Sharp function and weighted L^p estimates for a class of pseudo-differential operators

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1. Introduction

The purpose of this paper is to prove pointwise inequalities and to establish the boundedness on L^p (weighted) spaces for pseudo-differential operators with symbols $\sigma(x, \xi)$ in the class $S^{-na/2}_{\varrho,\delta}$, $0<a<1$. The prototype of our results is a theorem of Chanillo [1] for the particular case $\sigma(x, \xi)=e^{i|\xi|^a}\theta(\xi)|\xi|^{-na/2}$, where $\theta(\xi)$ is a smooth cut-off function vanishing in a neighborhood of the origin. We point out, however, that whereas Chanillo's results make extensive use of the kernel formula, the method used here is to break up the symbol instead, in a manner compatible with the decomposition of the function. This idea may also be used to establish weak-type (1, 1) inequalities as has been done by Chanillo, Kurtz and Sampson [2].

In order to state our results we begin by introducing the relevant notations and definitions. We say that a symbol $\sigma(x, \xi)$ is in the class $S^m_{\varrho,\delta}$, or that $\sigma\in S^m_{\varrho,\delta}$, if for x, ξ in $\mathbf{R}^n$,

$$\left|\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial \xi^\beta}\sigma(x, \xi)\right| \leqq c_{\alpha,\beta}(1+|\xi|)^{m-\varrho|\beta|+\delta|\alpha|}.$$

We will consider in this paper pseudo-differential operators (ψ.d.o.) with symbols $\sigma(x, \xi)\in S^m_{\varrho,\delta}$, that is we consider operators T given by

$$Tf(x) = \int_{\mathbf{R}^n} e^{i(x,\xi)}\sigma(x, \xi)\hat{f}(\xi)\,d\xi,$$

where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f.

In addition to the well-known L^2 results for some classes of ψ.d.o. we mention here that more recently sharp L^p boundedness results for operators with symbols in the class $S^{-\beta}_{1-a,\delta}$ with $0\leqq\delta<a-1<1$ and $\beta<na/2$ have been established by C. Fefferman [4].

We set

$$M_p f(x) = \sup_{x\in Q}\left(\frac{1}{|Q|}\int_Q |f(y)|^p dy\right)^{1/p},$$

where Q is a cube with sides parallel to the coordinate axes. This is the generalized Hardy—Littlewood maximal function of f. We also need the sharp maximal function $f^{\#}$ of f which is given by

$$f^{\#}(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)-f_Q|dy,$$

where $f_Q=\frac{1}{|Q|}\int_Q f(y)\,dy$.

For a general symbol $\sigma(x,\xi)\in S^{-na/2}_{1-a,\delta}$, $0<\delta<1-a$, $0<a<1$, we will show that for $f\in C_0^\infty(\mathbf{R}^n)$,

$$(1.1)\qquad (Tf)^{\#}(x) \leqq cM_2 f(x).$$

A proof of this assertion may be found in Section 2, Lemma (2.4). An interesting open question is whether (1.1) is best possible, i.e. whether 2 is the smallest index that may be used in the right hand side of the inequality. If one is willing to specialize the symbols $\sigma(x,\xi)$ then it is possible to obtain sharper results. The main result proved in this paper is

Theorem (1.2). *Let* $0<a<1$, *and* $\sigma(x,\xi)\in S^{-na/2}_{1,0}$. *Consider the* ψ*.d.o.*

$$Tf(x) = \int_{\mathbf{R}^n} e^{i(\langle x,\xi\rangle+|\xi|^a)}\sigma(x,\xi)\hat{f}(\xi)\,d\xi.$$

Then for $1<r<\infty$ *and* $f\in C_0^\infty(\mathbf{R}^n)$

$$(Tf)^{\#}(x) \leqq c_r M_r f(x).$$

Theorem (1.2), and of course Lemma (2.4), lead to various weighted L^p inequalities. We list some of them as a theorem but do not prove them as the proof technique, once we have the pointwise estimates, is by now well-known. We refer to Kurtz and Wheeden [7] and Miller [8], for instance, for further details.

We adopt the usual notation that

$$\|f\|_{p,w} = \left(\int_{R^n} |f(x)|^p\, w(x)\,dx\right)^{1/p}$$

and we say that $w\in A_p$ if

$$\sup_Q\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leqq c.$$

Again, as usual, $A_\infty=\bigcup_{p\geqq 1}A_p$. We then have

Theorem (1.3). *Let Tf be a ψ.d.o. with symbol* $\sigma(x, \xi)\in S^{-na/2}_{1-a,\delta}$, $\delta<1-a$, $0<a<1$.

(a) *Let* $w\in A_\infty$, *then, for* $0<p<\infty$ *and* $f\in C_0^\infty(\mathbf{R}^n)$,

$$\|Tf\|_{p,w} \leqq c_p\|M_2 f\|_{p,w}.$$

(b) *Let* $w\in A_{p/2}$, *then, for* $2\leqq p<\infty$, *and* $f\in C_0^\infty(\mathbf{R}^n)$,

$$\|Tf\|_{p,w} \leqq c_p\|f\|_{p,w}.$$

Theorem (1.4). *Let Tf be a ψ.d.o. with symbol* $e^{i|\xi|^a}\sigma(x, \xi)$, $\sigma(x, \xi)\in S^{-na/2}_{1,0}$ *and* $0<a<1$.

(a) *Let* $w\in A_\infty$, *then, for* $1<r<\infty$, $0<p<\infty$ *and* $f\in C_0^\infty(\mathbf{R}^n)$,

$$\|Tf\|_{p,w} \leqq c_{r,p}\|M_r f\|_{p,w}.$$

(b) *Let* $w\in A_p$, *then, for* $1<p<\infty$,

$$\|Tf\|_{p,w} \leqq c_p\|f\|_{p,w}.$$

We would like to point out that it is possible to prove weighted weak type (1, 1) estimates for $w\in A_1$, for ψ.d.o.'s with symbol $e^{i|\xi|^a}\sigma(x, \xi)$, $\sigma(x, \xi)\in S^{-na/2}_{1,0}$. The technique of the proof is based on the decomposition lemma of Chanillo in [1].

2. Preliminary lemmas

We begin by introducing a notation. By $|x|\sim t$ we denote the fact that the values of x in question lie in the annulus $\{x\in\mathbf{R}^n\colon at<|x|<bt\}$ where $0<a\leqq 1<b<\infty$, and the precise values of a and b are irrelevant.

Lemma (2.1). *Let* $\sigma(x, \xi)\in S^{-na/2}_{1-a,\delta}$, $0<\delta\leqq 1$ *and* $0<a<1$. *Let* $k(x, w)$ *denote the inverse Fourier transform, in the ξ-variable and in the distribution sense of* $\sigma(x, \xi)$, *that is informally*

$$k(x, w) = \int e^{i(\xi, w)}\sigma(x, \xi)\, d\xi.$$

Then for $|x-x_0|\leqq d\leqq 1/2$ *and* $N\geqq 1$,

$$\Big(\int_{|y-x_0|\sim(2^N d)^{1-a}} |k(x, x-y)-k(x_0, x_0-y)|^2 dy\Big)^{1/2}$$
$$\leqq c|x-x_0|^{(1-a)(m-n/2)}/(2^N d)^{m(1-a)}$$

where m is an integer such that $n/2<m<n/2+1/(1-a)$.

Proof. The idea behind the proof is by now fairly standard. However, we do point out that unlike the Calderón—Zygmund class of symbols, i.e. $\sigma(x, \xi)\in S^0_{1,0}$, the exponent 2 in the integrand seems to be the largest value that can be used and the proof below breaks down for exponents larger than 2.

Let $\sum_{j\geqq 0}\theta_j(\xi)\equiv 1$ be a smooth partition of unity such that $\theta_j(\xi)$ is supported in $|\xi|\sim 2^{dj}$, $j\geqq 1$, and $\theta_0(\xi)$ is supported in $|\xi|\leqq 2$. Let $\sigma(x,\xi)=\sum_j\sigma_j(x,\xi)$, $\sigma_j(x,\xi)=\sigma(x,\xi)\theta_j(\xi)$. Moreover, put

$$\int_{\mathbf{R}^n} e^{i(x,\xi)}\sigma_j(x,\xi)\hat{f}(\xi)\,d\xi=\int_{\mathbf{R}^n} k_j(x,x-y)f(y)\,dy,\quad j\geqq 0.$$

Thus $k(x,w)=\sum_{j\geqq 0}k_j(x,w)$. We now get

$$\Big(\int_{|y-x_0|\sim(2^Nd)^{1-a}}|k(x,x-y)-k(x_0,x_0-y)|^2dy\Big)^{1/2}$$
$$\leqq\sum_{j=0}^{\infty}\Big(\int_{|y-x_0|\sim(2^Nd)^{1-a}}|k_j(x,x-y)-k_j(x_0,x_0-y)|^2dy\Big)^{1/2}.$$

We choose j_0 so that $2^{j_0}|x-x_0|\sim 1$, and break up the sum on the right above as follows,

$$\sum_{j<j_0}\Big(\int_{|y-x_0|\sim(2^Nd)^{1-a}}|k_j(x,x-y)-k_j(x_0,x_0-y)|^2\,dy\Big)^{1/2}$$
$$+\sum_{j\geqq j_0}\Big(\int_{|y-x_0|\sim(2^Nd)^{1-a}}|k_j(x_0,x_0-y)|^2\,dy\Big)^{1/2}$$
$$+\sum_{j\geqq j_0}\Big(\int_{|y-x_0|\sim(2^Nd)^{1-a}}|k_j(x,x-y)|^2dy\Big)^{1/2}\equiv I_1+I_2+I_3,\quad\text{say}.$$

We discuss I_2 and I_3 first. Since both terms are treated similarly we only estimate I_3 here. Now

$$I_3\leqq c\sum_{j\geqq j_0}(2^N d)^{-m(1-a)}\Big(\int_{|y-x_0|\sim(2^Nd)^{1-a}}|k_j(x,x-y)|^2|y-x_0|^{2m}\,dy\Big)^{1/2}.$$

Since $|x-x_0|\leqq d\leqq 1/2$ and $|y-x_0|\sim(2^Nd)^{1-a}$ in the above integral we also have that for those y's in question $|y-x_0|\leqq c|y-x|$. Thus the right side above is bounded by

$$c\sum_{j\geqq j_0}(2^N d)^{-m(1-a)}\Big(\int_{|y-x_0|\sim(2^Nd)^{1-a}}|k_j(x,x-y)|^2|y-x|^{2m}dy\Big)^{1/2}.$$

We may now majorize the above expression by

$$c\sum_{j\geqq j_0}(2^N d)^{-m(1-a)}\sum_{|\beta|=m}\left(\int\left|\frac{\partial^\beta}{\partial\xi^\beta}\sigma_j(x,\xi)\right|^2d\xi\right)^{1/2}$$
$$\leqq c\sum_{j\geqq j_0}(2^N d)^{-m(1-a)}2^{j(-na/2-m(1-a)+n/2)}.$$

The choice of m assures us that $-na/2-m(1-a)+n/2<0$. Thus the sum above is dominated by

$$c(2^N d)^{-m(1-a)}2^{j_0(-na/2-m(1-a)+n/2)}\leqq c(2^N d)^{-m(1-a)}|x-x_0|^{(m-n/2)(1-a)}.$$

We have finished estimating I_3. We now consider I_1. In the first place

(2.2) $$I_1 \leqq \sum_{j<j_0}\Big(\int_{|y-x_0|\sim(2^N d)^{1-a}} |k_j(x, x-y)-k_j(x_0, x-y)|^2 dy\Big)^{1/2}$$
$$+\sum_{j<j_0}\Big(\int_{|y-x_0|\sim(2^N d)^{1-a}} |k_j(x_0, x-y)-k_j(x_0, x_0-y)|^2 dy\Big)^{1/2}.$$

We consider the first term on the right. Since as we observed above $|y-x_0|\leqq c|y-x|$ we can bound this expression by

$$\sum_{j<j_0}(2^N d)^{-m(1-a)}\Big(\int_{|y-x_0|\sim(2^N d)^{1-a}}\big(|y-x_0|^m |k_j(x, x-y)-k_j(x_0, x-y)|\big)^2 dy\Big)^{1/2}$$
$$\leqq c\sum_{j<j_0}(2^N d)^{-m(1-a)}\Big(\int_{|y-x_0|\sim(2^N d)^{1-a}}\big(|y-x|^m |k_j(x, x-y)-k_j(x_0, x-y)|\big)^2 dy\Big)^{1/2}$$
$$\leqq c\sum_{j<j_0}(2^N d)^{-m(1-a)}\sum_{|\beta|=m}\Big(\int \sup_{\eta}\Big|\frac{\partial}{\partial\eta}\frac{\partial^\beta}{\partial\xi^\beta}\sigma_j(\eta, \xi)\Big|^2 d\xi\Big)^{1/2}|x-x_0|$$
$$\leqq c\Big(\sum_{j<j_0}(2^N d)^{-m(1-a)}2^{j(-na/2-m(1-a)+n/2+1)}\Big)|x-x_0|.$$

The choice of m and the fact that $2^{j_0}|x-x_0|\sim 1$ yields that the sum above is bounded by $c(2^N d)^{-m(1-a)}|x-x_0|^{(m-n/2)(1-a)}$.

We now consider the second term on the right in (2.2). In this case we first dominate this term by

$$\sum_{j<j_0}(2^N d)^{-m(1-a)}\Big(\int_{|y-x_0|\sim(2^N d)^{1-a}}\big(|y-x_0|^m |k_j(x_0, x-y)-k_j(x_0, x_0-y)|\big)^2 dy\Big)^{1/2}.$$

By Leibniz's formula we can majorize this expression by

$$c\sum_{j<j_0}(2^N d)^{-m(1-a)}\Big(\int \sum_{|\beta|+|\gamma|=m}\Big|\frac{\partial^\beta}{\partial\xi^\beta}\sigma_j(x_0, \xi)\frac{\partial^\gamma}{\partial\xi^\gamma}\big(e^{i(x-x_0, \xi)}-1\big)\Big|^2 d\xi\Big)^{1/2}$$
$$\leqq c\sum_{j<j_0}(2^N d)^{-m(1-a)}\sum_{|\beta|+|\gamma|=m, |\gamma|\neq 0}|x-x_0|^{|\gamma|}2^{j(-na/2-|\beta|(1-a)+n/2)}$$
$$+c\sum_{j<j_0}(2^N d)^{-m(1-a)}|x-x_0|2^{j(-na/2-m(1-a)+n/2+1)}.$$

But $|x-x_0|\leqq 1/2$ and since $m<n/2+1/(1-a)$, it is easily seen that the second sum above dominates the first one. Since $2^{j_0}|x-x_0|\sim 1$, this second term is readily seen to be bounded by $c(2^N d)^{-m(1-a)}|x-x_0|^{(m-n/2)(1-a)}$, as desired. This completes the proof. Note that we only needed derivatives up to a finite order to obtain the conclusion.

We shall recall one more fact about symbols in $S^m_{\varrho,\delta}$, namely that the convolution kernels are essentially compactly supported.

Lemma (2.3). *Let* $\sigma(x, \xi)\in S^0_{\varrho,\delta}$, $0<\varrho<1$, *and let as usual*

$$k(x, w) = \int_{\mathbf{R}^n} e^{i(w, \xi)}\sigma(x, \xi)\, d\xi.$$

Then for $|w|\geqq 1/4$ *and arbitrarily large* M,

$$|k(x, w)| \leqq c_M |w|^{-2M}.$$

Proof. Choose an integer M so that $n<2M\varrho$. Then

$$|w|^{2M} k(x, w) = c\int_{\mathbf{R}^n} e^{i(w,\xi)}(-\Delta_\xi)^M \sigma(x, \xi)\, d\xi,$$

where Δ_ξ denotes the Laplacian in the ξ-variable. Now $|(-\Delta_\xi)^M \sigma(x, \xi)|\leqq c_M(1+|\xi|)^{-2M\varrho}$ and since $n<2M\varrho$ it follows easily that

$$|w|^{2M}|k(x, w)| \leqq c_M$$

as we wanted to prove.

We now wish to show the sharp function estimate for general symbols $\sigma(x, \xi)\in S^{-na/2}_{1-a, \delta}$.

Lemma (2.4). *Let* $\sigma(x, \xi)\in S^{-na/2}_{1-a,\delta}$, $0<a<1$ *and* $\delta<1-a$. *Let* T *be the pseudo-differential operator with symbol* σ, *i.e.*

$$Tf(x) = \int_{\mathbf{R}^n} e^{i(x,\xi)}\sigma(x, \xi)\hat{f}(\xi)\, d\xi.$$

Then for $f\in C_0^\infty(\mathbf{R}^n)$,

$$(Tf)^{\#}(x) \leqq cM_2 f(x).$$

Proof. The proof follows the lines of the argument in Theorem 1 of Feffermann—Stein [5]. Fix a point x_0 and a cube I centered at x_0 of side length d. The non-trivial case is when $d\leqq 1$, which we consider first.

Let $f(x)=f_1(x)+f_2(x)$ with $f_1(x)=f(x)\chi_J(x)$, where J is a cube concentric with I of sidelength d^{1-a}. Let $\sigma(x, \xi)=\sigma(x, {}_m)|\xi|^{na/2}|\xi|^{-na/2}=q(x, \xi)|\xi|^{-na/2}$, say. We note that $q(x, \xi)\in S^0_{1-a,\delta}$ and by a result of Hörmander [6], the ψ.d.o. with symbol $q(x, \xi)$ is bounded on $L^2(\mathbf{R}^n)$. We denote this operator by G. We also let $1/p=1/2-a/2$. Then by the usual Hardy—Littlewood—Sobolev fractional integration theorem

$$\int_I |Tf_1(x)|dx \leqq cd^{n/p'}\Big(\int_{\mathbf{R}^n} |Tf_1(x)|^p\, dx\Big)^{1/p} \leqq cd^{n/p'}\Big(\int_{\mathbf{R}^n} |Gf_1(x)|^2 dx\Big)^{1/2}.$$

Thus by the L^2 boundedness of G we get

$$\int_I |Tf_1(x)|\, dx \leqq cd^{n/p'}\Big(\int_J |f_1(x)|^2 dx\Big)^{1/2} \leqq cd^n M_2 f(x_0).$$

We now estimate the term involving $Tf_2(x)$. Since

$$Tf_2(x) = \int_{\mathbf{R}^n} k(x, x-y) f_2(y)\, dy,$$

letting $c_I=\int_{\mathbf{R}^n} k(x_0, x_0-y)f_2(y)\,dy$ we get, for $x\in I$,

$$|Tf_2(x)-c_I| \leqq \int_{\mathbf{R}^n} |k(x, x-y)-k(x_0, x_0-y)|\,|f_2(y)|\,dy$$

$$\leqq \sum_{N=1}^{\infty}\Big(\int_{|y-x_0|\sim(2^N d)^{1-a}} |k(x, x-y)-k(x_0, x_0-y)|^2\,dy\Big)^{1/2}$$

$$\times\Big(\int_{|y-x_0|\sim(2^N d)^{1-a}} |f_2(y)|^2\,dy\Big)^{1/2}.$$

Using Lemma (2.1) for the first term in the summands on the right above we get, for $|x-x_0|\leqq d$,

$$|Tf_2(x)-c_I| \leqq c\sum_{N=1}^{\infty} d^{(1-a)(m-n/2)}(2^N d)^{-m(1-a)+n(1-a)/2} M_2 f(x_0)$$

$$\leqq c\sum_{N=1}^{\infty} 2^{N(n/2-m)(1-a)} M_2 f(x_0) \leqq cM_2 f(x_0)$$

since $m>n/2$. This concludes the case $d\leqq 1$.

In case $d>1$ we proceed as follows. Let $2I$ denote the cube concentric with I but with sidelength twice that of I. Put $f(x)=f_1(x)+f_2(x)$, where $f_1(x)=f(x)\chi_{2I}(x)$. By the boundedness of T in $L^2(\mathbf{R}^n)$

$$\int_I |Tf_1(x)|\,dx \leqq d^{n/2}\Big(\int |Tf_1(x)|^2\,dx\Big)^{1/2} \leqq cd^{n/2}\Big(\int_{2I} |f(x)|^2\,dx\Big)^{1/2} \leqq cd^n M_2 f(x_0).$$

To estimate $Tf_2(x)$ we simply use the rapid decay of $k(x, y)$. Indeed we note by Lemma (2.3) that $|k(x, x-y)|\leqq c|x-y|^{-2n}$. Therefore

$$|Tf_2(x)| \leqq c\int_{|y-x_0|>2d} |f(y)|\,|x-y|^{-2n}\,dy$$

and since $|x-x_0|\leqq d$ a well-known argument, similar to that of Theorem 2 in Chapter 3 in Stein's book [9], shows that

$$|Tf_2(x)| \leqq cM_2 f(x_0)$$

in this case. Combining all these estimates, and since I is arbitrary, we obtain the desired conclusion.

Lemma (2.5). *Given* $\sigma(x, \xi)\in S^{-na/2}_{1-a,\delta}$, $\delta<1-a$, *then for* $1<p<\infty$ *we have*

$$\|Tf\|_p \leqq c_p\|f\|_p.$$

Proof. From Lemma (2.4) it follows by Theorem 5 of Fefferman and Stein [5] that the result is valid for $2<p<\infty$. We now consider T^*, the adjoint of T. It is also a ψ.d.o. with principal symbol in $S^{-na/2}_{1-a,\delta}$ if $\delta<1-a$, by a result of Hörmander [6]. Thus T^* is bounded on $L^p(\mathbf{R}^n)$, $2<p<\infty$, by Lemma (2.4). This means that T is bounded on $L^p(\mathbf{R}^n)$, $1<p<2$, and consequently on $L^2(\mathbf{R}^n)$ as well, by interpolation. This completes the proof.

We shall now proceed to prove some lemmas which are to be used in the proof of the main result. $\theta(\xi)$ denotes a non-negative, smooth radial function so that

$$\theta(\xi) = \begin{cases} 1 & |\xi| \geqq 1 \\ 0 & |\xi| \leqq 1/2. \end{cases}$$

Lemma (2.6). *Let* $\sigma(x, \xi) \in S^0_{1,0}$ *and define*

$$G_p f(x) = \int_{\mathbf{R}^n} e^{i(\langle x,\xi\rangle + |\xi|^a)} \theta(\xi) |\xi|^{-n(2-a)(1/p-1/2)} \sigma(x, \xi) \hat{f}(\xi)\, d\xi.$$

Then, for $1<p\leqq 2$ *and* $1/p+1/q=1$,

$$\|G_p f\|_q \leqq c_p \|f\|_p.$$

Proof. We will prove this theorem by using a complex family of operators which we define, for the complex parameter z, as follows,

$$G_z f(x) = \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} \theta(\xi) \frac{e^{i|\xi|^a}}{|\xi|^z} \sigma(x, \xi) \hat{f}(\xi)\, d\xi.$$

In the first place observe that if $\operatorname{Re} z=0$, then by the L^2 continuity alluded to before we clearly have that

$$\|G_z f\|_2 \leqq c\|f\|_2. \tag{2.7}$$

Next let $\operatorname{Re} z = n(2-a)/2$, $\operatorname{Im} z = y$. In this case

$$\begin{aligned} G_z f(x) &= \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} \frac{\theta(\xi) e^{i|\xi|^a}}{|\xi|^{n(2-a)/2}} |\xi|^{-iy} \sigma(x, \xi) \hat{f}(\xi)\, d\xi \\ &= \int e^{i\langle x,\xi\rangle} \hat{k}(\xi) q(x, \xi) \hat{f}(\xi)\, d\xi, \end{aligned} \tag{2.8}$$

where $\hat{k}(\xi) = \theta(\xi) e^{i|\xi|^a}/|\xi|^{n(2-a)/2}$ and $q(x, \xi) = |\xi|^{-iy}\sigma(x, \xi)$, say. Let A denote the ψ.d.o. with symbol $\hat{k}(\xi)$ and B the ψ.d.o. with symbol $q(x, \xi)$. As is well-known the symbol $\sigma_{A\circ B}(x, \xi)$ corresponding to composition of A and B is given by the asymptotic expansion

$$\sum_{\alpha\geqq 0} c_\alpha \frac{\partial^\alpha}{\partial\xi^\alpha} \hat{k}(\xi) \frac{\partial^\alpha}{\partial x^\alpha} q(x, \xi)$$

$$= \hat{k}(\xi)\, q(x, \xi) + \sum_{0<|\alpha|<N} c_\alpha \frac{\partial^\alpha}{\partial\xi^\alpha} \hat{k}(\xi) \frac{\partial^\alpha}{\partial x^\alpha} q(x, \xi) + r(x, \xi).$$

Because of the explicit form of the kernels involved the reader can readily verify that we have

$$|r(x, \xi)| \leqq c(1+|\xi|)^{-(n+1)},$$

provided N is chosen sufficiently large, in fact $N \geqq (1+na/2)/(1-a)$ will do. Consequently, and referring back to formula (2.8), in order to obtain the desired estimates in this case it will suffice to consider the ψ.d.o. with symbol

$$\sigma_{A\circ B}(x,\xi)-\sum_{0<|\alpha|<N} c_\alpha \frac{\partial^\alpha}{\partial\xi^\alpha}\hat{k}(\xi)\frac{\partial^\alpha}{\partial x^\alpha}q(x,\xi)-r(x,\xi). \tag{2.9}$$

We readily see that

$$\left\|\int_{\mathbf{R}^n} e^{i(x,\xi)} r(x,\xi)\hat{f}(\xi)\,d\xi\right\|_\infty \leqq c\int (1+|\xi|)^{-(n+1)}\,d\xi\|f\|_1 \leqq c\|f\|_{H^1}, \tag{2.10}$$

where $H^1(\mathbf{R}^n)$ denotes the Hardy space, see Fefferman and Stein [5]. Similarly, since $q(x,\xi)\in S^0_{0,1}$ by Theorem 26 in Chapter 4 of Coifman and Meyer's book [3] it follows that B maps $H^1(\mathbf{R}^n)$ into $L^1(\mathbf{R}^n)$ boundedly. From Theorem 9 in part II of Wainger [11] we also know that the kernel $k(x)$ corresponding to A is in $L^\infty(\mathbf{R}^n)$. Therefore

$$\left\|\int_{R^n} e^{i(x,\xi)}\sigma_{A\circ B}(x,\xi)\hat{f}(\xi)\,d\xi\right\|_\infty \leqq c\,\|k\|_{L^\infty}\left\|\int_{\mathbf{R}^n} e^{i(x,\xi)}q(x,\xi)\hat{f}(\xi)\,d\xi\right\|_1$$
$$\leqq c\|k\|_{L^\infty}(1+|y|)^M\|f\|_{H^1}$$

as well.

To treat each summand in the sum (2.9) we proceed in an identical fashion. To illustrate this point fix α, $0<j=|\alpha|<N$. We have to deal with a term of the form

$$\frac{\partial^\alpha}{\partial\xi^\alpha}\hat{k}(\xi)\frac{\partial^\alpha}{\partial x^\alpha}q(x,\xi)\equiv\hat{k}_\alpha(\xi)q_\alpha(x,\xi),\ \text{say}.$$

Again by the results of Wainger we know that the kernel $k_\alpha(x)$ is in $L^\infty(\mathbf{R}^n)$ and since $q_\alpha(x,\xi)\in S^0_{1,0}$, we can write a similar expression to that appearing in (2.9) but now with the sum consisting of terms of the form

$$\sum_{0<|\beta|<N-j} c_\beta\frac{\partial^\beta}{\partial\xi^\beta}\hat{k}_\alpha(\xi)\frac{\partial^{\beta+\alpha}}{\partial x^{\beta+\alpha}}q(x,\xi).$$

Iterating this procedure, after a finite number of steps, we obtain only principal terms which can be treated as $\sigma_{A\circ B}$ and remainder terms which can be treated as it was done in (2.10). Collecting our estimates we finally obtain that for

$$\operatorname{Re} z = n(2-a)/2,\quad \operatorname{Im} z = y$$
$$\|G_z f\|_\infty \leqq c(1+|y|)^M\|f\|_{H^1}. \tag{2.11}$$

We are now in a position to interpolate between the inequalities (2.7) and (2.11). A simple argument, outlined on p. 159 of Fefferman and Stein [5], shows how we can apply Corollary 1 on p. 156 of that paper to obtain the desired result. Our proof is thus complete.

We would like to point out a simpler version of the above lemma which proves helpful later on.

Lemma (2.12). *Let* $\sigma(x,\xi)\in S^0_{1,0}$ *and define, for fixed* $\tau\in\mathbf{R}^n$,

$$G_p f(x)=\int_{\mathbf{R}^n} e^{i(x,\xi)} e^{i|\xi|^a}\theta(\xi)|\xi|^{-n(2-a)(1/p-1/2)}\sigma(\tau,\xi)\hat f(\xi)\,d\xi.$$

Then for $1<p\leqq 2$ *and* $1/p+1/q=1$,

$$\|G_p f\|_q\leqq c_p\|f\|_p.$$

Proof. The proof uses the same ideas as that of Lemma (2.6) and may be deduced from it by treating τ as a fixed parameter.

Lemma (2.13). *Let* $\eta(\xi)\in C_0^\infty(\mathbf{R}^n)$ *and radial. Assume that* $\operatorname{supp}\eta\subseteqq\{\xi: 0<r_0<|\xi|<r_1\}$, r_0, r_1 *are some fixed real numbers. For* $\lambda\geqq 1$ *and* δ *real we define*

$$(R_\lambda f)\hat{}\,(\xi)=|\xi|^\delta\eta(\xi/\lambda)\hat f(\xi).$$

Then

$$\|R_\lambda f\|_p\leqq c_p\lambda^\delta\|f\|_p,\quad 1\leqq p\leqq\infty.$$

Proof. Rewriting $(R_\lambda f)\hat{}\,(\xi)=\lambda^\delta(|\xi|/\lambda)^\delta\eta(\xi/\lambda)\hat f(\xi)$, we see that $R_\lambda f(x)= =\lambda^\delta(\psi_\lambda * f)(x)$, where $\psi(x)$ is a Schwartz function and $\psi_\lambda(x)=\lambda^n\psi(\lambda x)$. Thus

$$\|R_\lambda f\|_p\leqq\lambda^\delta\|\psi_\lambda * f\|_p\leqq\lambda^\delta\|\psi_\lambda\|_1\|f\|_p\leqq c\lambda^\delta\|f\|_p.$$

Lemma (2.14). *Given* $\varphi(\xi)\in C_0^\infty(\mathbf{R}^n)$ *such that* $\operatorname{supp}\varphi\subseteqq\{\xi: 1<|\xi|<2\}$ *define* $\varphi_\lambda(\xi)=\varphi(\xi/\lambda)$ *and let*

$$g(x)=\int_{\mathbf{R}^n} e^{i(x,\xi)}|\xi|^\delta\varphi_\lambda(\xi)\,d\xi.$$

Then

$$\|g\|_p\leqq c\lambda^{\delta+n/q},\quad 1/p+1/q=1.$$

Proof. We note that $|\xi|^\delta\varphi_\lambda(\xi)=\lambda^\delta(|\xi|/\lambda)^\delta\varphi(\xi/\lambda)$. Thus $g(x)=\lambda^{\delta+n}\psi(\lambda x)$, where ψ is a Schwartz function; the conclusion then follows by a simple change of variables.

The next lemma deals with a gradient estimate. We choose a function ϱ_1 of a single, non-negative variable supported away from the origin and in $C_0^\infty(\mathbf{R}^+)$. We extend ϱ_1 radially to $\mathbf{R}^n$, call this extension again ϱ_1, i.e. $\varrho_1(\xi)=\varrho_1(|\xi|)$, and we can now consider $\varrho_1\in C_0^\infty(\mathbf{R}^n)$. More precisely assume $\operatorname{supp}\varrho_1(\xi)\subseteqq\{\xi: (1/8)^{1/(1-a)}<|\xi|<50^{1/(1-a)}\}$, and let $\varrho_{1,j}(\xi)=\varrho_1((a/2^j d)^{1/(a-1)}|\xi|)$, $j\geqq 1$. We can now state the lemma; in applications we only need the result for $d\leqq 1$.

Lemma (2.15). *Let* $\sigma(x,\xi)\in S^{-na/2}_{1,0}$, *and define*

$$K_j(x,x-y)=\int_{\mathbf{R}^n} e^{i(x-y,\xi)} e^{i|\xi|^a}\theta(\xi)\varrho_{1,j}(\xi)\sigma(x,\xi)\,d\xi.$$

Let $|x-x_0|\leqq d$, *then for* $1<p\leqq 2$, $1/p+1/q=1$,

$$\Big(\int_{\mathbf{R}^n}|K_j(x_0,x_0-y)-K_j(x,x-y)|^q\,dy\Big)^{1/q}\leqq c\big(2^{-jn/p}d^{1-n/p}+d(2^j d)^{1/(a-1)-n/p}\big).$$

Proof. We break up the integrand into two terms,

$$\Big(\int_{\mathbf{R}^n} |K_j(x_0, x_0-y)-K_j(x, x_0-y)|^q\, dy\Big)^{1/q}$$
$$+\Big(\int_{\mathbf{R}^n} |K_j(x, x_0-y)-K_j(x, x-y)|^q\, dy\Big)^{1/q} = I_1+I_2, \quad \text{say.}$$

We will estimate I_1 first. We write the integrand as

$$K_j(x_0, x_0-y)-K_j(x, x_0-y) = -\sum_{k=1}^n \int_0^{|x-x_0|} u_k \frac{\partial}{\partial x_k} K_j(x_0+tu, x_0-y)\, dt,$$

where $u=(u_1, \dots, u_n)$ is the unit vector $(x-x_0)/|x-x_0|$. Thus

$$I_1 \leqq \sum_{k=1}^n \Big(\int_{\mathbf{R}^n} \Big|\int_0^{|x-x_0|} u_k \frac{\partial}{\partial x_k} K_j(x_0+tu, x_0-y)\, dt\Big|^q\, dy\Big)^{1/q}.$$

By Minkowski's integral inequality and Hölder's inequality with $1/p+1/q=1$, we can further majorize this by

$$\sum_{k=1}^n \int_0^{|x-x_0|} \Big(\int_{\mathbf{R}^n} \Big|\frac{\partial}{\partial x_k} K_j(x_0+tu, x_0-y)\Big|^q\, dy\Big)^{1/q} dt \tag{2.16}$$
$$\leqq \sum_{k=1}^n \Big(\int_0^{|x-x_0|} \int_{\mathbf{R}^n} \Big|\frac{\partial}{\partial x_k} K_j(x_0+tu, x_0-y)\Big|^q\, dy\, dt\Big)^{1/q} |x-x_0|^{1/p}$$
$$\leqq d^{1/p} \sum_{k=1}^n \Big(\int_0^{|x-x_0|} \int_{\mathbf{R}^n} \Big|\frac{\partial}{\partial x_k} K_j(x_0+tu, w)\Big|^q\, dw\, dt\Big)^{1/q}.$$

As the estimates are independent of the coordinates we carry them out for an arbitrary k. Put $\tau=x_0+tu$. Then the integrand of the above expressions is

$$\frac{\partial}{\partial x_k} K_j(\tau, w) = \int_{\mathbf{R}^n} e^{i(w,\xi)} e^{i|\xi|^a} \theta(\xi) \frac{\partial}{\partial \tau_k} \sigma(x, \xi) \varrho_{1,j}(\xi)\, d\xi,$$

and since $|\xi|^{na/2}\, \partial\sigma/\partial\tau_k \in S^0_{1,0}$, Lemma 2.12 applies. We may thus choose $\hat{f}_j(\xi)= \varrho_{1,j}(\xi)|\xi|^{n(2-a)(1/p-1/2)-na/2}$ and estimate (2.16) by

$$d^{1/p} \sum_{k=1}^n \Big\{\int_0^{|x-x_0|} \Big(\int_{\mathbf{R}^n} \Big|\frac{\partial}{\partial x_k} K_j(x_0+tu, w)\Big|^q\, dw\Big)^{q/q} dt\Big\}^{1/q} \tag{2.17}$$
$$\leqq cd^{1/p} \sum_{k=1}^n \Big(\int_0^{|x-x_0|} \|f_j\|_p^q\, dt\Big)^{1/q} \leqq cd^{1/p} \|f_j\|_p d^{1/q} = cd\, \|f_j\|_p.$$

With a minor adjustment of the constants involved because of the support of ϱ_1, we may employ Lemma 2.14 with $\delta=n(2-a)(1/p-1/2)-na/2$, $\lambda=(2^j d)^{1/(a-1)}$

to get that (2.17) is bounded by $cd(2^j d)^{(\delta+n/q)/(a-1)}$. A simple computation shows that $(\delta+n/q)/(a-1)=-n/p$. Thus

$$I_1 \leqq cd^{1-n/p}2^{-jn/p}. \tag{2.18}$$

We now estimate the main term, i.e. I_2. Now,

$$K_j(x, x_0-y)-K_j(x, x-y) = \sum_{k=1}^n \int_0^{|x-x_0|} u_k \frac{\partial}{\partial y_k} K_j(x, x_0+tu-y)\,dt,$$

where $u=(x-x_0)/|x-x_0|$ is the unit vector with components $(u_1, \dots, u_n)$. Consequently I_2 is majorized by

$$d^{1/p} \sum_{k=1}^n \left(\int_0^{|x-x_0|} \int_{\mathbf{R}^n} \left|\frac{\partial}{\partial y_k} K_j(x, x_0+tu-y)\right|^q dy\,dt\right)^{1/q}$$

$$= d^{1/p} \sum_{k=1}^n \left(\int_0^{|x-x_0|} \int_{\mathbf{R}^n} \left|\frac{\partial}{\partial w_k} K_j(x, w)\right|^q dw\,dt\right)^{1/q}.$$

Note that

$$\frac{\partial}{\partial w_k} K_j(x, w) = \int_{\mathbf{R}^n} e^{i(w,\xi)} e^{i|\xi|^a} \theta(\xi)\xi_k \sigma(x, \xi)\varrho_{1,j}(\xi)\,d\xi$$

$$= \int_{\mathbf{R}^n} e^{i(w,\xi)} e^{i|\xi|^a} \frac{\theta(\xi)}{|\xi|^{n(2-a)(1/p-1/2)}} \sigma_k(x, \xi)\hat{f}_j(\xi)\,d\xi,$$

where $\sigma_k(x, \xi)=\xi_k|\xi|^{na/2-1}\sigma(x, \xi)$ and $\hat{f}_j(\xi)=\varrho_{1,j}(\xi)|\xi|^{1-na/2}|\xi|^{n(2-a)(1/p-1/2)}$. Now $\sigma_k(x, \xi)\in S^0_{1,0}$ and as above we may apply Lemmas (2.12) and (2.14) to get

$$\left(\int_{\mathbf{R}^n} \left|\frac{\partial}{\partial w_k} K_j(x, w)\right|^q dw\right)^{1/q} \leqq c(2^j d)^\mu$$

with

$$\mu = [n/q-1-na/2+n(2-a)(1/p-1/2)]/(a-1) = 1/(a-1)-n/p.$$

Thus I_2 is majorized by $cd(2^j d)^{1/(a-1)-n/p}$. Combining this estimate with that for I_1 we obtain the desired conclusion.

The next sequence of lemmas deals with asymptotic expansions. Let ϱ_2 be a smooth function of the positive real numbers vanishing on $[(1/4)^{1/(1-a)}, \infty)$. Let $\varrho_2\in C_0^\infty(\mathbf{R}^n)$ also denote the radial extension of the above function, i.e. $\varrho_2(\xi)=\varrho_2(|\xi|)$.

Lemma (2.19). *Let* $\varrho_2(\xi)\in C_0^\infty(\mathbf{R}^n)$ *be as above and put* $\varrho_{2,j}(\xi)=\varrho_2\big((a/2^j d)^{1/(a-1)}|\xi|\big)$; $j\geqq 1$ *and* $d\leqq 1$. *For* $\sigma(x, \xi)\in S^{-na/2}_{1,0}$ *put*

$$K_j(x, y) = \int_{\mathbf{R}^n} e^{i(y,\xi)} e^{i|\xi|^a} \theta(\xi)\sigma(x, \xi)\varrho_{2,j}(\xi)\,d\xi.$$

Then for $|y|\sim 2^j d$, *more precisely* $2^{j-1}d<|y|<2^j d$, $2^j d\leqq 1$, *we have for* $\varepsilon=\varepsilon(a)>0$,

$$|K_j(x, y)| \leqq c|y|^{-n+\varepsilon}.$$

Proof. We express $K_j(x, y)$ in polar coordinates. Thus for $y=|y|y'$, $\xi=|\xi|\xi'=r\xi'$,

$$K_j(x, y) = \int_{S^{n-1}}\int_0^\infty e^{i(r^a+|y|r(y',\xi'))}\theta(r)\varrho_{2,j}(r)\sigma(x, r\xi')r^{n-1}dr\, d\xi'.$$

We now integrate by parts the inner integral. Before we proceed we note that $\theta(r)\varrho_{2,j}(r)$ is supported in $\{r: 0\leqq r<(2^{j+2}d/a)^{1/(a-1)}\}$. Since $0<a<1$ we have for $|y|\sim 2^j d$ that

$$(2.20)\qquad |ar^{a-1}+|y|(y', \xi')| \geqq ar^{a-1}-|y| \geqq cr^{a-1}.$$

Thus integration by parts yields

$$(2.21)\qquad \int_0^\infty e^{i(r^a+|y|r(y',\xi'))}\theta(r)\varrho_{2,j}(r)\sigma(x, r\xi')r^{n-1}dr$$

$$= i\int_0^\infty e^{i(r^a+|y|r(y',\xi'))}\frac{d}{dr}\big[(ar^{a-1}+|y|(y', \xi'))^{-1}\theta(r)\varrho_{2,j}(r)\sigma(x, r\xi')r^{n-1}\big]dr.$$

This process may be carried out repeatedly depending on the value of a. Moreover in view of the support of $\varrho_{2,j}(r)$ the range of integration extends only to $(2^j d/a)^{1/(a-1)}\sim c|y|^{1/(a-1)}$. We now observe that each integration by parts yields an extra factor of r^{-a} in the integrand. For e.g. if we estimate the integrand on the right in (2.21) we note that

$$\frac{d}{dr}\big[(ar^{a-1}+|y|(y', \xi'))^{-1}\theta(r)\varrho_{2,j}(r)\sigma(x, r\xi')r^{n-1}\big]$$
$$= a(1-a)r^{a-2}(ar^{a-1}+|y|(y', \xi'))^{-2}\theta(r)\varrho_{2,j}(r)\sigma(x, r\xi')r^{n-1}$$
$$+(ar^{a-1}+|y|(y', \xi'))^{-1}\theta'(r)\varrho_{2,j}(r)\sigma(x, r\xi')r^{n-1}$$
$$+(ar^{a-1}+|y|(y', \xi'))^{-1}\theta(r)\varrho'_{2,j}(r)\sigma(x, r\xi')r^{n-1}$$
$$+(ar^{a-1}+|y|(y', \xi'))^{-1}\theta(r)\varrho_{2,j}(r)\frac{d}{dr}\sigma(x, r\xi')r^{n-1}$$
$$+(n-1)(ar^{a-1}+|y|(y', \xi'))^{-1}\theta(r)\varrho_{2,j}(r)\sigma(x, r\xi')r^{n-2} = I_1+I_2+I_3+I_4+I_5, \quad \text{say}.$$

We now apply (2.20) to I_1 and use the fact that $|\sigma(x, r\xi')|\leqq cr^{-na/2}$. Thus I_1 is dominated by

$$(2.22)\qquad c\theta(r)r^{a-2+2(1-a)+n-1-na/2} \leqq c(1+r)^{n-1-a-na/2}.$$

As for I_2 we note that $\theta'(r)$ is supported in $\{r: 1/2<r<4\}$. Thus again by (2.20), (2.22) holds trivially. For I_3 we note that $|\varrho'_{2,j}(r)|\leqq c(2^j d)^{1/(1-a)}\leqq cr^{-1}$. Again by (2.20), I_3 is bounded by $c(1+r)^{n-1-a-na/2}$. For I_4, $|d/dr\,\sigma(x, r\xi')|\leqq cr^{-na/2-1}$, and we again arrive at (2.22). For the last term, I_5, because $|\sigma(x, r\xi')|\leqq cr^{-na/2}$

and (2.20) hold, we get immediately (2.22). Thus the right side of (2.21) may be bounded by

$$c\int_0^{c|y|^{1/(a-1)}}(1+r)^{n-1-a-na/2}\,dr \leqq c|y|^{-n+a(1-n/2)/(1-a)}.$$

When $n=1$, $a(1-n/2)/(1-a)=a/2(1-a)=\varepsilon(a)>0$. For $n>1$ we integrate the right side of (2.21) by parts once more to get

$$i\int_0^{\infty} e^{i(r^a+|y|r(y',\xi'))}\frac{d}{dr}\big[(ar^{a-1}+|y|(y',\xi')^{-1}$$
$$\times\frac{d}{dr}\{(ar^{a-1}+|y|(y',\xi'))^{-1}\theta(r)\varrho_{2,j}(r)\sigma(x,r\xi')r^{n-1}\}\big]\,dr.$$

The gain in the integrand is now r^{-2a}. Therefore such repeated integration by parts gives the requisite decay at infinity and thus the lemma. In fact one may show that $|K_j(x,y)|\leqq c$, but for our purpose it is enough to prove that $|K_j(x,y)|\leqq c|y|^{-n+\varepsilon}$, $\varepsilon>0$. This is what we have just done.

We shall now need to perform a stationary phase computation. To do so we reguire the following lemma. Its proof was supplied to us by Ravi Kulkarni.

Lemma (2.23). *Let $\mu_1,\dots,\mu_n$ be real numbers such that $\sum_{j=1}^n \mu_j^2=1$. Let A be the $n\times n$ matrix given by*

$$A=\begin{pmatrix}\mu_1^2+\mu_n^2 & \mu_1\mu_2 & \cdots & \mu_1\mu_{n-1}\\ \mu_2\mu_1 & \mu_2^2+\mu_n^2 & \cdots & \mu_2\mu_{n-1}\\ \vdots & & & \\ \mu_{n-1}\mu_1 & \mu_{n-1}\mu_2 & \cdots & \mu_{n-1}^2+\mu_n^2\end{pmatrix}.$$

Then $\det A=\mu_n^{2(n-2)}$.

Proof. We construct the $n\times n$ matrix B as follows

$$B=\begin{pmatrix}1 & \mu_1 & \mu_2 & \cdots & \mu_{n-1}\\ 0 & \mu_1^2+\mu_n^2 & \mu_1\mu_2 & \cdots & \mu_1\mu_{n-1}\\ 0 & \mu_2\mu_1 & \mu_2^2+\mu_n^2 & \cdots & \mu_2\mu_{n-1}\\ \vdots & & & & \\ 0 & \mu_{n-1}\mu_1 & \mu_{n-1}\mu_2 & \cdots & \mu_{n-1}^2+\mu_n^2\end{pmatrix}.$$

It is evident that $\det B=\det A$. Now in B we perform the row operations $R_i-\mu_{i-1}R_1$, $i=2,3,\dots,n$, to get

$$\det B=\det\begin{pmatrix}1 & \mu_1 & \mu_2 & \cdots & \mu_{n-1}\\ -\mu_1 & \mu_n^2 & 0 & \cdots & 0\\ \vdots & 0 & \mu_n^2 & \cdots & 0\\ \vdots & \vdots & & & \vdots\\ -\mu_{n-1} & 0 & 0 & \cdots & \mu_n^2\end{pmatrix}.$$

Expanding by the first row we get

$$\det A = \det B = \mu_n^{2(n-1)} + \mu_n^{2(n-2)} \sum_{j=1}^{n-1} \mu_j^2.$$

But $\sum_{j=1}^n \mu_j^2 = 1$, thus $\det A = \mu_n^{2(n-1)} + \mu_n^{2(n-2)}(1-\mu_n^2) = \mu_n^{2(n-2)}$, as we wanted to show.

We now need to recall the asymptotic formula for the principle of stationary phase. The proof of this proposition may be found in Chapter 7, volume II, of Treves' book [10]. To state the proposition let us introduce some notation. Points in $\mathbf{R}^k$ are denoted by σ. $f(\sigma) \in C^\infty(\mathbf{R}^k)$ and is real-valued. $g(x, \sigma) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^k)$ and is compactly supported in the variable σ. Moreover, and in order to satisfy the hypothesis of the principle of stationary phase, we shall assume that σ_0 is the unique critical value of $f(\sigma)$ in the support of $g(x, \sigma)$ in the variable σ. That is, $\nabla f(\sigma_0) = 0$, but $\det H_0 \neq 0$, where H_0 denotes the Hessian of $f(\sigma)$ at σ_0. Moreover, we shall assume that $g(x, \sigma)$ is uniformly bounded along with its partial derivatives in both x and σ variables.

With the usual notation $D_\sigma = 1/i \left(\dfrac{\partial}{\partial \sigma_1}, \dots, \dfrac{\partial}{\partial \sigma_k}\right)$ we have

Proposition (2.24). *For* $t \to \infty$, *and for any* $M > 0$,

$$\int_{\mathbf{R}^k} e^{itf(\sigma)} g(x, \sigma)\, d\sigma = \left(\frac{2\pi}{t}\right)^{k/2} |\det H_0|^{-1/2} \exp\left\{i \left[tf(\sigma_0) + \frac{\pi}{4} \operatorname{sgn} H_0\right]\right\}$$

$$\times \sum_{j=0}^M \left(\frac{i/2}{j}\right)^j (H_0^{-1} D_\sigma D_\sigma)^j g(x, \sigma_0) t^{-j} + O(t^{-M-1-k/2}).$$

The O bound is uniform and does not depend on x, because by assumption $g(x, \sigma)$ and all its derivatives in both x and σ variables are in $L^\infty(\mathbf{R}^n \times \mathbf{R}^k)$. For the next lemma we need one more new notation. Let ϱ_3 be a cut-off function on the positive real line. More precisely, ϱ_3 is a C^∞ function which vanishes for $0 \leqq t < (1/2)^{1/(a-1)}$ and equals 1 for $t > 3^{1/(1-a)}$. Extend ϱ_3 radially to $\mathbf{R}^n$, and for $j \geqq 1$ put $\varrho_{3,j}(\xi) = \varrho_3((a/2^j d)^{1/(a-1)} |\xi|)$. Then $\operatorname{supp} \varrho_{3,j} \subseteqq \{\xi \colon |\xi| > (2^{j-1} d/a)^{1/(a-1)}\}$ and $\varrho_{3,j} \equiv 1$ for $|\xi| > 3^{1/(a-1)} (2^j d/a)^{1/(a-1)}$.

Lemma (2.25). *Let* $\sigma(x, \xi) \in S_{1,0}^{-na/2}$ *and define*

$$K_j(x, y) = \int_{\mathbf{R}^n} e^{i(\langle y, \xi\rangle + |\xi|^a)} \theta(\xi) \varrho_{3,j}(\xi) \sigma(x, \xi)\, d\xi.$$

Then for $|y| \sim 2^j d$, *more precisely for* $2^{j-1} d < |y| < 2^j d \leqq 1$, $|K_j(x, y)| \leqq c$.

Proof. We first express the integral defining $K_j(x,y)$ in polar coordinates to get, for $y'\in S^{n-1}$,

(2.26)
$$K_j(x, y) = \int_{1/2}^{\infty} e^{ir^a} \varrho_{3,j}(r) r^{-na/2+n-1} \times \int_{S^{n-1}} e^{ir|y|(y',\xi')} \theta(r) r^{na/2} \sigma(x, r\xi')\, d\xi'\, dr.$$

Our aim now is to apply the stationary phase principle to the inner integral. Thus our goal is to convert the inner integral into a form where Proposition (2.24) applies. Put $y'=(\mu_1, \ldots, \mu_n)$, $\sum_{j=1}^n \mu_j^2=1$. Since those $y'\in S^{n-1}$ with one of the μ_j's$=0$, $1\leqq j\leqq n$, form a set of measure zero and can thus be disregarded, we only consider those y's such that $\mu_j\neq 0$ for all j, $1\leqq j\leqq n$. We will first show that there exist an integer M and C^∞ functions φ_m, ζ_m, $1\leqq m\leqq M$ such that

(2.27)
$$\int_{S^{n-1}} e^{ir|y|(y',\xi')} \theta(r) r^{na/2} \sigma(x, r\xi')\, d\xi'$$
$$= ce^{ir|y|}(r|y|)^{-(n-1)/2} \sum_{m=0}^{M} (r|y|)^{-m} \varphi_m(x, r, \mu_1, \ldots, \mu_n)$$
$$+ce^{-ir|y|}(r|y|)^{-(n-1)/2} \sum_{m=0}^{M} (r|y|)^{-m} \zeta_m(x, r, \mu_1, \ldots, \mu_n) + O\big((r|y|)^{-M-(n+1)/2}\big).$$

Moreover, for $1\leqq m\leqq M$ and $j=0, 1, 2, \ldots$

(2.28)
$$\left|\frac{\partial^j}{\partial r^j} \varphi_m(x, r, \mu_1, \ldots, \mu_n)\right| \leqq cr^{-j},$$
$$\left|\frac{\partial^j}{\partial r^j} \zeta_m(x, r, \mu_1, \ldots, \mu_n)\right| \leqq cr^{-j},$$

and the O constant may be taken uniformly independent of x, r and $\mu_1, \ldots, \mu_n$.

We choose a finite and smooth partition of unity as follows. First construct a band around the equator of S^{n-1} and cover this band by a finite number of surface balls. Together with the semi-hemispheres of $S^{n-1}\setminus$band containing the North and South poles of S^{n-1}, we obtain N regions which we call $\sum_k$, $1\leqq k\leqq N$. The illustration on the next page will clarify this situation. Let now $\{\psi_k\}_{k=1}^N$ be a smooth partition of unity on the surface of S^{n-1} such that supp $\psi_k \subseteq \sum_k$.

The choice of $\sum_k$ is made in a manner so that the projection of $\sum_k$ onto one of the coordinate hyperplanes $\sigma_i=0$, $i=1, \ldots, n$, is non-singular, i.e. the Jacobian of the projection of $\sum_k$ onto one of the hyperplanes $\sigma_i=0$ is non-zero there. Now

$$\int_{S^{n-1}} e^{ir|y|(y',\xi')} \theta(r) r^{na/2} \sigma(x, r\xi')\, d\xi'$$
$$= \sum_{k=1}^{N} \int_{S^{n-1}} e^{ir|y|(y',\xi')} \theta(r) \psi_k(\xi') r^{na/2} \sigma(x, r\xi')\, d\xi' \equiv \sum A_k, \quad \text{say.}$$

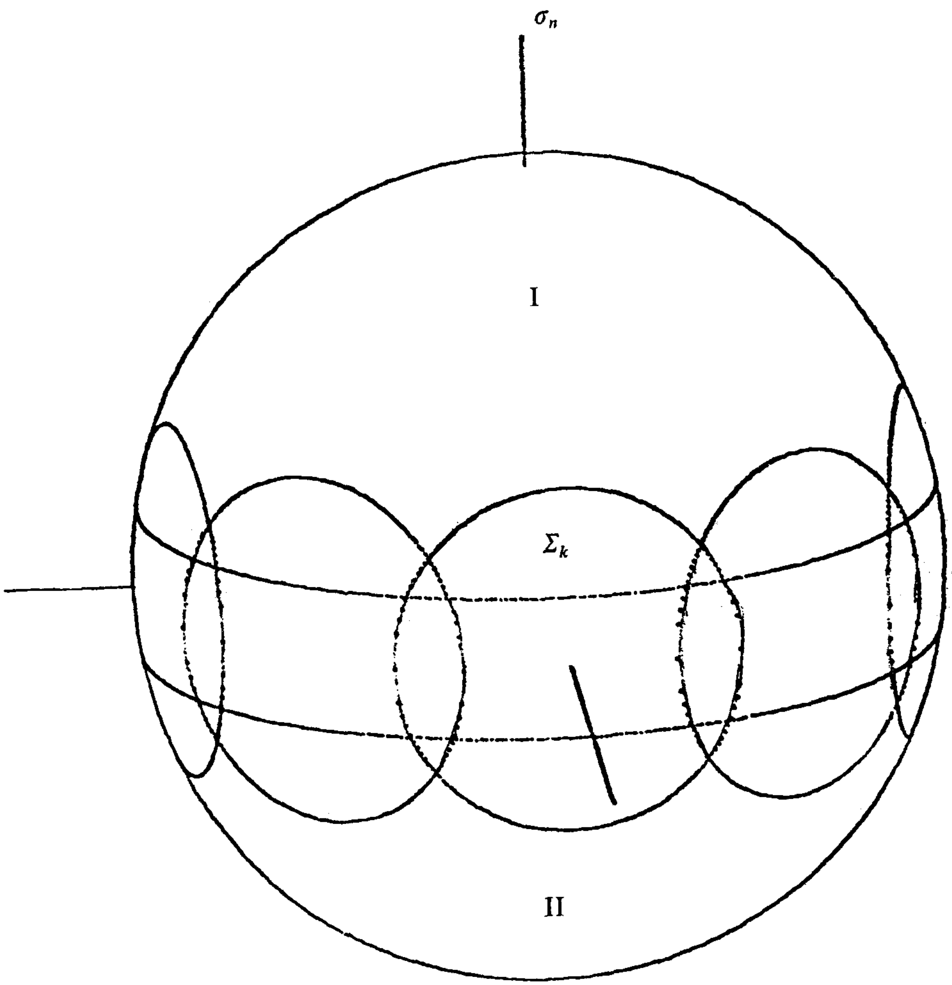

Figure 1

From now on we set $r|y|=t$. Since $|y|\sim 2^j d$ and $r>c(2^j d)^{1/(a-1)}$, we note that $t>c(2^j d)^{a/(a-1)}\geqq 1$ since $2^j d\leqq 1$. Thus as $j\to\infty$, $t\to\infty$. Let us consider one term in the right side above. Assume that we are considering the region I, and that ψ_1 is the function corresponding to this region. Clearly we may project region I onto $\sigma_n=0$ and the projection is non-singular. Thus we have

$$A_1=\int_{\mathbf{R}^{n-1}} e^{it\left(\sum_{j=1}^{n-1}\sigma_j\mu_j+\mu_n\sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2}\right)}\psi_1\left(\sigma_1,\dots,\sigma_{n-1},\sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2}\right)$$
$$\times\varphi(x,r,\sigma_1,\dots,\sigma_{n-1})\,d\sigma_1\dots d\sigma_{n-1},\quad\text{with}$$

$$\varphi(x,r,\sigma_1,\dots,\sigma_{n-1})=\theta(r)r^{na/2}\sigma\left(x,r\sigma_1 r\sigma_2,\dots,r\sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2}\right)J(\sigma_1,\dots,\sigma_{n-1}),$$

where J is the Jacobian of the projection. We notice that in the projection of the support of ψ_1 onto $\sigma_n=0$, we have $\sum_{j=1}^{n-1}\sigma_j^2 \leqq c<1$. Thus, if we define

$$\varphi_1(x, r, \sigma_1, \ldots, \sigma_{n-1}) = \psi_1\left(\sigma_1, \ldots, \sigma_{n-1}, \sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2}\right) \varphi(x, r, \sigma_1, \ldots, \sigma_{n-1}),$$

then it follows that $\varphi_1(x, r, \sigma_1, \ldots, \sigma_{n-1})$ is compactly supported and C^∞ in $\sigma_1, \ldots$ $\ldots, \sigma_{n-1}$, and bounded in x and r. Thus A_1 equals

(2.29)

$$\int_{\sum_{j=1}^{n-1}\sigma_j^2 \leqq c<1} e^{it\left(\sum_{j=1}^{n-1}\sigma_j\mu_j+\mu_n\sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2}\right)} \varphi_1(x, r, \sigma_1, \ldots, \sigma_{n-1})\, d\sigma_1 \ldots d\sigma_{n-1}.$$

We may now apply Proposition (2.24). To do so we compute the critical point σ_0 of $\sum_{j=1}^{n-1}\sigma_i\mu_i+\mu_n\sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2}$, and arrive at the system of equations

$$\mu_i = \mu_n\sigma_i\Big/\sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2}, \quad i = 1, 2, \ldots, n-1.$$

Assuming that $\mu_n>0$ (the case $\mu_n<0$ can be treated in an entirely analogous manner) and since $\sum_{j=1}^{n}\mu_j^2=1$, the solution of the above system is $\sigma_i=\mu_i$. Thus $\sigma_0=(\mu_1, \ldots, \mu_{n-1})$. Now the Hessian H_0 of $\sum_{j=1}^{n-1}\sigma_j\mu_j+\mu_n\sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2}$ at σ_0 is

$$-(\mu_n^{-2})\begin{pmatrix} \mu_1^2+\mu_n^2 & \mu_1\mu_2 & \ldots \mu_1\mu_{n-1} \\ \mu_1\mu_2 & \mu_2^2+\mu_n^2 & \ldots \mu_2\mu_{n-1} \\ \vdots & & \\ \mu_1\mu_{n-1} & \mu_2\mu_{n-1} & \ldots \mu_{n-1}^2+\mu_n^2 \end{pmatrix}.$$

So by Lemma (2.23), $\det H_0=(-1)^{n-1}\mu_n^{2(n-2)}/\mu_n^{2(n-1)}=(-1)^{n-1}\mu_n^{-2}$. We now see by virtue of Proposition (2.24) that

$$A_1 = (2\pi/t)^{(n-1)/2}|\mu_n| \exp\{it\pm\pi/4\}$$
$$\times\sum_{m=0}^{M}\frac{(i/2)^m}{m!}(\mu_n^2 D_\sigma D_\sigma)^m\varphi_1(x, r, \sigma_0)t^{-m}+O(t^{-M-1-(n-1)/2}).$$

We let

$$|\mu_n|\frac{(i/2)^m}{m!}(\mu_n^2 D_\sigma D_\sigma)^m\varphi_1(x, r, \sigma_0) = \varphi_m(x, r, \mu_1, \ldots, \mu_n).$$

Since $\theta(r)r^{na/2}\sigma(x, r\xi')\in S_{1,0}^0$, we can establish (2.28) by a direct computation. Recalling that $t=r|y|$, we see at once that

$$A_1 = ce^{ir|y|}(r|y|)^{(n-1)/2}\sum_{m=0}^{M}(r|y|)^{-m}\varphi_m(x, r, \mu_1, \ldots, \mu_m)+O\left((r|y|)^{-M-(n+1)/2}\right).$$

We repeat this process for each of the functions ψ_k in the partition arriving in each case at integrals as in (2.29). Moreover in each case, depending on which hyperplane we project, the phase function will be

$$\sum_{j=1, j\neq k}^{n}\sigma_j\mu_j\pm\mu_k\sqrt{1-\sum_{j=1, j\neq k}^{n}\sigma_j^2}.$$

The negative sign arises if we consider a region such as II in the picture, where $\sigma_n = -\sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2}$ and thus

$$(y', \sigma) = \sum_{j=1}^{n-1}\sigma_i\mu_i - \mu_n\sqrt{1-\sum_{j=1}^{n-1}\sigma_j^2},$$

for $\sigma\in$II. Integrals arising from such a region give rise to the second term on the right in (2.27). We have thus proved (2.27).

Now choose M so that $n-1-na<2M$. With this choice of M we substitute the right side of (2.27) for the inner integral of (2.26) to get

$$|K_j(x, y)| \leqq c\Big(|y|^{-(n-1)/2}\Big|\int_{1/2}^{\infty} e^{i(r^a+r|y|)}\varrho_{3,j}(r)r^{-na/2+(n-1)/2} \times \sum_{m=0}^{M}(r|y|)^{-m}\varphi_m(x, r, \mu_1, \dots, \mu_n)\,dr\Big|\Big)$$
$$+c\Big(|y|^{-(n-1)/2}\Big|\int_{1/2}^{\infty} e^{i(r^a-r|y|)}\varrho_{3,j}(r)\,r^{-na/2+(n-1)/2}\sum_{m=0}^{M}(r|y|)^{-m}\zeta_m(x, r, \mu_1, \dots, \mu_n)\,dr\Big|\Big)$$
$$+c|y|^{-M-(n+1)/2}\int_{1/2}^{\infty}\varrho_{3,j}(r)r^{-na/2-M+(n-1)/2-1}\,dr \equiv |I_1|+|I_2|+|I_3|, \quad \text{say.}$$

Recalling that $\operatorname{supp}\varrho_{3,j}\subset\{r\colon r>c|y|^{1/(a-1)}\}$ we see that the term $|I_3|$ above is at most

$$c|y|^{-M-(n+1)/2}\int_{|y|^{1/(a-1)}}^{\infty} r^{-na/2-M+(n-1)/2-1}\,dr = c|y|^{(M+1/2)a/(1-a)-n}.$$

Thus if $n<(M+1/2)a/(1-a)$, a choice which is compatible with the earlier determination of M, then because $|y|\leqq 1$ the term $|I_3|$ is uniformly bounded.

We handle I_1 and I_2 by repeated integration by parts. The technique for both terms is the same and thus for brevity we only consider I_1. We first note that the derivative of the phase function in I_1 for r in the support of $\varrho_{3,j}(r)$ satisfies the estimate

$$(2.30) \qquad \big|ar^{a-1}+|y|\big| \geqq |y|-ar^{a-1} \geqq c|y|.$$

Thus performing an integration by parts in I_1 we get

$$(2.31) \qquad I_1 = c|y|^{-(n-1)/2}\int_{1/2}^{\infty} e^{i(r^a+r|y|)}\frac{\partial}{\partial r}\big[(ar^{a-1}+|y|)^{-1} \times \varrho_{3,j}(r)r^{-na/2+(n-1)/2}(r|y|)^{-m}\varphi_m(x, r, \mu_1, \dots, \mu_n)\big]\,dr.$$

The process above is a typical step and we may carry it out repeatedly. At each step the integrand decays by a factor of $(r|y|)^{-1}$ over the previous step. Let us show this for the first step. We now make use of the estimates (2.28) and (2.30) to estimate the integrand above. Now

$$\frac{\partial}{\partial r}[(ar^{a-1}+|y|)^{-1}\varrho_{3,j}(r)r^{-na/2+(n-1)/2}(r|y|)^{-m}\varphi_m(x, r, \mu_1, \dots, \mu_n)]$$

$$= a(1-a)r^{a-2}(ar^{a-1}+|y|)^{-2}\varrho_{3,j}(r)r^{-na/2+(n-1)/2}(r|y|)^{-m}\varphi_m(x, r, \mu_1, \dots, \mu_n)$$

$$+\varrho'_{3,j}(r)(ar^{a-1}+|y|)^{-1}r^{-na/2+(n-1)/2}(r|y|)^{-m}\varphi_m(x, r, \mu_1, \dots, \mu_n)$$

$$+(ar^{a-1}+|y|)^{-1}\varrho_{3,j}(r)\frac{\partial}{\partial r}(r^{-na/2+(n-1)/2}(r|y|)^{-m}\varphi_m(x, r, \mu_1, \dots, \mu_n))$$

$$= J_1+J_2+J_3, \quad \text{say.}$$

Consider J_1 first. Now for $r\in\operatorname{supp}\varrho_{3,j}(r)$, we have (2.30), thus $r^{a-2}(ar^{a-1}+|y|)^{-2}\leqq c(r^{a-1}|y|^{-1})(r|y|)^{-1}\leqq c(r|y|)^{-1}$. The other factors appearing in the integrand of J_1 coincide with the original factors in the integrand of I_1. Thus we have verified our claim in this case.

For J_2 we simply note that $|\varrho'_{3,j}(r)|\leqq cr^{-1}$ and thus again because of (2.30) our claim is verified. In view of (2.28) the claim follows for J_3, too. Thus performing $M-m+1$ integration by parts for the m^{th} summand in (2.31), we get that I_1 reduces to the case I_3, or in other words,

$$|I_1| \leqq c|y|^{-(n-1)/2}\sum_{m=0}^{M}\int_{|y|^{1/(a-1)}}^{\infty}(r|y|)^{-M+m-1}r^{-na/2+(n-1)/2}(r|y|)^{-m}\,dr$$

$$= c|y|^{-M-(n+1)/2}\int_{|y|^{1/(a-1)}}^{\infty}r^{-na/2-M+(n-1)/2-1}\,dr = c|y|^{(M+1/2)a/(1-a)-n}.$$

But because $|y|\leqq 1$ the choice of M yields that $|I_1|\leqq 1$.

To handle I_2 we note that the derivative of the phase function is $ar^{a-1}-|y|$, and for $r\in\operatorname{supp}\varrho_{3,j}(r)$ we again have $|ar^{a-1}-|y||\geqq c|y|$ and we proceed again as we did for I_1. This completes our proof.

3. The basic estimate

We are now ready to prove Theorem (1.2) of the introduction. Before beginning the proof we note that if needed we may assume that $\sigma(x, \xi)$ is supported in $|\xi|\geqq 1/2$, in the ξ-variable. The reason being that we may write

$$e^{i|\xi|^a}\sigma(x, \xi) = \theta(\xi)e^{i|\xi|^a}\sigma(x, \xi)+(1-\theta(\xi))e^{i|\xi|^a}\sigma(x, \xi),$$

where $\theta(\xi)\in C_0^\infty(\mathbf{R}^n)$ is a cut-off radial function, which equals 1 at infinity and vanishes near the origin. Then a direct computation yields that $(1-\theta(\xi))e^{i|\xi|^a}\sigma(x, \xi)$ has a kernel which satisfies an L^q-Hörmander condition, for $1<q<\infty$. This in turn yields, see for instance Kurtz and Wheeden [7], that the operator induced by the symbol $(1-\theta(\xi))e^{i|\xi|^a}\sigma(x, \xi)$ indeed satisfies the sharp function estimate of Theorem (1.2). Thus the main point is to prove the estimate for the ψ.d.o. induced by $\theta(\xi)e^{i|\xi|^a}\sigma(x, \xi)$.

Proof of Theorem (1.2). Fix a point x_0 and a cube Q centered at x_0. The proof of the basic estimate breaks up into the analysis of two cases according to the size of Q. We let diameter $Q=d$ and consider the trivial case first.

Case 1. Suppose $d\geqq 1/4$. Let $3Q$ denote the cube concentric with Q but with diameter $3d$ and let $f_1(x)=f(x)\chi_{3Q}(x)$, $f_2(x)=f(x)-f_1(x)$. Thus supp f_2 lies outside $3Q$. Now

$$d^{-n}\int_Q |Tf(x)|\,dx \leqq d^{-n}\int_Q |Tf_1(x)|\,dx+d^{-n}\int_Q |Tf_2(x)|\,dx.$$

For the first term on the right above we have

$$d^{-n}\int_Q |Tf_1(x)|\,dx \leqq \left(d^{-n}\int_{\mathbf{R}^n} |Tf_1(x)|^p\,dx\right)^{1/p}, \quad 1 \leqq p < \infty.$$

But by Lemma (2.5), for any p, $1<p<\infty$,

$$d^{-n}\int_{\mathbf{R}^n} |Tf_1(x)|^p\,dx^{1/p} \leqq c\left(d^{-n}\int_{3Q} |f_1(x)|^p\,dx\right)^{1/p} \leqq cM_pf(x_0).$$

Now, there is a constant c_1 (depending only on the dimension n) so that supp $f_2\subseteqq \{y\colon |x_0-y|>c_1d\}$ and consequently for $x\in Q$, $f_2(x-y)$ vanishes unless $|x-y|> c_2d$, where c_2 is another (dimensional) constant. Consequently

$$Tf_2(x) = \int_{|x-y|>c_2d} k(x, x-y)f(y)\,dy$$

and since $d\geqq 1/4$ by Lemma (2.3), for $x\in Q$ we have

$$|Tf_2(x)| \leqq \int_{|x_0-y|>c_2d} |f(y)|/|x-y|^{2n}dy \leqq cM_pf(x_0).$$

Thus in this case $d^{-n}\int_Q |Tf(x)|dx\leqq cM_pf(x_0)$, as we wanted to show.

Case 2. The case when $d\leqq 1/4$. Let $\chi_1(x)$ denote the characteristic function of the set $\{x\colon |x-x_0|<2d\}$, $\chi_2(x)$ that of the set $\{x\colon 2d\leqq|x-x_0|<d^{1-a}\}$, $\chi_3(x)$ that of the set $\{x\colon d^{1-a}\leqq|x-x_0|<1/2\}$ and $\chi_4(x)$ that of the set $\{x\colon |x-x_0|\geqq 1/2\}$. Put $f_j(x)=f(x)\chi_j(x)$, $1\leqq j\leqq 4$. Thus $f(x)=f_1(x)+f_2(x)+f_3(x)+f_4(x)$. We first consider $f_1(x)$ and $f_4(x)$.

Now, by Lemma (2.5) for any p, $1<p<\infty$,

$$d^{-n}\int_Q |Tf_1(x)|\,dx \leqq \left(d^{-n}\int_Q |Tf_1(x)|^p\,dx\right)^{1/p}$$
$$\leqq c\left(d^{-n}\int_{|x-x_0|<2d} |f_1(x)|^p\,dx\right)^{1/p} \leqq cM_pf(x_0),$$

which ends the estimate for $f_1(x)$. Now for $x\in Q$, by Lemma (2.3)

$$|Tf_4(x)| \leqq c\int_{|x_0-y|>1/2} |f_4(y)|/|x-y|^{2n}\,dy$$
$$\leqq c\int_{|x_0-y|>1/2} |f(y)|/|x_0-y|^{2n}\,dy \leqq cM_pf(x_0).$$

As this is a uniform estimate for $x\in Q$, we see that

$$d^{-n}\int_Q |Tf_4(x)|\,dx \leqq cM_pf(x_0).$$

We are thus left with estimating $Tf_2(x)$ and $Tf_3(x)$. We begin by estimating $Tf_2(x)$. We break up $f_2(x)$ by setting $f_2(x)=\sum_{j=2}^{j_0} f_j(x)$, where $f_j(x)=f_2(x)\varkappa\{x: 2^{j-1}d<|x-x_0|<2^jd\}$ with $2^{j_0}d\sim d^{1-a}$. Now,

$$Tf_2(x)=\sum_{j=2}^{j_0}\int_{\mathbf{R}^n} k(x,x-y)f_j(y)\,dy.$$

We will now break up $k(x,x-y)$ into three pieces, with a decomposition depending upon j. To do so we need to construct a partition of unity for $\mathbf{R}^n$. Let $\eta_1(\xi)$, $\eta_2(\xi)$ be radial and in $C_0^\infty(\mathbf{R}^n)$ with the property that

$$\eta_1(\xi)=\begin{cases}1 & (1/7)^{1/(1-a)}<|\xi|<40^{1/(1-a)}\\ 0 & |\xi|<(1/8)^{1/(1-a)} \quad\text{or}\quad |\xi|>50^{1/(1-a)},\end{cases}$$

and

$$\eta_2(\xi)=\begin{cases}1 & |\xi|<(1/5)^{1/(1-a)}\\ 0 & |\xi|>(1/4)^{1/(1-a)}.\end{cases}$$

Now define $\psi(\xi)$ radial, $C_0^\infty(\mathbf{R}^n)$, be such that

$$\psi(\xi)=\begin{cases}1 & |\xi|<30^{1/(1-a)}\\ 0 & |\xi|>40^{1/(1-a)}.\end{cases}$$

We let $\varrho_1(\xi)=\psi(\xi)\eta_1(\xi)/(\eta_1(\xi)+\eta_2(\xi))$ and $\varrho_2(\xi)=\psi(\xi)\eta_2(\xi)/(\eta_1(\xi)+\eta_2(\xi))$. Because of this construction ϱ_1 and ϱ_2 are essentially the cut-off functions in the statements of Lemmas (2.15) and (2.19), respectively. Also, if we set $\varrho_{3,j}(\xi)=1-(\varrho_{1,j}(\xi)+\varrho_{2,j}(\xi))$, the $\varrho_{3,j}(\xi)$'s basically satisfy the conditions imposed on the cut-off function in Lemma (2.25). Thus, for each j, the new functions $\varrho_{1,j}(\xi)$, $\varrho_{2,j}(\xi)$ and $\varrho_{3,j}(\xi)$ form a partition of unity for $\mathbf{R}^n$ and we have

$$Tf_2(x)=\sum_{j=2}^{j_0}\Big[\int_{\mathbf{R}^n} e^{i(\langle x,\xi\rangle+|\xi|^a)}\theta(\xi)\sigma(x,\xi)\big(\varrho_{1,j}(\xi)+\varrho_{2,j}(\xi)+\varrho_{3,j}(\xi)\big)\hat{f}_j(\xi)\,d\xi\Big]$$
$$=\sum_{j=2}^{j_0}\int_{\mathbf{R}^n}\big(K_{j,1}(x,x-y)+K_{j,2}(x,x-y)+K_{j,3}(x,x-y)\big)f_j(y)\,dy,$$

say. By Lemmas (2.19) and (2.25), for $x\in Q$,

$$|Tf_2(x)|\leqq\sum_{j=2}^{j_0}\int_{|x-y|\sim 2^jd}|f_j(y)|/|x-y|^{n-\varepsilon}\,dy+\sum_{j=2}^{j_0}\Big|\int_{\mathbf{R}^n}K_{j,3}(x,x-y)f_j(y)\,dy\Big|.$$

Since $|x_0-y|\sim|x-y|$ for $x\in Q$ and $d\geqq 1$, it follows that

$$|Tf_2(x)|\leqq c\sum_{j=2}^{j_0}\int_{|x_0-y|\sim 2^jd}|f_j(y)|/|x_0-y|^{n-\varepsilon}\,dy$$

$$+\sum_{j=2}^{j_0}\Big|\int_{\mathbf{R}^n}K_{j,3}(x,x-y)\,f_j(y)\,dy\Big|\leqq cM_pf(x_0)+\sum_{j=2}^{j_0}\Big|\int_{\mathbf{R}^n}K_{j,3}(x,x-y)f_j(y)\,dy\Big|.$$

Thus

(3.1)

$$d^{-n}\int_Q|Tf_2(x)|\,dx\leqq cM_pf(x_0)+\sum_{j=2}^{j_0}d^{-n}\int_Q\Big|\int_{\mathbf{R}^n}K_{j,3}(x,x-y)f_j(y)\,dy\Big|\,dx.$$

We need further to estimate the second term above. By Hölder's inequality this term does not exceed

$$A=\sum_{j=2}^{j_0}\Big(d^{-n}\int_Q\Big|\int_{\mathbf{R}^n}K_{j,3}(x,x-y)\,f_j(y)\,dy\Big|^q dx\Big)^{1/q}.$$

Now letting $\delta=n(2-a)(1/p-1/2)$ we get

$$\int_{\mathbf{R}^n}K_{j,3}(x,x-y)f_j(y)\,dy$$

$$=\int_{\mathbf{R}^n}e^{i(\langle x,\xi\rangle+|\xi|^a)}\frac{\theta(\xi)\sigma(x,\xi)}{|\xi|^\delta}|\xi|^{na/2}\varrho_{1,j}(\xi)|\xi|^{-na/2+\delta}\hat{f}_j(\xi)\,d\xi.$$

Since $\sigma(x,\xi)|\xi|^{na/2}\in S^0_{1,0}$, and we may thus use Lemma (2.6), we get, for $1/p+1/q=1$,

$$\Big(\int_Q\Big|\int_{\mathbf{R}^n}K_{j,3}(x,x-y)f_j(y)\,dy\Big|^q dx\Big)^{1/q}$$

$$\leqq c\big\|\big(|\xi|^{-na/2+\delta}\varrho_{1,j}(|\xi|\hat{f}_j(\xi)\big)^\vee\big\|_p=c\|F_j\|_p,\quad\text{say}.$$

We now apply Lemma (2.13) and since $\varrho_{1,j}(\xi)=\varrho_1\big((a/2^jd)^{1/(a-1)}|\xi|\big)$, we have, for $1<p<\infty$, that

$$\|F_j\|_p\leqq c(2^jd)^{(\delta-na/2)/(a-1)}\|f_j\|_p$$

$$\leqq c(2^jd)^{(\delta-na/2)/(a-1)+n/p}M_pf(x_0).$$

Thus for such p's,

$$A\leqq cd^{-n/q}M_pf(x_0)\sum_{j=2}^{j_0}(2^jd)^{(\delta-na/2)/(a-1)+n/p}.$$

Now $(\delta-na/2)/(a-1)+n/p=n/q(1-a)$. Because $2^{j_0}d\sim d^{1-a}$ it readily follows that

$$A\leqq cd^{-n/q}M_pf(x_0)\sum_{j=2}^{j_0}(2^jd)^{n/q(1-a)}\leqq cM_pf(x_0).$$

In view of (3.1) we easily have

$$d^{-n}\int_Q|Tf_2(x)|\,dx\leqq cM_pf(x_0).$$

We now consider $Tf_3(x)$. We again write $f_3(x)=\sum_{j=j_0}^{j_1}f_j(x)$, $f_j(x)=f_3(x)\chi\{x\colon 2^{j-1}d<|x-x_0|<2^jd\}$ and $2^{j_0}d\sim d^{1-a}$, $2^{j_1}d\sim 1$. Defining $\varrho_{1,j}$, $\varrho_{2,j}$ and $\varrho_{3,j}$

exactly as we did while considering $f_2(x)$, we get,

$$Tf_3(x) = \sum_{j=j_0}^{j_1} \int_{\mathbf{R}^n} (K_{j,1}(x, x-y)+K_{j,2}(x, x-y)+K_{j,3}(x, x-y)) f_j(y)\, dy.$$

We also let

$$c_Q = \sum_{j=j_0}^{j_1} \int_{\mathbf{R}^n} K_{j,3}(x_0, x_0-y) f_j(y)\, dy.$$

Thus

$$|Tf_3(x)-c_Q| \leqq \sum_{j=j_0}^{j_1} \Big(\int_{\mathbf{R}^n} |K_{j,1}(x, x-y)|\, |f_j(y)|\, dy+\int_{\mathbf{R}^n} |K_{j,2}(x, x-y)|\, |f_j(y)|\, dy\Big)$$
$$+\sum_{j=j_0}^{j_1} \int_{|x_0-y|\sim 2^j d} |K_j(x, x-y)-K_j(x_0, x_0-y)|\, |f_j(y)|\, dy \equiv B+C+D, \quad \text{say.}$$

Using Lemmas (2.19) and (2.25), we see that for $x\in Q$,

$$B+C \leqq c \sum_{j=j_0}^{j_1} \int |f_j(y)|/|x-y|^{n-\varepsilon}\, dy \leqq c \sum_{j=j_0}^{j_1} \int |f_j(y)|/|x_0-y|^{n-\varepsilon} dy \leqq cM_p f(x_0).$$

We now estimate D. We have, again for $1/q+1/p=1$, that D is dominated by

$$\sum_{j=j_0}^{j_1} \Big(\int_{|x_0-y|\sim 2^j d} |K_{j,3}(x, x-y)-K_{j,3}(x_0, x_0-y)|^q\, dy\Big)^{1/q} \Big(\int |f_j(y)|^p\, dy\Big)^{1/p}.$$

Using Lemma (2.15) for the first term on the right above we see that

$$D \leqq c \sum_{j=j_0}^{j_1} (2^{-jn/p} d^{1-n/p}+d(2^j d)^{1/(a-1)-n/p})(2^j d)^{n/p} M_p f(x_0)$$
$$\leqq cM_p f(x_0) \Big(d \sum_{j=j_0}^{j_1} 1+d^{-a/(1-a)} \sum_{j=j_0}^{j_1} 2^{-j/(1-a)}\Big).$$

Since $2^{j_1} d\sim 1$, $2^{j_0} d\sim d^{1-a}$, we see that the expression above is at most

$$cdM_p f(x_0)(-d \log (d^{a-1})+1).$$

But $d\leqq 1/4$, thus $D\leqq cM_p f(x_0)$ as well. Combining these estimates we have the uniform bound

$$|Tf_3(x)-c_Q| \leqq cM_p f(x_0),$$

thus arriving at

$$d^{-n} \int_Q |Tf_3(x)-c_Q|\, dx \leqq cM_p f(x_0).$$

This finishes our proof.

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