Random walks on groups. Applications to Fuchsian groups

N. Th. Varopoulos

§ 0. Introduction

Let G be a discrete group and let $\mu \in \mathbf{P}(G)$ be a probability measure on G. I shall define three random walks on G by the following three doubly stochastic matrices.

$$P_l^\mu(G)\colon\ P_l(x, y) = \mu(\{y^{-1}x\});\quad x, y, \in G,$$

$$P_r^\mu(G)\colon\ P_r(x, y) = \mu(\{xy^{-1}\});\quad x, y \in G,$$

$$P_s^\mu(G)\colon\ P_s(x, y) = \frac{P_l(x, y)+P_r(x, y)}{2};\quad x, y \in G.$$

From the general theory (cf. [1]) it is an easy matter to verify that if $\sum_{n\geqq 0}\mu^n(\{e\})<+\infty$ (μ^n indicates the convolution power of μ and $e\in G$ is the neutral element of G) then the above three walks are transient and if $\sum_{n\geqq 0}\mu^n(\{e\})=+\infty$ the above three walks are reccurent. What is also true but less well-known (cf. [2], [3]) is that if we restrict our attention to these measures $\mu \in \mathbf{P}(G)$ that satisfy:

(i) supp μ is finite,

(ii) $\mu=\check{\mu}$ (i.e. $\check{\mu}(\{g\})=\mu(\{g^{-1}\})$, $g\in G$),

(iii) Gp (supp μ)$=G$,

then the transience or recurrence of the above three walks is independent of the particular choice of μ and only depends on G. We say that G is transient if these walks (for μ that satisfy (i), (ii) and (iii)) are transient, otherwise we say that G is recurrent. The following seems to be a reasonable:

Conjecture. *Let G be a finitely generated group then G is recurrent if and only if there exists $G^*\subset G$ such that the index $[G\colon G^*]<+\infty$ is finite and* $G^*\cong\{e\}$, $\mathbf{Z}$ or $\mathbf{Z}^2$.

So far the conjecture has been proved when G is either soluble or linear (cf. [3], [6]. Observe incidentally that the linear case can be reduced to the soluble case by a theorem of Tits cf. [7] §10.16). One of the results that will be proved in this paper is the following:

Theorem 1. *Let G be a finitely generated group and let $G\supset H_1\supset H_2$ be two finitely generated subgroups such that*

$$|G:H_1|=|H_1:H_2|=|H_2|=+\infty.$$

Then G is transient.

This theorem shows that if the above conjecture is false then the conterexample must be close to the "Tarski monsters" that have been constructed only recently (cf. [8]).

Randoms walks on groups are closely related to Brownian motion on manifolds (cf. [4], [5]) and to the "convergence type" of Fuchsian groups (cf. [9]).

Let Γ be a Fuchsian group acting on $U=\{z\in\mathbf{C},\ |z|<1\}$ (assume that it is of the first kind for otherwise the problems in question do not even arise). We say that Γ is of convergent type if:

$$\sum_{\gamma\in\Gamma}(1-|\gamma 0|)<+\infty,$$

otherwise we say that it is of divergent type.

Let now Γ_0 be a finitely generated Fuchsian group and let $\Gamma\lhd\Gamma_0$ be a normal subgroup. I shall distinguish the following three cases:

Case (A): Either there are no parabolic elements in Γ_0 or for every Z cyclic parabolic subgroup of Γ_0 (i.e. Z consists entirely of parabolic elements, we have then $Z\cong\mathbf{Z}$) we have

$$|\Gamma\cap Z|=+\infty.$$

Case (B): We are not in case (A) and the group Γ_0/Γ is finite or a finite extension of a cyclic group (we say then that Γ_0/Γ is "cyclic by finite").

Case (C): We are not in Case (A) and the group Γ_0/Γ is *not* a finite extension of a cyclic group. We shall prove the following

Theorem 2. *Let Γ_0 be a finitely generated Fuchsian group and let $\Gamma\subset\Gamma_0$ be a* normal *subgroup, then:*

In case (A): Γ is of convergent type if and only if the group Γ_0/Γ is transient.

In case (B): Γ is of divergent type.

In case (C): Γ is of convergent type.

An equivalent way to present the above is to say:

(i) If Γ_0/Γ is cyclic by finite then Γ is of divergent type.

(ii) If Γ_0/Γ is not cyclic by finite and each parabolic element in Γ_0 generates a finite subgroup in Γ_0/Γ then Γ is of convergent type if and only if Γ_0/Γ is transient.

(iii) If Γ_0/Γ is not cyclic by finite and there exists one parabolic element in Γ_0 that generates an infinite subgroup of Γ_0/Γ then Γ is of convergent type.

§ 1. Statement of the results

The main technical result on which everything else rests is the following

The Step-up Theorem. *Let G be a group generated by the finite symmetric set $S=\{g_1, \dots, g_s\}=S^{-1}$ and let $H\subset G$ be a subgroup of infinite index $|G\colon H|=+\infty$. Let $\xi_n=\int_0^1 \lambda^n\, d\xi(\lambda)$ where $\xi\in\mathbf{P}([0,1])$ is a probability measure on* $[0, 1]$. Let *$\mu=\check{\mu}\in\mathbf{P}(H)$ be a symmetric probability measure on H that satisfies*

$$\sum_{n\geqq 1} n^{-1/2}\xi_n\mu^n(\{e\})<+\infty. \tag{1.1}$$

Then for every $\nu\in\mathbf{P}(G)$ satisfying:

(i) *$\alpha\mu\leqq\nu$ for some $\alpha>0$*

(ii) *$\nu(\{e\})>0$ and $\nu(g_j)>0$ $\forall g_j\in S$*

we have

$$\sum_{n\geqq 0}\xi_n\nu^n(\{e\})<+D. \tag{1.2}$$

Corollary 1. *Let G be an infinite finitely generated group and let ν be a symmetric measure such that Gp {supp ν}$=G$. Then $\nu^n(\{e\})=0(n^{-1/2+\varepsilon})$ $(\forall\varepsilon>0)$ (and also $=0(n^{1/2}(\log)\, n^{1+\varepsilon})$, $\varepsilon>0$ etc.).*

Proof: If we take in the step-up theorem $H=\{e\}$, $\mu=\delta_e$ and $\xi_n=Cn^{-1/2-\varepsilon}$ we deduce that $\sum_{n\geqq 1} n^{-1/2-\varepsilon}\nu^n(\{e\})<+\infty$. The result follows because $\nu^{2n}(\{e\})$ is a decreasing sequence in n (observe that $\nu^2(\{e\})=\sup_{g\in G}\nu^2(\{g\})$).

Corollary 2. *Let G be a finitely generated group and let $H\subset G$ be a subgroup that is also finitely generated and such that $|H|=|G\colon H|=+\infty$. Then for every symmetric probability measure $\nu\in\mathbf{P}(G)$ that satisfies Gp (supp ν)$=G$ we have $\nu^n(\{e\})=0(n^{-1+\varepsilon})$, $\forall\varepsilon>0$, (and also $0(n^{-1}(\log n)^{1+\varepsilon})$, $\varepsilon>0$ etc.).*

Proof: If we take $\xi_n=n^{-1/2-\varepsilon}$ in the step-up theorem and $\mu=\check{\mu}\in\mathbf{P}(H)$ some measure that satisfies $\mu^n(\{e\})=n^{-1/2+\varepsilon/2}$ (by Cor. 1) we deduce that $\sum_{n\geqq 1} n^{-\varepsilon}\nu^n(\{e\})<+\infty$. The result follows as before.

Proof of Theorem 1. In the step-up theorem take $\xi_n\equiv 1$, $H=H_1$ and $\mu=\check{\mu}\in\mathbf{P}(H_1)$ such that $\mu^n(\{e\})=0(n^{-2/3})$ (by Cor. 2). It follows that $\sum_{\nu\geqq 0}\nu^n(\{e\})<+\infty$.

§ 2. The tools for the proof of the step-up theorem

The proof of the step-up theorem is based on the same analytic principle that was already used in [4] and [5] and which appeared in a special form for the first time in [2].

Let X be a discrete space and let $K_1(x, y)$, $K_2(y, x)$ be two *doubly* stochastic kernels on X (i.e. $K_i(x, y)$ and $K_i^*(x, y)=K_i(y, x)$ $i=1, 2$, are all Markovian). Let us also assume that for some $\alpha>1$ we have $K_1(y, x)\leqq\alpha K_2(x, y)$ and also that $K_1(x, y)=K_1(y, x)$ (i.e. K_1 is symmetric). Let further $0\leqq f\in l^2(X)$ and $\xi_n=\int_0^1\lambda^n d\xi(\lambda)$ where $\xi\in\mathbf{P}([0, 1])$ is a probability measure on $[0, 1]$ (e.g. $\xi_n\sim n^{-\beta}$ of $n^{-\beta}(\log n)^A$, $\beta>0$). The conclusion is that

$$\sum_{n\geqq 0}\xi_n\langle K_2^n f, f\rangle \leqq \alpha\sum_{n\geqq 0}\xi_n\langle K_1^n f, f\rangle \tag{2.1}$$

($K_i^n f$ indicates the corresponding l^2-operator and $\langle\,\rangle$ is the scalar product in l^2).

The other main ingredient in the proof of the step-up theorem is essentially of geometric nature. Let G be a discrete group generated by the symmetric finite set $S=\{g_1, ..., g_s\}$. Let $H\subset G$ be a subgroup such that $|G: H|=+\infty$. Let now $P=\{\gamma_0=e, \gamma_1, \gamma_2, ...\}$ be a sequence of points in G. We say that P is a path (relative to H and S) if

(i) $\gamma_k^{-1}\gamma_{k+1}\in S$, $k=0, 1, ...$

(ii) the cosets $H\gamma_j$, $j=0, 1, ...$, are distinct.

The only thing that we shall need is that paths exist. Indeed let $\dot{d}$ be the canonical quotient distance induced on G/H by the left invariant distance d_l on G relative to the set of generators S (cf. [11], [5]). Let $\dot{g}\in G/H$ be such that $\dot{d}(\dot{e}, \dot{g})=N$ ($\dot{e}=H\in G/H$) and let $g\in\dot{g}$ be such that $d_l(e, g)=N$. We have then $g=g_{i_1}g_{i_2}\cdots g_{i_N}$ with $g_{i_s}\in S$ $s=1, ..., N$. But then the sequence $\{\gamma_j\}_{j=0}^N$ given by $\gamma_0=e$, $\gamma_n=g_{i_1}g_{i_2}\cdots g_{i_n}$ $(n\geqq 1)$ is clearly a "path of length N". A standard Tychonov diagonal process gives then "infinite" paths as required.

One more construction will be needed from the theory of random walks. Let X_i $(i=1, 2)$ and $K_i(x, y)$ be two Markovian matrices generating random walks W_i on X_i, then we can clearly define the product walk $W_1\otimes W_2$ on the space $X=X_1\times X_2$ by the matrix $K(x, y)=K_1(x_1, y_1)K_2(x_2, y_2)$ with $x=(x_1, x_2)$, $y=(y_1, y_2)\in X$.

§ 3. Proof of the step-up theorem

Let G, H, μ, ν, S and $\{\xi_n\}$ be as in the statement of the step-up theorem. Let also $P=\{\gamma_0, \gamma_1, \ldots\}$ be a path in G relative to H and S. Let us now define a random walk on G/H by the following symmetric stochastic matrix:

$$W(\dot{x}, \dot{y}) = \begin{cases} 1/2 & \text{if } \dot{x} = H \text{ and } \dot{y} = H \text{ or } H\gamma_1 \\ 1/2 & \text{if } \dot{x} = H\gamma_k \text{ and } \dot{y} = H\gamma_{k\pm 1} \quad (k \geqq 1) \\ 1 & \text{if } \dot{x} = \dot{y} \neq H\gamma_k \quad (\forall k \geqq 1) \\ 0 & \text{in all other cases.} \end{cases}$$

The above walk is essentially a reflecting standard coin tossing game on the image of the path $\dot{P}=\{\dot{\gamma}_k, k\geqq 0\}\subset G/H$ and it is clear that $W^n(\dot{e}, \dot{e})\sim Cn^{-1/2}$.

Let us now observe that, at least as a set, we can identify G with $H\times(G/H)$ by identifying $(h, \dot{g}_k)$ with $hg_k\in G$ where $(\dot{g}_k;\, k\geqq 0)$ is an enumeration of G/H and $\Gamma=(g_k\in \dot{g}_k)$ is a system of coset representative. We shall assume that $P\subset\Gamma$.

We can now define on $H\times(G/H)$ the Cartesian product walk $K=P_r^\mu(H)\otimes W$ where $P_r^\mu(H)$ is the right walk defined by μ on H as in §0. Using the above identification we can then identify K with a symmetric random walk on G, and that walk satisfies

(3.1) $$K^n(e, e) = \mu^n(\{e\})W^n(\dot{e}, \dot{e}) \sim Cn^{-1/2}\mu^n(\{e\}).$$

The pivot of the proof lies in the simple observation that:

(3.2) $$K = P_r^\mu(H)\otimes W \leqq \alpha(P_s^\nu(G))^2 = \alpha \text{ (the square of the matrix } P_s^\nu)$$

for some positive α. In fact this is the "raison d'être" of P_s^ν. The verification of (3.2) is immediate and rests on the conditions (i) and (ii).

The estimate (3.1) together with (2.1) and the hypothesis (1.1) on μ gives then that $\sum_{n\geqq 0} \xi_n \nu^{2n}(\{e\})$ and completes the proof of the Theorem.

§4. Proof of theorem 2

The proof of theorem 2 is based on the step-up theorem and on the results of [9]. For $\Gamma\subset\Gamma_0$ as in Theorem 2 let us proceed as in [9] and let us fix F_i $(i=1, \ldots, k)$ a complete set of representatives of inequivalent under conjugation maximal cyclic parabolic subgroups of Γ_0 (cf. [10] § 10-3) and let us also fix symmetric measures $\mu_i\in\mathbf{P}(\Gamma_0)$, $i=0, 1, \ldots, k$. Where supp μ_0 is finite with $Gp(\text{supp } \mu_0)=\Gamma_0$ and where μ_i, $1\leqq i\leqq k$, is the Cauchy distribution $\mu_i(\xi_i^n)=C(1+n^2)^{-1}$ $(n\in\mathbf{Z})$ on $Gp(\xi_i)=F_i$. Let us denote by $\nu=\alpha\left(\frac{1}{k+1}\sum_{j=0}^k \mu_j\right)$ where α: $\Gamma_0\to\Gamma_0/\Gamma$ is the canonical homomorphism.

What emerges from the main theorem of [9] is that Γ is of convergent type if and only if the random walk $P_l^\nu(\Gamma_0/\Gamma)$ is transient.

Proof of Theorem 2:

Case (A): ν is compactly supported and the result follows from the previous few lines.

Case (B): Let Z be a cyclic parabolic subgroup such that $|\Gamma\cap Z|<+\infty$; we have then $\Gamma\cap Z=\{e\}$ and the group $\Gamma_1=Gp(\Gamma, Z)$ is then of finite index in Γ_0 (this follows from the algebraic hypothesis on Γ_0/Γ). Γ_1 is then also finitely generated, and we can therefore assume that $\Gamma_0=Gp(\Gamma, Z)=\Gamma_1$ and choose $F_1\supset Z$. It follows that with the obvious identification, $\nu(p)\geqq\alpha\nu_1(p)$ $(p\in\mathbf{Z})$, where $\nu_1(p)=C(1+p^2)^{-1}$ is the Cauchy distribution on Z. But then since $\nu_1^n(0)\sim\frac{1}{n}$ the estimate (2.1) gives $\sum_{n\geqq 0}\nu^n(0)=+\infty$ and proves our assertion.

Case (C): Arguing as on Case (B) we can choose F_1 such that $A=\alpha(F_1)\cong\mathbf{Z}$ and $|\Gamma_0/\Gamma\colon A|=+\infty$. But then with the obvious identification we have $\nu|_A\geqq\alpha\nu_1$ for some $\alpha>0$ (ν_1 is the Cauchy distribution on $A\cong\mathbf{Z}$ as above).

The step-up theorem applies then with $G=\Gamma_0/\Gamma$, $H=A$, $\mu=\nu_1$, $\xi_n\equiv 1$ and gives $\sum_{n\geqq 0}\nu^n(\{e\})<+\infty$.

The proof is complete.

References

1. Feller, W., *An introduction to Probability Theory and its Applications.* J. Wiley 3rd Edition.
2. Baldi, P., Lohoue, N. et Peyriere, J., *C. R. Acad. Sci. Paris t* **285** (A) (1977), p. 1103—1104.
3. Guivarch, Y. et Al., Marches aléatories sur les groupes le Lie. *Lectures Notes n° **624** Springer-Verlag.*
4. Varopoulos, N., Brownian Motion and Transient Groups. *Ann. Inst. Fourier.* **XXXIII** (2) (1983), p. 241—261.
5. Varopoulos, N., Brownian motion and random walks on manifolds Institut Mittag—Leffler Repat n° 7 1983 and Ann. Inst. Fourier 1984 t. XXXIV (2) (1984), p. 243—269.
6. Varopoulos, N., Random walks on soluble groups. *Bull. Sc. Math.* 2e serie **(107)** 1983.
7. Wehrfritz, B. A. F., *Infinite linear groups.* Ergebnisse der Mathematic n° 76.
8. Ol'šanskiĭ, A. Ju., An infinite group with subgroups of prime orders. *Math. U.S.S.R. Izvestija* Vol. **16** (1981) n° 2 p. 276—289.
9. Varopoulos, N., A characterization of Fuschian Groups of Convergent type. *Arkiv för math.* (1984), p. 293—307.
10. Beardon, A. F., *The Geometry of Discrete Groups.* Springer-Verlag.
11. Gromov, M., Groups of Polynomial growth and expanding maps. *I.H.E.S. Pub. Math.* **(53)** 1981.

Received November 2, 1983

N. Th. Varopoulos
Université Paris VI
Analyse Complexe et
Geometrie
4 Place Jussien
F-75230 PARIS Cedex 05
FRANCE