Holomorphic functions, measures and BMO

V. Nestoridis

Introduction

The central subject of this paper is an extension of the following result:

Theorem 1 ([2], [3]). *Let f be in the disc algebra (more generally in H^1) and let z_0 be a point in the open unit disc. Then there is an interval I on the unit circle* **T** *with length $|I|$, $0<|I|\leqq 2\pi$, such that $f(z_0)=1/|I| \int_I f\,d\sigma$, where σ denotes the Lebesgue measure on* **T**.

We extend the above theorem to the general case of finite strictly positive continuous measures on **T**, under the supplementary restriction that $f(z_0)\notin f(\mathbf{T})$. In the particular case where μ is the Lebesgue measure, Theorem 1 implies that the hypothesis "$f(z_0)\notin f(\mathbf{T})$" is not needed. However, this restriction is not superfluous in the general case; see § 4, prop. 19 for a relevant counterexample.

The above extension is purely topological in nature. We prove that for any complex continuous function f on **T** and any complex number $w\notin f(\mathbf{T})$, the following are equivalent:

a) For every finite strictly positive continuous measure μ on **T**, there is an interval $I\subset\mathbf{T}$ such that $w=1/\mu(I)\int_I f\,d\mu$.

b) f has non-zero winding number with respect to w.

This equivalence enables us to determine the range of the BMO norm of $\varphi\circ U$, where φ is any given continuous unimodular function on **T** and U varies in the set of all homeomorphisms of **T** onto itself. In the case where φ has non-zero winding number with respect to 0, we show that $\varphi\circ U$ has BMO norm equal to 1 for all U. If φ has zero winding number with respect to 0, then the BMO norm of $\varphi\circ U$ can be made arbitrarily close to zero and does not exceed

$$\frac{1}{2}\cdot \sup_{x,y} |\varphi(e^{ix})-\varphi(e^{iy})|.$$

The basic background needed in this paper can be found in [7], [12], [9].

§ 1. Averages of holomorphic functions

In this section we prove the extension of theorem 1 mentioned in the introduction.

We denote by $\mathbf{C}=\mathbf{R}^2$ the plane and by $\mathbf{T}$ the unit circle. M denotes the set of finite strictly positive measures on $\mathbf{T}$, which are continuous in the sense that they do not have point masses. We recall that a Borel measure μ is called strictly positive on a topological space X, if $\mu(V)>0$ for all non-empty open subsets V of X. We refer to [8], [11], [12] for basic information concerning measures.

By the term interval of $\mathbf{T}$ we mean any arc of T with strictly positive length less than or equal to 2π. We reserve the letter I for such intervals and $|I|$ denotes the length of I, $0<|I|\leqq 2\pi$.

If a function f defined on $\mathbf{T}$, or on a larger set, is integrable with respect to some measure $\mu\in M$ (i.e. $f\in L^1_{(\mu)}$), then $f_{I,\mu}$ denotes the μ-average of f on the interval I of $\mathbf{T}$: $f_{I,\mu}=1/\mu(I)\int_I f\,d\mu$. The set of all interval averages of f with respect to μ is denoted by $A_\mu(f)$: $A_\mu(f)=\{f_{I,\mu}\colon I\subset\mathbf{T}$ interval with length $|I|$, $0<|I|\leqq 2\pi\}$.

If f is a complex continuous function on $\mathbf{T}$ and w a complex number in $\mathbf{C}\setminus f(\mathbf{T})$, then the winding number of f with respect to w is an integer counting how many times f wraps around w; see [4], [9], [13].

The winding number of a constant function is obviously zero. Any two functions which are homotopic in $\mathbf{C}\setminus\{w\}$ have the same winding number with respect to w.

Now we prove:

Proposition 2. *Let f be a complex continuous function on the unit circle* $\mathbf{T}$ *and w a point in $\mathbf{C}\setminus f(\mathbf{T})$. If f has non-zero winding number with respect to w, then w is a μ-interval average of f for all $\mu\in M$.*

Proof. Let $\mu\in M$. For $\varepsilon\in(0,\pi]$ and $e^{ix}\in\mathbf{T}$ we denote by $I_{\varepsilon,x}$ the interval $I_{x,\varepsilon}=\{e^{i\theta}\colon x-\varepsilon<\theta<x+\varepsilon\}$. We define $F(\varepsilon, e^{ix})=f_{I_{x,\varepsilon},\mu}$ for $0<\varepsilon\leqq 2\pi$ and $F(0, e^{ix})=f(e^{ix})$. Since $\mu\in M$ and f is uniformly continuous on $\mathbf{T}$, the map F is continuous on $[0,2\pi]\times\mathbf{T}$. Therefore it defines a homotopy between the constant function $F(\pi, e^{ix})$ and $F(0, e^{ix})=f(e^{ix})$.

If $w\notin A_\mu(f)$, then the homotopy F takes values in $\mathbf{C}\setminus\{w\}$. It follows that $F(\pi, e^{ix})$ and f have the same winding number with respect to w. Since $F(\pi, e^{ix})$ is constant, f must have zero winding number with respect to w. This contradicts the hypothesis. ∎

A complex function belongs to the disc algebra $A(D)$, if it is continuous on the closure $\bar{D}$ of D and holomorphic in D; see [5], [7], [10], [12] for information concerning the disc algebra, Blaschke products, inner function and H^∞ functions.

Suppose $f \in A(D)$. If z_0 is a point of D such that $f(z_0) \notin f(\mathbf{T})$, then according to the argument principle, $f_{|\mathbf{T}}$ has non-zero winding number with respect to $f(z_0)$. Proposition 2 implies now $f(z_0) \in A_\mu(f)$ for all $\mu \in M$. Thus, we obtain the desired extension of theorem 1:

Theorem 3. *Let $f \in A(D)$ and $z_0 \in D$ such that $f(z_0) \notin f(\mathbf{T})$. Then for every finite strictly positive continuous measure μ on $\mathbf{T}$, there is an interval $I \subset \mathbf{T}$ such that $f(z_0) = 1/\mu(I) \int_I f\, d\mu$.* ∎

For all $\mu \in M$, $z \in \mathbf{T}$ and all continuous functions f on $\mathbf{T}$, the average $f_{I,\mu}$ converges to $f(z)$, as I shrinks to z. This observation together with theorem 3 proves the following

Corollary 4. *$f(D) \subset \overline{A_\mu(f)}$ for all $\mu \in M$ and $f \in A(D)$.* ∎

Another corollary of theorem 3 is the fact that the μ-BMO norm of non-constant finite Blaschke products equals 1.

For $\mu \in M$, $\varphi \in L^1(\mu)$ and $p \in [1, +\infty)$ the p-BMO norm of φ with respect to μ is defined by

$$_{p,\mu}|||\varphi||| = \sup_I \left[\frac{1}{\mu(I)} \int_I |\varphi - \varphi_{I,\mu}|^p \, d\mu\right]^{1/p}.$$

If φ is unimodular μ-almost everywhere (i.e. $|\varphi(e^{it})| = 1$ μ-a.e.), then an easy computation shows that

$$_{2,\mu}|||\varphi||| = \{1 - [\inf_I |\varphi_{I,\mu}|]^2\}^{1/2}.$$

On applying the triangular inequality we also find

$$_{1,\mu}|||\varphi||| \geqq 1 - \inf_I |\varphi_{I,\mu}|.$$

Since $_{p,\mu}|||\varphi|||$ increases with p we have

$$1 - \inf_I |\varphi_{I,\mu}| \leqq {}_{p,\mu}|||\varphi||| \leqq \{1 - \inf_I |\varphi_{I,\mu}|^2\}^{1/2} \leqq 1,$$

for all $p \in [1, 2]$ and unimodular functions φ.

Suppose that B is a non-constant finite Blaschke product. Then $B \in A(D)$ and $0 \in B(D) \setminus B(T)$. Theorem 3 implies that $\min_I |B_{I,\mu}| = 0$ for all $\mu \in M$. Since B is unimodular on T we obtain:

$$1 = 1 - \inf_I |B_{I,\mu}| \leqq {}_{p,\mu}|||B||| \leqq \{1 - \inf_I |B_{I,\mu}|^2\}^{1/2} = 1, \quad 1 \leqq p \leqq 2.$$

Thus we have proved:

Proposition 5. *$_{p,\mu}|||B||| = 1$ for every non-constant finite Blaschke product B, $p \in [1, 2]$ and $\mu \in M$.* ∎

The proof of Proposition 5 applies more generally to any continuous unimodular function φ on $\mathbf{T}$ with non-zero winding number. Therefore ${}_{p,\mu}|||\varphi|||=1$ for any such φ, $\mu\in M$ and $p\in[1, 2]$.

In the particular case, where μ is the Lebesgue measure on $\mathbf{T}$, the hypothesis "$f(z_0)\notin f(\mathbf{T})$" is not needed in theorem 3 and proposition 5 holds for any non constant inner function (see [2], [3]).

In § 4 below we offer counterexamples related to the results of the present section. In particular we show that the hypothesis "$f(z_0)\notin f(t)$" is not superfluous in theorem 3.

§ 2. The converse of proposition 2

In this section we prove the converse of proposition 2.

Proposition 6. *Suppose that a complex continuous function f on the unit circle $\mathbf{T}$ has zero winding number with respect to some point $w\in\mathbf{C}\setminus f(\mathbf{T})$. Then for every $\alpha<\inf_{|z|=1}|f(z)-w|$ there is $\mu\in M$ such that $|w-f_{I,\mu}|>\alpha$ for all averages $f_{I,\mu}$ on intervals $I\subset T$.* ∎

Propositions 6 and 2 imply theorem 7.

Theorem 7. *Let f: $\mathbf{T}\to\mathbf{C}$ be a continuous function and w a point in $\mathbf{C}\setminus f(\mathbf{T})$. Then $w\in A_\mu(f)$ for all $\mu\in M$ if and only if f has non-zero winding number with respect to w.* ∎

For the proof of proposition 6 we approximate the function $f-w/|f-w|$ by unimodular step functions, that is, functions of the form: $g=\sum_0^N e^{i\lambda k}\chi_{I_K}$, where $\lambda_k\in\mathbf{R}$, χ_I denotes the characteristic function of I and $I_K\subset\mathbf{T}$, $k=0, ..., N$ is a finite family of two-by-two disjoint intervals covering $\mathbf{T}$.

We omit the elementary proofs of lemmas 8 and 9 below, which will be used in the proof later on.

Lemma 8. *Let g be a complex unimodular step function and $\mu\in M$. Then we have:*

a) *The set $A_\mu(g)$ is a compact subset of $\bar{D}$.*

b) *If $|g_{I,\mu}|>\delta$ for all intervals I, then there is $\bar{\delta}>\delta$ such that $|g_{I,\mu}|\geqq\bar{\delta}$ for all I's.* ∎

Lemma 9. *Suppose $A\in\mathbf{C}$, $\theta\in\mathbf{R}$, $\delta<\bar{\delta}\leqq|A|\leqq 1$ and $0\leqq t<1/2(\bar{\delta}-\delta)$. Then $|A+te^{i\theta}/1+t|>\delta$.* ∎

Let $0<r<\pi/4$ and $N\geqq 1$ be an integer. Then $\Lambda_N(r)$ will denote the set of complex unimodular step functions $\varphi=\sum_0^{N-1} e^{i\lambda_k}\chi_{I_K}$ such that $I_k=\{e^{i\theta}\colon\ Y_k\leqq\theta<Y_{k+1}\}$

with $Y_0 < Y_1 < \ldots < Y_{N-1} < Y_N = Y_0 + 2\pi$, $\lambda_k \in \mathbf{R}$, $|\lambda_0 - \lambda_{N-1}| < r$ and $|\lambda_k - \lambda_{k+1}| < r$ for all $k = 0, \ldots, N-2$.

Let $\varphi \in \Lambda_N(r)$, $N \geqq 3$. Without loss of generality we may assume that $\lambda_{N-1} \geqq \lambda_k$ for all $k = 0, \ldots, N-1$. Then we consider the maps:

$$\omega(\theta) = Y_{N-2} + (\theta - Y_{N-2}) \frac{Y_{N-1} - Y_{N-2}}{Y_N - Y_{N-2}},$$

$$F(e^{i\theta}) = e^{i\omega(\theta)} \quad \text{for} \quad Y_{N-2} \leqq \theta \leqq Y_N,$$

$$F(e^{i\theta}) = e^{i\theta} \quad \text{for} \quad Y_0 \leqq \theta < Y_{N-2}.$$

We observe that F maps I_k onto itself for $k = 0, \ldots, N-3$ and F maps $I_{N-2} \cup I_{N-1}$ onto I_{N-2}.

Lemma 10. *Let $\varphi \in \Lambda_N(r)$, $N \geqq 3$ and F be the map associated to φ as above. Then we have:*

a) *F: $\mathbf{T} \to \mathbf{T} \setminus I_{N-1}$ is a measurable bijection.*
b) *For any interval I of the form $I = \{e^{i\theta}: \eta \leqq \theta < \xi\}$, $\eta < \xi \leqq \eta + 2\pi$, the set $\tilde{I} = F^{-1}(I) = F^{-1}(I - I_{N-1})$ is either an interval or the empty set.*
c) *The function $g = \varphi \circ F$ belongs to $\Lambda_{N-1}(r)$.*

Proof. Parts a) and b) can be easily verified. We prove part c).

The function g has the form $g = \sum_0^{N-2} e^{i\lambda_k} \chi_{\tilde{I}_k}$, where $\tilde{I}_k = \{e^{i\theta}: \tilde{Y}_k \leqq \theta < \tilde{Y}_{k+1}\}$, $\tilde{Y}_k = Y_k$ for $0 \leqq k \leqq N-2$ and $\tilde{Y}_{N-1} = Y_N = \tilde{Y}_0 + 2\pi$. The inequalities $|\lambda_k - \lambda_{k+1}| < r$ for $0 \leqq k \leqq N-3$ hold because $\varphi \in \Lambda_N(r)$. Since $|\lambda_{N-2} - \lambda_{N-1}| < r$ and $|\lambda_{N-1} - \lambda_0| < r$ the assumption $\lambda_{N-1} \geqq \lambda_k$ for all $k = 0, \ldots, N-1$ implies that $\lambda_{N-2}, \lambda_0 \in (\lambda_{N-1} - r, \ \lambda_{N-1}]$. Therefore $|\lambda_{N-2} - \lambda_0| < r$. It follows $g \in \Lambda_{N-1}(r)$. ∎

Lemma 11. *Let $\varphi \in \Lambda_N(r)$, $0 < r < \pi/4$, $N \geqq 3$ and $g = \varphi \circ F$ as above. We suppose that there is a measure $v \in M$ such that $|g_{I,v}| > \cos r$ for all intervals $I \subset \mathbf{T}$. Then the measure μ defined by $d\mu(e^{i\theta}) = \chi_{\mathbf{T} - I_{N-1}} \, dv(F^{-1}(e^{i\theta})) + \nu/|I_{N-1}| \, \chi_{I_{N-1}} \, d\theta$ belongs to M for all $\nu > 0$. If $\nu > 0$ is close enough to 0, then $|\varphi_{I,\mu}| > \cos r$ for all intervals I.*

Proof. The map F^{-1}: $\mathbf{T} \setminus I_{N-1} \to \mathbf{T}$ is a measurable bijection by lemma 10a. Since $v \in M$, the measure μ_1, $d\mu_1(e^{i\theta}) = \chi_{\mathbf{T} - I_{N-1}} \, dv(F^{-1}(e^{i\theta}))$, is strictly positive on $\mathbf{T} \setminus I_{N-1}$ and does not have any point masses. Since $d\mu = d\mu_1 + \nu/|I_{N-1}| \, \chi_{I_{N-1}} \, d\theta$ it follows that $\mu \in M$ for all $\nu > 0$.

By hypothesis $\cos r < |g_{I,v}| \leqq 1$ for all I's. Hence, using lemma 8b, there is $\tilde{\delta} > \cos r$ such that $|g_{I,v}| \geqq \tilde{\delta}$ for all I's. Let $\tilde{I}_{N-2} = I_{N-2} \cup I_{N-1}$, $\tilde{I}_k = I_k$ for

$k=0,\dots,N-3$ and let $\varrho=\min\{v(\tilde{I}_K);0\leqq k\leqq N-2\}>0$. We shall show $|\varphi_{I,\mu}|>\cos r$ for all I's, provided that $0<\nu<1/2(\delta-\cos r)\varrho$.

First we consider the case where the interval I contains at least one I_j with $0\leqq j\leqq N-2$. Then the set $\tilde{I}=F^{-1}(I)=F^{-1}(I\setminus I_{N-1})$ contains $\tilde{I}_j=F^{-1}(I_j)$; it follows that $v(\tilde{I})\geqq v(\tilde{I}_j)\geqq\varrho$. We also have $|g_{I,v}|\geqq\tilde{\delta}$, because $\tilde{I}$ is an interval (lemma 10b).

One can easily verify that:

$$\varphi_{I,\mu}=\frac{\int_I g\,dv+\frac{|I\cap I_{N-1}|}{|I_{N-1}|}\nu e^{i\lambda_{N-1}}}{v(\tilde{I})+\frac{|I\cap I_{N-1}|}{|I_{N-1}|}\nu}.$$

It follows that

$$\varphi_{I,\mu}=\frac{g_{I,v}+te^{i\lambda_{N-1}}}{1+t},\quad\text{with}\quad 0\leqq t=\frac{|I\cap I_{N-1}|}{|I_{N-1}|}\frac{\nu}{v(\tilde{I})}<\frac{1}{2}(\hat{\delta}-\cos r).$$

Since $\tilde{\delta}\leqq|g_{\tilde{I},v}|\leqq 1$ lemma 9 implies $|\varphi_{I,\mu}|>\cos r$.

We consider now the case where the interval I does not contain any I_j with $0\leqq j\leqq N-2$. Then either $I\subset I_k\cup I_{k+1}$ with $0\leqq k\leqq N-3$ or $I\subset I_{N-2}\cup I_{N-1}\cup I_0$. In both cases $\varphi_{I,\mu}$ is of the form

$$\varphi_{1,\mu}=\frac{\alpha e^{i\theta_0}+\beta e^{i\theta_1}+\gamma e^{i\theta_2}}{\alpha+\beta+\gamma}\qquad\text{with}\quad \alpha,\beta,\gamma\geqq 0,\quad \beta>0$$

$$\text{and}\quad |\theta_0-\theta_1|<r,\quad |\theta_1-\theta_2|<r.$$

It follows that $\varphi_{I,\mu}$ belongs to the convex hull of an arc of $\mathbf{T}$ with opening strictly less than $2r$. Therefore $|\varphi_{I,\mu}|>\cos r$ and the proof is complete. ∎

Proposition 12. *Suppose $\varphi\in A_N(r)$ for some $N\geqq 1$ and $0<r<\pi/4$. Then there is $\mu\in M$ such that $|\varphi_{I,\mu}|>\cos r$ for all intervals $\mathbf{T}I\subset$.*

Proof. For $N=1$ the function φ is unimodular and constant on $\mathbf{T}$. Therefore $|\varphi_{I,\mu}|=1>\cos r$ for all $\mu\in M$ and all I's. For $N=2$ the function φ takes at most two values $e^{i\lambda_0}$, $e^{i\lambda_1}$ with $|\lambda_0-\lambda_1|<r$, $\lambda_0,\lambda_1\in\mathbf{R}$. Therefore for any $\mu\in M$ and any interval I we have $|\varphi_{I,\mu}|>\cos r/2>\cos r$.

Let $N\geqq 3$. By induction we assume the lemma to be true for $N-1$ and we prove it for N.

Let $\varphi\in A_N(r)$ and $F\colon\mathbf{T}\to\mathbf{T}\setminus I_{N-1}$ be associated to φ as in lemma 10. Then $g=\varphi\circ F\in A_{N-1}(r)$ according to lemma 10c.

By the induction hypothesis there is $v\in M$ such that $|g_{I,v}|>\cos r$ for all I's. Now lemma 11 gives a measure $\mu\in M$ such that $|\varphi_{I,\mu}|>\cos r$ for all I's and the proof is complete. ∎

We are ready now to prove proposition 6.

Proof of proposition 6. Without loss of generality we may assume $w=0$. We also have $f(e^{it})=|f(e^{it})|e^{ib(t)}$ with b some real function continuous on $[0, 2\pi]$. Since f has zero winding number with respect to $w=0$, we have $b(0)=b(2\pi)$.

Obviously $\lim_{\varepsilon\to 0}(-\varepsilon+\cos 2\varepsilon)=1>\alpha/\inf|f|$. Therefore we may choose $0<\varepsilon<\pi/8$ such that $-\varepsilon+\cos 2\varepsilon>\alpha/\inf|f|$.

Let λ be a real step function on $[0, 2\pi)$ such that

$$\lambda(0)=\lim_{t\to 2\pi}\lambda(t)=b(0) \quad\text{and}\quad |\lambda(e^{i\theta})-b(e^{i\theta})|<\varepsilon$$

for all $\theta\in[0, 2\pi)$. We may also assume that λ is right continuous.

We consider the unimodular step function $\varphi(e^{it})=e^{i\lambda(t)}$. One can easily check that $\varphi\in\Lambda_N(2\varepsilon)$ for some $N\geqq 1$. By proposition 12 there is a measure $v\in M$ such that $|\varphi_{I,v}|>\cos 2\varepsilon$ for all intervals $I\subset\mathbf{T}$.

Since $\big|f/|f|-\varphi\big|\leqq|b-\lambda|<\varepsilon$, it follows that

$$\left|\frac{1}{v(I)}\int_I\frac{f}{|f|}\,dv\right|\geqq|\varphi_{I,v}|-\left|\frac{1}{v(I)}\int_I\left(\frac{f}{|f|}-\varphi\right)dv\right|>\cos 2\varepsilon-\varepsilon>\frac{\alpha}{\inf|f|}.$$

We consider now the measure $\mu\in M$ defined by $d\mu=dv/|f|$. Then $\mu(I)=\int_I 1/|f|\,dv<v(I)/\inf|f|$. It follows that

$$|f_{I,\mu}|=\left|\frac{1}{\mu(I)}\int_I\frac{f}{|f|}\,dv\right|>(\inf|f|)\cdot\left|\frac{1}{v(I)}\int_I\frac{f}{|f|}\,dv\right|>\alpha$$

and the proof is complete. ∎

Remarks. a) A slight modification in the proof shows that the measure μ in proposition 6 can be chosen so that $d\mu(e^{i\theta})=h(\theta)\,d\theta$, with h a C^∞ strictly positive 2π-periodic function.

b) Let $f\colon\mathbf{T}\to\mathbf{C}\setminus\{0\}$ be a continuous function and $\mu\in M$. We define $\gamma(\theta)=\int_0^\theta f(e^{it})\,d\mu(e^{it})$, $\theta\in\mathbf{R}$. Obviously $\gamma(2\pi n+\theta)=n\gamma(2\pi)+\gamma(\theta)$ for every integer n and $0\leqq\theta\leqq 2\pi$. The map γ defines a continuous (locally) rectifiable curve whose length s satisfies $ds(\theta)=|f(e^{i\theta})|\,d\mu(e^{i\theta})$. We also have $d\gamma/d\theta=f(e^{i\theta})\,d\mu/d\theta$, $d\theta$-almost everywhere. Since $\mu\in M$, we have $d\mu/d\theta\geqq 0$. Therefore $\operatorname{Arg} d\gamma/d\theta=\operatorname{Arg} f(e^{i\theta})$, $d\theta$-almost everywhere on the set $0\neq d\mu(e^{i\theta})/d\theta$. In particular $\operatorname{Arg} d\gamma/d\theta=\operatorname{Arg} f(e^{i\theta})$ for all θ's, provided that μ is of the form $d\mu=h\,d\theta$, with h a strictly positive 2π-periodic continuous function. Therefore the tangent of γ follows the argument of f.

We also have the inequalities:

$$\frac{1}{\|f\|_\infty}|f_{I,\mu}|\leqq\left|\frac{\gamma(\theta_2)-\gamma(\theta_1)}{s(\theta_2)-s(\theta_1)}\right|\leqq\frac{1}{\inf|f|}|f_{I,\mu}|$$

for all intervals $I=\{e^{i\theta}\colon\theta_1<\theta<\theta_2\}$, $\theta_1<\theta_2\leqq\theta_1+2\pi$.

It is obvious now that the condition $|f_{I,\mu}|>\alpha>0$ for all I's is equivalent to a local chord-arc condition

$$\left|\frac{\gamma(\theta_2)-\gamma(\theta_1)}{s(\theta_2)-s(\theta_1)}\right| > \tilde{\alpha} > 0 \quad \text{for all} \quad \theta_1 \leqq \theta_2 \leqq \theta_1+2\pi.$$

Thus, proposition 6 and remark a) imply that for every continuous 2π-periodic real function b, there are C^1 curves γ such that $\operatorname{Arg}\gamma'=b$ and

$$1 \geqq \left|\frac{\gamma(\theta_2)-\gamma(\theta_1)}{s(\theta_2)-s(\theta_1)}\right| > \alpha > 0 \quad \text{for} \quad \theta_1 < \theta_2 < \theta_1+2\pi.$$

Conversely, an alternative proof of proposition 6 could be based on the existence of a curve γ with the above properties. This is more or less the approach in the proof of proposition 19 (§ 4).

c) A slight modification in our proofs yields the best possible inequality $|\varphi_{I,\mu}|>\cos r/2$ instead of $|\varphi_{I,\mu}|>\cos r$ (proposition 12), which is actually enough for our purposes in proposition 6 and theorem 7.

§ 3. BMO norm of unimodular functions

For any $\mu\in M$ and $\varphi\in L_1(\mu)$ the 2-BMO norm of φ with respect to μ is defined as follows:

$$_{\mu}|||\varphi||| = {}_{2,\mu}|||\varphi||| = \sup_I \left[\frac{1}{\mu(I)}\int_I |\varphi-\varphi_{I,\mu}|^2 d\mu\right]^{1/2}.$$

In the particular case of the Lebesgue measure σ on $\mathbf{T}$ we write $|||\varphi|||$ instead of $_{\sigma}|||\varphi|||$ We refer to [1], [6], [7] for information about BMO.

Let L be the set of topological homeomorphisms of $\mathbf{T}$ onto itself. For φ any continuous function on $\mathbf{T}$, we denote $L\varphi=\{|||\varphi\circ U|||;\ U\in L\}$. Then one can easily see that $L\varphi=\{_{\mu}|||\varphi|||;\ \mu\in M\}$.

Our purpose in this section is to determine the set $L\varphi$ for any continuous unimodular function φ (see prop. 15). Towards this end we use results from the previous sections and lemmas 13 and 14 below.

Let φ be a continuous unimodular function on $\mathbf{T}$ and let $\mu\in M$. As in § 1, $_{\mu}|||\varphi|||=\{1-|\inf_I|\varphi_{I,\mu}||^2\}^{1/2}$ and $0\leqq {}_{\mu}|||\varphi|||\leqq 1$, i.e. $L\varphi\subset[0, 1]$.

Lemma 13. *Let φ be a continuous unimodular function on* $\mathbf{T}$ *and $\mu, v\in M$. For any $t\in[0, 1]$ we denote $\mu_t=tv+(1-t)\mu$. Then the map $t\to g_{\mu,v}(t)=\inf_I|\varphi_{I,\mu_t}|$ is continuous on* $[0, 1]$. *It follows that $L\varphi$ is a subinterval of* $[0, 1]$.

Proof. Obviously $\mu_t\in M$. Suppose that for all $\mu, v\in M$ the function $g_{\mu,v}$ is continuous on $[0, 1]$. Then the map $t\to {}_{\mu_t}|||\varphi|||=\sqrt{1-|g_{\mu,v}(t)|^2}\in L\varphi$ is also continuous. The intermediate value theorem implies that the range of this map is an interval containing the values ${}_{\mu}|||\varphi||| = {}_{\mu_0}|||\varphi|||$ and ${}_{v}|||\varphi||| = {}_{\mu_1}|||\varphi|||$. Since $L\varphi = \{{}_{\mu}|||\varphi|||;\ \mu\in M\}\subset[0, 1]$, it follows that $L\varphi$ is a subinterval of $[0, 1]$.

The proof will be completed if we show the continuity of the map $g_{\mu,v}$.

Let $\varepsilon>0$. The uniform continuity of φ on $\mathbf{T}$ implies the existence of a positive integer n such that $|\varphi(e^{i\theta})-\varphi(e^{it})|<\varepsilon/2$ for all $|\theta-t|\leqq 2\pi/n$. For such an n, we split $\mathbf{T}$ into $2n$ intervals $I_1, \ldots, I_{2n}$ of equal lengths π/n. Let δ be the minimum of $\mu(I_1), \ldots, \mu(I_{2n}), v(I_1), \ldots, v(I_{2n})$. Since μ and v are strictly positive measures, δ is strictly positive. We denote

$$K = \frac{\mu(\mathbf{T})+v(\mathbf{T})}{\delta}+\frac{|\mu(\mathbf{T})+v(\mathbf{T})|^2}{\delta^2}\in(0, +\infty).$$

We shall show that for all intervals $I\subset\mathbf{T}$ and all $t_1, t_2\in[0, 1]$ with $|t_1-t_2|<\varepsilon/K$ the following inequalities hold: $|\varphi_{I,\mu_{t_1}}|-\varepsilon<|\varphi_{I,\mu_{t_2}}|<|\varphi_{I,\mu_{t_1}}|+\varepsilon$. Then taking the infima over all I's we obtain

$$g_{\mu,v}(t_1)-\varepsilon \leqq g_{\mu,v}(t_2) \leqq g_{\mu,v}(t_1)+\varepsilon,$$

which proves the continuity of $g_{\mu,v}$.

Let $I\subset\mathbf{T}$ be an interval of length $|I|$, $0<|I|\leqq 2\pi$. We distinguish two cases: $0<|I|<2\pi/n$ and $2\pi/n\leqq|I|\leqq 2\pi$.

In the first case, we choose a point α in I. Then $|\varphi(z)-\varphi(\alpha)|<\varepsilon/2$ for all $z\in I$ and

$$|\varphi_{I,\mu_t}-\varphi(\alpha)| \leqq \frac{1}{\mu_t(I)}\int_I |\varphi(\alpha)-\varphi(z)|\,d\mu_t(z) < \frac{\varepsilon}{2}$$

for all $t\in[0, 1]$. Therefore

$$|\varphi_{I,\mu_{t_1}}-\varphi_{I,\mu_{t_2}}| \leqq |\varphi_{I,\mu_{t_1}}-\varphi(\alpha)|+|\varphi_{I,\mu_{t_2}}-\varphi(\alpha)| < \varepsilon.$$

It follows that

$$|\varphi_{I,\mu_{t_1}}|-\varepsilon < |\varphi_{I,\mu_{t_2}}| < |\varphi_{I,\mu_{t_1}}|+\varepsilon.$$

In the case $2\pi/n\leqq|I|\leqq 2\pi$, the interval I contains at least one of the intervals $I_1, \ldots, I_{2n}$. It follows that $\mu_t(I)\geqq\delta$ for all $t\in[0, 1]$. Therefore

$$|\varphi_{I,\mu_{t_1}}-\varphi_{I,\mu_{t_2}}| = \left|\frac{1}{\mu_{t_1}(I)}\int_I \varphi\, d(\mu_{t_1}-\mu_{t_2})+\left(\frac{1}{\mu_{t_1}(I)}-\frac{1}{\mu_{t_2}(I)}\right)\int_I \varphi\, d\mu_{t_2}\right|$$

$$\leqq |t_1-t_2|\,\frac{\mu(\mathbf{T})+v(\mathbf{T})}{\delta}+\frac{|t_1-t_2|\,|\mu(\mathbf{T})+v(\mathbf{T})|}{\delta^2}\,|\mu(\mathbf{T})+v(\mathbf{T})| = K(t_1-t_2).$$

Since $|t_1-t_2|<\varepsilon/K$, we have $|\varphi_{I,\mu_{t_1}}-\varphi_{I,\mu_{t_2}}|<\varepsilon$, which implies $|\varphi_{I,\mu_{t_1}}|-\varepsilon<|\varphi_{I,\mu_{t_1}}|<|\varphi_{I,\mu_{t_1}}|+\varepsilon$. ∎

Lemma 14. *Let φ be a complex continuous unimodular function φ on* **T**. *We denote* $A=\{w\in C\colon |w|<1$, and $w=1/\mu(\mathbf{T})\int_T \varphi\, d\mu$ for some $\mu\in M\}$ *and* $B=\{w\in C\colon |w|<1$ and $w=1/\mu(I)\int_I \varphi\, d\mu$ for some $\mu\in M$ and some interval $I\subset\mathbf{T}\}$. *Let Γ be the interior of the convex hull of the arc $\varphi(\mathbf{T})$. Then $A=B=\Gamma$.*

Proof. The inclusion $A\subset B$ is obvious. To show $B\subset\Gamma$ let $w=\varphi_{I,\mu}\in B$. We consider v the measure defined by $v(X)=\mu[\varphi^{-1}(X)]$ for all Borel sets $X\subset\mathbf{T}$. Then v is supported on $\varphi(\mathbf{T})$ and it is strictly positive on it. We also have $w=1/v(\varphi(I))\int_{\varphi(I)} z\, dv(z)$. Therefore w belongs to the convex hull of $\varphi(I)$. Since $|w|<1$, the arc $\varphi(I)\subset\mathbf{T}$ must have strictly positive length. The bary-center w of a strictly positive measure $v_{|\varphi(I)}$ on an arc φ(I) with strictly positive length, belongs always to the interior of the convex hull of $\varphi(I)\subset\varphi(\mathbf{T})$. Thus $w\in\Gamma$ and we proved $B\subset\Gamma$.

It remains to show $\Gamma\subset A$. Let $w\in\Gamma$. Then there are points $w_i=\varphi(z_i)$, $z_i\in\mathbf{T}$, $i=1, 2, 3$ such that w is in the interior of the triangle with vertices w_1, w_2, w_3. One can easily find discs D_i centered at w_i, $i=1, 2, 3$ with the following property: for any choice $w_i'\in D_i$, $i=1, 2, 3$, the point w is in the interior of the triangle with vertices w_1', w_2', w_3'.

For any $z\in D$ we denote by μ_z the (normalized) Poisson kernel associated with z (see [5], [7], [10], [12]). We extend φ from **T** to $\bar{D}$ setting $\varphi(z)=\int \varphi\, d\mu_z$ for all $z\in D$. This extension is the harmonic extension of φ and is continuous on $\bar{D}$. Therefore there are points $z_i'\in D$ close enough to z_i such that $\varphi(z_i')\in D_i$, $i=1, 2, 3$. It follows that w is a convex combination of $\varphi(z_i')$, $i=1, 2, 3$:

$$w=\sum_{i=1}^{3} t_i\varphi(z_i') \quad \text{with} \quad 0\leqq t_i,\ \sum_{i=1}^{3} t_i=1.$$

Consider the measure $\mu=\sum_{i=1}^{3} t_i\mu_{z_i'}$. Then $\mu\in M$, $\mu(\mathbf{T})=1$ and $1/\mu(\mathbf{T})\int \varphi\, d\mu=\sum_{i=1}^{3} t_i\varphi(z_i')=w$. Since $w\in\Gamma$ we have $|w|<1$. It follows that $w\in A$. Thus we proved $\Gamma\subset A$. ∎

Proposition 15. *Let $\varphi\colon \mathbf{T}\to\mathbf{T}$ be a continuous unimodular function on* **T**. *If φ has non-zero winding number with respect to 0, then $L\varphi=\{1\}$. If φ is constant then $L\varphi=\{0\}$. In the case where φ is non-constant with zero winding number with respect to 0, we denote by $\varepsilon\in(0, 2\pi]$ the length of the arc $\varphi(\mathbf{T})$. Then $L\varphi=(0, \sin\varepsilon/2)$ for $0<\varepsilon\leqq\pi/2$ and $L\varphi=(0, 1]$ for $\pi<\varepsilon\leqq 2\pi$.*

Proof. Obviously $L\varphi=\{0\}$ when φ is constant. If φ has non-zero winding number, then proposition 2 implies that $\min_I |\varphi_{I,\mu}|=0$ for all $\mu\in M$. It follows ${}_{\mu}|||\varphi|||=1$ for all $\mu\in M$. Therefore $L\varphi=\{1\}$.

We consider now the case of a non-constant φ with zero winding number with respect to 0. Lemma 13 assures that $L\varphi$ is a subinterval of [0, 1]. Proposition 6 implies that for every $\eta>0$, there is $\mu\in M$ with ${}_{\mu}|||\varphi|||<\eta$; therefore $\inf L\varphi=0$. Lemma 14 implies that $\sup L\varphi=\sup_{w\in\Gamma}\sqrt{1-|w|^2}$. It follows that $\sup L\varphi=1$ for $\pi<\varepsilon\leqq 2\pi$ and $\sup L\varphi=\sin\varepsilon/2$ for $0<\varepsilon\leqq\pi$.

Since φ is non-constant, we have $0\notin L\varphi$. If $0<\varepsilon\leqq\pi$, then $\cos\varepsilon/2\notin\{|w|: w\in\Gamma\}$ and $\sin\varepsilon/2\neq{}_{\mu}|||\varphi|||$ for all $\mu\in M$; therefore $L\varphi=(0,\sin\varepsilon/2)$. If $\pi<\varepsilon\leqq 2\pi$, then $0\in\Gamma$ and ${}_{\mu}|||\varphi|||=1$ for some $\mu\in M$. It follows that $1\in L\varphi$ and $L\varphi=(0,1]$. ▮

§ 4. Counterexamples

This section contains comments and couterexamples related to the results of § 1. Propositions 16 and 17 give examples of functions f not in $A(D)$ such that for some $\mu\in M$ the set $A_\mu(f)$ is not dense in $f(D)$. The example in proposition 16 is in H^∞, while the one in proposition 17 is not holomorphic but it is open in D and continuous on $\bar{D}$. Finally proposition 19 gives an example of a function $f\in A(D)$ such that $A_\mu(f)$ does not contain $f(D)$ for some $\mu\in M$, although $f(D)\in\overline{A_\mu(f)}$ as expected.

Proposition 16. *There are an infinite Blaschke product f and an absolutely continuous measure μ strictly positive on* **T** *such that $A_\mu(f)$ is not dense in $f(D)$ and ${}_{2,\mu}|||f|||<1$.*

Proof. Consider f an infinite Blaschke product, whose zeros accumulate everywhere on **T**. Let $0<\delta<1$ and $E=\{e^{i\theta}\in\mathbf{T}: \operatorname{Re} f(e^{i\theta})>\delta\}$. Then it is known that $|E\cap I|>0$ for all intervals $I\subset\mathbf{T}$; see [15], chapter VII for a related result. It follows that the absolutely continuous measure μ, $d\mu=X_E\,d\theta$, is a strictly positive measure on **T**.

Obviously $|f_{I,\mu}|\geqq\operatorname{Re} f_{I,\mu}>\delta$ for all intervals I. Therefore ${}_{2,\mu}|||f|||=\{1-\inf_I|f_{I,\mu}|^2\}^{1/2}\leqq(1-\delta^2)^{1/2}<1$. Since $0\in f(D)$ and $|f_{I,\mu}|>\delta>0$ for all I's, we see that $A_\mu(f)$ is not dense in $f(D)$. ▮

The above counterexample, communicated to the author by W. Rudin, shows that corollary 4 and proposition 5 do not extend to H^∞ functions and to general absolutely continuous measures.

Theorem 3 and corollary 4 extend easily in the case of functions $g\circ U$ with $g\in A(D)$ and U any homeomorphism of $\bar{D}$ onto itself. For any non-constant function $g\in A(D)$ the composition $g\circ U$ is continuous on $\bar{D}$, open in D and light, i.e. for any $w\in\mathbf{C}$ the set $(g\circ U)^{-1}(w)$ does not have accumulation points in D (see [14]). Our next proposition shows that theorem 3 and corollary 4 are not in general true for open-continuous functions which are not light.

Proposition 17. *There is a function $f\colon \bar{D}\to\mathbf{C}$ continuous on $\bar{D}$ and open in D such that $A_\sigma(f)$ is not dense in $f(D)$, where σ denotes the Lebesgue measure on* $\mathbf{T}$.

Proof. Consider the function $h(x+iy)=|y|\exp(x+i/|y|)$ for $x, y\in\mathbf{R}$, $y\neq 0$ and $h(x)=0$ for $x\in\mathbf{R}$. Then without great difficulty, one can check that the map $h\colon \mathbf{C}\to\mathbf{C}$ is continuous and open. It is also easy to see that $h(z_0)\notin h(\mathbf{T})$ for some $z_0\in D$.

Since $h(e^{-i\theta})=h(e^{i\theta})$ for all $\theta\in\mathbf{R}$, it follows that $h_{|\mathbf{T}}$ has zero winding number with respect to any point in $\mathbf{C}\setminus h(\mathbf{T})$. In particular $h_{|\mathbf{T}}$ has zero winding number with respect to $h(z_0)$. Proposition 6 implies now the existence of a measure $\mu\in M$ such that $h(z_0)\notin\overline{A_\mu(h)}$.

Since $\mu\in M$, there is a homeomorphism U of $\bar{D}$ onto itself such that $\mu(\mathbf{T})\,d\theta/2\pi=d\mu(U(e^{i\theta}))$. This implies $A_\mu(h)=A_\sigma(h\circ U)$, where σ denotes the Lebesgue measure on $\mathbf{T}$. Let $f=h\circ U$ and $z=U^{-1}(z_0)\in D$. Then f is continuous on $\bar{D}$, open in D and $f(D)\ni f(z)=h(z_0)\notin\overline{A_\mu(h)}=\overline{A_\sigma(f)}$. It follows that $A_\sigma(f)$ is not dense in $f(D)$. ∎

Theorem 1 shows that in the particular case of the Lebesgue measure the hypothesis "$f(z_0)\notin f(\mathbf{T})$" is not needed in theorem 3. Proposition 19 below gives a counterexample of a function $f\in A(D)$ and a measure $\mu\in M$ such that $f_{I,\mu}\neq 0$ for all intervals $I\subset\mathbf{T}$, although $0\in f(D)$. Certainly $0\in f(T)$, by theorem 3. We see, therefore, that the hypothesis "$f(z_0)\notin f(\mathbf{T})$" is not superfluous in theorem 3. Equivalently theorem 1 fails in the general case of measures $\mu\in M$.

In the example of proposition 19 the set $A_\mu(f)$ is dense in $f(D)$, by corollary 4. Therefore although $A_\mu(f)$ avoids 0, it must meet every disc centered at 0. This is an essential difference with the previous counterexamples and we expect a more delicate construction. The idea of this construction follows from lemma 18 whose straightforward proof is ommited; see also remark b in § 2.

Lemma 18. *Let $f\colon \mathbf{T}\to\mathbf{C}$ be a continuous function and $\mu\in M$. We denote $\tilde{\gamma}(\theta)=\int_0^\theta f(e^{it})\,d\mu(e^{it})$ for $0<\theta\leqq 4\pi$. Then we have:*

a) *$\tilde{\gamma}$ is continuous on $[0, 4\pi]$ and defines a rectifiable curve.*

b) *$f_{I,\mu}\neq 0$ for all intervals $I\subset\mathbf{T}$, if and only if $\tilde{\gamma}(A)\neq\tilde{\gamma}(B)$ for all $0\leqq A< B\leqq A+2\pi<4\pi$.*

c) *If $\tilde{\gamma}$ is one-to-one on $[0, 4\pi]$, then $f_{I,\mu}\neq 0$ for all intervals $I\subset\mathbf{T}$.*

d) *Let $I\subset\mathbf{T}$ be an open interval and suppose that $\chi_I\,d\mu(e^{i\theta})=\chi_I h(e^{i\theta})\,d\theta$ with h a strictly positive continuous function. Then $\tilde{\gamma}'(\theta)=f(e^{i\theta})h(e^{i\theta})$ for all $e^{i\theta}\in I$. Moreover, if $f(e^{i\theta})\neq 0$ on I, there are continuous determinations of $\operatorname{Arg}\tilde{\gamma}'$ and $\operatorname{Arg} f$ on I such that $\operatorname{Arg} f=\operatorname{Arg}\tilde{\gamma}'$.*

To construct the desired counterexample we will start with a function $f\in A(D)$ satisfying: $0\in f(D), f(1)=0, f(e^{i\theta})\neq 0$ on $T-\{1\}$ and $\lim_{\theta\to 0^+}\operatorname{Arg} f(e^{i\theta})=$

$\lim_{\theta\to 2\pi^-} \operatorname{Arg} f(e^{i\theta}) = -\infty$, where $\operatorname{Arg} f(e^{i\theta})$ is a continuous determination of the argument of $f(e^{i\theta})$ on $(0, 2\pi)$.

Then we construct a curve γ: $[0, 2\pi]\to \mathbf{C}$ such that $\operatorname{Arg}\gamma' = \operatorname{Arg} f$, $\gamma(0)=0$ and $\tilde{\gamma}$ is one-to-one on $[0, 4\pi]$; where $\tilde{\gamma}=\gamma$ on $[0, 2\pi]$ and $\tilde{\gamma}(\theta)=\gamma(2\pi)+\gamma(\theta-2\pi)$ on $[2\pi, 4\pi]$. This is possible because $\lim_{\theta\to 0} \operatorname{Arg} f(e^{i\theta}) = \lim_{\theta\to 2\pi^-} \operatorname{Arg} f(e^{i\theta}) = -\infty$.

Next we try to find a measure μ on $\mathbf{T}$ such that $\tilde{\gamma}(\theta)=\int_0^\theta f\,d\mu$. Then $f_{I,\mu}\neq 0$ for all I's according to lemma 18c.

We state now proposition 19 and we give a more detailed proof.

Proposition 19. *There are $f\in A(D)$, $\mu\in M$ and $z_0\in D$ such that $f(z_0)=0\notin A\mu(f)$.*

Proof. We consider the function $h(z)=z(z-1)\exp(z+1/z-1)$; then $h\in A(D)$. Let Ω denote the simply connected domain containing 0 and bounded by the Jordan curve

$$\left\{e^{i\theta}\colon 0 \leqq \theta \leqq 2\pi-\frac{\pi}{3}\right\} \cup \left\{\frac{1}{2}+it\colon \frac{-\sqrt{3}}{2} \leqq t \leqq \frac{1}{2}\right\} \cup \left\{\frac{1}{2}(1+e^{i\theta})\colon 0 \leqq \theta \leqq \frac{\pi}{2}\right\};$$

then $\Omega\subset D$ (see figure 1). Let φ: $D\to\Omega$ be a conformal mapping from D onto Ω, such that $\varphi(1)=1$. Then the function $f=h\circ\varphi$ is in $A(D)$ and $f(1)=f(z_0)=0$, where $z_0=\varphi^{-1}(0)\in D$. One can also easily check that f satisfies the conditions:

i) $f(e^{i\theta})\neq 0$ for all $e^{i\theta}\in\mathbf{T}\setminus\{1\}$. A continuous determination of $\operatorname{Arg} f(e^{i\theta})$, $0<\theta<2\pi$, satisfies $\lim_{\theta\to 0^+} \operatorname{Arg} f(e^{i\theta}) = \lim_{\theta\to 0^-} \operatorname{Arg} f(e^{i\theta}) = -\infty$. There is $\theta_0\in(0, 2\pi)$ such that $\operatorname{Arg} f(e^{i\theta})$ is strictly increasing on $(0, \theta_0]$ and strictly decreasing on $[\theta_0, 2\pi)$.

ii) There is $\delta>0$ such that $|f(e^{i\theta})|\,|\operatorname{Arg} f(e^{i\theta})|\geqq\delta$ on $(0, 2\pi)$.

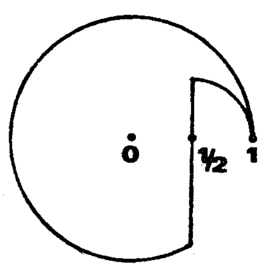

Figure 1

Let now γ be a continuous rectifiable curve in $\mathbf{C}$ starting from 0. We denote by $K\in(0, +\infty)$ its total length and we parametrize γ by arc-length: γ: $[0, K]\ni S\to \gamma(S)\in\mathbf{C}$, $\gamma(0)=0$. We suppose that γ has continuous derivative $\gamma'(S)$ on $(0, K)$. Then $|\gamma'(S)|=1$ and $\gamma'(S)=\exp(i\operatorname{Arg}\gamma'(S))$ for all $S\in(0, K)$. We define $\tilde{\gamma}$: $[0, 2K]\to C$ as follow: $\tilde{\gamma}=\gamma$ on $[0, K]$ and $\gamma(S)=\gamma(K)+\gamma(S-K)$ for $S\in[K, 2K]$.

We suppose that the conditions iii), iv) and v) below are satisfied.

iii) The map $\tilde{\gamma}$ is one-to-one on $[0, 2K]$.

iv) There is $S_0 \in (0, K)$ such that a continuous determination of $\operatorname{Arg} \gamma'(S)$ is strictly increasing on $(0, S_0]$, strictly decreasing on $[S_0, K)$ and satisfies $\lim_{S\to 0+} \operatorname{Arg} \gamma'(S) = \lim_{S\to K-} \operatorname{Arg} \gamma'(S) = -\infty$ and $\operatorname{Arg} \gamma'(S_0) = \operatorname{Arg} f(e^{i\theta_0})$.

v) $\int_0^K |\operatorname{Arg} \gamma'(S)|\, ds < \infty$, where $\operatorname{Arg} \gamma'$ is the determination of the argument of γ' in iv).

We assume for the moment the existence of a curve γ with the above properties. At the end of the proof we shall give an example of such a curve.

Properties i) and iv) imply the existence of a unique increasing homeomorphism $[0, 2\pi] \ni \theta \to S(\theta) \in [0, K]$ such that $S(0)=0$, $S(\theta_0)=S_0$, $S(2\pi)=K$ and $\operatorname{Arg} f(e^{i\theta}) = \operatorname{Arg} \gamma'(s(\theta))$ for all $\theta \in (0, 2\pi)$. We define $S(\theta) = S(2\pi) + S(\theta - 2\pi) = K + S(\theta - 2\pi)$, for $2\pi \leqq \theta \leqq 4\pi$. Obviously $S\colon [0, 4\pi] \to [0, 2K]$ is an increasing homeomorphism such that $S(0)=0$, $S(2\pi)=K$, $S(4\pi)=2K$ and $\operatorname{Arg} \tilde{\gamma}'(S(\theta)) = \operatorname{Arg} f(e^{i\theta})$ for all $\theta \in (0, 4\pi)$, $\theta \neq 2\pi$.

We define μ by the relation $d\mu(e^{i\theta}) = dS(\theta)/|f(e^{i\theta})|$, $0 < \theta < 2\pi$. Since S is strictly increasing, μ is a strictly positive measure on $\mathbf{T}$. Moreover the continuity of S implies that μ does not have point masses. Properties ii) and v) imply that μ is a finite measure:

$$\int_0^{2\pi} d\mu(e^{i\theta}) = \int_0^{2\pi} \frac{dS(\theta)}{|f(e^{i\theta})|} \leqq \frac{1}{\delta} \int_0^{2\pi} |\operatorname{Arg} f(e^{i\theta})|\, dS(\theta)$$

$$= \frac{1}{\delta} \int_0^{2\pi} |\operatorname{Arg} \gamma'(S(\theta))|\, dS(\theta) = \frac{1}{\delta} \int_0^K |\operatorname{Arg} \gamma'(S)|\, dS < \infty.$$

We see, therefore, that $\mu \in M$.

Let $0 \leqq \theta \leqq 2\pi$. Then

$$\int_0^\theta f(e^{it})\, d\mu(e^{it}) = \int_0^\theta \frac{f(e^{it})}{|f(e^{it})|}\, dS(t)$$

$$= \int_0^\theta e^{i \operatorname{Arg} f(e^{it})}\, dS(t) = \int_0^\theta e^{i \operatorname{Arg} \gamma'(S(t))}\, dS(t)$$

$$= \int_0^{S(\theta)} e^{i \operatorname{Arg} \gamma'(s)}\, ds = \int_0^{S(\theta)} \gamma'(S)\, dS = \gamma(S(\theta)) - \gamma(S(0))$$

$$= \gamma(S(\theta)) = \tilde{\gamma}(S(\theta)).$$

Similarly $\int_0^\theta f(e^{it})\, d\mu(e^{it}) = \tilde{\gamma}(S(\theta))$ for $2\pi \leqq \theta \leqq 4\pi$. The map $\tilde{\gamma}$ is one-to-one on $[0, 4\pi]$ by condition iii). Since S is injective, it follows that the map $\theta \to \int_0^\theta f\, d\mu = \tilde{\gamma}(S(\theta))$ is one-to-one on $[0, 4\pi]$. Lemma 18c implies now that $f_{I,\mu} \neq 0$ for all intervals $I \subset \mathbf{T}$.

It remains to give an example of a double spiral γ satisfying all above requirements. Such a curve is represented in figure 2:

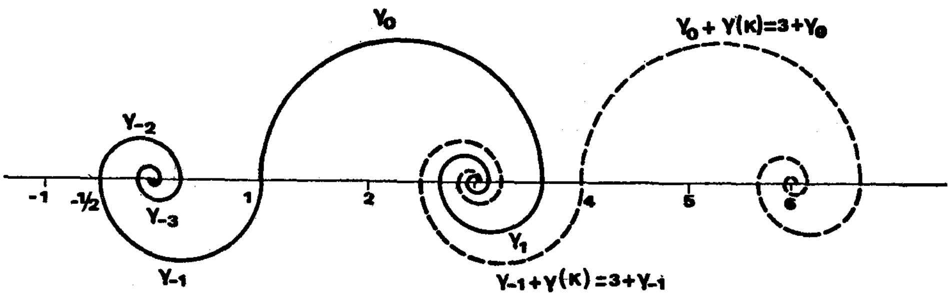

Figure 2

We denote by γ_n, $n=0, \pm 1, \pm 2, \ldots$ semicircles with centers on the real axis which are contained in the upper half-plane for even n and in the lower half-plane for odd n. The semicircle γ_0 has as diameter the segment $[1, 3+5/8]$. For $n=2, 4, 6, \ldots$ the diameter of γ_n is $[3-5/2^{n+2}, 3+5/2^{n+3}]$. For $n=1, 3, 5, \ldots$ the diameter of γ_n is $[3-5/2^{n+3}, 3+5/2^{n+2}]$. The diameter of γ_n is the segment $[-2^{n+1}, 2^n]$ for $n=-2, -4, -6, \ldots$. Finally for $n=-1, -3, -5, \ldots$ the diameter of γ_n is $[-2^n, 2^{n+1}]$.

We give to γ_n the positive orientation for $n<0$ and the negative one for $n \geqq 0$. Then one can check that a rotation of the curve $\sum_{-\infty}^{+\infty} \gamma_n$ satisfies all the requirements relative to the curve γ. ∎

Acknowledgement. I wish to express my thanks to A. Greenleaf, J. Harper, J. H. B. Kemperman, D. Marshall, S. Pappadopoulou, S. Pichorides, N. Varopoulos and many others not listed here, for helpful discussions.

References

1. Baernstein, A. II, Analytic functions of bounded mean oscillation, in: Brannan, D. A. and Clunie, J. G. (eds.), *Aspects of contemporary complex analysis,* pp. 3—36, Academic Press, London, 1980.
2. Danikas, N. and Nestoridis, V., Interval averages of H^1 functions and B.M.O. norm of inner functions, in: *Harmonic Analysis, Proceedings, Cortona, Italy 1981,* Lecture Notes in Mathematics 992, pp. 174—192, Springer-Verlag, Berlin etc., 1982.
3. Danikas, N. and Nestoridis, V., A property of H^1 functions, *Complex Variables Theory Appl.* 4 (1985), 277—284.
4. Dugundji, J., *Topology,* Allyn and Bacon, Boston, 1978.
5. Duren, P., *Theory of H^p spaces,* Academic Press, New York, 1970.
6. Fefferman, C. and Stein, E., H^p spaces of several variables, *Acta Math.* **129** (1972), 137—193.
7. Garnett, J., *Bounded analytic functions,* Academic Press, New York, 1981.
8. Halmos, P., *Measure theory,* Van Nostrand, New York, 1950.
9. Hocking, J. and Young, G., *Topology,* Addison-Wesley, Reading, 1961.
10. Hoffman, K., *Banach spaces of analytic functions,* Prentice-Hall, Englewood Cliffs, 1962.
11. Royden, H., *Real analysis,* Macmillan, New York, 1963.

12. RUDIN, W., *Real and complex analysis,* McGraw-Hill, New York, 1974.
13. WALLACE, A., *An introduction to algebraic topology,* Pergamon Press, London, 1957.
14. WHYBURN, G., *Topological analysis,* Princeton University Press, Princeton, 1964.
15. ZYGMUND, A., *Trigonometric Series,* vol. 1, Cambridge University Press, Cambridge, 1959.

Received December 3, 1984

V. Nestoridis
Department of Mathematics
University of Crete
Heraklion
Crete
Creece