Conformal mapping and Hausdorff measures

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Introduction

Let $f\colon \mathbf{D}\to\Omega$ be a conformal mapping of the unit disc $\mathbf{D}$ onto a Jordan domain Ω and let E be a Borel subset of the unit circle $\mathbf{T}=\partial\mathbf{D}$. Suppose we know that E is of positive p-dimensional Hausdorff measure. What can be said about metric properties of the set fE, the image of E under the boundary correspondence induced by f? In particular, what are the restrictions on its Hausdorff dimension $\dim fE$?

In the case $p=1$ this problem corresponds to a problem on the metric properties of harmonic measure. It had for a long time been known (see [25]) that if $|E|>0$, then

$$\dim fE \geqq \tfrac{1}{2} \tag{0.1}$$

but some efforts were required to improve this bound to the sharp result

$$\dim fE \geqq 1 \tag{0.2}$$

see [21]. The basic step was taken by Lennart Carleson [9] who proved that

$$\dim fE \geqq \beta < \tfrac{1}{2} \tag{0.3}$$

for an (unspecified) absolute constant β.

In fact, it is easy to establish the inequality

$$\dim fE \geqq \tfrac{1}{2}\dim E, \tag{0.4}$$

generalizing (0. 1), for all $p<1$ (see Section 1.2). The natural question whether the estimates (0.2) and (0.3) can also be extended to $p<1$ is the starting-point in our study. The answer is as follows. An estimate of Carleson type exists for all $p>0$, whereas an estimate of the type (0.2) holds only in case $p=1$. The present paper provides some quantitative amplifications of this answer. Some other relevant problems, including those concerning the relationship between boundary distortion and the behaviour of the derivative in conformal mappings, are also considered.

It should be noted that no nontrivial upper bound of $\dim fE$ in terms of $\dim E$ is possible in the whole class of conformal mappings onto Jordan domains. In fact, as was shown in [25], the image of a set of arbitrarily small Hausdoff dimension may have a positive area.

0.1. Notation related to Hausdorff measures. For $p>0$ the p-dimensional Hausdorff measure is denoted by Λ_p. We shall always consider Λ_p only on Borel subsets of $\mathbf{T}$ or of $\mathbf{C}$ (the complex plane). The measures Λ_p are particular cases (with $\varphi(t)=t^p$) of general Hausdorff measures Λ_φ corresponding to measure functions φ (i.e. continuous increasing functions on $[0, +\infty)$ satisfying $\varphi(0)=0$). For the definition and properties of Λ_φ, see [8], [13]. However, it will often be more convenient to use the set functions H_φ, which are defined by

$$H_\varphi(e) = \inf \sum \varphi(r_j)$$

the infimum being taken over all coverings of a plane set e with discs of radii r_j. The quantities H_φ enjoy all properties of general capacities and Borel sets are capacitable with respect to H_φ (see [8], Chapter 1 and 2). Obviously, H_φ are finite on bounded sets and

$$H_\varphi(e) = 0 \Leftrightarrow \Lambda_\varphi(e) = 0.$$

Similarly to the definition of the harmonic measure, we introduce the following notion.

Definition. Let $f\colon \mathbf{D}\to\Omega$ be a conformal mapping onto a Jordan domain Ω and $p>0$. The set function h_p^f is defined on Borel subsets e of $\partial\Omega$ by

$$h_p^f(e) = H_p(f^{-1}e). \tag{0.5}$$

When f is clear from the context, we simplify the notation to h_p. The set function h_p^f is said to be absolutely continuous with respect to H_φ (notation: $h_p^f \ll H_\varphi$) if

$$H_\varphi(e) = 0 \Rightarrow h_p^f(e) = 0.$$

Observe that

$$E\subset \mathbf{T},\quad H_p(E) > 0 \Rightarrow \dim fE \geqq q$$

is equivalent to

$$\forall q' < q\colon h_p^f \ll H_{q'}.$$

Sometimes it is convenient to consider (0.5) as the definition of $h_p^f(e)$ for arbitrary plane Borel sets e. Clearly, in this case

$$h_p^f(e) = h_p^f(e\cap\partial\Omega).$$

We also define a set function $\varkappa=\varkappa^f$ by

$$\varkappa^f(e) = \operatorname{cap} f^{-1}e$$

where cap is the logarithmic capacity. As follows from the well-known relationship between the capacities H_p and cap (see [18], p. 253),

$$h_p^f(e) \leqq C[\varkappa^f(e)]^p \tag{0.6}$$

where C is a universal constant.

0.2. Main results on boundary distortion. In the statements, for brevity, we introduce the following function.

Definition. Let $p\in(0,1]$. By $d(p)$ we denote the supremum of the set of numbers $q>0$ satisfying

$$E \subset \mathbf{T}, \quad H_p(E) > 0 \Rightarrow \dim fE \geqq q$$

for all f mapping $\mathbf{D}$ onto a Jordan domain. In other words

$$d(p) = \sup\{q\colon\ h_p^f \ll H_q\}.$$

In terms of $d(p)$, (0.2) means that

$$d(1) = 1.$$

I do not know the exact value of $d(p)$ for any other p. As it was noted, our purpose is to improve upon trivial bounds

$$\tfrac{1}{2}p \leqq d(p) \leqq p$$

(the left-hand inequality is just (0.4)).

Theorem 0.1. *If* $p>1$ *then*

$$\tfrac{1}{2}p < d(p) < p. \tag{0.7}$$

Moreover

$$\lim_{p\to\infty} \frac{d(p)}{p} = \frac{1}{2}, \tag{0.8}$$

$$\lim_{p\to 1} \frac{d(p)}{p} = 1. \tag{0.9}$$

Remark that (0.9) provides a generalization of (0.2):

$$\dim E = 1 \Rightarrow \dim fE \geqq 1.$$

Upper bounds of $d(p)$, including the right-hand inequality in (0.7), require constructions of the corresponding examples. Such examples already happen to exist in the class of starlike domains and they even provide the stronger bound

$$d(p) \leqq \frac{p}{2-p} \tag{0.10}$$

which implies (0.8).

Theorem 0.2. *For any* $p\in(0, 1]$, *there exist a conformal mapping* f *onto a Jordan starlike domain with rectifiable boundary and a subset* $E\subset \mathbf{T}$ *satisfying*

$$H_p(E) > 0, \quad \dim fE \leqq \frac{p}{2-p}.$$

It is interesting to note that this bound is sharp even in the wider class, that of close-to-convex domains.

Theorem 0.3. *Let* f *be a close-to-convex function and* $E\subset\mathbf{T}$. *Then*

$$\dim fE \geqq \frac{\dim E}{2-\dim E}.$$

On the other hand, (0.10) is not sharp for arbitrary Jordan domains, at least when p is close to one. An example will be constructed to show that the order of $p-d(p)$ is at most $\frac{1}{2}$ as $p\to 1-$, whereas $p-p(2-p)^{-1}$ is infinitesimal of first order. More precisely, the following is valid.

Theorem 0.4. *If* $p>\frac{19}{20}$, *then*

$$\frac{1}{30}\sqrt{1-p} \leqq p-d(p) \leqq \sqrt{12}\sqrt{1-p}.$$

The lower bound of $d(p)$ contained in Theorem 0.4 easily follows from some known properties of integral means of the derivative of a univalent function. This estimate implies (0.9) as well as the inequality $d(p)>\frac{p}{2}$ in (0.7) for p sufficiently close to one.

For arbitrary $p>0$, I do not dispose of such an elementary proof of $d(p)>\frac{p}{2}$. The proof (rather crude) is obtained by a famous device due to L. Carleson [9]. Our argument differs from that in [9] only in technical details. The main difference is that we avoid any modification of Ω, for it is difficult to track down the effect of Hausdorff measures under such modifications. At the same time, the approach of L. Carleson seems to be much deeper than the result we derive with its help. It would not be a surprise if the inequality $d(p)>\frac{p}{2}$ admits a more direct proof.

0.3. Connection with the behaviour of the derivative. The greater part of the results stated relies on the relationship between boundary distortion and the behaviour of the derivative in conformal mappings. The following two assertions occur to be most convenient in further applications.

Theorem 0.5. *Let* $\alpha>0$ *and*

$$\liminf_{r\to 1-}\frac{|f'(r\zeta)|}{(1-r)^{\alpha}}>0 \quad \textit{for } \Lambda_p\textit{-almost all } \zeta\in\mathbf{T}. \tag{0.11}$$

Then for any $q<\dfrac{p}{1+\alpha}$,

$$h_p^f \preccurlyeq H_q.$$

Theorem 0.6. *If*

$$h_p^f \preccurlyeq H_{p(1+\alpha)^{-1}},$$

then for all ζ *outside a possible exceptional set of* Λ_p*-measure zero,*

$$\limsup_{r\to 1-}\frac{|f'(r\zeta)|}{(1-r)^{\alpha}}>0.$$

These assertions are certainly very far from being invertible, for one of them contains lim inf while the other lim sup. No criterion expressed in terms of f' is known to me for the validity of

$$h_p^f \preccurlyeq H_q.$$

However, such a criterion can easily be obtained in the particular case when Ω is a quasidisc.

If I is a subarc of $\mathbf{T}$ with center at ζ, then by a_I we denote the point $(1-|I|)\zeta$; $|\cdot|$ denotes the normalized Lebesgue measure on $\mathbf{T}$.

Definition. Let f be a univalent function on $\mathbf{D}$ and φ be a measure function. Define the set function D_φ^f by

$$D_p^f(E)=D_\varphi(E)=\inf\textstyle\sum_\nu \varphi(|I_\nu||f'(q_\nu)|)$$

where $E\subset\mathbf{T}$, and the infimum is taken over all coverings of E with subarcs $\{I_\nu\}$ of $\mathbf{T}$, $a_\nu=a_{I_\nu}$. If $\varphi(t)=t^q$, for some $q>0$, D_φ^f is denoted by D_q^f. As above, the notation

$$H_p \preccurlyeq D_\varphi^f$$

means that

$$E\subset\mathbf{T},\quad D_\varphi^f(E)=0\Rightarrow H_p(E)=0.$$

If Ω is a quasidisc and $E\subset\mathbf{T}$, then (see Section 2.1)

$$D_q^f(E)\asymp H_q(fE)$$

and hence

$$h_p^f\preccurlyeq H_q \Leftrightarrow H_p\preccurlyeq D_q^f.$$

It is probable that also in the general case a criterion could be expressed in similar terms. I can prove only a weaker version of the necessity.

Theorem 0.7. *Let f be a conformal mapping onto a Jordan domain. If*

(0.12) $$H_p \ll D_q^f,$$

then

$$\forall q' < q\colon h_p^f \ll H_{q'}.$$

Observe that $D_\varphi^f(E)=0$ if and only if there exists a subset $\Lambda \subset \mathbf{D}$ such that E lies in the cluster set of Λ, and

(0.13) $$\sum_{\lambda \in \Lambda} \varphi[\operatorname{dist}(f(\lambda), \partial\Omega)] < \infty.$$

Sums of the form (0.13) were studied in [20], [23] in the context of dominating subsets. It was proved ([23], Lemma 2.3) that, for regular φ, the following two assertions are equivalent:

$$H_1 \ll D_\varphi^f$$

and

$$\liminf_{r\to 1-} \frac{|f'(r\zeta)|}{\psi(1-r)} > 0 \quad \text{for a.e. } \zeta\in\mathbf{T}$$

where $\psi(t)=t^{-1}\varphi^{-1}(t)$. Thus, for $p=1$, the conditions (0.11) and (0.12) in Theorems 0.5 and 0.7 are equivalent. This is no longer true for $p<1$.

Theorem 0.8. *For any $p\in(0,1)$, there exist $\alpha>0$ and a conformal mapping onto a Jordan domain such that, for some $q>\frac{p}{1+\alpha}$,*

$$h_p^f \ll H_q,$$

but

$$\lim_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} = 0, \quad \zeta\in E,$$

on some subset $E\subset\mathbf{T}$ of positive Λ_p-measure.

Consequently, Theorem 0.6 will be false for $p<1$ if we substitute lim sup by lim inf.

0.4. Strong absolute continuity. If $p=1$, the set function h_p^f coincides with the harmonic measure of Ω evaluated at $f(0)$. In this case $h_p^f \ll H_\varphi$ is equivalent to the condition

(0.14) $$\forall \varepsilon>0 \quad \exists\delta>0\colon H_\varphi(e)<\delta \Rightarrow h_p^f(e)<\varepsilon.$$

This motivates the following

Definition. The set function h_p^f is said to be strongly absolutely continuous with respect to H_φ (notation: $h_p^f \prec H_\varphi$) if it satisfies the condition (0.14). The notation $H_p \prec D_\varphi^f$ has a similar meaning.

It is clear that

$$h_p^f \prec H_\varphi \Rightarrow h_p^f \ll H_\varphi,$$

but the converse is false for $p<1$ (see Section 4.3). Therefore, it is interesting to look at the boundary distortion also from the viewpoint of the new notion. For the strong absolute continuity we are able to trace the relationship with the behaviour of the derivative even more precisely (compare with Theorem 0.7).

Theorem 0.9.

1) $$H_p \prec D_q^f \Rightarrow \forall\, q' < q\colon\ h_p^f \prec H_{q'}.$$

2) $$h_p^f \prec H_q \Rightarrow H_p \prec D_q^f.$$

Another important distinction between the two notions follows from the fact that the simple condition (0.11) in Theorem 0.5 is no longer sufficient for the strong absolute continuity (see Section 4.3).

On the other hand, we shall see that all estimates stated in Section 0.2 (including that of Carleson type) stay in force also for the strong absolute continuity.

0.5. A problem on dominating subsets. The methods employed in the paper to study the set functions D_φ^f enable us to make an advance in a problem on dominating subsets stated in [22], Section 3.1. Let Ω be a Jordan domain and φ be a measure function. It is known (see [22], Lemma 3.1) that if there exists a dominating subset Λ of Ω satisfying

$$\sum\nolimits_{\lambda\in\Lambda} \varphi[\operatorname{dist}(\lambda, \partial\Omega)] < \infty, \tag{0.15}$$

then the harmonic measure of Ω is singular with respect to H_φ. (The latter means that there exists a Borel subset $e\subset\partial\Omega$ of full harmonic but zero Λ_p-measure.) Is the converse true? We answer in the affirmative for measure functions of special type.

Theorem 0.10. *Let φ be a logarithmico-exponential function. The harmonic measure of Ω is singular with respect to the Hausdorff measure Λ_φ if and only if there exists a dominating subset of Ω satisfying* (0.15).

0.6. Organization of the paper. The paper consists of seven sections.

In Section 1 we derive several basic results on the boundary distortion (most of them are known) from a theorem due to A. Pfluger [27].

In Section 2 we study relations between boundary distortion and the derivative of the conformal mapping, and prove Theorems 0.5, 0.6, 0.7 and 0.9.

In Section 3 we study distortion properties of close-to-convex functions and prove Theorems 0.2 and 0.3.

In Section 4 two examples are provided. The first corresponds to Theorem 0.8. The second exhibits distinction between absolute and the strong absolute continuity.

In Section 5 we study the boundary distortion in dimensions close to one and prove Theorem 0.4.

As was noted, the results of Sections 3 and 5 imply all assertions of Theorem 0.1 except for the inequality $d(p) > \frac{p}{2}$. The proof of the latter is the subject of Section 6.

Section 7 is devoted to concluding remarks. First, we list the facts on the radial growth of the reciprocal of the derivative obtained in the previous sections and also state the counterparts of these results concerning the growth of the derivative itself. Secondly, we prove Theorem 0.10 on dominating subsets.

Some more notation. N is the set of positive integers; $\Delta(z_0, r)$ is the disc $\{z: |z-z_0|<r\}$, and $\bar{\Delta}(z_0, r)$ is its closure; $R(z_0; r_1, r_2)$ is the annulus $\Delta(z_0, r_2)\setminus\bar{\Delta}(z_0, r_1)$.

The letters c and C are used to denote various constants.

1. Some consequences of Pfluger's Theorem

In this section some auxiliary results on boundary distortion are provided. All of them admit simple proofs based on the technique of extremal lengths, mainly invoking a theorem due to A. Pfluger. For the most part, these results are well-known, but our approach may possibly deserve some interest.

1.1. Facts on extremal length. For the convenience of the reader, we recall the definition and some basic properties of extremal lengths. See [3], [2] and [26] for a more comprehensive account.

Definition. Let Γ be a family of locally rectifiable curves in **C**. Consider all non-negative Borel measurable functions ϱ on **C**, integrable with respect to the area measure m_2, and for each such ϱ define

$$L(\varrho) = \inf_{\gamma\in\Gamma} \int_\gamma \varrho(z)|dz|.$$

The supremum

$$\lambda(\Gamma) = \sup_\varrho \frac{[L(\varrho)]^2}{\iint \varrho^2 dm_2}$$

is called the extremal length of the family Γ.

Properties. 1) Extremal length is conformally invariant.

2) If each curve $\gamma_2\in\Gamma_2$ contains some curve $\gamma_1\in\Gamma_1$, then $\lambda(\Gamma_1)\leqq\lambda(\Gamma_2)$.

3) If the families $\{\Gamma_j\}$ lie in disjoint Borel sets and if $\Gamma=\bigcup\Gamma_j$, then

$$[\lambda(\Gamma)]^{-1} \geqq \sum [\lambda(\Gamma_j)]^{-1}.$$

4) If the families $\{\Gamma_j\}$ are as above and Γ is a family such that each $\gamma \in \Gamma$ contains at least one $\gamma_j \in \Gamma_j$ for any j, then

$$\lambda(\Gamma) \geqq \sum \lambda(\Gamma_j).$$

Example. Let Γ be the family of all curves in the annulus $R(z_0; r_1, r_2)$ that join the boundary circumferences. Then

$$\lambda(\Gamma) = \frac{1}{2\pi} \log \frac{r_2}{r_1}.$$

Pfluger's theorem ([27]). *Let K be a Jordan curve in $\Delta(0, \frac{1}{3})$ surrounding the origin. Let E be a Borel subset of* $\mathbf{T}$ *and Γ be the family of all curves in* $\mathbf{D}$ *joining E with K. Then*

$$c_K \exp\{-\pi\lambda(\Gamma)\} \leqq \operatorname{cap} E \leqq C_K \exp\{-\pi\lambda(\Gamma)\} \tag{1.1}$$

where

$$c_K = \inf_{z\in K} \frac{1-|z|}{\sqrt{|z|}}, \quad C_K = \sup_{z\in K} \frac{1+|z|}{\sqrt{|z|}}. \tag{1.2}$$

In the sequel we shall use only the right-hand inequality in (1.1). Taking into account the properties of the extremal length listed above, this inequality may be rewritten in the following conformally invariant form.

Corollary. *Let $f\colon \mathbf{D} \to \Omega$ be a conformal mapping onto a Jordan domain Ω and K be a continuum lying in Ω. Let e be a Borel subset of $\partial\Omega$ and Γ be the family of all curves in Ω joining K with e. Then*

$$\varkappa^f(e) \leqq C \exp\{-\pi\lambda(\Gamma)\} \tag{1.3}$$

where C does not depend on e.

By (1.3) and (0.6), we also have

$$h_p^f(e) \leqq C \exp\{-\pi p\lambda(\Gamma)\}. \tag{1.4}$$

1.2. Proposition. *Let f be a conformal mapping onto a Jordan domain Ω. There exists $C > 0$ such that for all $e \subset \partial\Omega$,*

$$\varkappa^f(e) \leqq C[\operatorname{diam} e]^{1/2}. \tag{1.5}$$

Proof. Fix a continuum K in Ω. Let $R = \operatorname{dist}(K, \partial\Omega)$. Clearly, it is enough to prove (1.5) only for e of diameter less than R. Let Γ be a family of all curves joining e and K. Then

$$\lambda(\Gamma) \geqq \frac{1}{2\pi} \log \frac{R}{\operatorname{diam} e}.$$

By (1.3),

$$\varkappa(e) \leqq C \exp\left\{-\frac{\pi}{2\pi} \log \frac{R}{\operatorname{diam} e}\right\} = C[R^{-1} \operatorname{diam} e]^{1/2}. \quad \square$$

Corollary 1. *For any* $p>0$,

$$h_p^f \succ H_{p/2}.$$

Proof. Let $e\subset\partial\Omega$ and $H_{p/2}(e)<\delta$. Then $e\subset\bigcup\Delta_\nu$ where $\{\Delta_\nu\}$ are discs of radii r_ν and

$$\sum r_\nu^{p/2}<\delta.$$

By (0.6) and (1.5),

$$h_p(e)\leqq\sum h_p(\Delta_\nu)\leqq C\sum[\varkappa(\Delta_\nu)]^p$$
$$\leqq C\sum(\operatorname{diam}\Delta_\nu)^{p/2}\leqq C\delta. \quad \square$$

Corollary 2.

$$d(p)\geqq\tfrac{1}{2}p. \quad \square$$

Remark. The inequality (1.5) is immediate from a result of Ch. Pommerenke [28] which asserts, in particular, the following. If g is a conformal mapping of $\mathbf{C}\setminus\overline{\mathbf{D}}$ onto the exterior of a Jordan domain and satisfies $g(\infty)=\infty$, $g'(\infty)=1$, then

$$[\operatorname{cap}E]^2\leqq\operatorname{cap}gE,\quad E\subset\mathbf{T}.$$

As to the latter, it may also be derived from Pfluger's theorem. One have to apply the formula (which follows from (1.1) and (1.2))

$$\operatorname{cap}E=\lim_{R\to\infty}\sqrt{R}\exp\{-\pi\lambda(\Gamma_R)\},$$

where Γ_R is the family of all arcs joining E and $\partial\Delta(0,R)$, and the estimate

$$\operatorname{cap}e\leqq\lim_{R\to\infty}R\exp\{-\pi\lambda(\Gamma_R')\} \tag{1.6}$$

valid for any bounded plane set e (Γ_R' is the family of all arcs joining e and $\partial\Delta(0,R)$). The inequality (1.6) readily follows from the estimate of $\lambda(\Gamma_R')$ arising by the choice $\varrho=|\operatorname{grad}u|$ in the definition of the extremal length, where u is the equilibrium potential of e (cf. [26], § 2.23).

1.3. Proposition. *Let f be a conformal mapping onto a Jordan domain, let I be a subarc of* $\mathbf{T}$ *and onto* $a=a_I$. *Let* $R>1$ *and set*

$$\Delta_R=\Delta(f(a),R|I|\,|f'(a)|).$$

Then

$$\operatorname{cap}(I\setminus f^{-1}\Delta_R)\leqq CR^{-1/2}|I| \tag{1.7}$$

where C is a universal constant.

Proof. Applying an appropriate Möbius transformation, we can reduce the problem to the case $a=0$, $f(0)=0$, $f'(0)=1$. Let $e=\partial\Omega\setminus\Delta(0,R)$. We must verify that

$$\varkappa(e)\leqq CR^{-1/2}.$$

Let Γ denote the family of all curves in Ω joining $\partial\Delta(0, \frac{1}{8})$ with e. Then

$$\lambda(\Gamma) \geqq \frac{1}{2\pi} \log 8R$$

and, by (1.3),

$$\varkappa(e) \leqq C \exp\{-\pi\lambda(\Gamma)\} \leqq C \exp\left\{-\tfrac{1}{2}\log 8R\right\} = CR^{-1/2}$$

where, by (1.2) and the distortion theorem, C can be chosen absolute. □

The inequality (1.7) is a particular case of another result due to Ch. Pommerenke [29], see also [30], Chapter 11.

1.4. Theorem. *Let $f\colon \mathbf{D} \to \Omega$ be a conformal mapping onto a Jordan domain Ω. For any $p \in (0, 1]$ and any $M > 0$ there are numbers $r_0 > 0$ and $k_0 \in \mathbf{N}$ satisfying the following. If Δ is a disc of radius r, $r \leqq r_0$, and Δ' is a disc of radius $2r$ concentric with Δ, then there exist N subarcs $\sigma_1, \ldots, \sigma_N$ of $\partial\Delta'$,*

$$N \leqq k_0 \log \frac{1}{r}, \tag{1.8}$$

which are crosscuts of Ω and separate from $f(0)$ the subarcs $\beta_1, \ldots, \beta_N$ of $\partial\Omega$ such that

$$h_p^f\left[\Delta \setminus \bigcup_{j=1}^{N} \beta_j\right] \leqq r^M.$$

Proof. Fix a small circle K centered at $f(0)$. We can assume that dist $(K, \partial\Omega) > 2r_0$. Let Δ be a disc of radius $r \leqq r_0$ such that $\partial\Omega \cap \Delta \neq \emptyset$. We carry out the following construction. See Figure 1.

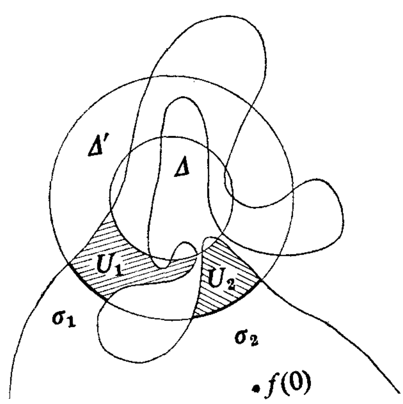

Fig. 1

Let Ω_0 denote the component of $\Omega\setminus\bar{\Delta}$ containing $f(0)$. Let $\{U_j\}$ be the set of those components of $\Omega_0\cap\Delta'$ whose boundary has an arc on $\partial\Delta$. It is clear that this set is not empty, that the components U_j are disjoint and lie in the annulus $\Delta'\setminus\bar{\Delta}$.

The set $(\partial U_j\cap\partial\Delta')\setminus\partial\Omega$ is relatively open with respect to $\partial\Delta'$. Let σ_j denote the component of this set that separates U_j from $f(0)$, and β_j denote the subarc of $\partial\Omega$ that is separated from $f(0)$ by the crosscut σ_j. Denote $e_j=\beta_j\cap\Delta$. Then

$$\partial\Omega\cap\Delta=\bigcup e_j. \tag{1.9}$$

Let Γ_j be the family of all arcs in Ω joining K with e_j, and $\tilde{\Gamma}_j$ be the family of all arcs in U_j which join the boundary circumferences of the annulus $\Delta'\setminus\bar{\Delta}$. By the properties of extremal lengths,

$$\lambda(\Gamma_j)\geqq\lambda(\tilde{\Gamma}_j)$$

and

$$\sum[\lambda(\Gamma_j)]^{-1}\leqq\left[\frac{1}{2\pi}\log 2\right]^{-1}.$$

By (1.4)

$$h_p(e_j)\leqq A\exp\{-\pi p\lambda(\Gamma_j)\}$$

with a constant A depending only of f, K and p. The two last inequalities imply that for all $k\in\mathbf{N}$

$$\operatorname{card}\{j:\ h_p(e_j)\geqq Ar^{\pi pk}\}\leqq\frac{2\pi}{\log 2}k\log\frac{1}{r}. \tag{1.10}$$

Choose $k_0\in\mathbf{N}$ large enough to satisfy

$$\pi pk_0>M.$$

Then

$$\sum\nolimits_{\{j:h_p(e_j)<Ar^{\pi pk_0}\}}h_p(e_j)\leqq\sum\nolimits_{k\geqq k_0}\sum\nolimits_{\{j:Ar^{\pi p(k+1)}\leqq h_p(e_j)\leqq Ar^{\pi pk}\}}h_p(e_j)$$
$$\leqq\frac{2\pi A}{\log 2}\log\frac{1}{r}\sum\nolimits_{k\geqq k_0}(k+1)r^{\pi pk}\leqq r^M$$

provided $r\leqq r_0$ and r_0 is sufficiently small.

We have actually proved the theorem. In fact, put N equal to

$$\operatorname{card}\{j:\ h_p(e_j)\geqq Ar^{\pi pk_0}\}.$$

By (1.10), $N\leqq k_0\log\dfrac{1}{r}$. Suppose that the U_j are arranged in such a way that the numbers $h_p(e_j)$ come in decreasing order. By (1.9),

$$h_p\left(\Delta\setminus\bigcup\nolimits_{j=1}^N\beta_j\right)\leqq\sum\nolimits_{\{j:h_p(e_j)<Ar^{\pi pk_0}\}}h_p(e_j)\leqq r^M.\quad\square$$

Remark. In the case $p=1$ the theorem was established in [22], Lemma 2.3, and the present proof is quite similar to that. The method of proof is essentially due to L. Carleson [9].

Corollary. *Let $f\colon \mathbf{D}\to\Omega$ be a conformal mapping onto a Jordan domain Ω and K be a continuum in Ω. For any $p, q\in(0, 1]$ and any $q'>q$ there exists $r_0>0$ satisfying the following. If Δ is a disc of radius $r\leqq r_0$ with*

$$h_p^f(\Delta) \geqq r^{pq},$$

then there exists a subarc σ of $\partial\Delta'$ which constitutes a crosscut of Ω and satisfies

$$\lambda(\Gamma_\sigma) \leqq \frac{q'}{\pi}\log\frac{1}{r} \tag{1.11}$$

where Γ_σ is the family of all curves in Ω joining σ with K.

Proof. Applying the last theorem with a sufficiently large $M>0$, we obtain N subarcs β_j of $\partial\Omega$ such that

$$N \leqq k_0\log\frac{1}{r}$$

and

$$h_p\left(\Delta\setminus\bigcup_{j=1}^N \beta_j\right) \leqq \tfrac{1}{2}r^{pq}.$$

Hence

$$\sum_{j=1}^N h_p(\beta_j) \geqq \tfrac{1}{2}r^{pq}.$$

Consequently, there exists j_0 such that for $\beta=\beta_{j_0}$ we have

$$h_p(\beta) \geqq c|\log r|^{-1}r^{pq}.$$

By (1.4)

$$h_p(\beta) \leqq A\exp\{-\pi p\lambda(\Gamma_\sigma)\}$$

where $\sigma=\sigma_{j_0}$. Thus (if r_0 is small),

$$\exp\{-\pi p\lambda(\Gamma_\sigma)\} \geqq c|\log r|^{-1}r^{pq} \geqq r^{pq'} = \exp\left\{-pq'\log\frac{1}{r}\right\}$$

which implies (1.11). □

1.5. Proposition. *Let $f\colon \mathbf{D}\to\Omega$ be a conformal mapping onto a Jordan domain Ω, I be a subarc of $\mathbf{T}$ and σ be a crosscut of Ω joining the endpoints of $f(I)$. Then*

$$\operatorname{diam}\sigma \geqq c|I||f'(a_I)|$$

where $c>0$ is a universal constant.

Proof. By means of an appropriate Möbius transformation we can reduce the problem to the case $f(0)=0$, $f'(0)=1$, $I=\mathbf{T}_+$ (the upper semicircle) and $a_I=0$. We should verify that in this case

$$\operatorname{diam}\sigma \geqq c.$$

Let $\Delta=\Delta\left(0, \frac{1}{2}\right)$. Then $\operatorname{dist}(f\Delta, \partial\Omega)\geqq\frac{1}{54}$ by the distortion theorem. If $\sigma\cap f\Delta\neq\emptyset$,

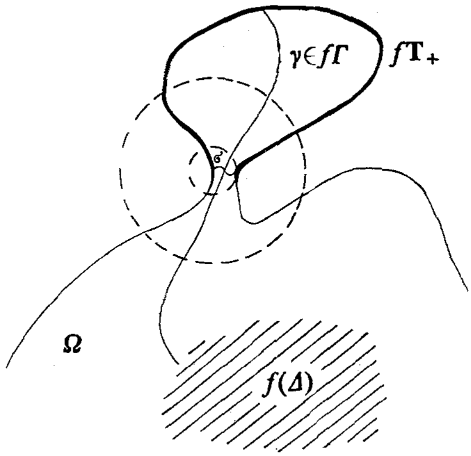

Fig. 2

then $\operatorname{diam}\sigma \geqq \frac{1}{54}$. Also if $\sigma\cap f\Delta=\emptyset$, then the crosscut $f^{-1}\sigma$ separates a semicircle, say $\mathbf{T}_+$, from Δ. Denote the family of all arcs in $\mathbf{D}$ joining $\mathbf{T}_+$ with $\partial\Delta$ by Γ. On the one hand,

$$\lambda(\Gamma) \leqq \frac{1}{\pi}\log 2.$$

On the other hand (see Figure 2),

$$\lambda(\Gamma) \geqq \frac{1}{2\pi}|\log(54\operatorname{diam}\sigma)|.$$

Hence $\operatorname{diam}\sigma \geqq \frac{1}{216}$. □

The last result is certainly well-known (cf. [30], Exercise 2, p. 318). Lemma 2.2 in [22] is a consequence of Proposition 1.5.

2. Boundary distortion and the behaviour of the derivative

2.1. It is instructive first to consider the case of quasidiscs. Recall that a Jordan domain Ω is said to be a quasidisc if there exists a number $M>0$ such that for any two points w_1 and w_2 on $\partial\Omega$,

$$\min(\operatorname{diam}\beta_1, \operatorname{diam}\beta_2) \leqq M|w_1-w_2|$$

where β_1 and β_2 are the components of $\partial\Omega\setminus\{w_1, w_2\}$. See [1] for the relation with quasiconformal mappings.

Lemma. *Let f be a conformal mapping of* $\mathbf{D}$ *onto a quasidisc* Ω. *There exists* $C>0$ *such that if* I *is a subarc of* $\mathbf{T}$, *then*

$$C^{-1}|I||f'(a_I)| \leqq \operatorname{diam} fI \leqq C|I||f'(a_I)|. \tag{2.1}$$

Proof. The left-hand inequality (with a universal C) follows from Proposition 1.5. To prove the second inequality, let I be a subarc of $\mathbf{T}$ with $|I|\leqq\frac{1}{4}$. From both endpoints of I we draw the subarcs I_1 and I_2 congruent to I. By Proposition 1.3 there are points $z_1\in I_1$ and $z_2\in I_2$ such that

$$|f(a_I)-f(z_1)|, \quad |f(a_I)-f(z_2)| \leqq C|I||f'(a_I)|$$

with a universal C. Hence

$$|f(z_1)-f(z_2)| \leqq C_1|I||f'(a_I)|.$$

By $\tilde{I}$ we denote the subarc containing I with endpoints z_1 and z_2. Then either

$$\operatorname{diam} f\tilde{I} \leqq MC_1|I||f'(a_I)|$$

and (2.1) is true, or

$$\operatorname{diam} f(\mathbf{T}\setminus\tilde{I}) \leqq MC_1|I||f'(a_I)|. \tag{2.2}$$

Since $\mathbf{T}\setminus\tilde{I}$ contains a semicircle, by Proposition 1.5,

$$\operatorname{diam} f(\mathbf{T}\setminus\tilde{I}) \geqq c|f'(0)|$$

where c is a universal constant. Therefore, by (2.2),

$$|I||f'(a_I)| \geqq c_1 > 0$$

where c_1 does not depend on I. Hence

$$\operatorname{diam} f\tilde{I} \leqq \operatorname{diam}\Omega \leqq (c_1^{-1}\operatorname{diam}\Omega)|I||f'(a_I)|,$$

and the right-hand inequality in (1.1) follows. □

Proposition. *Let f be a conformal mapping onto a quasidisc and* $q>0$. *There exists* $C>0$ *such that for any* $E\subset\mathbf{T}$,

$$C^{-1}D_q^f(E) \leqq H_q(fE) \leqq CD_q^f(E). \tag{2.3}$$

Proof. First we establish the right-hand inequality. Cover E by arcs $\{I_\nu\}$ with

$$\sum [|I_\nu||f'(a_\nu)|]^q < D_q(E) = \varepsilon.$$

Then the right-hand inequality in (2.1) implies that

$$H_q(fE) \leqq \sum [C|I_\nu||f'(a_\nu)|]^q \leqq CD_q(E)+C\varepsilon.$$

To prove the second inequality in (2.3), we consider a covering of fE with discs $\{\Delta_\nu\}$ of radii r_ν such that

$$\sum r_\nu^q < H_q(fE)+\varepsilon.$$

Let $e_\nu = fE\cap\Delta_\nu$. By the definition of a quasidisc, either $r_\nu > c > 0$ with c not depending on E (in this case, (2.3) already follows) or there is a subarc I_ν of $\mathbf{T}$ such that $e_\nu \subset fI_\nu$ and fI_ν is contained in a disc of radius Mr_ν. If the latter holds for all ν, then $E\subset \bigcup I_\nu$ and, by (2.1),

$$\sum [|I_\nu||f'(a_\nu)|]^q \leqq \sum [CMr_\nu]^q \leqq CH_q(fE)+C\varepsilon. \quad \square$$

Corollary. *If f is a conformal mapping onto a quasidisc and p, q are positive numbers, then*

$$h_p^f \preccurlyeq H_q \Leftrightarrow H_p \preccurlyeq D_q^f; \tag{2.4}$$

$$h_p^f \prec H_q \Leftrightarrow H_p \prec D_q^f. \tag{2.5}$$

2.2. Conjecture. *The equivalences* (2.4) *and* (2.5) *are valid for arbitrary Jordan domains.*

If this conjecture were true, the problem of the boundary distortion would be completely reduced to a question concerning derivatives of conformal mappings. Unfortunately, I can prove only partial results in this direction.

2.3. Theorem. *Let $f\colon \mathbf{D}\to\Omega$ be a conformal mapping onto a Jordan domain Ω. Let $p, q\in(0, 1]$ and*

$$\varphi(t) = t^q|\log t|^{1-q}.$$

Then

$$H_p \preccurlyeq D_q^f \Rightarrow h_p^f \preccurlyeq H_\varphi; \tag{2.6}$$

$$H_p \prec D_q^f \Rightarrow h_p^f \prec H_\varphi. \tag{2.7}$$

Proof. First we prove (2.7). Suppose that $H_p \prec D_q$, and let $e=fE\subset\partial\Omega$ satisfy $H_\varphi(e)<\delta$ where δ is a small number. We have to prove that $H_p(E)$ is also small.

Consider a covering of e with discs $\{\Delta_\nu\}$ of radii r_ν satisfying

$$\sum \varphi(r_\nu) < \delta.$$

To each disc we apply Theorem 1.4 with a fixed constant $M>q$. There exist crosscuts $\sigma_j^{(\nu)}$, $1\leqq j\leqq N(\nu)$,

$$N(\nu) \leqq k_0 \log\frac{1}{r_\nu}, \tag{2.8}$$

which lie on $\partial\Delta'_\nu$ and separate the subarcs $\beta_j^{(\nu)}$ of $\partial\Omega$ from $f(0)$ such that

$$h_p\Big(\Delta_\nu \setminus \bigcup_{j=1}^{N(\nu)} \beta_j^{(\nu)}\Big) \leqq r_\nu^M.$$

Denote

$$e_\delta = fE_\delta = e \cap \bigcup_\nu \bigcup_{j=1}^{N(\nu)} \beta_j^{(\nu)}.$$

Then

$$e \setminus e_\delta \subset \bigcup_\nu [\Delta_\nu \setminus \bigcup_{j=1}^{N(\nu)} \beta_j^{(\nu)}]$$

and

$$h_p(e \setminus e_\delta) \leqq \sum r_\nu^M \leqq \delta.$$

Also denote $I_j^{(\nu)} = f^{-1}\beta_j^{(\nu)}$ and $a_j^{(\nu)} = a_{I_j^{(\nu)}}$. Applying Hölder's inequality, (2.8) and Proposition 1.5, we have

$$\sum_{j=1}^{N(\nu)} [|I_j^{(\nu)}||f'(a_j^{(\nu)})|]^q \leqq [N(\nu)]^{1-q} [\sum_{j=1}^{N(\nu)} |I_j^{(\nu)}||f'(a_j^{(\nu)})|]^q$$

$$\leqq C\left[\log \frac{1}{r_\nu}\right]^{1-q} [\sum_{j=1}^{N(\nu)} \operatorname{diam} \sigma_j^{(\nu)}]^q.$$

Since $\sigma_j^{(\nu)}$ are disjoint, the latter does not exceed

$$C|\log r_\nu|^{1-q} r_\nu^q = C\varphi(r_\nu).$$

Since $E_\delta \subset \bigcup_{\nu,j} I_j^{(\nu)}$,

$$D_q(E_\delta) \leqq C \sum_\nu \varphi(r_\nu) \leqq C\delta.$$

Because of $H_p \prec D_q$, $H_p(E_\delta)$ tends to zero as $\delta \to 0$. Hence

$$h_p(e) = H_p(E) \leqq H_p(E_\delta) + H_p(E \setminus E_\delta) \leqq \delta + o(1) \quad \text{as} \quad \delta \to 0,$$

and $h_p \prec H_\varphi$.

Next we verify (2.6). Let $H_p \ll D_q$, and suppose that $E \subset \mathbf{T}$ satisfies $H_\varphi(fE) = 0$. We have to prove that $H_p(E) = 0$. Reasoning as above, for each $\delta > 0$, we obtain a subset $E_\delta \subset E$ such that

$$H_p(E \setminus E_\delta) < \delta, \quad D_q(E_\delta) < \delta.$$

Define

$$E_0 = \bigcap_{n \geqq 1} \bigcup_{k \geqq n} E_{2^{-k}}.$$

Then, for all $n \in \mathbf{N}$,

$$D_q(E_0) \leqq D_q\left(\bigcup_{k \geqq n} E_{2^{-k}}\right) \leqq \sum_{k \geqq n} D_q(E_{2^{-k}}) \leqq 2^{-n+1}.$$

Hence $D_q(E_0) = 0$ and

$$H_p(E_0) = 0. \tag{2.9}$$

For any $n \in \mathbf{N}$ we also have

$$E \setminus E_0 \subset \bigcup_{k \geqq n} E \setminus E_{2^{-k}}.$$

Therefore

$$H_p(E \setminus E_0) \leqq \sum_{k \geqq n} H_p(E \setminus E_{2^{-k}}) \leqq 2^{-n+1},$$

and $H_p(E \setminus E_0) = 0$. Combined with (2.9), this gives $H_p(E) = 0$. □

The result obtained implies Theorem 0.7 and the first assertion in Theorem 0.9. The second assertion will be proved in Section 2.4.

Remark. For $p=q=1$, Theorem 2.3 was established in [20], Theorem 4, where it was also noted that the implications (2.6) and (2.7) (coinciding when $p=1$) are reversible. In Section 7 we shall proceed with the discussion of the case $p=1$, that of the harmonic measure.

2.4. Theorem. *Let f be a conformal mapping onto a Jordan domain. Then*

$$h_p^f \prec H_q \Rightarrow H_p \prec D_q^f.$$

Proof. Suppose that $h_p \prec H_q$. To prove the theorem, it is sufficient to verify that for any $\varepsilon>0$ there exists $\delta>0$ satisfying:

$$E \text{ is compact}, \quad D_q(E) < \delta \Rightarrow H_p(E) \leqq \varepsilon. \tag{2.10}$$

In fact, if E is an arbitrary Borel subset of $\mathbf{T}$, then, by the properties of the "capacities" H_p (see [8], Ch. 2), there exists a sequence of compact subsets $E_n \subset E$ such that $H_p(E)=\lim H_p(E_n)$. If $D_q(E)<\delta$, then $D_q(E_n)<\delta$ and, by (2.10), $H_q(E_n) \leqq \varepsilon$. Hence $H_q(E) \leqq \varepsilon$, which provides $H_p \prec D_q$.

Thus, let E be a compact subset of $\mathbf{T}$ and $D_q(E)<\delta$. We can cover E by a finite number of arcs $\{I_\nu\}$ with

$$\sum [|I_\nu||f'(a_\nu)|]^q < \delta,$$

so that the multiplicity of the covering is at most two. Fix a large number $R>0$ and apply Proposition 1.3 to each arc I_ν. For each ν there exists a compact subset $F_\nu \subset I_\nu$ such that

$$|F_\nu| \geqq \tfrac{1}{2}|I_\nu|, \tag{2.11}$$

and fF_ν lies in a disc of radius $R|I_\nu||f'(a_\nu)|$. Denote the compact set $\bigcup F_\nu$ by F. Then $H_q(fF) \leqq R^q\delta$ and, since $h_p \prec H_q$, $H_p(F)=o(1)$ as $\delta \to 1$.

Now we prove that (2.11) implies that

$$H_p(F) \geqq cH_p(E) \tag{2.12}$$

with a universal c. This will yield (2.10). By the Frostman theorem (see, e.g., [8], Ch. 2, Theorem I), there exists a nonnegative measure μ supported by E such that

$$\int d\mu \geqq cH_p(E),$$

and for any subarc I of $\mathbf{T}$

$$\mu|I| \leqq |I|^p. \tag{2.13}$$

By means of μ we shall construct a measure η supported by F and satisfying

(2.14) $$\int d\eta \geqq cH_p(E); \quad \forall I:\ \eta(I) \leqq 10|I|^p.$$

Then (2.12) will trivially follow.

Let η be defined by

$$\eta = \sum \eta_\nu$$

where the measures η_ν are supported by F_ν and have the constant density $|F_\nu|^{-1}\mu(I_\nu)$ with respect to Lebesgue measure. Then

$$\int d\eta_\nu = \mu(I_\nu);$$
$$\int d\eta = \sum \mu(I_\nu) \geqq \int d\mu \geqq cH_p(E).$$

To prove (2.14), let I denote a subarc of $\mathbf{T}$ with endpoints ζ_1 and ζ_2.

$$\eta(I) = \sum \eta_\nu(I) \leqq \sum_{\{\nu:\zeta_1\in I_\nu\}} + \sum_{\{\nu:\zeta_2\in I_\nu\}} + \sum_{\{\nu:I_\nu\subset I\}}.$$

The last sum does not exceed

$$\sum_{\{\nu:I_\nu\subset I\}} \mu(I_\nu) \leqq 2\mu(I) \leqq 2|I|^p.$$

The first and the second sums contain at most two terms. Let, for instance, $\zeta_1\in I_\nu$. If $|I_\nu|\leqq|I|$, then

$$\eta_\nu(I) = |F_\nu|^{-1}|F_\nu\cap I|\mu(I_\nu) \leqq \mu(I_\nu) \leqq |I_\nu|^p \leqq |I|^p.$$

If $|I_\nu|\geqq|I|$, then

$$\eta_\nu(I) \leqq 2|I_\nu|^{-1}|I||I_\nu|^p \leqq 2|I|^p$$

because of $p\leqq 1$. Hence (2.14) follows. □

I do not know whether Theorem 2.4 remains true with $\preccurlyeq$ replaced by $\prec$. Sometimes the following partial result turns out to be useful.

2.5. Proposition. *Let f be a conformal mapping onto a Jordan domain and $h_p^f \preccurlyeq H_q$. Then for any subset $E\subset\mathbf{T}$ of positive Λ_p-measure and for any $C>0$*

(2.15) $$\inf\left\{\sum [|I_\nu||f'(a_\nu)|]^q:\ E\subset\bigcup I_\nu,\ \sum|I_\nu|^p\leqq C\right\} > 0.$$

Proof. Assume that the infimum in (2.15) is zero for some $C>0$ and $E\subset\mathbf{T}$ with $H_p(E)>0$. Fix $\delta>0$ and consider a covering $E\subset\bigcup I_\nu$ satisfying

$$\sum|I_\nu|^p\leqq C, \quad \sum[|I_\nu||f'(a_\nu)|]^q < \varepsilon.$$

Applying Proposition 1.3 (with $R=\varepsilon^{-1/2}$) to each I_ν, we obtain subsets $F_\nu\subset I_\nu$ such that

$$H_p(I_\nu\setminus F_\nu) \leqq C\varepsilon^{p/4}|I_\nu|^p,$$

and fF_ν lies in a disc of radius $\varepsilon^{-1/2}|I_\nu||f'(a_\nu)|$. Define

$$F = F^{(\varepsilon)} = \bigcup F_\nu.$$

Then

$$H_q(fF^{(\varepsilon)}) \leqq \varepsilon^{-q/2} \sum [|I_\nu||f'(a_\nu)|]^q \leqq \varepsilon^{1-q/2}$$

and

$$H_p(E\setminus F^{(\varepsilon)}) \leqq \sum H_p(I_\nu\setminus F_\nu) \leqq C\varepsilon^{p/4} \sum |I_\nu|^p \leqq C\varepsilon^{p/4}.$$

Let

$$E_0 = \bigcap_{n\geqq 1} \bigcup_{k\geqq n} F^{(2^{-k})}.$$

For any $n\in\mathbf{N}$

$$H_q(fE_0) \leqq \sum_{k\geqq n} H_q(fF^{(2^{-k})}) \leqq \sum_{k\geqq n} 2^{-k(1-q/2)}.$$

Hence

(2.16) $$H_q(fE_0) = 0.$$

Also, for any $n\in\mathbf{N}$,

$$E\setminus E_0 \subset \bigcup_{k\geqq n} (E\setminus F^{(2^{-k})}),$$

and

$$H_p(E\setminus E_0) \leqq C \sum_{k\geqq n} 2^{-kp/4},$$

which implies $H_p(E_0)>0$. Combined with (2.16), this contradicts the assumption $h_p \lessdot H_q$. □

As consequences of Theorem 2.3 and Proposition 2.5, we shall prove Theorems 0.5 and 0.6 (see Introduction).

2.6. *Proof of Theorem 0.5.* Because of (2.6), it is sufficient to prove $H_p \lessdot D_{p(1+\alpha)^{-1}}$. Assume the contrary — that there is a subset $E\subset\mathbf{T}$ satisfying $H_p(E)>0$ and $D_{p(1+\alpha)^{-1}}(E)=0$. By hypothesis, there exists a subset $E_0\subset E$ of positive Λ_p-measure such that, for some $c>0$,

(2.17) $$|f'(r\zeta)| \leqq c(1-r)^\alpha$$

for all $\zeta\in E_0$ and $r\in(0, 1)$. For any $\varepsilon>0$, there exists a covering $E_0\subset \bigcup I_\nu$ with

$$\sum [|I_\nu||f'(a_\nu)|]^{p/1+\alpha} < \varepsilon.$$

Clearly, we can assume that all I_ν meet E_0. Then, by (2.17) and the distortion theorem,

$$|f'(a_\nu)| \geqq c|I_\nu|^\alpha.$$

Consequently,

$$\sum |I_\nu|^p \leqq C \sum [|I_\nu||f'(a_\nu)|]^{p/1+\alpha} \leqq C\varepsilon,$$

and hence $H_p(E_0)=0$. □

Remark. The hypothesis of Theorem 0.5 does not, in general, imply the strong absolute continuity, see Section 4.3. Reasoning as above, one can easily verify the following sufficient condition: *If*

$$\liminf_{n\to\infty} H_p\{\zeta\colon n|f'(r\zeta)| \leqq (1-r)^\alpha \quad \textit{for some} \quad r\in(0, 1)\} = 0,$$

then

$$h_p^f \prec H_{p(1+\alpha)^{-1}}.$$

2.7. *Proof of Theorem 0.6.* Assume the contrary — that there is a subset $E_0 \subset \mathbf{T}$ of positive Λ_p-measure satisfying

$$\lim_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} = 0, \quad \zeta \in E_0.$$

Then there exists a subset $E \subset E_0$ of finite positive Λ_p-measure with the uniform estimate

(2.18) $$|f'(r\zeta)| \leqq [\delta(1-r)](1-r)^\alpha, \quad \zeta \in E,$$

where $\delta(t) = o(1)$ as $t \to 0$, holding on it. Fix $\varepsilon > 0$ and consider a covering of E with arcs I_ν such that

$$|I_\nu| < \varepsilon; \; H_p(E) \leqq \sum |I_\nu|^p \leqq \Lambda_p(E) + \varepsilon.$$

By (2.18) and the distortion theorem

$$|f'(a_\nu)| \leqq \delta(\varepsilon) I_\nu^\alpha.$$

Therefore

$$\sum [|I_\nu| \, |f'(a_\nu)|]^{p/1+\alpha} \leqq [\delta(\varepsilon)]^q [\Lambda_p(E) + \varepsilon] \xrightarrow[\varepsilon\to 0]{} 0.$$

Hence the infimum in (2.15) is zero for $q = p(1+\alpha)^{-1}$ and $C = H_p(E)$. By Proposition 2.5 this contradicts the assumption $h_p \lessdot H_q$. □

2.8. *Remark.* As it has been noted, for $p=1$, $h_p \lessdot H_{p(1+\alpha)^{-1}}$ implies that

$$\liminf_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} > 0$$

for a.e. $\zeta \in \mathbf{T}$. For $p < 1$, this is no longer true, see Section 4.1. The reason is that the lower density (with respect to H_p, $p < 1$) of a set of positive Λ_p-measure may be zero everywhere. But if the set is subject to some density condition, we can claim more

Proposition. *Let the subset* $E \subset \mathbf{T}$ *satisfy* $0 < \Lambda_p(E) < \infty$ *and*

$$\liminf_{t\to 0} \frac{H_p(E \cap \Delta(\zeta, t))}{t^p} \geqq c > 0, \quad \zeta \in E,$$

and let

$$\liminf_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} = 0, \quad \zeta \in E.$$

Then $h_p^f \not\lessdot H_{p(1+\alpha)^{-1}}$.

Proof. Fix $\varepsilon > 0$. For any $\zeta \in E$ there is a subarc I_ζ of $\mathbf{T}$ with center at ζ such that

$$|f'(a_{I_\zeta})| < \varepsilon |I_\zeta|^\alpha;$$
$$H_p(E \cap I_\zeta) \geqq c|I_\zeta|^p.$$

Applying the covering lemma to $\{I_\zeta\}_{\zeta\in E}$, we obtain a covering $\{I_{\zeta_\nu}\}$ of finite multiplicity. For brevity, we shall write I_ν instead of I_{ζ_ν}. Then

$$\sum |I_\nu|^p \leqq C \sum \Lambda_p(E\cap I_\nu) \leqq C\Lambda_p(E) = C.$$

On the other hand

$$\sum [|I_\nu||f'(a_\nu)|]^{p/1+\alpha} \leqq \varepsilon^q \sum |I_\nu|^p \to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Therefore, the infimum in (2.15) is zero for $q=p(1+\alpha)^{-1}$. By Proposition 2.5 $h_p \not\prec H_q$.

The argument is applicable, for example, to standard Cantor sets of constant ratio.

3. Boundary distortion of close-to-convex functions

To illustrate the results obtained in the previous section, we consider a question on the boundary distortion in the class of close-to-convex functions. This class plays an important role in the theory of conformal mappings. On the one hand, it is large enough to contain many interesting examples of univalent functions. On the other, this class is often much easier to deal with, and many problems, open for arbitrary univalent functions, admit a complete solution for close-to-convex functions.

Recall that a function f analytic in the unit disc is said to be close-to-convex if there is a starlike function g such that

$$\text{Re}\,\frac{zf'(z)}{g(z)} > 0 \quad \text{for all} \quad z\in\mathbf{D}. \tag{3.1}$$

Also recall that an analytic function g satisfying $g(0)=0$, $g'(0)>0$ is a starlike function if

$$\text{Re}\,\frac{zg'(z)}{g(z)} > 0, \quad z\in\mathbf{D}. \tag{3.2}$$

Starlike functions are obviously close-to-convex. Close-to-convex functions are univalent (see [12] §§ 2.5 and 2.6).

Our main result is as follows. If f is a close-to-convex function, then

$$\dim fE \geqq \frac{\dim E}{2-\dim E}, \quad E\subset\mathbf{T},$$

and this bound is sharp, see Theorem 0.2 and 0.3.

3.1. *Proof of Theorem 0.2.* Let $0<p<1$. Take an appropriate Cantor set $E\subset\mathbf{T}$ statisfying

$$0 < \Lambda_p(E) < \infty$$

and denote by μ the probability measure which is a multiple of the restriction $\Lambda_p|E$. Let u denote the Poisson integral of μ and $\tilde{u}$ the conjugate function with $\tilde{u}(0)=0$. Define

$$w = \frac{2}{1+u+i\tilde{u}}.$$

Then $w(0)=1$ and

$$\operatorname{Re} w = 2\frac{1+u}{(1+u)^2+\tilde{u}^2} > 0. \tag{3.3}$$

If f is defined by

$$f(z) = z\exp\left\{\int_0^z \frac{w(z)-1}{z}\,dz\right\}, \quad z\in\mathbf{D},$$

then $\dfrac{zf'}{f}=w$ and (3.2) is valid. Hence f is univalent and maps $\mathbf{D}$ onto a domain Ω starlike with respect to the origin.

We shall verify that $\partial\Omega$ is a rectifiable Jordan curve. Since

$$|f'(z)| \leqq 2|f(z)||w(z)| \leqq 4|f(z)|, \quad |z| > \tfrac{1}{2},$$

the derivative f' is bounded on $\mathbf{D}$. Consequently, f is continuous up to the boundary and $\partial\Omega$ has a finite length. The injectivity of $f|\mathbf{T}$ follows from the well-known identity

$$\frac{\partial}{\partial\theta}\arg w(re^{i\theta}) = \operatorname{Re} w(re^{i\theta})$$

and also from (3.3) and the fact that $\operatorname{Re} w\neq 0$ a.e. on $\mathbf{T}$.

To check the distortion properties of f, we make use of Theorem 0.6. Because of the homogeneity of the Cantor set E,

$$\liminf_{t\to 0}\frac{\mu\Delta(\zeta, t)}{t^p} \geqq c > 0$$

for all $\zeta\in E$. By a simple estimate of the Poisson integral, this implies that

$$u(r\zeta) \geqq c(1-r)^{p-1}$$

for all $\zeta\in E$ and $r\in(0, 1)$. Hence, for $\zeta\in E$,

$$|f'(r\zeta)| \leqq C[u(r\zeta)]^{-1} \leqq C(1-r)^{1-p}.$$

Applying Theorem 0.6 with $\alpha<1-p$, we have

$$h_p \nless H_{p(1+\alpha)^{-1}}.$$

The assertion now easily follows. □

Remark. The idea to use the Poisson integral of a singular measure for the construction of an example in boundary distortion is due to A. Lohwater and G. Piranian [19].

3.2. Proposition. *Let f be a close-to-convex function mapping* **D** *onto a Jordan domain. If* $0<p<1$ *and* $0<q<\frac{p}{2-p}$, *then*

$$h_p^f \prec H_q.$$

Proof. Having in view to apply the assertion contained in Section 2.6, we prove that if

$$F_n = \{\zeta\in\mathbf{T}\colon\ n|f'(r\zeta)| \leqq (1-r)^{1-p} \quad \text{for some} \quad r\in(0,1)\},$$

then

$$H_p(F_n) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.4}$$

Let $w=\frac{zf'}{g}$, where g is a starlike function satisfying (3.1). Since (see [30], Ch. 11 Theorem 9)

$$\operatorname{cap}\left\{\zeta\in\mathbf{T}\colon \int_0^1 |g'(r\zeta)|\,dr > R\right\} = o(1) \quad \text{as} \quad R\to\infty,$$

for the validity of (3.4) it is enough to prove the following. If Φ is an analytic function in **D** with positive real part and

$$E_n = \{\zeta\in\mathbf{T}\colon\ |\Phi(r\zeta)| \geqq n(1-r)^{p-1} \quad \text{for some} \quad r\in(0,1)\},$$

then

$$H_p(E_n) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.5}$$

Let μ be a positive measure on **T** such that

$$\Phi(z) = \int \frac{z+\zeta}{z-\zeta}\,d\mu(\zeta), \quad z\in\mathbf{D}.$$

If $\zeta\in E_n$, then

$$\sup_{t>0} \frac{\mu\Delta(\zeta,t)}{t^p} \geqq cn \tag{3.6}$$

with c depending only on p. In fact, if $\mu\Delta(\zeta,t)\leqq At^p$ for all $t>0$, then

$$|\Phi(r\zeta)| \leqq CA\int_0^1 \frac{t^p\,dt}{|r-e^{it}|^2}\,dt \leqq CA(1-r)^{p-1}$$

where C depends only on p.

By (3.6), for each $\zeta\in E_n$, there is an interval $I_\zeta\subset\mathbf{T}$ with center at ζ satisfying

$$\mu I_\zeta \geqq cn|I_\zeta|^p.$$

Applying the covering lemma to $\{I_\xi\}_{\xi\in E_n}$, we can select a subcovering $\{I_{\xi_\nu}\}$ of multiplicity at most two. Then

$$H_p(E_n) \leqq \sum_\nu |I_{\zeta_\nu}|^p \leqq (cn)^{-1} \sum_\nu \mu I_{\zeta_\nu} \leqq 2(cn)^{-1}\mu E$$

wich implies (3.5). □

Theorem 0.3 is a consequence of the last result.

4. Two examples

We shall apply the method of the previous section to construct two examples which shed more light on the relationship between boundary distortion and the behaviour of the derivative. The first example (it corresponds to Theorem 0.8) shows that, for $p<1$, in contrast with $p=1$, the condition

$$H_p \ll D_q^f$$

is strictly weaker than the condition

$$\liminf_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} > 0 \quad \text{for } \Lambda_p\text{-a.e. } \zeta$$

where $q=p(1+\alpha)^{-1}$. The second example shows that, for $p<1$,

$$h_p^f \ll H_q \not\Rightarrow h_p^f \prec H_q. \tag{4.1}$$

4.1. *First example.* Choose a rapidly increasing sequence $\{\nu_k\}$ of positive integers and set

$$N_n = \prod_{k=1}^n \nu_k, \quad l_n = N_n^{-2}.$$

Consider ν_1 closed subarcs $I^{(1)}$ of $\mathbf{T}$ which are of length l_1 and equidistributed on $\mathbf{T}$. At the n-th stage of the construction, we place ν_n closed arcs $I^{(n)}$ of length l_n into each arc $I^{(n-1)}$ obtained at the previous stage, in such a manner that the distance between any two neighbouring arcs $I^{(n)}$ is at least

$$\frac{1}{2}\frac{l_{n-1}}{\nu_n} = \frac{1}{2}(l_{n-1}\, l_n)^{1/2}. \tag{4.2}$$

The union of all N_n arcs $I^{(n)}$ constitutes the closed subset $E^{(n)}\subset \mathbf{T}$. Define the compact set $\mathbf{E}$ by

$$E = \bigcap\nolimits_{n\geqq 1} E.$$

A standard argument shows that

$$0 < \Lambda_{1/2}(E) \leqq 1. \tag{4.3}$$

For each $I^{(n)}$, by $\zeta(I^{(n)})$ we denote a point at the distance $l_n^{3/4}$ from $I^{(n)}$ and consider the corresponding point measure of magnitude $l_n^{5/8}$. Let μ_n be the sum of all these point measures. Clearly,

$$\int d\mu_n = l_n^{1/8}.$$

Define the probability measure μ by

$$\mu = C \sum \mu_n$$

with an appropriate constant C. As in Section 3.1, let u denote the Poisson integral of the measure μ,

$$w = 2(1+u+i\tilde{u})^{-1} \tag{4.4}$$

$$f(z) = z \exp\left\{\int_0^z \frac{w(z)-1}{z}\, dz\right\}. \tag{4.5}$$

Then f maps $\mathbf{D}$ onto a starlike Jordan domain Ω.

Theorem. *The conformal mapping f and the set E constructed above satisfy the conditions:*

$$H_{1/2} \leqslant D^f_{4/9}; \tag{4.6}$$

$$\forall \zeta \in E\colon \liminf_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^{1/6}} < \infty. \tag{4.7}$$

Remarks. 1) From (4.3) and the inequality

$$\frac{4}{9} > \frac{1/2}{1+1/6} = \frac{3}{7},$$

we obtain (4.1) for $p=1/2$. Observing that, by Theorem 0.7, (4.6) implies that

$$\forall q < 4/9\colon h^f_{1/2} \leqslant H_q, \tag{4.8}$$

we also obtain Theorem 0.8 for $p=1/2$. Similar arguments are applicable for any $p<1$. We omit the details.

2) In addition to (4.8), we will see that

$$\forall q > \frac{4}{9}\colon h^f_{1/2} \leqslant H_q. \tag{4.9}$$

4.2. *Proof of Theorem.* Let $\zeta \in I^{(n)}$ and $r=1-l_n^{3/4}$. Then

$$u(r\zeta) \geqq c(1-r)^{-1}\mu_n\{\zeta(I^{(n)})\} = c(1-r)^{-1/6}.$$

Consequently,

$$|f'(r\zeta)| \leqq C(1-r)^{1/6},$$

and (4.7) follows.

The rest of the proof is devoted to the verification of (4.6). We need a lemma.

Lemma. *Let I be a subarc of* $\mathbf{T}$. *Then*

$$H_{1/2}(E\cap I) \leqq \max\{|I|^{8/9}, C|f'(a_I)|^4\}$$

where C does not depend on I.

Proof of Lemma. Let $H_{1/2}(E\cap I)=I^{\alpha}$ with $\alpha\in[\frac{1}{2}, \frac{8}{9}]$. It is sufficient to check the inequality

$$|(u+i\tilde{u})(a_I)| \leqq C|I|^{-\alpha/4}.$$

Choose n so that $l_n\leqq|I|<l_{n-1}$. We consider two cases.

If $|I|<\frac{1}{2}(l_{n-1}l_n)^{1/2}$, then, by (4.2), $H_{1/2}(E\cap I)\leqq H_{1/2}(I^{(n)})=l_n^{1/2}$, and

$$|I| \leqq l_n^{1/2\alpha}.$$

We estimate

$$|(u+i\tilde{u})(a_I)| \leqq C\sum_{k\geqq 0}\int\frac{d\mu_k(\zeta)}{|\zeta-a_I|}.$$

Observe that

$$\sum_{k<n} \leqq C\sum_{k<n} l_k^{((1/8)-(3/4))} \leqq C|I|^{-\alpha/4},$$

$$\sum_{k>n} \leqq \frac{1}{|I|}\sum_{k>n}\|\mu_k\| \leqq l_n^{-1}\sum_{k>n} l_k^{1/8} \leqq C.$$

Finally, by (4.2)

$$\int\frac{d\mu_n(\zeta)}{|\zeta-a_I|} \leqq Cl_n^{5/8}\left[l_n^{-3/4}+(l_{n-1}l_n)^{-1/2}\left(1+\frac{1}{2}+\ldots+\frac{1}{N_n}\right)\right]$$
$$\leqq Cl_n^{-1/8} \leqq C|I|^{-\alpha/4}.$$

If $|I|>\frac{1}{2}(l_{n-1}l_n)^{1/2}$ then, by (4.2), I meets at most $1+2|I|(l_{n-1}l_n)^{-1/2}$ intervals $I^{(n)}$. Hence

$$H_{1/2}(E\cap I) \leqq C|I|(l_{n-1}l_n)^{-1/2}l_n^{1/2} = C|I|\,l_{n-1}^{-1/2}$$

and

$$|I| \geqq cl_{n-1}^{1/2(1-\alpha)}.$$

This case is further analysed along similar lines. □

Corollary. *If I is a subarc of* $\mathbf{T}$ *then*

$$H_{1/2}(E\cap I) \leqq C|I|\,|f'(a_I)|^{4/9}. \tag{4.10}$$

Proof of Corollary. If $H_{1/2}(E\cap I)\leqq|I|^{8/9}$ then

$$[H_{1/2}(E\cap I)]^{9/4} \leqq |I|^2 \leqq |I|\,|f'(a_I)|.$$

If $H_{1/2}(E\cap I)\leqq C|f'(a_I)|^4$ then

$$[H_{1/2}(E\cap I)]^{9/4} \leqq |I|[H_{1/2}(E\cap I)]^{1/4} \leqq C|I|\,|f'(a_I)|. \quad □$$

Proof of (4.6). Let $F \subset \mathbf{T}$. Since the derivative f' has nonzero radial limits on the complement to the support of the measure μ and since the set $\operatorname{supp} \mu \setminus E$ is countable,

$$H_{1/2}(F \setminus E) > 0 \Rightarrow D_{1/2}(F \setminus E) > 0;$$

see the proof of Theorem 0.5 in Section 2.6. Therefore, for the validity of (4.6) it is sufficient to show that

$$F \subset E, \quad D_{4/9}(F) = 0 \Rightarrow H_{1/2}(F) = 0.$$

If $D_{4/9}(F)=0$ then, for any $\varepsilon>0$, there exists a covering of F with intervals I_ν such that

$$\sum [|I_\nu| \, |f'(a_\nu)|]^{4/9} < \varepsilon.$$

By (4.10) we have

$$H_{1/2}(F) \leqq \sum [H_{1/2}(E \cap I_\nu)] < C\varepsilon.$$

This concludes the proof of the theorem. □

Proof of (4.9). Observe that

$$u(a_{I^{(n)}}) \geqq c l_n^{((5/8)-(3/4))} = c l_n^{-1/8},$$

which implies

$$|f'(a_{I^{(n)}})| \leqq C l_n^{1/8}.$$

If $q > \frac{4}{9}$, then

$$\sum [|I^{(n)}| \, |f'(a_{I^{(n)}})|]^p \leqq N_n l_n^{9/8q} = l_n^{(9/8q-1/2)} \to 0 \quad \text{as} \quad n \to \infty,$$

the summation being carried out over all intervals $I^{(n)}$. Since

$$\sum |I^{(n)}|^{1/2} = 1,$$

the infimum in (2.15) is zero for $p=\frac{1}{2}$ and $C=1$. Now (4.9) follows from Proposition 2.5. □

4.3. *Second example.* We fix

(4.11) $$p = (\log 9)^{-1} \log 2$$

until the end ofthe section. Let E_0 be the standard ternary Cantor set on $\mathbf{T}$. In the construction of E_0, 2^n closed intervals $I_0^{(n)}$ of length 3^{-n} arise at the n-th step. Let $I^{(n)}$ denote the interval of length 9^{-n} concentric with $I_0^{(n)}$, and let $E^{(n)}$ be the union of all such $I^{(n)}$. It is easy to see that for all n

(4.12) $$H_p(E^{(n)}) \geqq c > 0.$$

Let μ_n be the measure supported on $E^{(n)}$ with constant density such that $\int d\mu_n = n^{-2}$, and let $\mu = C \sum \mu_n$ be a probability measure. We shall consider the conformal mapping f defined by the formulae (4.4) and (4.5).

Theorem. 1) *If* $\alpha > 1-(\log 3)^{-1}\log 2$ *then*

$$\liminf_{r\to 1-}\frac{|f'(r\zeta)|}{(1-r)^\alpha} > 0, \quad \zeta\in\mathbf{T}. \tag{4.13}$$

2) *If* $q > \left(\log\frac{81}{2}\right)^{-1}\log 2$, *then*

$$D_q^f(E^{(n)}) \to 0, \quad n\to\infty. \tag{4.14}$$

Corollary. *If*

$$\left(\log\frac{81}{2}\right)^{-1}\log 2 < q < \left(\log\frac{81}{4}\right)^{-1}\log 2$$

then

$$h_p^f \ll H_q, \quad h_p^f \not\prec H_q.$$

Proof of Corollary. The absolute continuity follows from Theorem 0.5. The lack of strong absolute continuity follows from Theorem 0.9 and the fact that (4.12) and (4.14) imply $H_p \not\prec D_q^f$. □

Remark. One can easily obtain similar results for any $p<1$. We omit the details.

Proof of Theorem. 1) Obviously, the inequality (4.13) requires a proof only for ζ in E_0. Let $\zeta\in E_0$ and

$$1-r=3^{-k}.$$

We estimate

$$|(u+i\tilde{u})(r\zeta)| \leqq C\sum_{n\geq 1}\int\frac{d\mu_n(\eta)}{|\eta-r\zeta|}.$$

To this end we show that

$$\int\frac{d\mu_n(\eta)}{|\eta-r\zeta|} \leqq Cn^{-2}k(3/2)^k. \tag{4.15}$$

Then

$$|(u+i\tilde{u})(r\zeta)| \leqq C(1-r)^{-\alpha}$$

for $\alpha > 1-\dfrac{\log 2}{\log 3}$ and (4.13) follows.

To prove (4.15) consider two cases.

Suppose that $n\leqq k$. Taking into account that $\operatorname{dist}(\zeta, \operatorname{supp}\mu_n) \geqq 3^{-(n+1)}$ and that the distance between any two intervals $I^{(n)}$ is at least $3^{-(n+1)}$, we have

$$\int\frac{d\mu_n(\eta)}{|\eta-r\zeta|} \leqq Cn^{-2}2^{-n}3^{n+1}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^n}\right) \leqq Cn^{-2}n(3/2)^n \leqq Cn^2k(3/2)^k$$

Suppose that $n>k$. If I is an interval of length 3^{-k}, then it meets at most two intervals $I_0^{(k)}$. Therefore

$$\mu_n(I) \leqq n^{-2}2^{-k}.$$

Hence

$$\int \frac{d\mu_n(\eta)}{|\eta - r\zeta|} \leqq n^{-2}2^{-k}3^k\left(1+\frac{1}{2}+\ldots+\frac{1}{3^k}\right) \leqq Cn^{-2}k(3/2)^k.$$

2) The set $E^{(n)}$ is covered by 2^n intervals $I^{(n)}$. Observe that

$$u(a_{I^{(n)}}) \geqq n^{-2}2^{-n}3^{2n}.$$

Therefore

$$D_q(E^{(n)}) \leqq 2^n[9^{-n}n^2 2^n 9^{-n}]^q$$

and (4.14) follows. □

5. Boundary distortion in dimensions close to one

In this section we prove Theorem 0.4 which asserts, in particular, that

$$p - d(p) \asymp (1-p)^{1/2} \quad \text{as} \quad p \to 1-.$$

First we derive the lower estimate of $d(p)$.

5.1. Lemma. *Let γ, σ and α be positive numbers. If a univalent function f satisfies the condition*

(5.1) $$\int |f'(r\zeta)|^{-\gamma}|d\zeta| = O((1-r)^{-\sigma}), \quad r \to 1-,$$

then

(5.2) $$\dim\left\{\zeta \in \mathbf{T}: \liminf_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} < \infty\right\} \leqq \max\{0, 1+\sigma-\alpha\gamma\}.$$

Proof. For any $M>0$ and $\nu \in \mathbf{N}$ we define

$$E_\nu(M) = \{\zeta \in \mathbf{T}: |f'((1-e^{-\nu})\zeta)| < Me^{-\alpha\nu}\};$$

$$E(M) = \left\{\zeta \in \mathbf{T}: \liminf_{r\to 1-}\frac{|f'(r\zeta)|}{(1-r)^\alpha} < M\right\}.$$

By the distortion theorem

(5.3) $$E(2M) \subset \bigcap_{n\geqq 1}\bigcup_{\nu \geqq n} E_\nu(M).$$

We have to estimate the Hausdorff dimension of the set

$$E = \bigcup_{M>0} E(M).$$

Assume that $p>0$ and $H_p(E)>0$. Then

$$H_p(E(2M)) > 0$$

for some $M>0$. Fixing this value of M we will write, for brevity, E_ν instead of $E_\nu(M)$. By (5.3),

$$\sum_{\nu\geqq 1} H_p(E_\nu) = \infty.$$

Hence there exists an increasing sequence $\{\nu_j\}$ of positive integers such that for any j,

$$H_p(E_{\nu_j}) \geqq [\nu_j]^{-2}.$$

Now we fix $\nu=\nu_j$. By G we denote the $e^{-\nu}$-neighbourhood of E_ν. The set G consists of disjoint open intervals of length at least $e^{-\nu}$. Subdivide G into a union of N disjoint intervals, not necessarily open, of length between $\frac{1}{2}e^{-\nu}$ and $e^{-\nu}$. Then

$$Ne^{-\nu p} \geqq H_p(G) \geqq H_p(E_\nu) > \nu^{-2}$$

and

(5.4) $$|G| \geqq \frac{1}{2}e^{-\nu}N \geqq \frac{1}{2}e^{-\nu}e^{p\nu}\nu^{-2}.$$

Observe now that if $r=1-e^{-\nu}$ and $\zeta\in G$, then

$$|f'(r\zeta)| \leqq CMe^{-\nu\alpha}.$$

Combined with (5.1) and (5.4), this yields

$$Ce^{\sigma\nu} \geqq \int_G |f'(r\zeta)|^{-\gamma}|d\zeta| \geqq ce^{-\nu}e^{p\nu}\nu^{-2}e^{\alpha\gamma\nu}.$$

For large ν, the last estimate is possible only if

$$\sigma \geqq -1+p+\alpha\gamma,$$

so (5.2) follows. □

The application of the lemma is based on a recent result due to Ch. Pommerenke [31] which improves an earlier estimate of J. Clunie and Ch. Pommerenke [11].

Pommerenke Theorem. *Let f be a function univalent in* $\mathbf{D}$ *and* $\lambda\in\mathbf{R}$. *Then*

$$\int |f'(r\zeta)|^{\lambda}|d\zeta| = o\big((1-r)^{-\sigma}\big), \quad r\to 1-,$$

for any σ satisfying

(5.5) $$\sigma > -\tfrac{1}{2}+\lambda+\big(\tfrac{1}{2}-\lambda+4\lambda^2\big)^{1/2}.$$

Notice that the expression on the right is $\sim 3\lambda^2$ as $\lambda\to 0$ and it does not exceed $3\lambda^2$ provided $\lambda<0$. Thus, for any $\gamma>0$,

(5.6) $$\int |f'(r\zeta)|^{-\gamma}|d\zeta| = o\big((1-r)^{-3\gamma^2}\big).$$

Corollary 1. *If f is a univalent function and $0<\alpha\leqq 1$ then*

$$\dim\left\{\zeta\in\mathbf{T}\colon \liminf_{r\to 1-}\frac{|f'(r\zeta)|}{(1-r)^\alpha}<\infty\right\} \leqq 1-\frac{\alpha^2}{12}.$$

Proof. By (5.6) and the established lemma, for any $\gamma>0$ we have

$$\dim\{\dots\} \leqq 1+3\gamma^2-\alpha\gamma.$$

The minimum of the right-hand side is obtained by the choice $\gamma=\frac{1}{6}\alpha$ and is equal to $1-\frac{1}{12}\alpha^2$. □

Corollary 2.

$$d(p) \geqq \frac{p}{1+\sqrt{12}(1-p)^{1/2}}.$$

Proof. By the previous corollary we have

$$\liminf_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^{\alpha}} > 0 \quad \text{for } \Lambda_p\text{-a.e. } \zeta\in\mathbf{T}$$

for any $p\in(0, 1)$ and any $\alpha>\sqrt{12}(1-p)^{1/2}$. It remains to apply Theorem 0.5. □

Remark 1. By Corollary 2, for any $p\in(0, 1)$,

$$p-d(p) \leqq \frac{\sqrt{12}\, p(1-p)^{1/2}}{1+\sqrt{12}\,(1-p)^{1/2}} \leqq \sqrt{12}\,(1-p)^{1/2}.$$

This yields the right-hand inequality in Theorem 0.4.

Remark 2. Also by Corollary 2 we obtain the inequality $d(p)>\frac{p}{2}$ for

$$p\in(\frac{11}{12}, 1].$$

This result could easily be improved to

$$p\in(\frac{3}{4}, 1]$$

by referring to (5.5) in place of (5.6). Moreover, by a similar argument one can obtain the bound

$$p\in(0.601, 1]$$

if one applies the second theorem of Ch. Pommerenke in [31]: if f is a univalent function then

$$\int |f'(r\zeta)|^{-1}|d\zeta| = o((1-r)^{-0.601}), \quad r\to 1-.$$

On the other hand, I cannot extend this method to prove $d(p)>p/2$ for all $p\in(0, 1)$. See the next section for a proof based on the method due to L. Carleson.

5.2. Now we turn to the proof of the upper estimate of $d(p)$ for p close to one. To obtain an upper bound of $d(p)$ one should provide an example. We construct the corresponding example with the help of the lacunary power series

$$b(z) = \sum_{\nu\geqq 0} z^{2^\nu}, \quad z\in\mathbf{D}, \tag{5.7}$$

which gives rise to the univalent function

$$f(z) = \int_0^z \exp\left\{\frac{i}{5} b(z)\right\} dz, \quad z\in \mathbf{D}, \tag{5.8}$$

mapping $\mathbf{D}$ onto a Jordan domain (see, e.g., [30], Ch. 10, § 2).

Theorem. *Let f be a function defined by* (5.8) *and* (5.7), *and let* $p\in\left(\frac{19}{20}, 1\right)$ *and* $\alpha=\frac{1}{25}(1-p)^{1/2}$. *There exists a subset* $E\subset\mathbf{T}$ *of positive* Λ_p*-measure such that*

$$\lim_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} = 0, \quad \zeta\in E. \tag{5.9}$$

Corollary. *If* $p>\frac{19}{20}$ *then*

$$d(p) \leqq \frac{p}{1+\frac{1}{25}(1-p)^{1/2}}. \tag{5.10}$$

Proof of Corollary. Apply Theorem 0.6. □

Remark. If $p\in\left(\frac{19}{20}, 1\right)$ then, by (5.10),

$$p-d(p) \geqq \frac{p}{1+\frac{1}{25}\sqrt{1-p}}\,\frac{1}{25}(1-p)^{1/2} \geqq \frac{1}{30}(1-p)^{1/2},$$

which proves the left-hand inequality in Theorem 0.4.

For the proof of the theorem, we need a result on the boundary behaviour of the lacunary series (5.7) which was established in [24] as a slight amplification of a theorem due to J. Hawkes [16]. For the sake of completeness the proof is included.

Lemma. *Let* $\delta<\frac{1}{10}$. *There exists a subset* $E\subset\mathbf{T}$ *of Hausdorff dimension greater than* $1-5\delta^2$ *such that*

$$\liminf_{r\to 1-} \frac{\operatorname{Im} b(r\zeta)}{\log_2 \frac{1}{1-r}} > \frac{\delta}{3}, \quad \zeta\in E.$$

Proof of lemma. On the segment [0, 1] we define the functions

$$S_n(t) = \sum_{\nu=1}^n \sin(2^\nu\cdot 2\pi t).$$

It is easy to verify that if $n=[|\log_2 (1-r)|]$ then

$$|\operatorname{Im} b(re^{2\pi ti})-S_n(t)| \leqq C.$$

On [0, 1] we also consider the probability measure μ with respect to which the functions $t\mapsto t_\nu$ (=the ν-th figure in the dyadic expansion of t) are independent random variables with distribution

$$\mu\{t\colon t_\nu = o\} = \tfrac{1}{2}+\delta, \quad \mu\{t\colon t_\nu = 1\} = \tfrac{1}{2}-\delta.$$

The measure μ is invariant under the dyadic transformation

$$T(t) = 2t (\mathrm{mod}\ 1)$$

and ergodic with respect to it (see [5], Example 3.5). By the ergodic theorem, for μ-a.e.

$$\frac{1}{n} S_n(t) = \frac{1}{n} \sum_{\nu=1}^n \sin (2\pi T^\nu t) \to \int_0^1 \sin (2\pi t)\, d\mu(t).$$

By the Eagleston—Billingsley theorem ([5], § 14), the measure μ is absolutely continuous with respect to the Hausdorff measure Λ_α provided that

$$\alpha < \frac{\operatorname{Ent} T}{\log 2},$$

where

$$\operatorname{Ent} T = \left(\tfrac{1}{2}+\delta\right)\left|\log \left(\tfrac{1}{2}+\delta\right)\right| + \left(\tfrac{1}{2}-\delta\right)\left|\log \left(\tfrac{1}{2}-\delta\right)\right|$$

is the entropy of T. It remains only to note that

$$1-\frac{\operatorname{Ent} T}{\log 2} < 5\delta^2$$

provided $\delta<\frac{1}{10}$, and that

$$\int_0^1 \sin (2\pi t)\, d\mu(t) = 2\delta \int_0^{1/2} \sin (2\pi t)\, d\mu(t)$$

$$> 2\delta \frac{\sqrt{2}}{2} \mu\left[\frac{1}{8}, \frac{3}{8}\right] = \frac{\sqrt{2}\,\delta}{4} (1+2\delta)^2(1-2\delta) > \frac{\delta}{3}. \quad \square$$

Proof of theorem. Fix $p\in\left(\frac{19}{20}, 1\right)$ and define $\delta=\frac{1}{\sqrt{5}}(1-p)^{1/2}$. Since $\delta<\frac{1}{10}$ we can apply the lemma. Because of the identity

$$|f'(z)| = \exp\left\{-\frac{1}{5} \operatorname{Im} b(z)\right\},$$

we have

$$|f'(r\zeta)| \leqq (1-r)^{\delta/(15\log 2)}$$

for all ζ in the set E obtained in the lemma and all r sufficiently close to one. Hence (5.8) is valid for all $\alpha<\delta(15\log 2)^{-1}$, in particular, for

$$\alpha = \frac{1}{25}(1-p)^{1/2}. \quad \square$$

6. A version of a theorem by Carleson

This section is devoted to the proof of the inequality

$$d(p) > \frac{p}{2}, \quad p\in(0, 1]. \tag{6.1}$$

In fact, we shall establish two stronger results, any of which implies (6.1).

6.1. Theorem. *For any* $p\in(0, 1]$ *there is a number* $q>\frac{p}{2}$ *such that if* f *is a conformal mapping of the unit disc onto a Jordan domain, then*

$$h_p^f \prec H_q. \tag{6.2}$$

6.2. Theorem. *For any* $p\in(0, 1]$ *there is a number* $\alpha<1$ *such that if* f *is a conformal mapping of the unit disc onto a Jordan domain, then the inequality*

$$\liminf_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} > 0 \tag{6.3}$$

holds for all $\zeta\in\mathbf{T}$ *outside an exceptional set of* Λ_p*-measure zero.*

Both assertions are consequences of the following basic result essentially due to Lennart Carleson.

6.3. Theorem. *Let* $p\in(0, 1]$. *For any* $\varepsilon>0$ *there is a* $\delta>0$ *satisfying the following. If* f *is a conformal mapping onto a Jordan domain, then there exists a positive number* $r_0=r_0(\varepsilon, f)$ *such that for any* r, $0<r<r_0$, *the maximal number of discs* Δ *of radius less than* r *with*

$$h_p^f(\Delta) \geqq [H_p(\Delta)]^{1/2+\delta}$$

and with centers separated by $2r$ *does not exceed* $r^{-\varepsilon}$.

First we derive Theorems 6.1 and 6.2 from the latter result.

6.4. *Proof of Theorem 6.1.* Fix $p\in(0,1]$. Define $\varepsilon=p/4$ and let δ be a corresponding number in Theorem 6.3. We shall establish (6.2) with

$$q=\frac{p}{2}(1+\delta).$$

Let $E\subset\mathbf{T}$ and $H_q(fE)<\eta$ where η is a sufficiently small number. There exists a covering of fE by discs Δ_j of radii r_j satisfying

$$\sum r_j^q<\eta, \tag{6.4}$$

$$\max_j r_j<2^{-N}<r_0=r_0(\varepsilon,f),$$

where $N\in\mathrm{N}$ and $N=N(\eta)\to\infty$ as $\eta\to 0$. Introduce the notation:

$$J_0=\{j\colon h_p(\Delta_j)<r_j^q\},$$

$$J(r)=\{j\colon h_p(\Delta_j)\geqq r_j^q,\frac{r}{2}<r_j\leqq r\},\quad r>0.$$

Then

$$H_p(E)\leqq \sum_{j\in J_0}h_p(\Delta_j)+\sum_{k\geqq N}h_p(\bigcup_{j\in J(2^{-k})}\Delta_j). \tag{6.5}$$

By (6.4), the first sum does not exceed η. To estimate the second, observe that if $0<r<r_0$ then

$$h_p(\bigcup_{j\in J(r)}\Delta_j)\leqq Cr^{p/4}.$$

To this end, we apply the covering lemma to the collection of discs

$$\{2\Delta_j\colon j\in J(r)\}$$

and choose m disjoint discs $2\Delta_{(\nu)}$, $1\leqq\nu\leqq m$, such that

$$\bigcup_{j\in J(r)}\Delta_j\subset\bigcup_{\nu=1}^m 10\Delta_{(\nu)}.$$

(For a disc Δ and $k>0$, by $k\Delta$ we denote the concentric disc of radius k times the radius of Δ.) Since the centers of $\Delta_{(\nu)}$ are separated by $2r$ and

$$h_p(\Delta_{(\nu)})\geqq 2^{-q}r^q\geqq r^{p(1/2+\delta)},$$

we have by Theorem 6.3

$$m\leqq r^{-p/4}.$$

Hence

$$h_p(\bigcup_{j\in J(r)}\Delta_j)\leqq\sum_{\nu=1}^m h_p(10\Delta_{(\nu)})\leqq Cr^{p/2}r^{-p/4}=Cr^{p/4}$$

(in the last inequality we have applied (1.5) and (0.6)). Returning to (6.5), we have

$$H_p(E)\leqq\eta+C\sum_{k\geqq N}(2^{-k})^{p/4}\to 0\quad\text{as}\quad\eta\to 0.\quad\square$$

Remark. Our deduction of Theorem 6.1 from Theorem 6.3 coincides essentially with the corresponding part of Carleson's proof ([9], § 8).

6.5. *Proof of Theorem 6.2.* Let $p\in(0, 1]$, $\varepsilon=\frac{1}{4}p$, and δ be the corresponding number in Theorem 6.3. The assertion will be proved with

$$\alpha = \frac{1-2\delta}{1+2\delta}.$$

Assume that there is a compact subset $E\subset\mathbf{T}$ of positive Λ_p-measure such that (6.3) is false everywhere on E. For an arbitrary $N\in\mathbf{N}$ we carry out the following construction. Let $\zeta\in E$. Then, by assumption, there exists an interval I_ζ with center at ζ such that

$$|I_\zeta| \leqq 2^{-N}, \quad |f'(a_{I_\zeta})| \leqq |I_\zeta|^\alpha.$$

Applying the covering lemma, we choose a finite subcovering $\{I_{\zeta_j}\}$ of E of multiplicity at most two. In the sequel we write I_j instead of I_{ζ_j}. By Proposition 1.3, applied with a sufficiently large R, for any j there is a closed subset $F_j\subset I_j$ satisfying

$$H_p(F_j) \geqq \tfrac{1}{2}|I_j|^p, \tag{6.6}$$

$$fF_j \subset \Delta_j \overset{\text{def}}{=} \Delta(f(a_j), r_j),$$

where

$$r_j = R|I_j||f'(a_j)| < R|I_j|^{1+\alpha}. \tag{6.7}$$

Define

$$F^{(N)} = \bigcup F_j.$$

As in the proof of Theorem 2.4, (6.6) implies

$$H_p(F^{(N)}) > cH_p(E).$$

On the other hand the inclusion $fF^{(N)}\subset \cup\Delta_j$ implies the inequality

$$H_p(F^{(N)}) \leqq \textstyle\sum_{k\geqq N} h_p\left(\bigcup_{2^{-(k+1)}\leqq r_j\leqq 2^{-k}}\Delta_j\right). \tag{6.8}$$

From (6.6) and (6.7) we have successively

$$h_p(\Delta_j) \geqq H_p(F_j) \geqq \tfrac{1}{2}|I_j|^p \geqq \tfrac{1}{2}(R^{-1}r_j)^{p/1+\alpha} \geqq r_j^q$$

where

$$q = \frac{p}{2}\left(\frac{1}{2}+\delta\right).$$

Similarly to the proof of Theorem 6.1, the latter implies that the expression on the right in (6.8) tends to zero as $N\to\infty$. This contradicts the assumption $H_p(E)>0$.

Remark. The idea to apply Carleson's technique to questions concerning the derivative of a conformal mapping is due to J. Brennan [7].

6.6. *The beginning of the proof of Theorem 6.3.* Fix $p\in(0,1]$ and $\varepsilon>0$. Let A be the least positive number greater than $20\,\varepsilon^{-1}$ with $\exp A\in\mathbf{N}$. We shall verify the assertion for

$$\delta = \tau A^{-2} \tag{6.9}$$

where $\tau=\tau(A)$ is the number appearing in the following lemma borrowed from Carleson's proof ([9], § 6).

Lemma. *For any $A>0$ there is $\tau>0$ satisfying the following. Let U be a Jordan domain with two distinguished boundary arcs lying on different components of $\partial R(0; 1, e^A)$. Suppose that*

$$\lambda(\Gamma) < (2\pi)^{-1}A+\tau$$

where Γ is the family of all curves joining these arcs in U. Then U contains a sector of the annulus $R(0; 1, e^A)$ with central angle greater than $7\pi/4$.

Carleson's elegant proof is based on a normal families argument. We shall not reproduce it here.

Now we fix a conformal mapping f. For convenience, we assume that f maps $\mathbf{D}$ onto the exterior of a Jordan domain. Denote $\Omega=f(\mathbf{D})$. Without loss of generality, we may also assume that $f(0)=\infty$ and

$$\operatorname{diam}\partial\Omega = \frac{1}{4}. \tag{6.10}$$

Fix a number $r\leqq r_0$ with r_0 small. In the course of the proof we shall obtain a finite number of restrictions on the magnitude of r_0.

Definition. Let $z_0\in\mathbf{C}$ and $\nu\in\mathbf{N}$. The pair (z_0, ν) is said to be exceptional if Ω contains no sector of the annulus $R(z_0; re^{\nu A}, re^{(\nu+1)A})$ of apperture $7\pi/4$. If $z_0\in\mathbf{C}$, $n(z_0)$ denotes the number of ν's for which the pair (z_0, ν) is exceptional.

6.7. Lemma. *If*

$$h_p^f(\Delta(z_0, r)) \geqq r^{p(1/2+\delta)} \tag{6.11}$$

then

$$n(z_0) \leqq A^{-2}\log\frac{1}{r}.$$

Proof. By (6.10) we can assume that

$$\partial\Omega \subset \Delta\left(0, \tfrac{1}{2}\right).$$

Suppose that (6.11) is valid. We apply Corollary to Theorem 1.4 with $K=\mathbf{T}$, $q=\frac{1}{2}+\delta$ and $q'=\frac{1}{2}+2\delta$. If r_0 is small, then there exists a subarc σ_0 of $\partial\Delta(z_0, 2r)$ which is a crosscut of Ω and satisfies

$$\lambda(\Gamma) \leqq \pi^{-1}\left(\frac{1}{2}+2\delta\right)\log\frac{1}{r} \tag{6.12}$$

where Γ is the family of all curves in Ω joining $\mathbf{T}$ with σ_0. Let

$$N = \left[\frac{1}{A}\log\frac{1}{r}\right]. \tag{6.13}$$

On each circle $\partial\Delta(z_0, re^{\nu A})$, $\nu=1, \ldots, N$, we choose a subarc σ_ν, which is a crosscut of Ω, in such a way that for $\nu=0, \ldots, N-1$, $\sigma_{\nu+1}$ separates σ_ν from infinity. Let Γ_ν denote the family of all curves in Ω joining σ_ν with $\sigma_{\nu+1}$. By properties of extremal length,

$$\lambda(\Gamma) \geqq \sum \lambda(\Gamma_\nu). \tag{6.14}$$

For every ν we have the trivial estimate

$$\lambda(\Gamma_\nu) \geqq (2\pi)^{-1}A,$$

whereas, by Lemma in Section 6.6,

$$\lambda(\Gamma_\nu) \geqq (2\pi)^{-1}A+\tau$$

provided that (z_0, ν) is an exceptional pair. Taking these estimates into account, from (6.12), (6.13) and (6.14), we have

$$\frac{1}{\pi}\left(\frac{1}{2}+2\delta\right)\log\frac{1}{r} \geqq [N-n(z_0)]\frac{A}{2\pi}+n(z_0)\left(\frac{A}{2\pi}+\tau\right)$$

$$= \frac{NA}{2\pi}+n(z_0)\tau \geqq \frac{1}{2\pi}\log\frac{1}{r}+n(z_0)\tau.$$

Hence

$$\delta\log\frac{1}{r} \geqq n(z_0)\tau,$$

and by (6.8)

$$n(z_0) \leqq \frac{\delta}{\tau}\log\frac{1}{r} = A^{-2}\log\frac{1}{r}. \quad \square$$

6.8. *End of the proof of Theorem 6.3.* The argument used at the same stage in [9] (see § 10) applies verbatim. We shall therefore only supply few additional elucidations.

Suppose there are m discs $\Delta_j=\Delta(z_j, r_j)$, $1\leqq j\leqq m$, satisfying $r_j\leqq r\leqq r_0$,

(6.15) $$|z_j-z_k|\geqq 2r \quad (j\neq k), \quad h_p(\Delta_j)\geqq r_j^{p(1/2+\delta)}.$$

We have to prove that

(6.16) $$m\leqq r^{-\varepsilon}.$$

Without loss of generality we can assume that

(6.17) $$z_j\in\partial\Omega, \quad j=1, \dots, m;$$

and A is large enough to satisfy the following geometrical condition.

Let a, b, z be three points in a sector of the annulus $\bar{R}(0; 1, e^A)$ with central angle $\frac{\pi}{4}$ and $|a|=1$, $|b|=e^A$. Then the angle $\sphericalangle(azb)$ is greater than $\frac{\pi}{4}$. (See Figure 3.)

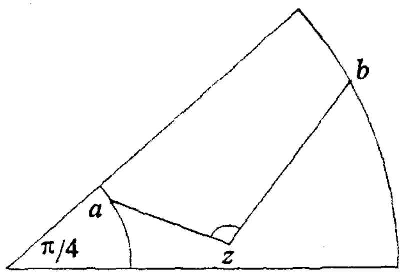

Fig. 3

Lemma 1. *Let*

(6.18) $$2\leqq\frac{|z_j-z_k|}{re^{\nu A}}\leqq e^A-1$$

and

(6.19) $$re^{(\nu+1)A}<\tfrac{1}{2}\operatorname{diam}\partial\Omega.$$

Then at least one of the two pairs (z_j, ν) *and* (z_k, ν) *is exceptional.*

Proof. Suppose that the pair (z_j, ν) is not exceptional. By hypothesis, $\partial\Omega$ meets both components of $\mathbf{C}\setminus R$, $R=R(z_j; e^{\nu A}, e^{(\nu+1)A})$ so the curve $\partial\Omega$ should pass inside a sector of R of apperture $\frac{\pi}{4}$. Let a and b be two points of $\partial\Omega$ lying on different circumferences of ∂R. By (6.17) and (6.18), z_k lies in the sector so we have

$$\sphericalangle(az_kb)>\frac{\pi}{4}.$$

Hence the pair (z_k, ν) is exceptional. $\square$

For $\nu=0, 1, \dots, N$,

$$N = 1+\left[A^{-1}\log\frac{1}{r}\right],$$

let $\mathcal{G}_\nu$ denote the lattice of all squares Q_ν of side $s_\nu=\frac{1}{2}re^{\nu A}$ such that the coordinates of the vertices are multiples of s_ν. Since e^A is an integer, $\mathcal{G}_{\nu+1}$ is a sublattice of $\mathcal{G}_\nu$. We will assume that no point z_j lies on the boundary of a square $Q_0\in\mathcal{G}_0$. By (6.15) each Q_0 contains at most one point z_j. By (6.10) we can also assume that all z_j lie in a single square $Q_N\in\mathcal{G}_N$.

Given $Q_{\nu+1}\in\mathcal{G}_{\nu+1}$ we define a set $S(Q_{\nu+1})$ consisting of some squares $Q_\nu\subset Q_{\nu+1}$ as follows

Definition. If, for all $z_j\in Q_{\nu+1}$, the pair (z_j, ν) is exceptional, we put $S(Q_{\nu+1})=\emptyset$. Otherwise,

$$S(Q_{\nu+1}) = \{Q_\nu\in\mathcal{G}_\nu\colon Q_\nu\subset Q_{\nu+1} \text{ and } Q_\nu\cap\Delta(z_j, 2re^{\nu A})\neq\emptyset \text{ for any } z_j\in Q_{\nu+1}$$

such that (z_j, ν) is non-exceptional$\}$.

Since a disc of radius $2re^{\nu A}=4s_\nu$ meets at most 100 squares of $\mathcal{G}_\nu$, we have

$$\text{card}\, S(Q_{\nu+1}) \leqq 100. \tag{6.20}$$

Lemma 2. *Let $\nu<N-1$ and $z_k\in Q_\nu\subset Q_{\nu+1}$. If the pair (z_k, ν) is non-exceptional, then*

$$Q_\nu \subset S(Q_{\nu+1}).$$

Proof. Let $z_j\in Q_{\nu+1}$ and let the pair (z_j, ν) be non-exceptional. We should verify that

$$Q_\nu\cap\Delta(z_j, 2re^{\nu A}) \neq \emptyset. \tag{6.21}$$

Since (6.10) and the inequality $\nu<N-1$ imply (6.19), we can make use of Lemma 1. We thus have either

$$|z_j-z_k| < 2re^{\nu A},$$

or

$$|z_j-z_k| > re^{(\nu+1)A}-re^{\nu A}.$$

The first inequality implies (6.21) while the second is ruled out because z_j and z_k lie in the same square $Q_{\nu+1}$. □

Each point z_j defines (and is determined by) a sequence of squares

$$Q_0\subset Q_1\subset\dots\subset Q_N \tag{6.22}$$

with $z_j\in Q_\nu$. According to Lemma 2

$$Q_\nu \subset S(Q_{\nu+1})$$

for all $\nu < N-1$ for which the pair (z_j, ν) is non-exceptional. By Lemma 6.7 the number of exceptional pairs does not exceed

$$A^{-2}\log\frac{1}{r} \leqq \frac{N}{A}.$$

Thus m does not exceed the number of all posible sequences (6.22) with $Q_\nu \subset S(Q_{\nu+1})$ except for at most $\left(\frac{N}{A}+1\right)$ indices ν. Therefore by (6.20),

$$m \leqq \sum_{k=0}^{[A^{-1}N]+1} \binom{N}{k} e^{2Ak} 100^{N-k} \leqq \exp\left\{2A\left(\frac{N}{A}+1\right)\right\} \sum_{k=0}^{N} \binom{N}{k} 100^{N-k}$$
$$= \exp\{2N+N\log 101+2A\} \leqq \left(\frac{1}{r}\right)^{20A^{-1}}$$

provided that $r \leqq r_0$ and r_0 is sufficiently small. By the choice of A, we have (6.16). □

7. Concluding remarks

7.1. *Radial growth of f' and $1/f'$.* Most of the results on boundary distortion obtained in this paper are consequences of the corresponding results on the radial growth of the reciprocal of the derivative of a univalent function. Now we wish to list the latter results explicitly and briefly discuss their counterparts for the derivative itself. The problem is to estimate the maximal dimension of the set on which the order of growth of the derivative (or its reciprocal) is greater than the given one. This problem admits different versions. As variable we prefer to choose the dimension of the exceptional set. For $p \in (0, 1]$ we define the following quantities. ((S) denotes the usual class of univalent functions.)

$$\alpha(p) = \sup\Big\{\alpha \geqq 0\colon \exists f \in (S)\ \exists E \subset \mathbf{T} \text{ such that } H_p(E) > 0 \text{ and } \liminf_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} = 0 \text{ for } \zeta \in E\Big\};$$

$$\alpha_1(p) = \sup\Big\{\alpha \geqq 0\colon \exists f \in (S)\ \exists E \subset \mathbf{T} \text{ such that } H_p(E) > 0 \text{ and } \lim_{r\to 1-} \frac{|f'(r\zeta)|}{(1-r)^\alpha} = 0 \text{ for } \zeta \in E\Big\};$$

$$\beta(p) = \sup\Big\{\beta \geqq 0\colon \exists f \in (S)\ \exists E \subset \mathbf{T} \text{ such that } H_p(E) > 0 \text{ and } \limsup_{r\to 1-} |f'(r\zeta)|(1-r)^\beta = \infty \text{ for } \zeta \in E\Big\};$$

$$\beta_1(p) = \sup\Big\{\beta \geqq 0;\ \exists f \in (S)\ \exists E \subset \mathbf{T} \text{ such that } H_p(E) > 0 \text{ and } \lim_{r\to 1-} |f'(r\zeta)|(1-r)^\beta = \infty \text{ for } \zeta \in E\Big\}.$$

Obviously, all four functions are decreasing and $\alpha(p) \geqq \alpha_1(p)$, $\beta(p) \geqq \beta_1(p)$. It is probable that they are continuous, strictly decreasing and that $\alpha(p)=\alpha_1(p)$, $\beta(p)=\beta_1(p)$, but I do not know the proof. Some estimates of these functions can be traced in the literature.

The distortion theorem trivially implies that

$$\alpha(p) \leqq 1, \quad \beta(p) \leqq 3.$$

A result of A. Beurling [4] shows that

$$|f'(r\zeta)| = o\left(\frac{1}{1-r}\right)$$

outside possible exceptional sets of logarithmic capacity zero, and hence

$$\beta(p) \leqq 1 \quad \text{for all} \quad p > 0.$$

W. Seidel and J. Walsh [33] proved that

$$\beta(1) \leqq \tfrac{1}{2},$$

and this result has recently been improved up to

$$\beta(1) = 0, \quad \alpha(1) = 0$$

by J. Clunie and T. MacGregor [10] (see also [17] and [22]). The only lower bound I know goes back to A. Lohwater and G. Piranian [19]. They constructed an example of $f \in (S)$ with

$$\varlimsup_{r \to 1-} |f'(r\zeta)|(1-r)^{1/2} = \infty$$

for all ζ in a set of positive capacity, but the proof shows, in fact, that

$$\alpha_1(p) > 0 \quad \text{and} \quad \beta_1(p) > 0 \quad \text{for all} \quad p < 1.$$

The results of the present paper provide estimates of $\alpha(p)$ and $\alpha_1(p)$. Although these results have only been established for conformal mappings onto Jordan domains, they can easily be extended to arbitrary univalent functions. We do not include the details. Remark that $\alpha(p)$ and $\alpha_1(p)$ are connected with the function $d(p)$ through the inequalities

$$\frac{p}{1+\alpha(p)} \leqq d(p) \leqq \frac{p}{1+\alpha_1(p)},$$

cf. Theorems 0.5 and 0.6. Our estimates of $\alpha(p)$ and $\alpha_1(p)$ are the following.

(7.1) $\qquad \alpha(p) \geqq \alpha_1(p) \geqq 1-p \quad$ (see Section 3.1).

(7.2) $\qquad \alpha(p) \asymp \alpha_1(p) \asymp (1-p)^{1/2} \quad \text{as} \quad p \to 1- \quad$ (see Section 5).

In particular

$$\alpha(0+) = \alpha_1(0+) = 1, \tag{7.3}$$

$$\alpha(1-) = \alpha_1(1-) = 0. \tag{7.4}$$

Finally

$$\alpha_1(p) \leqq \alpha(p) < 1 \quad \text{for} \quad p > 0 \quad \text{(Theorem 6.2)}. \tag{7.5}$$

An alternative way to prove (7.5) is to verify that

$$\inf_{\gamma \geqq 0} x(\gamma) = 1, \tag{7.6}$$

where

$$x(\gamma) = \inf \{x \geqq 0 \colon \int |f'(r\zeta)|^{-\gamma} |d\zeta| = O\big((1-)n^{x-\gamma}\big) \quad \text{for any } f \in (S)\}$$

(cf. Lemma 5.1). A stronger conjecture is that $x(2)=1$, cf. [7].

Arguments similar to that used in the paper allow to obtain some estimates of β and β_1:

$$\beta(p) \geqq \beta_1(p) \geqq 1-p, \tag{7.1'}$$

$$\beta(p) \asymp \beta_1(p) \asymp (1-p)^{1/2} \quad \text{as} \quad p \to 1, \tag{7.2'}$$

$$\beta(0+) = \beta_1(0+) = 1, \tag{7.3'}$$

$$\beta(1-) = \beta_1(1-) = 0. \tag{7.4'}$$

As to the counterpart of (7.5),

$$\beta_1(p) \leqq \beta(p) < 1 \quad \text{for} \quad p > 0, \tag{7.5'}$$

it admits a considerable amplification

$$\beta(p) \leqq 1 - \frac{p}{2} \tag{7.7}$$

with a very elementary proof. Together with (7.1) this gives a suitable approximation to $1-\beta(p)$ as $p \to 0+$:

$$1-\beta(p) \asymp 1-\beta_1(p) \asymp p.$$

I do not know whether a similar result is also true for $\alpha(p)$.

Proof of (7.7). With the help of an elementary transformation, any simply-connected domain (other than **C**) can be mapped onto a bounded domain. The composition with an elementary function does not have an influence on the dimension of the set on which the derivative of a univalent function has a given order of growth. Thus we can assume that

$$\iint |f'|^2 \, dm_2 < \infty.$$

Then from the distortion theorem or from the maximal theorem, it follows that

$$\int |f'(r\zeta)|^2 |d\zeta| = O\left(\frac{1}{1-r}\right) \quad \text{as} \quad r \to 1-. \tag{7.8}$$

Repeating the reasoning in the proof of Lemma 5.1, we obtain from (7.8)

$$\dim \{\zeta \in \mathbf{T}\colon \limsup_{r\to 1-} |f'(r\zeta)|(1-r)^\beta > 0\} \leqq 1+1-2\beta = 2(1-\beta).$$

By definition of $\beta(p)$ (7.7) follows. □

7.2. *On dominating subsets.* In [22], [23] the following question on dominating subsets was studied. Let Ω be a simply-connected domain and φ be a measure function. Does there exist a dominating subset Λ of Ω satisfying

$$\sum_{\lambda\in\Lambda} \varphi(\delta_\lambda) < \infty \tag{7.9}$$

where $\delta_\lambda = \operatorname{dist}(\lambda, \partial\Omega)$? See [23] for the background of the problem and [32] for the definition and properties of dominating subsets. Recall that the property of a set to be dominating is a conformal invariant and that $\Lambda \subset \mathbf{D}$ is a dominating subset of the unit disc iff there exists a subset $E \subset \mathbf{T}$ of full Lebesgue measure such that the intervals $I(\lambda)$, $\lambda \in \Lambda$, cover E with infinite multiplicity. (Here, for $\lambda \in \mathbf{D}$, $I(\lambda)$ stands for the interval I such that $\lambda = a_I$.) This characterization of dominating subsets of $\mathbf{D}$ makes obvious the following observation.

Let $f\colon \mathbf{D} \to \Omega$ be a conformal mapping onto Ω and φ be a measure function. There exists a dominating subset Λ of Ω satisfying (7.9) if and only if there exists a subset $E \subset \mathbf{T}$ such that $|E| = 1$ and $D^f_\varphi(E) = 0$.

It was proved in [22], Lemma 3.1 that the existence of a dominating subset satisfying (7.9) implies the singularity of the harmonic measure $\omega = h^f_1$ with respect to the Hausdorff measure Λ_φ and it was asked whether the converse is true. A (positive) answer has only been known for $\varphi(t) = t$ (see [20], Theorem 4).

There exists another version of the problem under consideration. The argument in Section 2.4 implies in fact that

$$\omega \not\ll H_\varphi \Rightarrow H_1 \not\ll D^f_\varphi \tag{7.10}$$

(see also [23], Lemma 2.3). The question is whether the converse of (7.10) is true. By Theorem 2.3 (or by [20], Theorem 4) the answer is "yes" for $\varphi(t) = t$.

The technique applied in the proof of Theorem 2.3 enables us to answer both questions in the affirmative for sufficiently regular measure functions φ. The restriction we impose on φ is certainly unnecessarilly severe and could easily be weakened considerably. At the same time, without any regularity condition, I am still unable to settle the question.

Theorem. *Let f be a conformal mapping of* $\mathbf{D}$ *onto a Jordan domain Ω and φ be a logarithmico-exponential measure function. The following are equivalent.*

1) $\omega \ll H_\varphi$.
2) $H_1 \ll D^f_\varphi$.
3) *For almost all* $\zeta \in \mathbf{T}$,

$$\liminf_{r\to 1-} \frac{|f'(r\zeta)|}{\psi(1-r)} > 0$$

where $\psi = t^{-1}\varphi^{-1}(t)$. *Similarly, the following three statements are equivalent.*

1) ω *is singular with respect to* H_φ.
2) *There exists a dominating subset Λ of Ω with*

$$\sum_{\lambda\in\Lambda} \varphi(\delta_\lambda) < \infty.$$

3) *For almost all* $\zeta \in \mathbf{T}$

$$\liminf_{r\to 1-} \frac{|f'(r\zeta)|}{\psi(1-r)} = 0.$$

We shall confine ourselves to the proof of only the first part of the theorem. First we make some preliminary remarks.

a) The class of logarithmico-exponentional functions (or L-functions) was introduced by G. H. Hardy [15], see also the excellent exposition in [6]. Roughly speaking, a measure function φ is an L-function if it is defined on an interval $(0, t_0)$ by an expression consisting of a finite number of log's and exp's together with some composition and arithmetic operations. By the main property of L-functions, any two of them are comparable, i.e. either $\varphi_1 = o(\varphi_2)$ or $\varphi_2 = o(\varphi_1)$, or $\varphi_1 \sim C\varphi_2$ as $t \to 0+$.

b) The equivalence $2 \Rightarrow 3$ follows from [23], Lemma 2.3. Hence the only assertion still to be proved is $2 \Rightarrow 1$.

c) The proof of $2 \Rightarrow 1$ will rely on certain results from [22], which we now recall. If $\varphi(t) = o(t)$ as $t \to 0$, any Ω admits a dominating subset satisfying (6.8), see [23], Lemma 3.1. In this case, $H_1 \not\ll D^f_\varphi$ and $2 \Rightarrow 1$ is trivial. On the other hand, if

$$t \exp\{|\log t|^{3/4}\} = o(\varphi(t)) \quad \text{as} \quad t \to 0,$$

then by [22], Theorem 1, $\omega \prec H_\varphi$, and $2 \Rightarrow 1$ is trivial again. Thus we can assume in the sequel that

$$ct \leqq \varphi(t) \leqq Ct \exp\{|\log t|^{3/4}\} \tag{7.11}$$

as $t \to 0$.

Proof of Theorem. For the reason pointed above, we shall only prove $2 \Rightarrow 1$ assuming (7.11) being valid. Define the L-function χ by

$$\varphi(t) = t\chi(t).$$

We derive from (7.11) that

$$c \leqq \chi(t) \leqq C \exp\{|\log t|^{3/4}\}.$$

It is easy to see that for L-functions the latter implies

$$\chi\left(\frac{t}{|\log t|}\right) \leqq C\chi(t) \quad \text{as} \quad t \to 0. \tag{7.12}$$

Assume that $\omega \not\prec H_\varphi$, i.e. there is a subset $e_0 \subset \partial\Omega$ with $\omega(e_0) > 0$ and $H_\varphi(e_0) = 0$. For any $\varepsilon > 0$ there exists a covering of e_0 by discs Δ_ν of radii $r_\nu \leqq r_0$ such that

$$\sum \varphi(r_\nu) < \varepsilon.$$

We proceed further as in the proof of Theorem 2.3. For any ν there are subarcs $\sigma_j^{(\nu)}$ of $\partial\Delta'_\nu$, $1 \leqq j \leqq N(\nu)$,

$$N(\nu) \leqq k_0 \log \frac{1}{r_\nu},$$

which are crosscuts of Ω and separate the subarcs $\beta_j^{(\nu)} = f(I_j^{(\nu)})$ of $\partial\Omega$ satisfying

$$\omega\left(\Delta_\nu \setminus \bigcup_{j=1}^{N(\nu)} \beta_j^{(\nu)}\right) \leqq \varphi(r_\nu). \tag{7.13}$$

By Proposition 1.5

$$\sum_{j=1}^{N(\nu)} |I_j^{(\nu)}| |f'(a_j^{(\nu)})| \leqq C \sum_{j=1}^{N(\nu)} \operatorname{diam} \sigma_j^{(\nu)} \leqq Cr_\nu.$$

Since φ is an L-function, (7.12) implies

$$\sum_{j=1}^{N(\nu)} \varphi\left(|I_j^{(\nu)}| |f'(a_j^{(\nu)})|\right) \leqq C\varphi(r_\nu). \tag{7.14}$$

Let e denote the set

$$e_0 \cap \left[\bigcup_\nu \bigcup_{j=1}^{N(\nu)} \beta_j^{(\nu)}\right]$$

and $E = f^{-1}e$. Then by (7.13)

$$\omega(e \setminus e_0) \leqq \varepsilon. \tag{7.15}$$

and

$$|E| > \tfrac{1}{2}\omega(e_0) > 0$$

provided ε is small. On the other hand, by (7.14),

$$D_\varphi^f(E) \leqq \sum_\nu \sum_{j=1}^{N(\nu)} \varphi(|I_j^{(\nu)}||f'(a_j^{(\nu)})|) \leqq C \sum_\nu r_\nu \leqq C\varepsilon. \tag{7.16}$$

Since ε is arbitrary, (7.15) and (7.16) imply that

$$H_1 \nless D_\varphi^f. \quad \square$$

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