\mathrm{Fr}\, \Omega) \cap V \subset M$ (respectively, the singularities with respect to the $\bar{\partial}_b$-equation on $(\mathrm{Fr}\, \Omega) \cap V$ are contained in M). For suitable M we shall continue the functions analytically into $\Omega \cap V_1$, V_1 being another neighborhood of M (respectively, we shall show that there are no singularities of the $\bar{\partial}_b$-equation on $(\mathrm{Fr}\, \Omega) \cap V$).

Definition 1. A manifold* $M \subset \mathrm{Fr}\, B$ ($B = B^n \subset \mathbf{C}^n$) is called removable if the following is true: for an arbitrary closed set $E \subset \mathrm{Clos}\, B$ with $E \cap \mathrm{Fr}\, B \subset M$ and $B \setminus E$ connected the envelope of holomorphy of $B \setminus E$ is B, i.e. every function f analytic in $B \setminus E$ is the restriction $\mathscr{F} | (B \setminus E)$ of a function $\mathscr{F}$ analytic in B.

Definition 2. A manifold $M \subset \mathrm{Fr}\, B$ is called $(L^\infty, \bar{\partial}_b)$-removable if every function $f \in L^\infty(\mathrm{Fr}\, B)$ with $\bar{\partial}_b f = 0$ in $(\mathrm{Fr}\, B) \setminus A$ (in the sense of distributions) for some $A = \mathrm{Clos}\, A \subset M$ satisfies $\bar{\partial}_b f = 0$ on the whole of $\mathrm{Fr}\, B$.

For the case of distributions on $\mathrm{Fr}\, B$ we have to change the definition a little bit (for example, the Dirac measure δ_p on $\mathrm{Fr}\, B$ at a point $p \in \mathrm{Fr}\, B$ satisfies the equation $\bar{\partial}_b \delta_p = 0$ on $\mathrm{Fr}\, B \setminus \{p\}$ but not on the whole of $\mathrm{Fr}\, B$, so even points are never removable in this sense).

Definition 3. A manifold $M \subset \mathrm{Fr}\, B$ is called $(\mathscr{E}', \bar{\partial}_b)$-removable if the following holds: for every distribution f on $\mathrm{Fr}\, B$ with $\bar{\partial}_b f = 0$ in $(\mathrm{Fr}\, B) \setminus A$ for some closed set $A \subset M$ there exists a distribution g on $\mathrm{Fr}\, B$ with support contained in M and such that $\bar{\partial}_b (f - g) = 0$ on the whole of $\mathrm{Fr}\, B$.

There is no obvious relationship between removability, $(L^\infty, \bar{\partial}_b)$-removability and $(\mathscr{E}', \bar{\partial}_b)$-removability.

We need also the following "local"

Definition 4. A manifold $M \subset \mathrm{Fr}\, B$ is said to be locally removable at a point $p \in M$ if there exists a neighbourhood $\mathscr{U}$ of p (in $\mathbf{C}^n$) such that $\mathscr{U} \cap M$ is removable in the sense of Definition 1.

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* In this work manifold (of dimension m) always means a topological space, every point of which has a neighborhood homeomorphic to an *open* set of $\mathbf{R}^m$.

In an analogous way local variants of Definitions 2 and 3 can be given. Note that, as we will see below, a manifold M which is locally removable at all points need not be removable.

Theorem 1. *Suppose* $M \subset \operatorname{Fr} B^n$ *is a manifold of class* C^2, $p \in M$. *If*

$$T_p^c(\operatorname{Fr} B) \setminus T_p M \neq \emptyset, \tag{1}$$

then M *is locally removable at* p.

($T_p M$ is the tangent space of M at the point p and $T_p^c(\operatorname{Fr} B)$ is the complex tangent space of the sphere at the point p (see [8], 5.4.2).)

If condition (1) fails, M is not necessarily locally removable at p as the following example shows.

Example 1. $p=(1, 0, \ldots, 0) \in \operatorname{Fr} B^n$, V is some small neighbourhood (in $\mathbf{C}^n$) of p and $M=\{z=(z_1, \ldots, z_n) \in V \cap \operatorname{Fr} B^n\colon \operatorname{Im} z_1=0\}$. Here $T_p M = T_p^c(\operatorname{Fr} B^n) = \{z \in \mathbf{C}^n\colon z_1=0\}$. Suppose g_ε is a bounded analytic function in the planar domain $\{z \in \mathbf{C}\colon |z|<1\} \setminus [1-\varepsilon, 1]$ for some small $\varepsilon>0$ which cannot be analytically continued to the whole unit disc $\{z \in \mathbf{C}\colon |z|<1\}$. The function $f_\varepsilon(z_1, z_2, \ldots, z_n) = g_\varepsilon(z_1)$ is then defined and holomorphic for $z=(z_1, \ldots, z_n) \in B^n \setminus \{\operatorname{Im} z_1=0,\ \operatorname{Re} z_1 \geqq 1-\varepsilon\}$, but is not holomorphically continuable to the whole of B^n. The singularity set $E=\{z \in \operatorname{Clos} B^n\colon \operatorname{Im} z_1=0, \operatorname{Re} z_1 \geqq 1-\varepsilon\}$ reaches the sphere $\operatorname{Fr} B^n$ along a subset of the set $\{z \in \operatorname{Fr} B^n\colon \operatorname{Im} z_1=0,\ |z-p|^2=|z_1-1|^2+|z_2|^2+\ldots+|z_n|^2 \leqq \varepsilon^2+(1-|z_1|^2) \leqq 2\varepsilon\}$ which is the intersection of M with a small neighbourhood of p (depending on ε). Note that f_ε is bounded, so M is not even locally $(L^\infty, \bar{\partial}_b)$-removable at p.

So, a manifold M of real dimension $\dim_r M < 2n-2$ is locally removable at every point. In the case of real dimension $2n-2$ the locally removability depends on the position of M with respect to the complex structure in the neighbourhood of the point p. So, real dimension $2n-2$ is the critical one for M to be locally removable or not. (If M has complete real dimension $2n-1$, it cannot be locally removable.)

The following example shows (with the help of Theorem 1) that a manifold which is locally removable at all points need not be removable.

As the referee pointed out to me other natural examples of sets which are locally removable at every smooth point but not globally removable are the zero sets of functions analytic in B^n and of class C^2 in $\operatorname{Clos} B^n$.

Example 2. Consider $\mathbf{C}^2$ and let $M \subset \operatorname{Fr} B^2$ be the torus

$$M = \{z = (z_1, z_2) \in \operatorname{Fr} B^2\colon |z_1| = |z_2| = 1/\sqrt{2}\}.$$

It is not hard to see from Theorem 1 that M is locally removable at every point.

On the other hand M divides $\operatorname{Fr} B^2$ into two parts $S_1=\{|z_1|>1/\sqrt{2}\}$ and $S_2=\{|z_1|<1/\sqrt{2}\}$. The function which is zero in $\{z\in B^2\colon |z_1|>1/\sqrt{2}\}$ and equals one in $\{z\in B^2\colon |z_1|<1/\sqrt{2}\}$ cannot be analytically continued to the whole of B^2. So, M is not removable and not even $(L^\infty, \bar\partial_b)$-removable.

A slight modification shows that such manifolds M do not necessarily disconnect $\operatorname{Fr} B^2$.

Example 3. $M=\{z=(z_1, z_2)\in \operatorname{Fr} B^2\colon |z_1|=|z_2|=1/\sqrt{2}, \arg z_1\in(-\alpha, \alpha)\}$, $0<\alpha<\pi$. A test function showing that M is not removable is $f(z_1, z_2)=g(z_1)$ ($z = (z_1, z_2)\in B^2\setminus\{|z_1| = 1/\sqrt{2}, \arg z_1\in(-\alpha, \alpha)\}$) with g analytic (and bounded) in $\{z\in \mathbf{C}\colon |z|<1\}\setminus\{|z|=1/\sqrt{2}, \arg z\in(-\alpha, \alpha)\}$ but not in the whole of $\{|z|<1\}$.

In this case M is a "ring". The condition that M contains a ring is, roughly speaking, the only "global obstruction" for M to be removable. The following theorem holds.

Theorem 2. (Main theorem.) *Suppose* $M\subset \operatorname{Fr} B^2$ *is a manifold of class* C^2 *and of real dimension* $\dim_r M=2$. *If* (1) *is satisfied for every point* $p\in M$ *(that means in this case that* M *is totally real) and*

(2) *M is diffeomorphic to the open unit disc in the plane*

then M is removable.

Corollary 1. *Suppose M is a simple (nonclosed) Jordan curve on* $\operatorname{Fr} B^2$ *of class* C^2. *Then M is removable.*

Note that the closed Jordan curve $\gamma=\{z=(z_1, z_2)\in \operatorname{Fr} B^2\colon z_1=a, |z_2|=\sqrt{1-|a|^2}\}$ (a being a complex number, $|a|<1$) is not removable as the function $f(z_1, z_2)=1/(z_1-a)$ shows.

This corollary can be proved easily by constructing a manifold $\tilde M\subset \operatorname{Fr} B^2$ satisfying the conditions of Theorem 2 and containing M. Corollary 1 can be proved also directly without using Theorem 2 (the direct proof is simpler than that of Theorem 2) and can be generalized to simple smooth Jordan curves on $\operatorname{Fr} B^n$ for arbitrary $n\geqq 2$. We will not do this here.

Corollary 1'. *Suppose M is as in Corollary* 1. *Then M is* $(\mathscr{E}', \bar\partial_b)$*-removable.* Smooth curves on $\operatorname{Fr} B^2$ are always $(L^\infty, \bar\partial_b)$-removable ([4]).

Corollary 2. *Suppose* $M\subset \operatorname{Fr} B^2$ *is as in Theorem* 2. *Then M is* $(L^\infty, \bar\partial_b)$*-removable.*

Corollary. 3. *Suppose* $M\subset \operatorname{Fr} B^2$ *is as in Theorem* 2. *Then M is* $(\mathscr{E}', \bar\partial_b)$*-removable.*

Examples 1, 2, 3 show that both conditions (1), (2) on M, namely, that $T_p^c(\mathrm{Fr}\, B^n)\setminus T_pM\neq\emptyset$ for all $p\in M$ and that M is diffeomorphic to a planar disc, are essential. Note that manifolds M of this kind are "large" in various senses. For example, they are uniqueness sets for the space $A(B^2)$ of functions analytic in B^2 and continuous in $\mathrm{Clos}\, B^2$ [16]. Note also that the results of [4] (Theorem 4.1(b)) concerning general differential operators of first order imply the $(L^\infty, \bar\partial_b)$-removability of a set $A\subset \mathrm{Fr}\, B^2$ if the two-dimensional Hausdorff measure of A is zero, $\Lambda_2(A)=0$. The condition $\Lambda_2(A)<\infty$ is not sufficient for general operators. Examples show that results of the type similar to Corollary 2 hold only for special operators (only for special differential operators of first order smooth simply connected manifolds transversal to the operator at each point are removable in the sense of Definition 2 or 3). (Indeed, even the Cauchy—Riemann operator $P=\dfrac{\partial}{\partial x}+i\dfrac{\partial}{\partial y}$, acting on functions defined for $(x, y, t)\in \mathbf{R}^3$ gives a counterexample. Take $M=\{(x, y, t)\in\mathbf{R}^3\colon y=0,\ x^2+t^2<3\}$. Then P is nowhere tangent with respect to M, but M is not removable (for the class L^∞). This can be seen by taking a test function g defined on $\mathbf{R}^3\setminus M$ in the following way. Let $f\in H^\infty(\hat{\mathbf{C}}\setminus[-1, 1])$, $f\neq\mathrm{const}$, $\hat{\mathbf{C}}$ being the extended complex plane. For $(x, y, t)\in\mathbf{R}^3\setminus M$ put $g(x, y, t)=f\left(\dfrac{x+iy}{(1-t^2)^2}\right)$ for $|t|<1$ and $g(x, y, t)=f(+\infty)$ for $|t|\geqq 1$. Note that $g\in C^1(\mathbf{R}^3\setminus M)$, so, obviously, $Pg=0$ on $\mathbf{R}^3\setminus M$. By fixing some t, $|t|<1$, it is easy to see that M is not removable.) So it would be interesting to give conditions for an operator (possibly different from the $\bar\partial_b$-operator for some smooth strongly pseudoconvex domain) to have Hartogs type removable singularities or not. It would also be interesting to give an operator theoretical proof of Corollaries 2 and 3 not using "complex methods".

Look now what Corollary 2 means in a special situation. Consider the domain $\Omega=\{z=(z_1, z_2)\in\mathbf{C}^2\colon |z_1|^2<\mathrm{Im}\, z_2\}$ (the Cayley transform of B^2). Its boundary $\mathrm{Fr}\,\Omega=H_2=\{(z_1, z_2)=(z, s+it)\colon t=|z|^2\}$ is the well-known Heisenberg group. Identify it with $\mathbf{C}\times\mathbf{R}$ by the mapping $(z, s)\to(z, s+i|z|^2)$. The $\bar\partial_b$-operator for H_2 is $L^0=2iz\dfrac{\partial}{\partial\bar w}-\dfrac{\partial}{\partial\bar z}$ $(w=s+it)$ (see [8], 18.2.4). Continuing $h\in C^1(H_2)$ onto $\mathbf{C}^2$ so that h does not depend on t we get $Lh=iz\dfrac{\partial}{\partial s}h-\dfrac{\partial}{\partial\bar z}h$. (Note that the formal adjoint operator with respect to $L^2(H^2)$ is the famous H. Lewy operator $\dfrac{\partial}{\partial x}-i\dfrac{\partial}{\partial y}+i(x-iy)\dfrac{\partial}{\partial s}$). Corollary 2 (more exactly, its slightly stronger variant which follows from the Theorem 2′ below) now gives the following: Suppose that $O\subset\mathbf{C}\times\mathbf{R}$ is an open set, $M\subset O$ is a C^2-manifold of $\dim M=2$ and M is diffeomorphic to an open planar disc. If for all $p=(z, s)\in M$ the inequality $T_pM\neq\{(Z, S)\in\mathbf{C}\times\mathbf{R}\colon S=\operatorname{Re}(2i\bar{z}Z)\}$ holds (that means L is not tangent for M at any point), then for any compact set $K\subset M$ and any $h\in L^\infty(O)$ the equality $Lh=0$ in $O\setminus K$ (in distributional sense) implies $Lh=0$ in O (in distributional sense). On the other hand there are C^2-manifolds of real dimension two and of arbitrarily small diameter without this property.

Now we come to the proof of the main theorem. An outline of the proofs of Corollaries 2 and 3 and also of the "local" Theorem 1 will be given later. Also some remarks will be given concerning the proof of the analogue of Theorem 2 for strongly pseudoconvex domains in $\mathbf{C}^2$ (instead of the ball B^2).

1. Proof of the main theorem. Denote for simplicity $B=B^2$, $S=\operatorname{Fr} B^2$. Suppose that $M\subset S$ is as in the theorem, E is a closed subset of $\operatorname{Clos} B$, $E\cap S\subset M$, and f is analytic in $B\setminus E$, $B\setminus E$ being connected. We have to prove the following. For every r, r being smaller than one and sufficiently close to one, f can be analytically continued into a neighbourhood of the sphere rS. Then f is analytic in some spherical layer $\{r_0<|z|<1\}$ and so by Hartogs' theorem also in the whole of B.

The idea of the proof is as follows. We take a point $\zeta\in E\cap rS$ and construct an analytic disc Δ_ζ (i.e. the holomorphic image of a planar domain) contained in B with $\zeta\in\Delta_\zeta$ and with boundary $\operatorname{Fr}\Delta_\zeta$ contained in $B\setminus E$. Our aim is to show that there is an analytic function in a neighbourhood of Δ_ζ which coincides with f in a neighbourhood of the boundary $\operatorname{Fr}\Delta_\zeta$. This will be done by using the continuity principle ([14], III § 17): f will be analytically continued "along a suitable family of analytic discs". Note that all constructed discs are situated in a small neighbourhood of M, so the following somewhat sharper form of Theorem 2 will be proved.

Theorem 2′. *Suppose $M\subset S$ is as in Theorem 2, V is some neighbourhood (in $\mathbf{C}^2$) of M and f is analytic in $(B\cap V)\setminus E$ for some closed E with $E\cap S\subset M$, $(B\cap V)\setminus E$ being connected. Then f can be analytically continued into the intersection of B with some neighbourhood of M. If f is bounded in $(B\cap V)\setminus E$, then so is the analytic continuation.*

For convenience of the reader we work out the details first in a simple particular case and then pass to the general case which needs some additional tools. This particular case is not needed in the proof of the general case.

Particular case. $M\subset S$ is diffeomorphic to the unit disc in the plane and is contained in the torus

$$\mathbf{T}^2 \stackrel{\text{def}}{=} \left\{z=(z_1, z_2)\in\mathbf{C}^2\colon |z_1|=|z_2|=\frac{1}{\sqrt{2}}\right\}\subset S.$$

Proof of Theorem 2′ *in the particular case.* Everything will be done in a small neighbourhood of M, so we can perform a logarithmic transformation of coordinates $(z_1, z_2)\to(\log z_1, \log z_2)$ and after a translation we come to the following situation. There is a manifold $M\subset\{\mathbf{0}\}+i\mathbf{R}^2$ ($\mathbf{0}$ is the origin in $\mathbf{R}^2$) which is diffeomorphic to the (open) unit disc in the plane. Write $M=\{\mathbf{0}\}+im$, where $m\subset\mathbf{R}^2$ can be assumed to be bounded, simply connected and with smooth boundary. Further V is some neighbourhood of M, say $V=v+im$, where v is a neighbourhood of zero in $\mathbf{R}^2$. Instead of B we have some domain Ω with smooth boundary such that $V\cap\Omega=(v\cap\omega)+im$, where ω is a domain in $\mathbf{R}^2$ with smooth boundary, such that the boundary points of ω belonging to v are points of strong convexity. Also $\mathbf{0}\in(\mathrm{Fr}\,\omega)\cap v$ (that means $M\subset(\mathrm{Fr}\,\Omega)\cap V$). Further there is a closed set E with $E\cap V\cap\mathrm{Fr}\,\Omega\subset M$ and a function f analytic in $(\Omega\cap V)\setminus E$, this set being connected. We have to show that analytic continuation of f gives a function which is analytic in all points of $\Omega\cap V$ sufficiently close to M. This is, of course, (after a unitary transformation) the situation of 18.1.7, 18.1.8. of the book by Rudin [8]. Instead of showing here that conditions 18.1.7. hold in our situation, we will carry out the proof in a somewhat different way which demonstrates the method used in the general case.

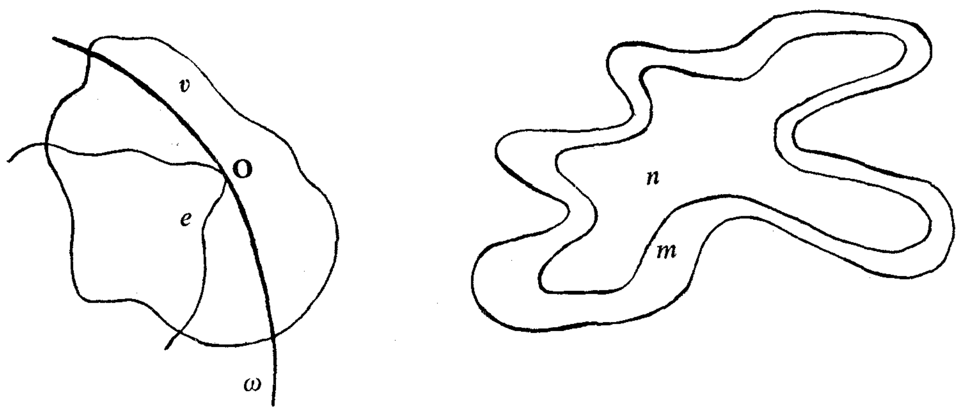

Fig. 1

Choosing v sufficiently small we may assume that $E\cap V\subset(e+in)\cap V$, where e and n are closed subsets of $\mathbf{R}^2$, $n\subset m$ and n is diffeomorphic to the closed unit disc in the plane and $e\cap v\subset\mathrm{Clos}\,\omega$, $e\cap\mathrm{Fr}\,\omega=\{\mathbf{0}\}$.

So, f is analytic in $\{(\omega+im)\setminus(e+in)\}\cap V$. We want to construct the analytic continuation of f into $(\omega+im)\cap(v_1+im)$ for some small neighbourhood v_1 of $\mathbf{0}$. Now it is not hard to construct the analytic disc Δ_ζ through a point $\zeta=\xi+i\eta\in(\omega\cap v_1)+im$. It will be a part of the complex line Π_ζ through the point ζ parallel to the complex tangent space to Ω at the points of M (this space is constant for points in M). Note that the projection of this complex line Π_ζ into the real space $\mathbf{R}^2$ is the line l_ξ containing $\xi\in\omega$ and being parallel to the tangent line for ω at the point $\mathbf{0}$. The projection of Π_ζ into the imaginary space $i\mathbf{R}^2$ is some line $i\mathscr{L}_\eta$ containing $i\eta$. So, the intersection of Π_ζ with $\Omega\cap V$ is $l_\xi\cap\omega\cap v+i(\mathscr{L}_\eta\cap m)$, the union of rectangles corresponding to the connected components of $\mathscr{L}_\eta\cap\omega$. ($l_\xi\cap\omega\cap v$ is a segment with the endpoints on $\operatorname{Fr}\omega$, if v_1 is small enough and $\xi\in v_1$.) The disc Δ_ζ is now the rectangle containing ζ. Obviously the intersection of Δ_ζ with a neighbourhood of its boundary $\operatorname{Fr}\Delta_\zeta$ is contained in $\Omega\cap V$. Now we write down the family of discs "along which we want to continue f". Take $\mathfrak{v}$, a unit vector in $\mathbf{R}^2$ orthogonal to $\mathscr{L}_\eta$ and pose $\mathfrak{V}=\mathbf{0}+i\mathfrak{v}$. By Π_ζ^s we denote the translation of the complex line Π_ζ by the vector $s\mathfrak{V}$ $(s\in(-\infty,\infty))$. (We translate in the imaginary direction orthogonal to Π_ζ). So, $\Pi_\zeta^s=l_\xi+i(\mathscr{L}_\eta+s\mathfrak{v})$ is parallel to Π_ζ and goes through $\xi+i(\eta+s\mathfrak{v})$. The intersection $\Pi_\zeta^s\cap\Omega\cap V$ is $\Pi_\zeta^s\cap\Omega\cap V=l_\xi\cap\omega\cap v+i\{(\mathscr{L}_\eta+s\mathfrak{v})\cap m\}$, the union of disjoint open rectangles $P_\zeta^{s,k}$ (the connected components of $\Pi_\zeta^s\cap\Omega\cap V$, corresponding to the connected components of $(\mathscr{L}_\eta+s\mathfrak{v})\cap m$).

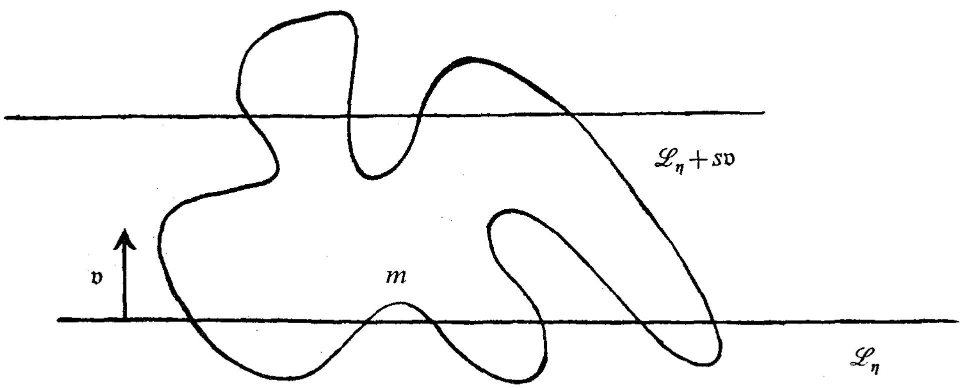

Fig. 2

For each rectangle its intersection with some neighbourhood of its boundary is situated in $\{(\omega+im)\setminus(e+in)\}\cap V$. The sets $\Pi_\zeta^s\cap\Omega\cap V$ form the desired family of analytic discs. We will show that for each $s\geqq 0$

(∗) there is an analytic function in some neighbourhood of

$$\Pi_\zeta^s\cap\Omega\cap V \text{ coinciding with } f \text{ near } \operatorname{Fr}(\Pi_\zeta^s\cap\Omega\cap V).$$

($\Pi_\zeta^0\overset{\text{def}}{=}\Pi_\zeta$ and $P_\zeta^{0,k}$ are the connected components of $\Pi_\zeta\cap\Omega\cap V$). So, we shall get a well-defined analytic continuation of the function f into a neighbourhood of the (arbitrary) point $\zeta\in\Omega\cap V$, which will prove Theorem 2′ in the particular case.

Denote $s_1=\sup\{s: (\mathscr{L}_\eta+s\mathfrak{v})\cap n\neq\emptyset\}$ and $s_2=\sup\{s: (\mathscr{L}_\eta+s\mathfrak{v})\cap m\neq\emptyset\}$. Then $-\infty<s_1<s_2<\infty$ and for $s\in(s_1, s_2)$ $(*)$ is obviously true. Suppose $(*)$ is not true for some $s\geqq 0$ and take $s_0\overset{\text{def}}{=}\sup\{s<s_2: (*) \text{ is not true for } s\}$. Obviously, $s_0<s_2$. We want to show that

$$(3) \qquad (*) \text{ is not true for } s_0.$$

This is based on the fact that the set of regularity of the function f is open but it needs some arguments because the boundaries $\operatorname{Fr}(\Pi_\zeta^s\cap\Omega\cap V)$ in general do not depend continuously on s. (There is only some semicontinuity.) Then we want to apply the continuity principle ([14], III § 17) and show that (3) is impossible, hence $(*)$ holds for all $s\geqq 0$. In this step the semicontinuity is easily seen to be sufficient. It is enough to choose suitable subsets of $\Pi_\zeta^s\cap\Omega\cap V$ $(s\geqq s_0)$ continuously depending on s. These two steps will be repeated also in the general situation. Now we will prove (3).

Suppose (3) does not hold: there is an analytic function in a neighbourhood W of $\Pi_\zeta^{s_0}\cap\Omega\cap V$ coinciding with f near $\operatorname{Fr}(\Pi_\zeta^{s_0}\cap\Omega\cap V)$. Recall that $\Pi_\zeta^s\cap\Omega\cap V= l_\xi\cap\omega\cap v+i(\mathscr{L}_\eta+s\mathfrak{v})\cap m$ for arbitrary s. Take a sufficiently large compact subset $K\subset\Pi_\zeta^{s_0}\cap\Omega\cap V$. Now for s sufficiently close to s_0 an arbitrary point $p\in\Pi_\zeta^s\cap\Omega\cap V$ is either close to K and therefore contained in W, or p is close to $\Pi_\zeta^{s_0}\setminus(\Omega\cap V)$ and therefore close to $\operatorname{Fr}(\Omega\cap V)$. But $\Pi_\zeta^s\cap\operatorname{Fr}(\Omega\cap V)\subset\{\operatorname{Fr}(l_\xi\cap\omega\cap v)+i(\mathscr{L}_\eta+s\mathfrak{v})\}\cup\{l_\xi\cap\omega\cap v+i\operatorname{Fr} m\}$, so points from $\Omega\cap V$ which are close to $\Pi_\zeta^s\cap\operatorname{Fr}(\Omega\cap V)$ belong to $\Omega\cap V\setminus E$. Now it is easy to see that for all s sufficiently close to s_0 the set $\Pi_\zeta^s\cap\Omega\cap V$ is contained in some open set into which the function f can be analytically continued in the desired way. Therefore $(*)$ holds for this s in contradiction to the definition of s_0, and (3) is proved.

Using the fact that $(*)$ is true for $s>s_0$ we will now get a contradiction to the continuity principle ([14], III § 17). Indeed, suppose $P_\zeta^{s_0,k}$ is a rectangle, one of the connected components of $\Pi_\zeta^{s_0}\cap\Omega\cap V$, for which f cannot be analytically continued to its neighbourhood in the desired way. It is easy to see that for each compact $K\subset P_\zeta^{s_0,k}$ we have $K+(s-s_0)\mathfrak{V}\subset\Pi_\zeta^s\cap\Omega\cap V$ for $|s-s_0|<\delta$ ($\delta>0$ depending on K). Taking sufficiently large closed rectangles for these K's we see that the points near $\operatorname{Fr}(K+(s-s_0)\mathfrak{V})$ are in $\Omega\setminus E$. This permits us to choose a family of (open) rectangles P_ζ^s ($s>s_0$, s is close to s_0) such that $\lim_{s\to s_0}P_\zeta^s=P_\zeta^{s_0,k}$, $\lim_{s\to s_0}\operatorname{Fr}P_\zeta^s=\operatorname{Fr}P_\zeta^{s_0,k}$, the points on P_ζ^s near $\operatorname{Fr}P_\zeta^s$ being contained in $\Omega\setminus E$ and such that the function f can be analytically continued into a neighbourhood of P_ζ^s in the desired way. (Recall that $(*)$ is true for $s>s_0$.) But this contradicts (3) by the continuity principle. Theorem 2′ is proved in the particular case.

2. Proof of Theorem 2′ in the general case. The main difficulty in this case is to construct the analytic discs mentioned above. The construction may be of some interest in itself, so we give the formulation of the corresponding result in Theorem 3 (see Section 4 below). We begin with a heuristic discussion to give an idea of how to do this. So, suppose $z \in B$ and z is close to M. We have to construct an analytic disc Δ_z, $z \in \Delta_z$, which is contained in B and whose boundary $\operatorname{Fr} \Delta_z$ lies in $B \setminus E$ and is close to M. We can expect that the discs look as in the special case: In one direction they are "long", which means they lie in some neighbourhood of a curve which is very close to M at all its points, and in the other (orthogonal) direction they are "short" and, roughly speaking,

(+) "reach the boundary $\operatorname{Fr} B$ very fast".

What does this mean for the tangent space of Δ_z at points $p \in \Delta_z$? The tangent space is, of course, a complex line (because Δ_z is analytic) and in view of (+) it must be close to the complex tangent space $T^c_{p'} S$ of S at some point $p' \in S$, p' being close to p. Now we proceed as follows. Suppose $r<1$, $z \in rM$. On rM we choose a smooth tangent vector field w such that the corresponding integral curve through z is almost the curve mentioned above. (So $w(p) \in T_p(rM)$ and $w(p)$ is "almost" in $T^c_{p/|p|} S$ for $p \in rM$.) By approximation (note that M is totally real) we get a vector field v which is analytic in some neighbourhood of $r \operatorname{Clos} M_1$ for some M_1, $M_1 \subset \operatorname{Clos} M_1 \subset M$, M_1 being of the same kind as M, and such that v is close to w on rM_1. Now we find a vector function $\mathscr{F} (=\mathscr{F}_z)$, analytic for ζ in some planar domain such that

$$\mathscr{F}(0) = z \tag{4}$$

$$\mathscr{F}'(\zeta) = v(\mathscr{F}(\zeta)). \tag{5}$$

The solution of this differential equation in a suitable planar domain gives us the desired disc. Condition (5) implies that $\{\mathscr{F}(x)\colon x \in I\}$ (I being some interval, $0 \in I \subset \mathbf{R}$) is the curve mentioned above.

The following lemma is needed to prove that in the direction iv the disc "reaches the regularity set of f very fast".

Lemma 1. *Suppose $E \subset \operatorname{Clos} B$ is a closed set, $B \setminus E$ is connected and $E \cap S \subset M$, where M is a C^2-manifold of real dimension $\dim_r M = 2$. Then there exists a closed set $A \subset E$ which touches M nontangentially and such that each function f analytic in $B \setminus E$ is the restriction to $B \setminus E$ of a function analytic in $B \setminus A$.*

A closed set $A \subset \operatorname{Clos} B$ is said to touch M nontangentially if $A \cap S \subset M$ and there are a neighbourhood U of M and a number $\alpha > 0$, such that

$$A \cap B \cap U \subset \bigcup_{\zeta \in M} K_\zeta(\alpha), \tag{6}$$

where $K_\zeta(\alpha) = \{\xi\colon |\xi - \zeta| < \alpha \operatorname{Re} \langle \xi - \zeta, \zeta \rangle\}$ is the nontangential cone (with respect to B) with vertex ζ and angle α. (Here $\langle a, b \rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2$ is the complex scalar product of the vectors $a, b \in \mathbf{C}^2$, $|a| = (\langle a, a\rangle)^{1/2}$ is the corresponding norm.) An equivalent definition is that there exists some $\beta > 0$ such that (with the U above)

$$A \cap B \cap U \subset \bigcup_{r_0 < r < 1} \{\xi \in rS\colon \operatorname{dist}(\xi, rM) < \beta(1-r)\} \tag{7}$$

(dist (p, A) is the distance of the point p from the set A with respect to the norm $|\cdot|$). Note that in (6) and (7) we can replace M by some open subset M_0 of M, $\operatorname{Clos} M_0 \subset M$ (because A is closed) and we can suppose that M_0 is diffeomorphic to a planar disc.

Proof of the lemma. Take an arbitrary point $\zeta \in M_0$. Denote by Π_ζ the (real) two-dimensional plane trough ζ orthogonal to M (that means orthogonal to $T_\zeta M$). So Π_ζ is the plane spanned by the two vectors ζ and n_ζ, where $n_\zeta \in T_\zeta S \cap (T_\zeta M)^\perp$, $|n_\zeta| = 1$. All we have to do is to find a set A satisfying

$$A \cap U \cap B \cap \Pi_\zeta \subset K_\zeta(\alpha) \cap \Pi_\zeta. \tag{8}$$

(Indeed, $\bigcup_{\zeta \in M_0} \Pi_\zeta$ covers some neighbourhood of M_0.) We use the following well known lemma (see [8], 18.1.8. or 18.1.12).

Lemma 2. *Suppose f is holomorphic in $B \cap U$, where U is some neighbourhood of the spherical cap $C_{\zeta,t} \overset{\text{def}}{=} \{z \in S\colon \operatorname{Re}\langle z, \zeta\rangle > t\}$ $(t < 1, \zeta \in S)$. Then f can be analytically continued into the sector $O_{\zeta,t} \overset{\text{def}}{=} \{z \in B\colon \operatorname{Re}\langle z, \zeta\rangle > t\}$.*

(Note that $O_{\zeta,t} \cup S_{\zeta,t}$ is the convex hull of the spherical cap $C_{\zeta,t}$.)

This lemma gives a well defined analytic function in $\bigcup_{C_{\zeta,t} \subset S \setminus M_0} O_{\zeta,t}$ coinciding with f near $S \setminus M_0$. Denote it by f as before. We have to show that the remaining exceptional set A satisfies (8). Suppose that $\zeta \in B_0$ and that B_ζ is some small ball centered at ζ. Then M divides $S \cap B_\zeta$ into two connected components. It follows easily from the fact that M is of class C^2 that each of them contains some spherical cap C_{ξ_j, t_j} $(j = 1, 2)$ such that their boundaries $\operatorname{Fr} C_{\xi_j, t_j}$ are tangent to M at the point ζ and $t_j = t_j(\zeta) \geqq t_0 > 0$ $(j = 1, 2)$ for some t_0 depending only on M, M_0, but not on $\zeta \in M_0$. This is enough. Indeed, $T_\zeta M = T_\zeta(\operatorname{Fr} C_{\xi_j, t_j})$ implies that $\xi_j - \zeta$ is orthogonal to $T_\zeta M$ $(j = 1, 2)$ and so Π_ζ is the plane spanned by ζ and ξ_j $(j = 1, 2)$. Now by Lemma 2 f is holomorphic at the points of the set $\{z \in B \cap \Pi_\zeta\colon \operatorname{Re}\langle z, \xi_j\rangle > t_j\}$ for $j = 1$ or 2, that is at the points of $B \cap \Pi_\zeta$ which belong to one of the halfplanes $P_j = \{z \in \Pi_\zeta\colon \operatorname{Re}\langle z, \xi_j\rangle > t_j\}$ $(j = 1, 2)$, ζ being a boundary point of both halfplanes and $\xi_j \in P_j$ $(j = 1, 2)$. Now the inequality $t_j \geqq t_0$ implies that near ζ the remaining set $(\Pi_\zeta \cap B) \setminus (P_1 \cup P_2)$ is contained in some nontangential angle $K_\zeta(\alpha) \cap \Pi_\zeta$ with α not depending on ζ. The lemma is proved.

In the following we will assume from the outset that the singularity set E is closed and touches M_0 nontangentially, where M_0 is diffeomorphic to an open planar disc, $M_0 \subset \operatorname{Clos} M_0 \subset M$ and we also fix a set M_1, $\operatorname{Clos} M_0 \subset M_1 \subset \operatorname{Clos} M_1 \subset M$ of the same kind as M_0.

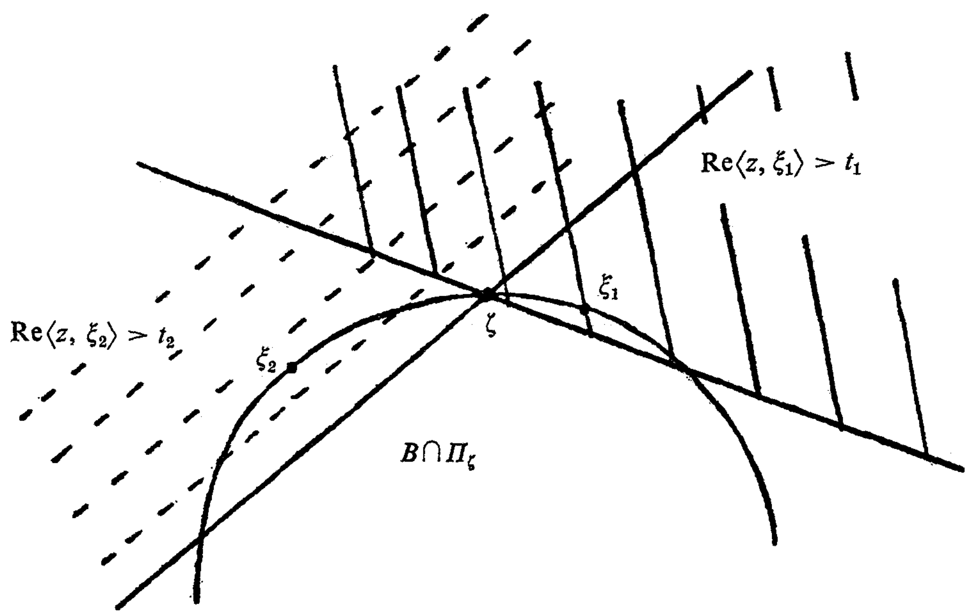

Fig. 3

3. We now come to the **definition of the vector field** w **and the construction of the holomorphic field** v. We construct a vector field w_1 on M and then put $w(p)=w_r(p)=rw_1\left(\frac{p}{r}\right)$ for $p\in rM$. The field w_1 has to be of class C^1 and to satisfy the condition $w_1(p)\in T_pM\cap T_p^cS$ for all $p\in M$. Both spaces T_pM and T_p^cS have real dimension two and are contained in the three-dimensional space T_pS, so their intersection is a (real) line (in view of the condition of the theorem $T_pM\neq T_p^cS$). All that remains to do is to choose an orientation on $T_pM\cap T_p^cS$ continuously depending on $p\in M$. With this aim we find a C^1 vector field v on M such that $v(p)$ is not orthogonal to $T_pM\cap T_p^cS$ for all p and $w_1(p)$ is the unit vector from $T_pM\cap T_p^cS$ satisfying $\mathrm{Re}\,\langle w_1(p), v(p)\rangle>0$. Then w_1 will be of class C^1 (because M and S are of class C^2). The vector $v(p)$ will be defined in the following way. There is a nonsingular C^1 vector field $\mathfrak{v}$ in $T_pS\ominus T_pM$ (the orthogonal complement of T_pM in T_pS) (recall that M is diffeomorphic to a planar disc). Take $v=i\mathfrak{v}$. We have to show that $i\mathfrak{v}(p)$ is not orthogonal to $T_p^cS\cap T_pM$ $(p\in M)$, or what is the same, that $\mathfrak{v}(p)$ is not orthogonal to $i(T_p^cS\cap T_pM)=T_p^cS\cap(iT_pM)$, that is $T_p^cS\cap iT_pM(\subset T_pS)$ is not contained in T_pM. This is obvious.

Now we approximate the vector field $w=w_r$ on rM $(r<1$ is sufficiently close to one) by a vector field $v=v_r$ holomorphic in a suitable neighbourhood of

rM. For this purpose we approximate w_1 on M by a suitable field v and then put $v_r(p)=r\mathrm{v}\left(\frac{p}{r}\right)$ for p near rM. This is possible in view of the following theorem ([1], [2], [3]).

Theorem A. *Suppose that $M\subset\mathbf{C}^n$ is a totally real manifold of class C^1, w is a vector field of class C^1 on M, and M_1 is an open subset of M with* $\mathrm{Clos}\, M_1\subset M$. *There exist a constant $C>0$ and a function γ on $[0, +\infty)$, $\gamma\geqq 0$, with $\lim_{\varepsilon\to 0}\gamma(\varepsilon)=0$, both depending only on M, M_1 and w, such that*

for every sufficiently small $\varepsilon>0$ there exists a vector field v_ε analytic in the ε-neighbourhood V_ε of $\mathrm{Clos}\, M_1$ *and such that*

$$\sup_{z\in \mathrm{Clos}\, M_1} |w(z)-\mathrm{v}_\varepsilon(z)| \leqq \varepsilon\gamma(\varepsilon) \tag{9}$$

and

$$|\mathrm{v}_\varepsilon(z_1)-\mathrm{v}_\varepsilon(z_2)| \leqq C|z_1-z_2|+\varepsilon\gamma(\varepsilon) \quad \text{for all} \quad z_1, z_2\in V_\varepsilon. \tag{10}$$

The *proof* follows from [3]. We will only sketch it. First construct a vector field of class C^1 in some fixed neighbourhood V of $\mathrm{Clos}\, M_1$ which coincides with w on M (denote it also by w) and such that $\bar\partial w(z)\to 0$ for $\mathrm{dist}\,(z, M)\to 0$ uniformly for $z\in V$. (This is Lemma 4.3 of Hörmander and Wermer [1], see also [2] Lemma 1.6.) So

$$|w(z_1)-w(z_2)| \leqq C|z_1-z_2| \quad \text{for} \quad z_1, z_2\in V. \tag{11}$$

Now put $\varepsilon>0$ and solve the $\bar\partial$-equation $\bar\partial u_\varepsilon=\bar\partial w$ in some suitable domain of holomorphy $T_\varepsilon\supset V_\varepsilon$. The function $\mathrm{v}_\varepsilon=w-u_\varepsilon$ is holomorphic in $T_\varepsilon\supset V_\varepsilon$ and the theorem follows if we can take u_ε with $\sup_{T_\varepsilon}|u_\varepsilon|\leqq\varepsilon\gamma(\varepsilon)$. This is essentially done in [3]. T_ε is defined by using the C^2 strictly plurisubharmonic function φ in V constructed in Lemma 1.2 of [2], such that $M\cap V=\{z\in V: \varphi(z)=0\}=\{z\in V: \mathrm{grad}\,\varphi(z)=0\}$. To solve the $\bar\partial$-problem on T_ε we use the kernel $K^\delta_{0,0}$ constructed in Section 3 of [3] and use the Koppelman formulas (1), (2) for $K^\delta_{0,0}$ (see also formula (10) in [3]) and estimate (a) with $\varrho=0$ from Lemma 4.4 of this work.

4. Now all is ready for **the construction of the discs.** Assume first that $z\in rM_1$ ($r<1$ is sufficiently close to one). (Later we will deal with the general case $z\in E\cap B$.)

Recall that $M\subset\mathrm{Fr}\, B^2$ is a C^2 totally real manifold, $M_1\subset\mathrm{Clos}\, M_1\subset M$ is a manifold diffeomorphic to the open unit disc. Further $E=\mathrm{Clos}\, E\subset\mathrm{Clos}\, B$ touches M nontangentially. Then analytic discs of the following kind can be constructed.

Theorem 3. *Suppose $r<1$ is sufficiently close to 1. Let $z\in rM_1$. There exists an analytic disc d_z of the following form:*

There are a planar rectangle

$$P = \{z = x+iy \in \mathbf{C}\colon x \in (a, b),\ y \in (-\varepsilon, \varepsilon)\}$$

($\varepsilon=$const$(1-r)$ for a suitable const) and an analytic mapping $\mathcal{F}=\mathcal{F}_z\colon P\to\mathbf{C}^2$ such that $\mathcal{F}(0)=z$ and $d_z=\mathcal{F}_z(P)$. Moreover, $\mathcal{F}$ is one to one and $0<c\leqq|\mathrm{grad}\,\mathcal{F}|\leqq C$ on P. The curve $\mathcal{F}_z((a, b))$ is the curve mentioned above which is very close to rM_1: $\mathrm{dist}\,(\zeta, rM_0)<(1-r)\gamma(1-r)$ for each $\zeta\in\mathcal{F}_z((a, b))$, where $\gamma(\varepsilon)=0(\varepsilon)$ for $\varepsilon\to 0$. Further $d_z\subset B$ and $\mathrm{Fr}\, d_z\subset B\setminus E$.

Proof of Theorem 3. Put $v=v_r$, $v_r(p)\overset{\mathrm{def}}{=} rv_\varepsilon\left(\frac{p}{r}\right)$, where v_ε is the holomorphic vector field in the $c(1-r)$-neighbourhood of M_1 (c is a sufficiently large constant depending on the constant α in (8)) constructed by Theorem A by taking $\varepsilon=c(1-r)$, $w=w_1$. We want to solve the differential equation

$$\mathcal{F}(0) = z \tag{4}$$

$$\mathcal{F}'(\zeta) = v(\mathcal{F}(\zeta)) \tag{5}$$

in some suitable planar region. By Cauchy's theorem (see, for example, [15], III § 13) there exists a unique solution in a neighbourhood of zero. Next we will continue the solution to the points of an interval of the real line and show that the image of this interval is a curve which is very close to rM, moreover, it is close to the integral curve of the vector field w_r on rM_1 through the point z. We need some information about the

integral curves of the field w_r.

Lemma 3. *Through every point $p\in M_1$ passes a unique integral curve of maximal length for the vector field $w_1|M_1$.* (Recall that w_1 is a nonsingular vector field of class C^1 defined on the C^2-manifold $M\supset\mathrm{Clos}\,M_1$ and w_1 is tangential to M at every point.) *All integral curves are simple Jordan curves of finite length and join two (distinct) points on the boundary* $\mathrm{Fr}\,M_1$. *The length of the curves can be estimated from above by a finite number depending only on M_1 and w_1, but not on the curve.*

Proof. We may assume that M_1 is the unit disc in the plane and w_1 is a C^1-vector field (tangent for M_1) in a neighbourhood of its closure, $|w_1|$ does not vanish there and so $0<c_1\leqq|w_1|\leqq c_2<\infty$ on $\mathrm{Clos}\,M_1$. The existence and the uniqueness of an integral curve through a given point z is well known (see, e.g. [9], II 1.1, II 3.1), which means that there exists a unique function $s(t)$ defined on the maximal interval $(a, b)\subset\mathbf{R}$, $0\in(a, b)$, such that

$$s(0) = z,\quad s'(t) = (w_1|M_1)(s(t))\quad (t\in(a, b)). \tag{12}$$

The length $\int_a^b |s'(t)|\, dt$ of the curve lies between $c_1(b-a)$ and $c_2(b-a)$, so we have to prove that $a \neq -\infty$, $b \neq +\infty$, $s(a+) \in \operatorname{Fr} M_1$, $s(b-) \in \operatorname{Fr} M_1$ and $s(a+) \neq s(b-)$.

Suppose, for example, that $b=+\infty$. Consider the equation (12) on the whole of M (instead of M_1) and recall that $w_1 \neq 0$ on M. We come to a contradiction with the theory of Poincaré—Bendixson: we have $\operatorname{Clos}\{s(t)\colon t \geqq 0\} \subset \operatorname{Clos} M_1$, this set is compact and is contained in M, a contradiction with the Theorems VII.4.3 and VII.3.1 of [9]. So, $b<+\infty$ and therefore $s(b-) \in \operatorname{Fr} M_1$ ([9], II 3.1). In the same way we prove that $a>-\infty$ and $s(a+) \in \operatorname{Fr} M_1$. The inequality $s(a+) \neq s(b-)$ also follows from Theorem VII.3.1 of [9]: we have to consider s on (a, b) as a part of the solution of (12) on the whole M (instead of M_1).

It remains to give uniform estimates for the length of the curves. To do this we choose some open set M_2 diffeomorphic to a planar disc and such that $\operatorname{Clos} M_1 \subset M_2 \subset \operatorname{Clos} M_2 \subset M$. Consider the integral curves for the field $w_1|M_2$ through a point $p \in M_2$, that is solve the differential equation

$$S_p'(t) = (w_1|M_2)(S_p(t)), \quad S_p(0) = p \tag{13}$$

on the maximal interval (A, B). Put $\Gamma_p = \{S_p(t)\colon t \in (A, B)\}$ and for $p \in M_1$ put $\gamma_p = \{s_p(t)\colon t \in (a, b)\} = \{S_p(t)\colon t \in (a, b)\}$ ((a, b) being as in (12)). Let Γ_p^0 be the connected component of $\Gamma_p \cap M_1$, containing p. Denote by $|\gamma|$ the length of the curve γ ($|\gamma| = \int_I |s'|$, if $\gamma = \{s(t)\colon t \in I\}$, I being an interval of $\mathbf{R}$). Then, obviously, $|\gamma_p| = |\Gamma_p^0| < |\Gamma_p|$ for $p \in M_1$. We shall show that for each $\zeta \in \operatorname{Clos} M_1$ there exists a neighbourhood V_ζ in M_2 such that for $z \in V_\zeta \cap M_1$ we have $|\Gamma_z^0| < |\Gamma_\zeta|$. By compactness of $\operatorname{Clos} M_1$ this is enough. But this is an easy consequence of the continuous dependence of the solution of the Cauchy problem on the initial data ([9], Theorem V.2.1). Indeed, if $\zeta \in \operatorname{Clos} M_1$ and $\Gamma_\zeta = \{S_\zeta(t)\colon t \in (A, B)\}$, then for some neighbourhood V_ζ of ζ and some small $\varepsilon > 0$ for $z \in V_\zeta \cap M_1$ the solution S_z of (13) is defined at least on $[A+\varepsilon, A-\varepsilon]$. Further $S_z(A+\varepsilon), S_z(B-\varepsilon) \in M_2 \setminus \operatorname{Clos} M_1$, $S_z(0) = z \in M_1$, so Γ_z^0 is contained properly in $\{S_z(t)\colon t \in (A+\varepsilon, B-\varepsilon)\}$. Moreover $S_z(t) - S_\zeta(t)$ ($t \in [A+\varepsilon, B-\varepsilon]$) is very small for $z \in V_\zeta$ and so is $S_z'(t) - S_\zeta'(t) = (w_1|M_2)(S_z(t)) - (w_1|M_2)(S_\zeta(t))$, therefore $|\Gamma_z^0| < |\Gamma_\zeta|$ for $z \in V_\zeta \cap M_1$, if V_ζ is small.

To construct the continuation of the solution of (4), (5) we need also

Gronwall's lemma. *Suppose f is a vector function of class C^1 on (a, b), $0 \in (a, b)$ and $f(0)=0$. Suppose that $|f'(t)| \leqq C|f(t)| + \varepsilon$ for all $t \in (a, b)$ and some $\varepsilon > 0$. Then*

$$|f(t)| \leqq \sqrt{\frac{2}{1+2C^2}} \cdot \varepsilon \cdot e^{(\frac{1}{2}+C^2)|t|} \leqq \varepsilon \operatorname{const} \quad \textit{for} \quad t \in (a, b), \tag{14}$$

where const *depends only on* C *and* $\max\{b, |a|\}$.

For convenience of the reader we give the short

Proof. Without loss of generality we assume that f is a real vector function. Then $|(|f|^2)'|=(2\sum f_j f_j')\leqq\sum(f_j)^2+\sum(f_j')^2\leqq|f|^2+(C|f|+\varepsilon)^2\leqq(1+2C^2)|f|^2+2\varepsilon^2$. Inequality (14) follows by integrating the inequality

$$\frac{(|f|^2)'(1+2C^2)}{(1+2C^2)|f|^2+2\varepsilon^2}\leqq 1+2C^2 \quad \text{for} \quad t>0 \quad \text{and the inequality}$$

$$-\frac{(|f|^2)'(1+2C^2)}{(1+2C^2)|f|^2+2\varepsilon^2}\leqq 1+2C^2 \quad \text{for} \quad t<0 \quad \text{and having in mind}$$

the condition $f(0)=0$.

Now it follows from Gronwall's lemma that (for sufficiently small $\varepsilon>0$) the solution $\mathscr{F}$ of (4), (5) exists for ζ in a neighbourhood of the interval $(a,b)\subset\mathbf{R}$, the maximal interval on which a solution of (12) exists with z/r instead of z (recall that $z\in rM_1$). Indeed, we can continue the function $\mathscr{F}$ as a solution of (4), (5) in a neighbourhood of all points t of an interval I as long as $\mathscr{F}(t)$ for $t\in I$ remains in rV_ε, i.e. the set where v is defined. We want to show that this is so on the whole of (a,b). Put $w_r(p)=rw_1(p/r)$ for $p\in rM_1$ and $s_z(t)=rs_{z/r}(t)$, the integral curve through $p\in rM_1$ for the field $w_r|rM_1$. Denote $f(t)=s_z(t)-\mathscr{F}(t)$. Then $f(0)=0$, $f'(t)=s_z'(t)-\mathscr{F}'(t)=w_r(s_z(t))-v(\mathscr{F}(t))$, so by the estimates (9) and (10) of Theorem A

$$|f'(t)|\leqq\left|w_r(s_z(t))-v(s_z(t))\right|+\left|v(s_z(t))-v(\mathscr{F}(t))\right| \tag{15}$$

$$=\left|rw_1(s_{z/r}(t))-rv_\varepsilon(s_{z/r}(t))\right|+\left|rv_\varepsilon(s_{z/r}(t))-rv_\varepsilon\left(\frac{\mathscr{F}(t)}{r}\right)\right|$$

$$\leqq r\varepsilon\gamma(\varepsilon)+r\varepsilon\gamma(\varepsilon)+rC\left|s_{z/r}(t)-\frac{\mathscr{F}(t)}{r}\right|\leqq 2\varepsilon\gamma(\varepsilon)+C|f(t)|,$$

as long as $\mathscr{F}(t)\in rV_\varepsilon$, so for those t by Lemma 3 and Gronwall's lemma

$$|f(t)|\leqq \text{const}\,\varepsilon\gamma(\varepsilon) \tag{16}$$

(const depending on C and the (uniform) estimate for $(b-a)$ from Lemma 3). Therefore (if $\varepsilon>0$ is sufficiently small) the inclusion $\mathscr{F}(t)\in rV_\varepsilon$ remains true for all $t\in(a,b)$ (because $s_z(t)\in rM_1$ for those t), and (16) holds on (a,b).

Now we continue the solution $\mathscr{F}$ of (4), (5) in imaginary directions of ζ. So we look at the differential equation

$$\frac{\partial}{\partial\tau}\mathscr{F}(t+i\tau)=i\frac{\partial}{\partial t}\mathscr{F}(t+i\tau)=iv(\mathscr{F}(t+i\tau)) \tag{17}$$

with initial value $\mathscr{F}(t)$ for $\tau=0$, t being some number from (a,b). Note that while $\mathscr{F}(t+i\tau)\in rV_\varepsilon$ we have (as in (15))

$$\left|iv(\mathscr{F}(t+i\tau))-iw_r(s_z(t))\right|\leqq 2\varepsilon\gamma(\varepsilon)+C|s_z(t)-\mathscr{F}(t+i\tau)|\leqq 2C\varepsilon \tag{18}$$

(for r close to one and therefore ε small), therefore

$$\mathscr{F}(t+i\tau)-\mathscr{F}(t) = iw_r(\sigma_z(t))\cdot\tau+O(\varepsilon|\tau|),$$

and so $\mathscr{F}(t+i\tau)\in rV_\varepsilon$ for $|\tau|\leqq\frac{\varepsilon}{2}$. (We assume that r is sufficiently close to one so that ε is small and also that $|w_r|$ is close to one.) Note that $\mathscr{F}(t)=\sigma_z(t)+O(\varepsilon\gamma(\varepsilon))$, $\sigma_z(t)\in rS$, $iw_r(\sigma_z(t))\tau\in T_{\sigma_z(t)}(rS)$, so for $|\tau|\leqq\frac{\varepsilon}{2}$ we have

$$\mathscr{F}(t)+iw_r(\sigma_z(t))\tau\in\bigl(r+O(\varepsilon\gamma(\varepsilon))+O(\varepsilon^2)\bigr)S = \bigl(r+o(\varepsilon)\bigr)S.$$

On the other hand $iw_r(\sigma_z(t))\notin T_{\sigma_z(t)}(rM)$, and by compactness we can assume that the angle between $iw_r(p)$ and $T_p rM$ is uniformly bounded away from zero for $p\in rM_1(\subset\operatorname{Clos} rM_1\subset M)$. Recall now that for the set E of singularities we have the inclusion (7) and that ε was defined by $\varepsilon=c(1-r)$. The choice of the constant c will be made precise now: we choose c depending on β in formula (7), and on the lower bound for the angle between $iw_r(p)$ and $T_p(rM)$ for $p\in rM_1$, and such that for $t\in(a,b)$ and for τ close to $+\frac{\varepsilon}{2}$ or $-\frac{\varepsilon}{2}$, $\mathscr{F}(t+i\tau)$ is contained in $B\setminus E$, the set where f is regular (recall that the function $\tau\to\mathscr{F}(t+i\tau)$ can be continued to the interval $|\tau|\leqq\frac{\varepsilon}{2}\bigr)$.

Recall now that $M_0\subset\operatorname{Clos} M_0\subset M_1$ is of the same kind as M_1, that is M_0 is diffeomorphic to an open disc and $E=\operatorname{Clos} E$ touches M_0 nontangentially. The following Lemma 4 will imply Theorem 3 and will also be useful in the following.

Lemma 4. *For some $\delta>0$ and all r sufficiently close to 1 the following is true. Suppose $z\in rM_1$ and (a,b) is the maximal interval for (12) as above. If $t\in(a,b)$ and $\operatorname{dist}\bigl(\sigma_z(t), r(M_1\setminus M_0)\bigr)<\delta$ (in particular, if $\operatorname{dist}\bigl(\sigma_z(t), \operatorname{Fr}(rM_i)\bigr)<\delta$, $i=0$ or 1) then $\left\{\mathscr{F}(t+i\tau)\colon |\tau|\leqq\frac{c}{2}(1-r)\right\}$ is contained in $B\setminus E$.*

The lemma follows from the facts that $E\cap S\subset M_0$, E is closed and M_0 is diffeomorphic to an open disc, so the intersection of B with some neighbourhood of $\operatorname{Clos}(M_1\setminus M_0)$ is contained in $B\setminus E$.

Applying now Theorem 3 to rM_0 instead of rM_1, $z\in rM_0$, we get analytic discs $\Delta_z\subset d_z$ (instead of d_z) with $z\in\Delta_z$. Note that $\Delta_z=\mathscr{F}_z(\{\zeta=x+iy\in\mathbf{C}\colon x\in(a_0,b_0), |y|<\varepsilon\})$, where (a_0,b_0) is the maximal interval on which a solution of the equation

$$(13_0)\qquad s_{z/r}(0) = z/r,\ s'_{z/r}(t) = (w_1|M_0)(s_{z/r}(t))$$

exists. By Lemma 4 we have $d_z\setminus\Delta_z\subset B\setminus E$.

5. Now we come to **the analytic continuation of the function f along a suitable family of discs of the form Δ_z**. Fix some r close to 1. The function f is analytic in the set $B\setminus E$, which contains some neighbourhood of $\operatorname{Fr}(rM_0)$. For every point $z\in rM_0$ there exists a unique analytic disc Δ_z of the type constructed above such that $z\in\Delta_z$ and the boundary $\operatorname{Fr}\Delta_z$ lies in $B\setminus E$. Our aim is to prove the following

Lemma 5. *For every disc Δ_z $(z\in rM_0)$ there exists a function which is analytic in some neighbourhood W_z of $\operatorname{Clos}\Delta_z$ and coincides with f near the boundary $\operatorname{Fr}\Delta_z$.*

From this lemma the proof of the Theorem 2′ follows. Indeed, by the lemma we have a well-defined analytic function $\tilde{f}$ in some neighbourhood of $\operatorname{Clos}(rM_0)$ which coincides with f in a neighbourhood of $\operatorname{Fr}(rM_0)$. The function is defined as follows: for ζ in a neighbourhood $\mathring{W}_z$ of a fixed disc $\operatorname{clos}\Delta_z$ ($\mathring{W}_z$ being much smaller than W_z) we define $\tilde{f}(\zeta)$ to agree with the function of Lemma 5 which is holomorphic in W_z and coincides with f in a neighbourhood of $\operatorname{Fr}\Delta_z$. Now it is not hard to see that the definition does not depend on the choice of Δ_z. Indeed, if $\zeta\in\mathring{W}_{z_1}\cap\mathring{W}_{z_2}$, then (if the $\mathring{W}_z$ are sufficiently small) we can assume that there is some small connected open part of $\mathring{W}_{z_2}$ contained in W_{z_1} which contains ζ and a part Γ of the boundary $\operatorname{Fr}\Delta_{z_2}$, Γ being close to a part of $\operatorname{Fr}\Delta_{z_1}$. This shows that the definition does not depend on the choice of Δ_z. So there is a function holomorphic in a neighbourhood of $\bigcup_{r_0<r<1}\operatorname{Clos}(rM_0)$ which coincides with f near $\bigcup_{r_0<r<1}\operatorname{Fr}(rM_0)$. (For a fixed r we can take the above defined function analytic in a neighbourhood of $\operatorname{Clos}(rM_0)$. Again it is easy to see that the definition is correct.) We have to continue the function also to points on rS in a $\beta(1-r)$-neighbourhood of rM_0 (see (7)). This can be done as follows. Suppose ζ lies in rS and in the $\beta(1-r)$-neighbourhood of rM_0. For every point $p\in S$ we consider the unitary operator u_p in $\mathbf{C}^2$ given by the matrix $\begin{pmatrix} p_1 & -\bar{p}_2 \\ p_2 & \bar{p}_1 \end{pmatrix}$ $(p=(p_1,p_2))$. Then it is not hard to see that there is some z with $|z-e_1|<\operatorname{const}\beta(1-r)$ $(e_1=(1,0)\in S)$ and such that $\zeta\in u_z(rM_0)$ and, moreover, $u_z(rM_0)$ is contained in the $\operatorname{const}\beta(1-r)$-neighbourhood of rM_0. We repeat now the construction for u_zM_0 instead of M_0 (the only difference is that we have

$$U\cap E\cap r'S\subset\{\zeta\in r'S\colon \operatorname{dist}(\zeta, ru_zM_1)<\operatorname{cost}\beta(1-r)\}$$

for $r'=r+o(1-r)$ instead of (7), so that we have to take c in the definition of $\varepsilon=c(1-r)$ somewhat larger). Now it is easy to see that this procedure gives a function analytic in the intersection of a neighbourhood of M with B and coinciding with f outside E.

It remains to give the

Proof of Lemma 5. Suppose the lemma is not true, so there is some disc Δ_z corresponding to a point $z \in rM_0$ such that there does not exist a function analytic in a neighbourhood of Δ_z and coinciding with f near $\operatorname{Fr} \Delta_z$. Such a disc we shall call a singular disc, and the point z we shall call a singular point. Consider the set of all integral curves γ with respect to the field $w_r | rM_0$, i.e. $\gamma = \{\sigma_z(t)\colon t \in (a_0, b_0)\}$ for some $z \in rM_0$ and the maximal interval (a_0, b_0) on which a solution of (13_0) exists. We shall call a curve γ a singular curve, if some point $\zeta \in \gamma$ is singular (so the corresponding disc Δ_ζ is singular). Now we define an ordering in the set of curves γ in the following way. Suppose rM_0 is oriented in some way and define on $\operatorname{Fr}(rM_0)$ the induced orientation. On a curve $\gamma = \{\sigma(t)\colon t \in (a_0, b_0)\}$ the real parameter t defines an orientation. To a curve γ correspond (in a unique way) its endpoints $p(=p(\gamma)) = \sigma(a_0+)$ and $q(=q(\gamma)) = \sigma(b_0-)$, $p, q \in \operatorname{Fr}(rM_0)$. Now $(rM_0) \setminus \gamma$ consists of two connected components, and also $\operatorname{Fr}(rM_0) \setminus (\{p\} \cup \{q\})$ consists of two components. Denote by $c_l(=c_l(\gamma))$ (the left component) that component of $\operatorname{Fr}(rM_0) \setminus (\{p\} \cup \{q\})$ for which $\gamma \cup \operatorname{Clos} c_l$ (with the previous orientations on γ and on the part c_l of $\operatorname{Fr}(rM_0)$) is an oriented curve and denote by $O_l(=O_l(\gamma))$ the corresponding component of $rM_0 \setminus \gamma$. The other components are denoted by $c_r(=c_r(\gamma))$ (the right component) or $O_r(=O_r(\gamma))$, respectively. We shall say that a curve γ_1 lies on the left of γ_2 if $c_l(\gamma_1) \subset c_l(\gamma_2)$ and that γ_1 lies strongly on the left of γ_2 if the inclusion is proper. Obviously, if γ_1 lies on the left of γ_2 and γ_2 lies on the left of γ_3 then γ_1 lies on the left of γ_3. Note that it is possible that neither γ_1 lies on the left of γ_2 nor γ_2 lies on the left of γ_1. Our aim is to prove the following lemma and to bring it to a contradiction with the continuity principle.

Lemma 6. *Suppose Lemma 5 is not true. Then there exists a singular curve γ^* such that there are no singular curves strongly on the left of γ^*.*

Proof. Suppose γ_0 is some singular curve. Define by induction singular curves γ_k in the following way. If for $k-1$ there are no singular curves strongly on the left of γ_{k-1} then we are done. Otherwise we choose a singular curve γ_k strongly on the left of γ_{k-1} such that

(19)
$$|c_l(\gamma_k)| < \frac{1}{k} + \inf\{|c_l(\gamma)|\colon \gamma \text{ being a singular curve strongly on the left of } \gamma_{k-1}\}$$

($|c_l(\gamma)|$ means the length of the curve $c_l(\gamma)$, we assume that $\operatorname{Fr} M_0$ is a C^1-curve.) Obviously, $c_l(\gamma_k) \subset c_l(\gamma_{k-1})$ for all k. Suppose that the number of k's is infinite and put $\bar{c} \overset{\text{def}}{=} \bigcap_k \operatorname{Clos} c_l(\gamma_k)$. Then $\bar{c}$ is a closed arc on $\operatorname{Fr}(rM_0)$ (possibly a single point). Denote by p, q its endpoints (possibly $p=q$). Obviously, $p(\gamma_k) \to p$, $q(\gamma_k) \to q$. Our aim is to show that $p \neq q$ and $p = p(\gamma^*)$, $q = q(\gamma^*)$ for some integral curve γ^*, where γ^* is the desired singular curve. To do this we denote by Γ the integral curve through the point $p(\in rM_1)$ for the field $w_r|rM_1$. Denote by Γ_k the corresponding curves through $p(\gamma_k)$. We choose the parametrization on Γ_k so that $\Gamma_k=\{\mathscr{S}_k(t)\colon t\in I_k\}$ ($I_k\subset\mathbf{R}$ is an interval), $0\in I_k$ and $\mathscr{S}_k(0)=p(\gamma_k)$. We parametrize also $\Gamma=\{\mathscr{S}(t)\colon t\in I\}$, $0\in I$, $\mathscr{S}(0)=p$. Suppose $t_k\in I_k$ satisfies $\mathscr{S}_k(t_k)=q(\gamma_k)$. Now $\mathscr{S}_k(0)\to\mathscr{S}(0)$, so by [9], Theorem V.2.1, we have $t\in I_k$ for an arbitrary $t\in I$ and sufficiently large k (depending on t). Also $\mathscr{S}_k(t)\to\mathscr{S}(t)$ uniformly for t on compact subsets of I. But $0\in I$ and $\mathscr{S}(t)$ can be continued into the segment $[0, t)$ (i.e. $[0, t)\subset I$) as long as $\mathscr{S}(t')$ remains in rM_1 for $t'\in[0, t)$. On the other hand $\mathscr{S}_k(t)\in rM_0$ for $t\in(0, t_k)$. Put $T=\lim t_{k_n}$ for some subsequence $\{k_n\}\subset\{k\}$ and prove that $[0, T]\subset I$. If not, then $\mathscr{S}$ can be defined on $[0, t)$ for some $t\leqq T$ and not on $[0, \tilde{t})$ if $\tilde{t}>t$. So by [9], II.3.1 $\mathscr{S}(t')\to\operatorname{Fr}(rM_1)$ for $t'\to t-0$. But $\mathscr{S}(t')=\lim_{n\geqq n_0}\mathscr{S}_{k_n}(t')\in\operatorname{Clos}(rM_0)$ for $0<t'<t\leqq T$, a contradiction. Now $\mathscr{S}(T)=q$ by [9], Theorem V.2.1 and the fact that $\mathscr{S}_{k_n}(t_{k_n})\to q$. Taking other subsequences $\{t_{k_m}\}$ this shows that $T=\lim t_k$.

Now we want to state that $p\neq q$ and that p, q are the endpoints of a singular curve γ^* (γ^* being an integral curve for the field $w_r|rM_0$). Take $\sigma>0$ so small that $[-\sigma, T+\sigma]\subset I$ and $\operatorname{dist}(\mathscr{S}(t), r(M_1\setminus M_0))<\delta$ for $t\in[-\sigma, \sigma]\cup[T-\sigma, T+\sigma]$. Therefore also $[-\sigma, T+\sigma]\subset I_k$ for $k\geqq k_0$ and also $\operatorname{dist}(\mathscr{S}_k(t), r(M_1\setminus M_0))<\delta$ for $t\in[-\sigma, \sigma]\cup[T-\sigma, T+\sigma]$ and $k\geqq k_0$. Denote by $\mathring{\Gamma}_k\subset\Gamma_k$ the curve

$$\mathring{\Gamma}_k=\{\mathscr{S}_k(t)\colon t\in(-\sigma, T+\sigma)\},$$

and by $\mathring{\Gamma}\subset\Gamma$, respectively, $\mathring{\Gamma}=\{\mathscr{S}(t)\colon t\in(-\sigma, T+\sigma)\}$. For $z=\mathscr{S}_k(t_0)$ (or $z=\mathscr{S}(t_0)$) with $t_0\in(-\sigma, T+\sigma)$ we denote by D_z the disc

$$D_z=\left\{\mathscr{F}_z(t+i\tau)\colon |\tau|<\frac{\varepsilon}{2},\ t_0+t\in(-\sigma, T+\sigma)\right\}$$

(i.e. the part of the disc d_z situated near $\mathring{\Gamma}_k$ (or $\mathring{\Gamma}$)). As for the discs Δ_z, a disc D_z is called singular if there is no function analytic in its neighbourhood and coinciding with f near its boundary. Now for each k there is some $z_k=\mathscr{S}_k(a_k)$, $a_k\in(0, t_k)$, such that the disc D_{z_k} is singular (because $\Delta_{z_k}\subset D_{z_k}$ is singular). Suppose z is a limit point of $\{z_k\}$. Then $z\in\mathring{\Gamma}\cap\operatorname{Clos}(rM_0)$. It is easy to see that D_z is singular (in the contrary case the equality $\lim D_{z_k}=D_z$ would imply that D_{z_k} is nonsingular for z_k close to z). The fact that D_z is singular implies by Lemma 4 that $\mathring{\Gamma}\cap rM_0\neq\emptyset$, so $T\neq 0$ (hence $p\neq q$). We need also the following.

Lemma 7. *For every $z\in\mathring{\Gamma}$ the disc D_z is singular.*

Proof. The set $\{z\in\mathring{\Gamma}\colon D_z \text{ is nonsingular}\}$ is open. By the continuity principle it is also closed. $\mathring{\Gamma}$ is connected, so the set of nonsingular discs is empty.

Take now $z\in\mathring{\Gamma}\cap rM_0$ The disc D_z is singular, so by Lemma 4 the corresponding disc $\Delta_z\subset D_z$ is singular. We want to show that the integral curve $\gamma=\gamma_z$ for the field $w_r|rM$ through the point z is the desired curve γ^*. We found already that this curve is singular. Further

(20) $$p(\gamma) = p, \quad q(\gamma) = q.$$

Indeed, put $\Gamma' = \{\mathcal{S}(t)\colon t\in[0, T]\}$, the part of $\mathring{\Gamma}$ between p and q; then $\Gamma' \subset \operatorname{Clos}(rM_0)$ (because $\Gamma' = \lim \Gamma'_k$ $(\Gamma'_k \overset{\text{def}}{=} \{\mathcal{S}_k(t)\colon t\in[0, t_k]\})$). For every k we have

$$p\in \operatorname{Clos} O_l(\gamma_k)\setminus\{\operatorname{Clos}\gamma_k\},$$

so $\Gamma' \subset \operatorname{Clos} O_l(\gamma_k)$ (if not, Γ' must intersect $\operatorname{Clos}\gamma_k$ in such a way that we get a contradiction with the uniqueness theorem for solutions of (12)). But $\gamma\subset\Gamma'\cap rM_0$, so $p(\gamma), q(\gamma)\in \operatorname{Clos} c_l(\gamma_k)$ for every k and so $|c_l(\gamma)|<|\bar{c}|=\lim|c_l(\gamma_k)|$ if one of the equalities (20) is false. But this contradicts (19). The inequality (19) implies also that there are no singular curves strongly on the left of γ. Lemma 6 is proved.

It is easy now to get a contradiction with the continuity principle. Indeed, take $\zeta_n\in O_l(\gamma^*)$, $\zeta_n\to z$. The integral curves γ_{ζ_n} of the field $w_r|rM$ through ζ_n are then strongly on the left of γ^* (taking into account that $\zeta_n\in O_l(\gamma^*)$ and the uniqueness property for solutions of (12)), and therefore they are nonsingular. By [9], Theorem V.2.1 there are parts $\tilde{\gamma}_{\zeta_n}$ of the curves γ_{ζ_n} such that $\tilde{\gamma}_{\zeta_n}\to\gamma^*$, $\zeta_n\in\tilde{\gamma}_{\zeta_n}$. We can assume that the endpoints of $\tilde{\gamma}_{\zeta_n}$ are near $\operatorname{Fr}(rM_0)$. (Compare with the end of the proof of the particular case.) So for the discs $\tilde{\Delta}_{\zeta_n} \overset{\text{def}}{=} \Big\{\mathcal{F}_{\zeta_n}(t+i\tau)\colon |\tau|<\frac{\varepsilon}{2},\ t$ is such that $\mathcal{S}_{\zeta_n}(t)\in\tilde{\gamma}_{\zeta_n}\Big\}$ the boundaries $\operatorname{Fr}\tilde{\Delta}_{\zeta_n}$ are contained in $V\cap B\setminus E$ and f can be continued into a neighbourhood of $\operatorname{Clos}\tilde{\Delta}_{\zeta_n}$ in the desired way (recall that γ_{ζ_n} are nonsingular). Now $\tilde{\Delta}_{\zeta_n}\to\Delta_z$, $\operatorname{Fr}\tilde{\Delta}_{\zeta_n}\to\operatorname{Fr}\Delta_z$ and the continuity principle gives a contradiction. Theorem 2′ is proved.

6. Outline of the proof of the local result for $n>2$. Take a point $p\in M$. After a unitary transformation of $\mathbf{C}^n$ we can assume that $p=(1, 0, \dots, 0)$ and that T_p^cM is contained in $\{z_1=z_2=0\}$. Lemma 1 is true for $n>2$ also. The idea of the proof of Theorem 1 is now to apply the construction of the discs from Theorem 2 to $B^n\cap\{z_3=z_3^0, \dots, z_n=z_n^0\}$ for small $z_3^0, \dots, z_n^0$.

Note that Theorem 1 can be strengthened in the same way that Theorem 2 was strengthened to obtain Theorem 2′. This is possible because the constructed discs are contained in a small neighbourhood of M.

7. Remarks about the proof of Theorem 2′ for strongly pseudoconvex domains $\Omega\subset\mathbf{C}^2$ with C^2-boundary. The proof follows the same scheme as the proof of Theorem 2′. Lemma 1 is of local character (for each boundary point $p\in\operatorname{Fr}\Omega$ there exists a biholomorphic mapping of some neighbourhood of p under which Ω becomes strongly convex near p, see [10] Theorem 1.4.14 or [8] 15.5.3). The other parts of the proof are not specific for the ball, except for the choice of $M_r=rM$ and approximation on M_r (by Theorem A) with a constant c and a function γ not depending on r. In the general case we choose the manifold M_r by the following procedure. Denote by $n_0(z)$ the unit inner normal (with respect to the domain Ω) at points $z\in M$, and denote by $v_0(z)$ a nondegenerate C^1-vector field, $v_0(z)\in T_zS\ominus T_zM$. Approximate these fields on a large compact part of M by fields $n(z)$ and $v(z)$ which are holomorphic in some neighbourhood V of this compact part and consider the holomorphic mapping $\psi(=\psi_{\varepsilon,A})$: $z\to z+\varepsilon n(z)+\varepsilon Av(z)$ $(z\in V)$. For small $\varepsilon>0$ the mapping is biholomorphic. Denote the image of $M\cap V$ by $M_{1-\varepsilon,A}$. The manifold $M_{r,A}$ for a suitable A will play the role of $U_z rM$ in Theorem 2′. Indeed, it is now enough (as in the proof of Theorem 2) to define the field w_1 on $M\cap V$ and to approximate it there by v_1. Then we can define the fields $w_{r,A}$ and $v_{r,A}$ on $M_{r,A}$ by the formula $w_{r,A}(z)=(\mathrm{grad}\,\psi)_{\psi^{-1}(z)}w_1(\psi^{-1}(z))$, $v_{r,A}(z)=(\mathrm{grad}\,\psi)_{\psi^{-1}(z)}v_1(\psi^{-1}(z))$ $(\psi=\psi_{\varepsilon,A})$ and the discs through $z\in M_{r,A}$ are defined by the formula $\psi(\mathcal{F}_{\psi^{-1}(z)}(\zeta))$ (with $\mathcal{F}_{\psi^{-1}(z)}$ defining the disc through $\psi^{-1}(z)$ which corresponds to the field v_1). (So

$$(\psi(\mathcal{F}_{\psi^{-1}(z)}(\zeta)))'=(\mathrm{grad}\,\psi)_{\mathcal{F}_{\psi^{-1}(z)}(\zeta)}\mathcal{F}'_{\psi^{-1}(z)}(\zeta)$$
$$=(\mathrm{grad}\,\psi)_{\psi^{-1}(\psi(\mathcal{F}_{\psi^{-1}(z)}(\zeta)))}v_1(\psi^{-1}(\psi(\mathcal{F}_{\psi^{-1}(z)}(\zeta))))=v_{r,A}(\psi(\mathcal{F}_{\psi^{-1}(z)}(\zeta))).)$$

8. Now we shall **deduce Corollaries 2 and 3 from Theorem 2.** (Note that they can be strengthened with the help of Theorem 2′.) Corollary 2 follows immediately from Theorem 2 and

Lemma 8. *For an (relatively) open set $\Gamma\subset \mathrm{Fr}\, B^2$ and a function $f\in L^\infty(\Gamma)$ the following facts are equivalent:*

a) $\bar\partial_b f=0$ *on Γ in the distributional sense;*

b) *f coincides a.e. on Γ with the radial boundary values of a bounded function $\mathcal{F}$ holomorphic in $B\cap V$ (V being some neighbourhood of Γ): $f(\zeta)=\lim_{r\uparrow 1}\mathcal{F}(r\zeta)$ a.e.*

Note that $\|f\|_{L^\infty(\Gamma)}=\|\mathcal{F}\|_{L^\infty(B\cap V)}$.

Indeed, Theorem 2 gives an analytic continuation of the function $\mathcal{F}$ obtained by Lemma 8 b) for $\Gamma=\mathrm{Fr}\, B\setminus A$ (the set where $\bar\partial_b f=0$) to the whole of the ball B. We get a function bounded and holomorphic in B (see Theorem 2′) with radial boundary values f a.e. on $\mathrm{Fr}\, B$, so $\bar\partial_b f=0$ on $\mathrm{Fr}\, B$, again by the lemma.

For the proof of Lemma 8 it is enough to continue the function to the sets $O_{\zeta,t}$ for all $C_{\zeta,t}\subset\Gamma$. This can be done by a refinement of the arguments at the end of the proof of Theorem 18.1.12 in [8] or by Lemma 15 and Theorem 16 of [13].

Corollary 3 follows from

Lemma 9. 1) *For an open set $\Gamma\subset\mathrm{Fr}\, B$ and a distribution f on Γ the following conditions are equivalent:*

a) *f is of finite order in Γ (i.e. $|f(\varphi)|\le c_f \max_{|\alpha|\le k_f}\|D^\alpha\varphi\|_{L^\infty(\Gamma)}$ for every function $\varphi\in C_0^\infty(\Gamma)$) and $\bar\partial_b f=0$ on Γ.*

b) *Denote by V the union of $O_{\zeta,t}$ for all spherical caps $C_{\zeta,t}\subset\Gamma$. There exists a function $\mathscr{F}$ analytic in V such that*

$$|\mathscr{F}(\zeta)| \leqq \tilde{c}_f(1-|\zeta|^2)^{-N_f}, \quad (\zeta\in V) \tag{21}$$

and the measures $\mathscr{F}^*(r\zeta)\,d\sigma(\zeta)$ on $\Gamma\subset\mathrm{Fr}\,B$ (σ being the normalized rotation invariant measure on $\mathrm{Fr}\,B$, $\mathscr{F}^*(r\zeta)=\mathscr{F}(r\zeta)$ if $r\zeta\in V$ and $\mathscr{F}^*(r\zeta)=0$ otherwise) tend to f as distributions.

2) *Moreover, if $\mathscr{F}$ is holomorphic in the set V of* 1b) *and satisfies* (21) *then there exists a distribution f on Γ of finite order such that $\mathscr{F}^*(r\zeta)\,d\sigma(\zeta)$ tend to f. If $f\equiv 0$ then $\mathscr{F}\equiv 0$.*

The proof of Corollary 3 follows immediately. Indeed, f is a distribution on the compact set $\mathrm{Fr}\,B$, so f is of finite order. Take the function $\mathscr{F}$ obtained by applying 1b) to $f|\Gamma$ and $\Gamma=\mathrm{Fr}\,B^2\setminus A$, the set where $\bar\partial_b f=0$. Theorem 2 gives an analytic continuation $\mathscr{F}_1$ of $\mathscr{F}$ to the whole of the ball B, satisfying (21) in the whole of the ball. (The estimate (21) in the whole of the ball follows from the maximum principle and the fact that each disc from the proof of Theorem 2′ is contained in some spherical layer of the form $\{z\in B\colon c_1(1-r^2)<(1-|z|^2)<c_2(1-r^2)\}$ $(0<r<1)$.) By Lemma 9 there is a distribution f_1 on $\mathrm{Fr}\,B$ corresponding to $\mathscr{F}_1$ and $f_1|\Gamma=f|\Gamma$.

Outline of the proof of Lemma 9. To prove point 1) of the lemma it is enough to prove it for all spherical caps C contained in Γ with unique constants c_f, $\tilde{c}_f$ and N_f and to use the uniqueness property (see 2)). We will prove now the implication a)$\Rightarrow$b). We want to consider "convolutions of the distribution f with smooth functions" and get continuations of these smoothed functions. For this we identify points ζ on S with unitary operators u_ζ by the formula $\zeta=\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}\to u_\zeta=\begin{pmatrix}\zeta_1 & -\bar\zeta_2\\ \zeta_2 & \bar\zeta_1\end{pmatrix}$. Take C^∞-mollifiers χ_n on S (or equivalently on the group $\mathfrak{U}$ of unitary operators). For $\zeta\in C_n$, some smaller spherical cap consisting of points $\zeta\in C$ with some distance (depending on n) from the boundary $\mathrm{Fr}\,C$, we define

$$f_n(\zeta) = \int_{\mathfrak{U}} f(u\zeta)\chi_n(u)\,du \tag{22}$$

(the integral is symbolic and means application of the distribution f to a suitable C_0^∞-function of $\zeta\in C$ (defined by χ_n)). Then $f_n\in C^\infty$ and it is easy to see that $\bar\partial_b f_n=0$ on C_n in the usual sense. Therefore ([8], 18.1.12) f_n extends to a continuous function in $C_n\cup O_n$, the convex hull of C_n (see the definition of O in Lemma 2), also denoted by f_n, which is holomorphic in O_n. The following lemma gives a convenient representation formula for $f_n(p)$ $(p\in O_n)$ by the boundary function $f_n|C_n$.

Lemma 10. *Suppose $C\subset S$ is some spherical cap, O is the corresponding subset of B. Suppose $g\in C(C\cup O)$ and g is analytic in O. Then for each $\zeta\in C$ and r suffi-*

ciently close to 1 *we have the representation*

$$g(r\zeta) = \int_\Gamma \psi(r, \langle\zeta, \xi\rangle) g(\xi)\, d\sigma(\xi) \tag{23}$$

where the function ψ *can be taken in the following form:*

$$\psi(r, \langle\zeta, \xi\rangle) = \frac{\mathrm{const}}{(1-r)^2} \frac{\varphi\left(\frac{|r-\langle\zeta, \xi\rangle|}{1-r}\right)}{\sqrt{1-|\langle\zeta, \xi\rangle|^2}} \tag{23a}$$

for some fixed function $\varphi \in C_0^\infty\left(\left(\frac{1}{4}, \frac{1}{2}\right)\right)$, $\varphi \geqq 0$,

$$\int_{\{z\in\mathbf{C}:\, |z|<\frac{1}{2}\}} \varphi(|z|)\, dm_2(z) = 1.$$

So $\psi(r, \langle\zeta, \xi\rangle)$ has the following properties: $\psi \geqq 0$; for fixed r, ζ the function $\xi \to \psi(r, \langle\zeta, \xi\rangle)$ is in $C_0^\infty(C_{\zeta, r-\frac{1}{2}(1-r)})$; $\int \psi(r, \langle\zeta, \xi\rangle)\, d\sigma(\xi) = 1$ for a suitable choice of const, so the integrals (23) give an approximation of the identity for $r \uparrow 1$; $\psi(r, \langle\zeta, \xi\rangle) = \psi(r, \langle\xi, \zeta\rangle)$.

Proof. Suppose $\zeta = \begin{pmatrix}1\\0\end{pmatrix} \overset{\mathrm{def}}{=} e_1$, so $\langle e_1, z\rangle = \bar{z}_1$ for $z \in \mathbf{C}^2$. For r sufficiently close to 1 we have by Cauchy's formula applied in the z_1 direction:

$$g(r, 0) = \int_{|z_1-r|<\frac{1}{2}(1-r)} dm_2(z_1) \frac{1}{(1-r)^2} \varphi\left(\frac{|r-z_1|}{1-r}\right) g(z_1, 0)$$

if $\varphi \in C_0^\infty\left(\left(\frac{1}{4}, \frac{1}{2}\right)\right)$, $\varphi \geqq 0$, $\int \varphi = 1$. Applying Cauchy's formula in the z_2 direction, we have

$$g(r, 0) = \int_{|z_1-r|<\frac{1}{2}(1-r)} (1-r)^{-2} \varphi\left(\frac{|r-z_1|}{1-r}\right) \frac{1}{\sqrt{1-|z_1|^2}}$$

$$\times \int_0^{2\pi} g\left(z_1, \sqrt{1-|z_1|^2} e^{i\theta}\right) dm_2(z_1) \frac{\sqrt{1-|z_1|^2}}{2\pi}\, d\theta.$$

The general case can be proved by rotation.

Continue now the proof of Lemma 9. The formula (23) applied to f_n shows that for all points $p \in O$, f_n $(n \geqq n(p))$ converge uniformly on a small neighbourhood of p to the function $\mathscr{F}$,

$$\mathscr{F}(r\zeta) = \int \psi(r, \langle\zeta, \xi\rangle) f(\xi)\, d\sigma(\xi), \quad r\zeta \in O. \tag{24}$$

(The integral is in distributional sense.) Further f_n are analytic in O_n, therefore $\mathscr{F}$ is analytic in O. Now the choice of ψ (see (23a)) together with the fact that f is of finite order gives the estimate $|\mathscr{F}(r\zeta)| \leqq c(1-r)^{-N}$ $(r\zeta \in O)$. We have to show now that

$$\int g(\zeta) \mathscr{F}^*(r\zeta)\, d\sigma(\zeta) \to \int g(\zeta) f(\zeta)\, d\sigma(\zeta) \tag{25}$$

for every $g\in C_0^\infty(C)$ (the right hand side being understood in distributional sense). This follows easily from (24): For r sufficiently close to 1 (in dependence on supp φ) we have

$$\int g(\zeta)\mathcal{F}^*(r\zeta)\,d\sigma(\zeta)=\int g(\zeta)\int\psi(r,\langle\zeta,\xi\rangle)f(\xi)\,d\sigma(\xi)\,d\sigma(\zeta)$$

(the last integral is also distributional). So

$$\int g(\zeta)\mathcal{F}^*(r\zeta)\,d\sigma(\zeta)=\int f(\xi)\,d\sigma(\xi)\Big(\int g(\zeta)\psi(r,\langle\zeta,\xi\rangle)\,d\sigma(\zeta)\Big).$$

It remains to use the facts that f is a distribution of finite order, $g\in C_0^\infty(C)$, and ψ is a smooth approximation of identity, $\psi(r,\langle\xi,\zeta\rangle)=\psi(r,\langle\zeta,\xi\rangle)$. The implication 1a)$\Rightarrow$1b) is proved.

Now we will prove 2), again for a spherical cap C and the corresponding set $O\subset B$. Assume that $C=C_{\zeta,t}$ where $\zeta=e_1=\begin{pmatrix}1\\0\end{pmatrix}$. Define $(J_2\mathcal{F})(z_1,z_2)=\int_0^{z_2}\mathcal{F}(z_1,\xi)\,d\xi$ for $(z_1,z_2)\in C_{\zeta,t}$, the integration being along a suitable curve joining 0 and z_2 (for example, along $\{\xi=rz_2\colon r\in(0,1)\}$). Let also $(J_1\mathcal{F})(z_1,z_2)=\int_{t+\varepsilon}^{z_1}\mathcal{F}(\xi,z_2)\,d\xi$ (integration along a suitable curve joining $t+\varepsilon$ and z_1, for example, a linear segment). Then $J_1\mathcal{F}$ and $J_2\mathcal{F}$ are analytic in $C_{\zeta,t+\varepsilon}$. For some l and k the function $J_1^lJ_2^k\mathcal{F}$ is analytic in $C_{\zeta,t+\varepsilon'}$ and bounded, as is not hard to see by a computation (l, k and the bound for $J_1J_2\mathcal{F}$ depend only on C_f and N_f from (21), not on ε and ε'). The point 2) can be proved now by taking derivatives of this bounded function along spheres of radius $r<1$. The uniqueness ($f\equiv0\Rightarrow\mathcal{F}\equiv0$) is obtained by considering the smooth function $\mathcal{F}_n^*(r\zeta)\overset{\text{def}}{=}\int_{\mathfrak{U}}\mathcal{F}^*(ur\zeta)\chi_n(u)\,du$ for a suitable C_0^∞-function χ_n on $\mathfrak{U}$ with small support near the identity. Then $\mathcal{F}_n^*(r\zeta)$ tend to $f_n(\zeta)\overset{\text{def}}{=}\int_{\mathfrak{U}}f(u\zeta)\chi_n(u)\,du$ as $r\uparrow1$ (the integral is distributional). The uniqueness theorem for smooth analytic functions and a suitable choice of χ_n give the desired result.

The remaining implication 1b)$\Rightarrow$1a) is now easy.

Remarks added April 15, 1987. During the preparation of this article the work of G. Lupacciolu "A theorem on holomorphic extension of CR-functions" appeared (*Pacific J. Math.* 124:1 Sept. 1986), where also removable singularities (in the sense of Definition 1) are considered. Lupacciolu considers arbitrary domains Ω (not necessarily strictly pseudoconvex domains) and shows that polynomial convexity of the singularity is sufficient for removability. For domains of holomorphy in $\mathbf{C}^n$ this condition was shown to be sufficient already by E. L. Stout ("Analytic continuation and boundary continuity of functions of several complex variables", *Proc. Roy. Soc. Edinburgh* **89A** (1981), 63—74.) For $n>2$ Stout showed that even rational convexity is sufficient.

It is in general not easy to give geometric conditions for polynomial or rational convexity. Compact subsets of smooth arcs are polynomially convex, so Corollary 1 follows from the works of Stout and Lupacciolu. Their proof uses integral formulas. It seems that this tool does not give Corollary 1′.

Every totally real manifold is locally polynomially convex, so Theorem 1 for the case $n=2$ follows from the works of Stout and Lupacciolu. But arbitrary compact sets on simply connected totally real manifolds in $\mathbf{C}^2$ are not necessarily polynomially convex (see the example of Wermer at p. 34 of the work of R. Nirenberg and R. O. Wells, *Trans. Amer. Math. Soc.* **142** (1969), 15—35), so theorem 2 is not contained in the works of Stout and Lupacciolu. It seems that the method of Lupacciolu and of Stout in the case of dimension $n=2$ does not give L^∞-estimates for obtaining removable singularities for CR-functions or CR-distributions (in the sense of Definitions 2 and 3). It seems also that their method does not allow to prove versions of the results "localized to a neighborhood of M" (in the sense of Theorem 2′) (such as, in particular, the analog of Corollary 2 for parts of Fr B^2 or arbitrary, not necessarily closed, smooth hypersurfaces in $\mathbf{C}^2$ with nondegenerate Levi form (instead of the whole of Fr B^2)).

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