A strongly nonlinear parabolic initial boundary value problem

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1. Introduction

Let Ω be a bounded open set in $\mathbf{R}^N$ and let Q_T be the cylinder $\Omega\times(0, T)$ with some given $T>0$. We shall consider the following parabolic initial boundary value problem

$$\text{(P)}\qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t}+Au(x,t)+g(x,t,u(x,t))=f(x,t) & \text{in } Q_T \\ \qquad\qquad u(x,t)=0 & \text{in } \partial\Omega\times(0,T) \\ \qquad\qquad u(x,0)=\psi(x) & \text{in } \Omega, \end{cases}$$

where A is an elliptic second order operator of the divergence form

$$(1)\qquad Au(x,t)=\sum\nolimits_{|\alpha|\leqq 1}(-1)^{|\alpha|}D^\alpha A_\alpha(x,t,u(x,t),Du(x,t))$$

for each $t\in[0, T]$ with the coefficients A_α satisfying the classical Leray—Lions conditions, and g is the strongly nonlinear part satisfying essentially only the condition

$$g(x,t,s)s\geqq -\lambda(x,t)\quad\text{for all}\quad (x,t)\in Q_T,\ s\in\mathbf{R},$$

where λ is some given function in $L^1(Q_T)$.

It was shown by P. Hess [3] that the corresponding Dirichlet problem for the elliptic equation

$$Au(x)+g(x,u(x))=f(x)\quad\text{in } \Omega$$

under similar conditions admits a weak solution. This result was recently generalised by J. R. L. Webb [11] also for higher order operators A by using new approximation results for Sobolev spaces obtained by L. I. Hedberg [2]. In this note we shall show that the problem (P) has a solution. This result is analogous to the elliptic case. The case where A is a higher order operator seems more complicated. Some results into this direction were obtained by H. Brezis and F. E. Browder [1], but stronger hypothesis on g was then needed.

There is also another approach to existence results for the problems involving the perturbation $g(u)$ with liberal growth. This approach makes use of a priori bounds in $L^\infty(Q_T)$-norm of u. But this implies restrictions which are somewhat unnatural in view of the general existence theory. We will discuss these restrictions when we introduce our conditions for the problem (P).

The case where even A may have liberal growth from above was studied by R. Landes [7]. We shall employ a Galerkin method which makes it possible to adapt some ideas of [7].

2. Prerequisites

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary. We start by introducing the conditions for the coefficients of the operator (1) (cf. [9], p. 323).

(A1) Each $A_\alpha(x, t, \eta, \zeta)$ as a function from $Q_T\times\mathbf{R}\times\mathbf{R}^N$ into $\mathbf{R}$ is measurable in (x, t) and continuous in η and ζ. For all $(x, t)\in Q_T$ and $(\eta, \zeta)\in\mathbf{R}^{N+1}$,

$$|A_\alpha(x, t, \eta, \zeta)| \leqq c_1\big(|\eta|^{p-1}+|\zeta|^{p-1}+k_1(x, t)\big)$$

with $1<p<\infty$, $c_1>0$ and $k_1\in L^{p'}(Q_T)$, $p'=p/(p-1)$.

(A2) For all $(x, t)\in Q_T$, $\eta\in\mathbf{R}$ and $\zeta\neq\zeta^*$ in $\mathbf{R}^N$,

$$\sum_{|\alpha|=1}\{A_\alpha(x, t, \eta, \zeta)-A_\alpha(x, t, \eta, \zeta^*)\}(\zeta_\alpha-\zeta_\alpha^*) > 0.$$

(A3) For all $(x, t)\in Q_T$ and $\xi=(\eta, \zeta)\in\mathbf{R}^{N+1}$,

$$\sum_{|\alpha|\leqq 1} A_\alpha(x, t, \xi)\,\xi_\alpha \geqq c_2|\zeta|^p-k_2(x, t)$$

with $c_2>0$ and $k_2\in L^1(Q_T)$.

The condition for g as a function from $Q_T\times\mathbf{R}$ into $\mathbf{R}$ reads as follows:

(G) $g(x, t, s)$ is measurable in (x, t) and continuous in s. For all $(x, t)\in Q_T$ and $s\in\mathbf{R}$ we have

$$g(x, t, s)\,s \geqq -\lambda(x, t),$$

where λ is a given function in $L^1(Q_T)$. For any $r\geqq 0$ there exists a function $h_r\in L^1(Q_T)$ such that for all $(x, t)\in Q_T$

$$\sup_{|s|\leqq r}|g(x, t, s)| \leqq h_r(x, t).$$

In the elliptic case a priori bounds in L^∞-norm for the generalized solutions are available for the operators satisfying the coercivity condition (A3) and additionally the assumption $k_2\in L^{q_1}$ with $q_1>N/p$. But even then the conditions on g and f are to be strengthened because of the inequality (7.2) of [4], p. 286. One has to assume $\lambda, f\in L^{q_2}$, $q_2>N/p$ instead of $\lambda\in L^1$ and f in the dual space of $W_0^{m,p}$.

In the parabolic case the boundedness of generalized solutions is shown in [5] only for $p=2$. The theory there is based on the inequality

$$\text{(N)} \qquad \|u\|_q \leqq C\|Du\|_p^{\alpha}\|u\|_2^{1-\alpha} \quad \text{for} \quad 0<\alpha<(1/2-1/q)(1/N-1/p+1/2)$$

due to Nirenberg and Golovkin, cf. (2.10) in [5], p. 62. In the interesting case $q>2$ we obtain positive values for α only if $p>2N/(2+N)$. For $p<2N/(2+N)$ no theory on the boundedness of generalized solutions seems to be available. If $p>2N/(2+N)$ the restrictions on the data are similar to the elliptic case and the initial value ψ has to be uniformly bounded (cf. inequalities (2.3) and (2.4) p. 424 and Theorem 6.6 p. 462 in [5]). We remark also that the inequality (4.26) in Theorem 6.7 means restriction on the growth of $g(u)$. In fact, $g(u)$ may be considered then as a bounded map and the theory of bounded pseudo-monotone operators is applicable. Finally we remark that our hypotheses are not optimal in the sense that Sobolev's imbedding theorem and (N) can be used to weaken our hypothesis (A1) in an obvious way.

The function spaces we shall be dealing with are denoted $V=W_0^{1,p}(\Omega)$, where p is given by (A1), $\mathscr{V}=L^p(0, T; V)$ with the usual norm

$$\|v\|_{\mathscr{V}} = \Big(\int_0^T \int_\Omega \sum_{|\alpha|\leqq 1} |D^\alpha v|^p \, dx\, dt\Big)^{1/p},$$

and $\mathscr{W}=\mathscr{V}\cap L^2(Q_T)$ with the norm

$$\|v\|_{\mathscr{W}} = \|v\|_{\mathscr{V}}+\|v\|_{L^2(Q_T)}.$$

For the Galerkin method we choose the sequence $\{w_1, w_2, \dots\}$ in $C_0^\infty(\Omega)$ such that $\bigcup_{n=1}^\infty V_n$ with $V_n=\operatorname{span}\{w_1, w_2, \dots, w_n\}$, is dense in $W_0^{j,p}(\Omega)$ where $j>N/p+1$. Since $W_0^{j,p}(\Omega)$ is continuously embedded in $C^1(\bar{\Omega})$, for any $v\in W_0^{j,p}(\Omega)$ there exists a sequence $(v_k)\subset\bigcup_{n=1}^\infty V_n$ such that $v_k\to v$ in $W_0^{j,p}(\Omega)$ and in $C^1(\bar{\Omega})$, too.

We denote further $\mathscr{V}_n=C([0, T]; V_n)$. It is easy to see that the closure of $\bigcup_{n=1} \mathscr{V}_n$ with respect to the norm

$$\|v\|_{C^{1,0}(Q_T)} = \sup_{\substack{|\alpha|\leqq 1\\ (x,t)\in Q_T}} \{|D^\alpha u(x, t)|\}$$

contains $C_0^\infty(Q_T)$. This implies that for any $f\in\mathscr{V}^*$ there exists a sequence $(f_k)\subset \bigcup_{n=1}^\infty \mathscr{V}_n$ such that $f_k\to f$ in $\mathscr{V}^*$ in the sense that

$$\int_{Q_T} \cdot f_k \, dx\, dt \to \langle f, \cdot\rangle_{\mathscr{V}}.$$

For any $\psi\in L^2(\Omega)$ there is a sequence $(\psi_k)\subset\bigcup_{n=1}^\infty V_n$ such that $\psi_k\to\psi$ in $L^2(\Omega)$. Then we have the following

Definition 1. A function $u_n \in \mathcal{V}_n$ $(n=1, 2, \ldots)$ is called a Galerkin solution of (P), if $\partial u_n/\partial t \in L^1(0; T; V_n)$, $u_n(0)=\psi_n$ and for all $\tau \in (0, T]$ we have

$$\int_{Q_\tau} \frac{\partial u_n}{\partial t} v\,dx\,dt + \int_{Q_\tau} \sum_{|\alpha|\leq 1} A_\alpha(x, t, u_n, Du_n) D^\alpha v\,dx\,dt$$
$$+\int_{Q_\tau} g(x, t, u_n) v\,dx\,dt = \int_{Q_\tau} f_n v\,dx\,dt$$

for all $v \in \mathcal{V}_n$, where $\psi_n \in \mathcal{V}_n$, $f_n \in \mathcal{V}_n$ and $\psi_n \to \psi$ in $L^2(\Omega)$ and $f_n \to f$ in $\mathcal{V}^*$.

We shall be interested in solutions u for (P) in the following sense.

Definition 2. A function $u \in \mathcal{W} \cap C^0([0, T]\, L^2(\Omega))$ is called a weak solution of (P), if

$$-\int_{Q_T} u \frac{\partial \varphi}{\partial t} dx\,dt + \int_\Omega u(t)\varphi(t)\,dx \Big|_0^T$$
$$+\int_{Q_T} \sum_{|\alpha|\leq 1} A_\alpha(x, t, u, Du) D^\alpha \varphi\,dx\,dt + \int_{Q_T} g(x, t, u)\varphi\,dx\,dt = \langle f, \varphi \rangle$$

for all $\varphi \in C^1([0, T]; C_0^\infty(\Omega))$ with $u(0)=\psi$, and $g(\cdot, \cdot, u) \in L^1(Q_T)$, $g(\cdot, \cdot, u)u \in L^1(Q_T)$.

We close this section by the following compactness result which will be needed in the proof of the existence theorem.

Proposition 1. *Let (u_n) be a bounded sequence in $L^p(0, T; W_0^{m-1,p}(\Omega)) \cap L^\infty(0, T; L^1(\Omega))$ with $1<p<\infty$ and $m \geq 1$. If $u_n(t) \rightharpoonup u(t)$ (weak convergence) in $L^1(\Omega)$ a.e. in $[0, T]$, then $u_n \to u$ in $L^p(0, T; W_0^{m-1,p}(\Omega))$ for some subsequence of (u_n).*

Remark. Usually the compactness in $L^p(0, T; W_0^{m,p}(\Omega))$ is obtained by a priori bounds of $\partial u_n/\partial t$ in some distribution spaces (cf. e.g. [1]). These bounds are replaced here by the hypothesis $u_n(t) \rightharpoonup u(t)$ in $L^1(\Omega)$ and $\|u_n(t)\|_{L^1(\Omega)} \leq C$, which can be verified easily for the sequence of Galerkin solutions.

Proof of Proposition 1. Let v_σ stand for the mollified function

$$v_\sigma(x, t) = \int_{\mathbf{R}^N} v(y, t) J_\sigma(x-y)\,dy, \quad \text{where} \quad v(y, t) = 0 \quad \text{for all} \quad y \notin \Omega.$$

Since $u_n(t) \rightharpoonup u(t)$ in $L^1(\Omega)$ we have $u_{n\sigma}(x, t) \to u_\sigma(x, t)$ a.e. in Q_T, moreover

$$\|u_{n\sigma}(t) - u_n(t)\|_{m-1,p} \leq \sigma \|u_n(t)\|_{m,p};$$

and for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that (cf. [10], p. 75 and 85)

$$\|D^\alpha u_n(t)\|_p \leq \varepsilon \|D^m u_n(t)\|_p + c_\varepsilon \|u_n(t)\|_1$$

for all $|\alpha| \leqq m-1$. Hence

$$\|D^\alpha u_n(t) - D^\alpha u_k(t)\|_p$$
$$\leqq \|D^\alpha u_n(t) - D^\alpha u_{n\sigma}(t)\|_p + \|D^\alpha u_{n\sigma}(t) - D^\alpha u_{k\sigma}(t)\|_p$$
$$+\|D^\alpha u_{k\sigma}(t) - D^\alpha u_k(t)\|_p \leqq \sigma\|D^m u_n(t)\|_p + \sigma\|D^m u_k(t)\|_p + \varepsilon\|D^m u_{n\sigma}(t) - D^m u_{k\sigma}(t)\|_p$$
$$+c_\varepsilon\|u_{n\sigma}(t) - u_{k\sigma}(t)\|_1.$$

Consequently, for all $|\alpha| \leqq m-1$,

$$\int_0^T \|D^\alpha u_n(t) - D^\alpha u_k(t)\|_p^p \, dt$$
$$\leqq c\Big\{\sigma^p \int_0^T \|D^m u_n(t)\|_p^p \, dt + \sigma^p \int_0^T \|D^m u_k(t)\|_p^p \, dt$$
$$+\varepsilon^p \int_0^T \|D^m u_{n\sigma}(t) - D^m u_{k\sigma}(t)\|_p^p \, dt + c_\varepsilon^p \int_0^T \|u_{n\sigma}(t) - u_{k\sigma}(t)\|_1^p \, dt\Big\},$$

c being some positive constant. By choosing first σ and ε small enough and then n and k large enough we can conclude that (u_n) is a Cauchy sequence in $L^p(0, T; W_0^{m-1,p}(\Omega))$, hence being convergent. Here we have used the fact that

$$\int_\Omega |u_{n\sigma}(x, t)| \, dx \leqq c \quad \text{for all } n \text{ and almost all } \quad t \in (0, T),$$

and $u_{n\sigma}(x, t) \to u_\sigma(x, t)$ a.e. in Q_T imply that

$$\int_0^T \|u_{n\sigma}(t) - u_{k\sigma}(t)\|_1^p \, dt \to 0 \quad \text{as} \quad n, k \to \infty$$

for any fixed σ.

3. Existence theorem

Our main result in this note is the following

Theorem 1. *Let Ω be a bounded smooth domain in $\mathbf{R}^N$ and $T>0$. Assume that the operator A defined by* (1) *satisfies the conditions* (A1), (A2) *and* (A3) *and the function g the condition* (G). *Then the problem* (P) *admits a weak solution u for any $f \in \mathscr{V}^*$ and $\psi \in L^2(\Omega)$.*

Proof. We shall give the proof in several steps. In many stages we can adopt the ideas of [7]. For convenience we assume that $\psi = 0$. The general case can be handled similarly without essential difficulties.

1°. To show the existence of Galerkin solutions u_n we proceed as follows. Applying Friedrichs' mollification with respect to space-time variables to the coefficient functions we obtain Galerkin solutions for the mollified problem. The coefficient functions A_α^ε and g^ε clearly meet the assumptions of Lemma 1 in [7]. Observing that

(A1), (A2), (A3) and (G) are true uniformly for the mollified problem we get a Galerkin solution u_n with $\varepsilon\to 0$, satisfying the conditions of Definition 1. Hence

$$\text{(2)}\qquad \begin{cases} \int_{Q_\tau} \frac{\partial u_n}{\partial t}\varphi\,dx\,dt + \int_{Q_\tau} \sum_{|\alpha|\le 1} A_\alpha(x,t,u_n,Du_n)D^\alpha\varphi\,dx\,dt + \\ \int_{Q_\tau} g(x,t,u_n)\varphi\,dx\,dt = \int_{Q_\tau} f_n\varphi\,dx\,dt \quad \text{for all}\quad \varphi\in\mathscr{V}_n,\quad \tau\in(0,T] \end{cases}$$

with $f_n\in\mathscr{V}_n$ and $f_n\to f$ in $\mathscr{V}^*$. Setting $\varphi=u_n$ we get by (A3) and (A1) that

$$\text{(3)}\qquad \begin{cases} \|u_n\|_{\mathscr{V}} \le c, \quad \|u_n\|_{L^\infty(0,T;L^2(\Omega))} \le c, \\ \|A_\alpha(\cdot,\cdot,u_n,Du_n)\|_{L^{p'}(Q_T)} \le c, \\ 0 \le \int_{Q_t} g(x,\tau,u_n)u_n\,dx\,d\tau \le c \quad \text{for all}\quad t\in[0,T]. \end{cases}$$

Consequently, for some subsequence of (u_n), $u_n\rightharpoonup u$ in $\mathscr{W}$ and $A_\alpha(\cdot,\cdot,u_n,Du_n)\rightharpoonup h_\alpha$ in $L^{p'}(Q_T)$ where $h_\alpha\in L^{p'}(Q_T)$ for all $|\alpha|\le 1$. Moreover, for any $\delta>0$,

$$|g(x,t,u_n(x,t))| \le \sup_{|s|\le 1/\delta} |g(x,t,s)| + \delta g(x,t,u_n(x,t))\,u_n(x,t)$$

which together with (G) and (3) means that $\{g(\cdot,\cdot,u_n)\}$ is weakly sequentially compact in $L^1(Q_T)$.

2°. We invoke Proposition 1 to get that $u_n(x,t)\to u(x,t)$ a.e. in Q_T for some further subsequence. Indeed, since (u_n) is bounded in $L^\infty(0,T;L^2(\Omega))$, it is bounded also in $L^\infty(0,T;L^1(\Omega))$ and we must show that $u_n(t)\rightharpoonup u(t)$ in $L^1(\Omega)$ for all $t\in[0,T]$. Let $\varphi\in\bigcup_{n=1}^\infty V_n$ be arbitrary and $t\in[0,T]$. Then by (2),

$$\begin{aligned} &\left|\int_\Omega (u_n(y,t)-u_k(y,t))\,\varphi(y)\,dy\right| \\ &= \left|\int_\Omega \int_0^t \frac{d}{dt}(u_n(y,\tau)-u_k(y,\tau))\,\varphi(y)\,d\tau\,dy\right| \\ &\le \left|\int_0^t\int_\Omega \sum_{|\alpha|\le 1}(A_\alpha(x,\tau,u_n,Du_n)) - A_\alpha(x,\tau,u_k,Du_k)\,D^\alpha\varphi\,dx\,d\tau\right| \\ &+ \left|\int_{Q_t}(g(x,\tau,u_n)-g(x,\tau,u_k))\,\varphi\,dx\,d\tau\right| + \left|\int_{Q_t}(f_n-f_k)\,\varphi\,dx\,dt\right| \to 0 \end{aligned}$$

as $n,k\to\infty$ because $\{A_\alpha(\cdot,\cdot,u_n,Du_n)\}$ and $\{g(\cdot,\cdot,u_n)\}$ are both weakly convergent in $L^1(Q_t)$.

Moreover, for any $z\in L^2(\Omega)$ there exists an approximating sequence $(\varphi_i)\subset\bigcup_{n=1}^\infty V_n$ such that $\varphi_i\to z$ in $L^2(\Omega)$. Hence we can conclude that $(u_n(t))$ is a Cauchy sequence in the weak topology of $L^2(\Omega)$, for all $t\in[0,T]$. Hence $u_n(t)\rightharpoonup\hat{u}(t)$ for all $t\in[0,T]$ in $L^2(\Omega)$, for some function $\hat{u}(t)\in L^2(\Omega)$.

But since $u_n\rightharpoonup u$ in $L^2(Q_T)$ it is easy to see that $\hat{u}(t)=u(t)$ a.e. in $[0,T]$. Thus Proposition 1 implies that $u_n\to u$ in $L^p(Q_T)$ and therefore $u_n(x,t)\to u(x,t)$ a.e. in Q_T for a further subsequence.

3°. By the continuity of g, $g(x, t, u_n(x, t)) \to g(x, t, u)$ a.e. in Q_T and Vitali's convergence theorem implies that $g(\cdot, \cdot, u_n) \to g(\cdot, \cdot, u)$ in $L^1(Q_T)$. By Fatou's lemma we get further

$$\int_{Q_T} g(x, t, u)\, u\, dx\, dt \leqq \liminf \int_{Q_T} g(x, t, u_n)\, u_n\, dx\, dt. \tag{4}$$

From (2) we get for all $\varphi \in C^1([0, T]; C_0^\infty(\Omega))$

$$\lim_{n\to\infty} \int_{Q_T} \frac{\partial u_n}{\partial t} \varphi\, dx\, dt + \int_{Q_T} \sum_{|\alpha|\leqq 1} h_\alpha D^\alpha \varphi\, dx\, dt + \int_{Q_T} g(x, t, u)\, \varphi\, dx\, dt = \langle f, \varphi \rangle_{\mathscr{V}}. \tag{5}$$

By Lemma 2 of [7] we also have that $\hat{u}\colon [0, T] \to L^2(\Omega)$ is weakly continuous and

$$\lim_{n\to\infty} \int_{Q_T} \frac{\partial u_n}{\partial t} \varphi\, dx\, dt = \int_\Omega \hat{u} \varphi\, dx \Big/_0^T - \int_{Q_T} u \frac{\partial}{\partial t} \varphi\, dx\, dt. \tag{6}$$

Hence the proof will be complete, if we can show that

$$\int_{Q_T} \sum_{|\alpha|\leqq 1} h_\alpha D^\alpha \varphi\, dx\, dt = \int_{Q_T} \sum_{|\alpha|\leqq 1} A_\alpha(x, t, u, Du)\, D^\alpha \varphi\, dx\, dt \tag{7}$$

for all $\varphi \in C^1([0, T]; C_0^\infty(\Omega))$, and

$$u \in C^0([0, T];\ L^2(\Omega)). \tag{8}$$

4°. For (7) it is sufficient to show that

$$\limsup_{n\to\infty} \int_{Q_T} \sum_{|\alpha|\leqq 1} A_\alpha(x, t, u_n, Du_n)(D^\alpha u_n - D^\alpha u)\, dx\, dt \leqq 0.$$

This is true because the mapping from V to V^* associated to the operator A is of the monotone type which can be verified by the assumptions (A1) and (A2) as in [8].

The above inequality however holds true, if at least for one subsequence $(v_k) \subset \bigcup_{n=1}^\infty \mathscr{V}_n$ with $v_k \rightharpoonup u$ in $\mathscr{V}$ we have

$$\lim_{k\to\infty} \limsup_{n\to\infty} \int_{Q_T} \sum_{|\alpha|\leqq 1} \{A_\alpha(x, t, u_n, Du_n) D^\alpha u_n - h_\alpha D^\alpha v_k\}\, dx\, dt \leqq 0. \tag{9}$$

By (5) we have for any fixed k,

$$-\int_{Q_T} \sum_{|\alpha|\leqq 1} h_\alpha D^\alpha v_k\, dx\, dt$$

$$= \lim_{n\to\infty} \int_{Q_T} \frac{\partial u_n}{\partial t} v_k\, dx\, dt + \int_{Q_T} g(x, t, u)\, v_k\, dx\, dt - \int_{Q_T} f v_k\, dx\, dt.$$

From (2) and (4) we therefore get

$$(10)\qquad \limsup_{n\to\infty}\int_{Q_T}\sum_{|\alpha|\le 1}\{A_\alpha(x,t,u_n,Du_n)D^\alpha u_n-h_\alpha D^\alpha v_k\}\,dx\,dt$$
$$\le \limsup_{n\to\infty}\int_{Q_T}(f_nu_n-fv_k)\,dx\,dt+\int_{Q_T}g(x,t,u)(v_k-u)\,dx\,dt+$$
$$+\limsup_{n\to\infty}\int_{Q_T}\frac{\partial u_n}{\partial t}(v_k-u_n)\,dx\,dt.$$

Since obviously

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_T}(f_nu_n-fv_k)\,dx\,dt=0,$$

it remains to show for (9) that

$$(11)\qquad \limsup_{k\to\infty}\int_{Q_T}g(x,t,u)(v_k-u)\,dx\,dt\le 0$$

and

$$(12)\qquad \limsup_{k\to\infty}\limsup_{n\to\infty}\int_{Q_T}\frac{\partial u_n}{\partial t}(v_k-u_n)\,dx\,dt\le 0$$

hold for some sequence $(v_k)\subset C^1([0,T];C_0^\infty(\Omega))$ such that $v_k\rightharpoonup u$ in $\mathscr{V}$.

The assertion (12) can be further modified

$$\int_{Q_T}\frac{\partial u_n}{\partial t}(v_k-u_n)\,dx\,dt$$
$$=-\frac{1}{2}\int_{Q_T}\frac{\partial}{\partial t}(u_n(t)-v_k(t))^2\,dx\,dt+\int_{Q_T}\frac{\partial v_k}{\partial t}(v_k-u_n)\,dx\,dt$$
$$=-\frac{1}{2}\|u_n(T)-v_k(T)\|^2_{L^2(Q)}+\int_{Q_T}\frac{\partial v_k}{\partial t}(v_k-u_n)\,dx\,dt.$$

Hence (12) is satisfied, if

$$(13)\qquad \limsup_{k\to\infty}\limsup_{n\to\infty}\int_{Q_T}\frac{\partial v_k}{\partial t}(v_k-u_n)\,dx\,dt\le 0.$$

The sequence (v_k) is then constructed as follows: Starting with the Galerkin sequence (u_n) we define for each $k\in\mathbf{N}$, $u_n^{(k)}$ the truncation at level k. Next we define for $\mu\in\mathbf{N}$

$$(u_n^{(k)})_\mu(x,t)=\mu\int_0^t u_n^{(k)}(x,s)\,e^{\mu(s-t)}\,ds.$$

Then $(u_n^{(k)})_\mu\to u_n^{(k)}$, $u_\mu^{(k)}\to u^{(k)}$ in $\mathscr{V}$ as $\mu\to\infty$ and

$$\frac{\partial}{\partial t}(n_n^{(k)})_\mu=\mu(u_n^{(k)}-(u_n^{(k)})_\mu).$$

(If $\psi\ne 0$ define

$$(u_n^{(k)})_\mu(x,t)=\mu\int_0^t\big(u_n^{(k)}(x,s)-u_n^{(k)}(x,0)\big)\cdot e^{\mu(s-t)}\,ds+u_n^{(k)}(x,0);$$

see [7]).

Finally we must take the mollification with respect to space variable, $[(u_n^{(k)})_\mu]_\sigma$ for $\sigma>0$ (cf. Proposition 1). It is obvious that this sequence is in $C^1([0, T]; C_0^\infty(\Omega))$. Moreover

$$\int_{Q_T} g(x, t, u)([(u_n^{(k)})_\mu]_\sigma - u)\, dx\, dt$$

$$= \int_{Q_T} g(x, t, u)([(u_n^{(k)})_\mu]_\sigma - u^{(k)})_\mu)\, dx\, dt + \int_{Q_T} g(x, t, u)((u_n^{(k)})_\mu - u_n^{(k)})\, dx\, dt$$

$$+ \int_{Q_T} g(x, t, u)(u_\mu^{(k)} - u^{(k)})\, dx\, dt + \int_{Q_T} g(x, t, u)(u^{(k)} - u)\, dx\, dt,$$

where each of the integrals converge to zero as $\sigma\to 0$, $n\to\infty$, $\mu\to\infty$, and $k\to\infty$, respectively. Thus we can choose (v_k) as a diagonal sequence of $\{[(u_n^{(k)})_\mu]_\sigma\}$ such that (11) holds true and $v_k \rightharpoonup u$ in $\mathcal{V}$. Since each $\partial v_k/\partial t$ is in $L^{p'}(Q_T)$, (13) follows from

$$\limsup_{k\to\infty} \int_{Q_T} \frac{\partial v_k}{\partial t}(v_k - u)\, dx\, dt \leqq 0. \tag{14}$$

Denote $A_k = \{(x, t)\in Q_T \colon |u(x, t)| \leqq k\}$, $k\in\mathbf{N}$. Then $u^{(k)} = u$ in A_k and

$$\operatorname{sgn}(u^{(k)} - u_\mu^{(k)}) = \operatorname{sgn}(u - u_\mu^{(k)}) \quad \text{in} \quad Q_T\setminus A_k; \quad \text{if} \quad u^{(k)} - u_\mu^{(k)} \neq 0,$$

because of $|u_\mu^{(k)}| \leqq k$. Since

$$\int_{Q_T} \frac{\partial v_k}{\partial t}(v_k - u)\, dx\, dt = \mu \int_{Q_T} \{(u_n^{(k)})_\sigma - [(u_n^{(k)})_\sigma]_\mu\}([(u_n^{(k)})_\sigma]_\mu - u)\, dx\, dt$$

$$\to \mu \int_{Q_T} \{(u^{(k)}) - (u^{(k)})_\mu\}((u^{(k)})_\mu - u)\, dx\, dt$$

$$= -\mu \int_{A_k} (u - (u^{(k)})_\mu)^2\, dx\, dt + \mu \int_{Q_T\setminus A_k} (u^{(k)} - (u^{(k)})_\mu)(u - (u^{(k)})_\mu)\, dx\, dt \leqq 0,$$

as $\sigma\to 0$ and $n\to\infty$, for any μ and k. Therefore we can conclude that (13) holds true, too.

5°. The final step of the proof is to show (8). We shall do this by showing that u is the limit of a Cauchy sequence in $C^0([0, T]; L^2(\Omega))$.

To begin with we note that we may choose v_k in such a manner that in addition to (11) and (14) we have for every $\tau\in(0, T]$

$$\lim_{n\to\infty} \left| \int_{Q_\tau} g(x, t, u_n) v_k\, dx\, dt - \int_{Q_\tau} g(x, t, u) u\, dx\, dt \right| \leqq \varepsilon_k' \tag{15}$$

and

$$\lim_{n\to\infty} \int_{Q_\tau} \frac{\partial v_k}{\partial t}(v_k - u_n)\, dx\, dt \leqq \varepsilon_k', \tag{16}$$

for some ε_k' not depending on τ and $\varepsilon_k'\to 0$ for $k\to\infty$. This is true because of the strong convergence of $[(u_n^{(k)})_\mu]_\sigma$ in $\mathcal{V}$ with respect to σ, n, μ, and of the strong convergence of u_n and $g(\cdot,\cdot,u_n)$ in $L^p(Q_T)$ and $L^1(Q_T)$ respectively. Furthermore by Fatou's Lemma we know that

$$\liminf_{n\to\infty}\int_{Q_\tau}\sum_{|\alpha|\leqq 1}A_\alpha(x,t,u_n,Du_n)D^\alpha u_n\,dx\,dt \geqq \int_{Q_\tau}\sum_{|\alpha|\leqq 1}A_\alpha(x,t,u,Du)D^\alpha u\,dx\,dt$$

and

$$\liminf_{n\to\infty}\int_{Q_\tau}g(x,t,u_n)u_n\,dx\,dt \geqq \int_{Q_\tau}g(x,t,u)u\,dx\,dt.$$

Thus we obtain the estimate

$$\begin{aligned}&\limsup_{n\to\infty}\int_{Q_\tau}\frac{\partial u_n}{\partial t}(u_n-v_k)\,dx\,dt\\ &=\limsup_{n\to\infty}\Big[\int_{Q_\tau}\sum_{|\alpha|\leqq 1}A_\alpha(x,t,u_n,Du_n)D^\alpha(v_k-u_n)\,dx\,dt\\ &\quad+\int_{Q_\tau}g(x,t,u_n)(v_k-u_n)\,dx\,dt+\int_{Q_\tau}f_n(u_n-v_k)\,dx\,dt\Big]\leqq \varepsilon_k,\end{aligned}$$

independently from $\tau\in(0,T]$. We have shown now that

$$(17)\qquad \begin{cases}\varepsilon_k+\varepsilon_k'\geqq \limsup\limits_{n\to\infty}\displaystyle\int_{Q_\tau}\frac{\partial}{\partial t}(u_n-v_k)(u_n-v_k)\,dx\,dt\\ \qquad\quad=\frac{1}{2}\limsup\limits_{n\to\infty}\|u_n(\tau)-v_k(\tau)\|^2_{L^2(\Omega)}\end{cases}$$

independently from $\tau\in(0,T]$, implying that v_k is a Cauchy sequence in $C^0([0,T], L^2(\Omega))$.

We close the paper by the following

Corollary. *The function u can be used as a testing function in* (P), *i.e.*

$$\begin{aligned}&\tfrac{1}{2}\int_\Omega (u(t))^2\,dx\Big/_0^\tau+\int_{Q_\tau}\sum_{|\alpha|\leqq 1}A_\alpha(x,t,u,Du)D^\alpha u\,dx\,dt\\ &\quad+\int_{Q_\tau}g(x,t,u)u\,dx\,dt=\int_{Q_\tau}f(x,t)\,u(x,t)\,dx\,dt\end{aligned}$$

for all $\tau\in(0,T]$.

Proof. By the results of the previous proof we have in view of (16)

$$\begin{aligned}0&\leqq \lim_{k\to\infty}\lim_{n\to\infty}\Big\{\int_{Q_\tau}g(x,t,u_n)(u_n-v_k)\,dx\,dt\\ &\quad+\int_{Q_\tau}\sum_{|\alpha|\leqq 1}A_\alpha(x,t,u_n,Du_n)(D^\alpha u_n-D^\alpha v_k)\,dx\,dt\Big\}\\ &=\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_\tau}\frac{\partial u_n}{\partial t}(v_k-u_n)\,dx\,dt+\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_\tau}f_n(u_n-v_k)\,dx\,dt\\ &\leqq \tfrac{1}{2}\lim_{k\to\infty}\limsup_{n\to\infty}\int_\Omega -(u_n(t)-v_k(t))^2\Big/_0^\tau\\ &\quad+\lim_{k\to\infty}\limsup_{n\to\infty}\int_{Q_\tau}\frac{\partial v_k}{\partial t}(v_k-u_n)\,dx\,dt\leqq 0.\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\int_{Q_\tau} g(x, t, u_n)u_n\,dx\,dt = \int_{Q_\tau} g(x, t, u)u\,dx\,dt \tag{18}$$

and

$$\lim_{n\to\infty}\int_{Q_\tau}\sum_{|\alpha|\le 1} A_\alpha(x, t, u_n, Du_n)D^\alpha u_n\,dx\,dt = \int_{Q_\tau}\sum_{|\alpha|\le 1} A_\alpha(x, t, u, Du)D^\alpha u\,dx\,dt. \tag{19}$$

In view of (18), (19) and the estimate just above (17) it is now apparent that we can replace lim sup by the limit in (17) which gives us

$$\frac{1}{2}\lim_{n\to\infty}\|u_n(\tau)-v_k(\tau)\|^2_{L^2(\Omega)} \leqq \varepsilon_k+\varepsilon_k'.$$

Since $v_k(\tau)\to u(\tau)$ we have also $u_n(\tau)\to u(\tau)$ in $L^2(\Omega)$ which implies that

$$\lim_{n\to\infty}\int_{Q_\tau}\frac{\partial u_n}{\partial t}u_n\,dx\,dt = \frac{1}{2}\int_\Omega u(t)^2\Big/_0^\tau. \tag{20}$$

The Corollary now follows from (18), (19) and (20) because u_n is a Galerkin solution.

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