On maximal functions generated by Fourier multipliers

H. Dappa and W. Trebels

1. Introduction

Denote by S the space of all infinitely differentiable, rapidly decreasing functions on $\mathbf{R}^n$ and by $L^p=L^p(\mathbf{R}^n)$, $1\leqq p\leqq\infty$, the standard Lebesgue spaces. Then $m\in L^\infty$ is said to be an M_p-multiplier if

$$\|m\|_{M_p} = \inf\{C\colon \|F^{-1}[mf^\wedge]\|_p \leqq C\|f\|_p,\ f\in S\}$$

is finite. Here $\hat{}$ or F denotes the Fourier transformation and F^{-1} its inverse. The purpose of this paper is to examine maximal functions of the type (A_t being a dilation matrix)

$$M_m f(x) = \sup_{t>0} |F^{-1}[m(A_{1/t}\xi)f^\wedge(\xi)](x)|,\ f\in S, \tag{1.1}$$

from the point of view of the Fourier multiplier m.

If $F^{-1}[m]$ is integrable and has an integrable decreasing radial majorant then it is well known (see e.g. Stein [18; p. 62] for isotropic dilations and for anisotropic ones see Madych [14]) that $M_m f$ can be dominated pointwise by the classical Hardy—Littlewood maximal function:

$$M_m f(x) \leqq CMf(x) \text{ a.e.},\ Mf(x) = \sup_{r>0} r^{-n}\int_{|y|\leqq r} |f(x-y)|\,dy. \tag{1.2}$$

(We use C as a constant, independent of f and x, which is not necessarily the same at each occurrence.) Moreover, for radial integrable $F^{-1}[m]$ there holds

$$\|M_m f\|_p \leqq C\|f\|_p \tag{1.3}$$

provided $n\geqq 3$ and $p>n/(n-1)$ which is a consequence of Stein's [20] result on spherical means. For the case $p=n=2$, Aguilera [1] has given sufficient conditions on radial $F^{-1}[m]$ to satisfy (1.3).

In [15] Peetre discusses multipliers m (which generate maximal functions via (1.1) with $A_t=tI$, I being the identity matrix) and, in particular, special cases of the quasi-radial multipliers $m(\varrho(\xi))$, m defined on $(0,\infty)$, introduced below, for which he gives conditions (via interpolation spaces) only on m. Both aspects were used (and Peetre's criteria modified) in Dappa and Trebels [6] where for the quasi-radial case a subordination argument is used; the resulting condition can easily be refined via square functions by a method in Stein and Weiss [22; p. 278]. For radial multipliers, the same idea is described in a paper by Carbery [3] which reached us during the preparation of this paper.

Here we want to specify conditions on the Fourier multiplier m so that the operator M_m is of weak or strong type (p,p); for $p=2$ see e.g. Stein and Wainger [21; p. 1271]. To this end, we discuss in Section 2 the case of quasi-radial Fourier multipliers via subordination (see e.g. Stein [19; p. 46]) with respect to appropriate Riesz kernels and give in Section 3 the relevant square function estimate.

A criterion of Zo [24] in Section 4 allows to weaken these conditions in the case weak $(1,1)$. In Section 5 we deal with not necessarily quasi-radial m and give a weak type $(1,1)$ and a strong type $(2,2)$ estimate. We conclude with an example of a maximal function which maps

$$L^p_\beta=\{f\in L^p\colon\ (1+|\xi|^2)^{\beta/2}f^\wedge\in[L^p]^\wedge\},\ \beta>0, \tag{1.4}$$

normed by $\|f\|_{L^p_\beta}=\|F^{-1}[(1+|\xi|^2)^{\beta/2}f^\wedge]\|_p$, boundedly into L^p.

2. The case of quasi-radial Fourier multipliers

Let P be a real $n\times n$ matrix with eigenvalues α_j, $\operatorname{Re}\alpha_j>0$; set $a=\min_{1\leqq j\leqq n}\operatorname{Re}\alpha_j$, $A=\max_{1\leqq j\leqq n}\operatorname{Re}\alpha_j$ and $\nu=\operatorname{tr}P$. Following Stein and Wainger [21] associate to P the dilation matrix $A_t=t^P$ and, slightly generalizing their notion, introduce positive, A_t-homogeneous distance functions $\varrho(\xi)$, i.e., continuous functions ϱ on $\mathbf{R}^n$ with

$$\varrho(\xi)>0 \quad\text{for}\quad \xi\neq 0,\ \varrho(A_t\xi)=t\varrho(\xi)\quad\text{for all}\quad t>0,\ \xi\in\mathbf{R}^n. \tag{2.1}$$

Then one can show (see Stein and Wainger [21], Dappa [5]) that for any $\varepsilon>0$ there are positive constants such that

$$\begin{aligned} c_\varepsilon|\xi|^{1/(a-\varepsilon)} &\leqq \varrho(\xi)\leqq C_\varepsilon|\xi|^{1/(A+\varepsilon)}, \quad |\xi|\to 0,\\ c'_\varepsilon|\xi|^{1/(A+\varepsilon)} &\leqq \varrho(\xi)\leqq C'_\varepsilon|\xi|^{1/(a-\varepsilon)}, \quad |\xi|\to\infty. \end{aligned} \tag{2.2}$$

If $m(r)$ is defined on $(0,\infty)$ we call its extension to $\mathbf{R}^n_0=\mathbf{R}^n\setminus\{0\}$ via $m(\varrho(\xi))$ a quasi-radial function and denote its corresponding maximal operator by $M_{m\circ\varrho}$.

In order to state our first result we need the notion of the fractional derivative. Following Gasper and Trebels [8], define for $0<\delta<1$ and a locally integrable function g on $\mathbf{R}$, which vanishes identically on $(-\infty, 0)$, the fractional integrals $(\omega>0)$

$$\Gamma(\delta) I_\omega^\delta(g)(t) = \begin{cases} \int_t^\omega (s-t)^{\delta-1} g(s)\,ds, & t<\omega \\ 0, & t \geqq \omega \end{cases}$$

and fractional derivatives of order γ, $\gamma=[\gamma]+\delta$ with $[\gamma]$ being the largest integer less than or equal γ, by

$$g^{(\delta)}(t) = \lim_{\omega\to\infty} -\frac{d}{dt} I_\omega^{1-\delta}(g)(t), \quad 0<\delta<1,$$

$$g^{(\gamma)}(t) = \left(\frac{d}{dt}\right)^{[\gamma]} g^{(\gamma-[\gamma])}(t), \quad \gamma>0,$$

whenever the right sides exist; $g^{(\gamma)}$, $\gamma\in\mathbf{N}$, denotes the classical derivative. In the following we assume that $I_\omega^{1-\delta} g$ is locally absolutely continuous for each $\omega>0$ as well as $g^{(\delta)}, \dots, g^{(\gamma-1)}$. (Note that a heuristic computation gives $[g^{(\gamma)}]^\wedge = (-1)^{[\gamma]}(-i\sigma)^\gamma g^\wedge$, $\sigma\in\mathbf{R}$.)

In the following theorem we discuss the maximal operator

$$M_{m\circ\varrho} f(x) = \sup_{t>0} \left| F^{-1}\left[m\left(\frac{\varrho(\xi)}{t}\right) f^\wedge \right](x) \right|, \quad f\in S.$$

Theorem 1. *Let* $\varrho\in C^{[n/2+1]}(\mathbf{R}_0^n)$ *be a positive,* A_t*-homogeneous distance function. Let* m *be a measurable function on* $(0,\infty)$ *which vanishes at infinity.*

a) *If* m *satisfies*

(2.3) $$\int_0^\infty t^{\lambda-1} |m^{(\lambda)}(t)|\,dt \leqq B,$$

(2.4) $$\left(\int_0^\infty |t^\lambda m^{(\lambda)}(t)|^2 \frac{dt}{t}\right)^{1/2} \leqq B$$

for $\lambda>n\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}$, *then* $M_{m\circ\varrho}$ *is of strong type* (p,p), $1<p<\infty$, *i.e.,* (1.3) *holds; also* $M_{m\circ\varrho}$ *is of weak type* (1,1).

b) *If* m *satisfies* (2.3) *for* $\lambda>(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)+1$, *then* $M_{m\circ\varrho}$ *is of strong type* (p,p), $2\leqq p\leqq\infty$.

Proof. b) In the case $p=\infty$ the hypothesis implies $m\circ\varrho\in[L^1]^\wedge$ by Dappa [5] and, therefore,

$$|F^{-1}[m(\varrho(\xi)/t) f^\wedge](x)|$$
$$\leqq t^\nu \int |F^{-1}[m\circ\varrho](A_t y)|\, \|f(\cdot - y)\|_\infty\,dy \leqq C\|f\|_\infty$$

uniformly in $t>0$, thus the assertion for $p=\infty$. The case $p=2$ follows from part a), $p=2$, since for $\nu-\lambda>1/2$, $\nu>1$, there holds (see Gasper and Trebels [8])

$$m^{(\lambda)}(t)=C\int_0^\infty (s-t)_+^{\nu-\lambda-1}m^{(\nu)}(s)\,ds$$

and therefore, by the integral Minkowski inequality,

$$\Big(\int_0^\infty |t^\lambda m^{(\lambda)}(t)|^2\frac{dt}{t}\Big)^{1/2}\leqq C\int_0^\infty |m^{(\nu)}(s)|\Big(\int_0^s t^{2\lambda-1}(s-t)^{2\nu-2\lambda-2}\,dt\Big)^{1/2}ds$$
$$\leqq C\int_0^\infty s^{\nu-1}|m^{(\nu)}(s)|\,ds.$$

Analogously

$$\int_0^\infty t^{\lambda-1}|m^{(\lambda)}(t)|\,dt\leqq C\int_0^\infty s^{\nu-1}|m^{(\nu)}(s)|\,ds,\quad \nu>\lambda,$$

and hence the hypothesis for $p=2$ in b) implies the hypothesis for $p=2$ in a). Thus, in particular, for $m_\mu(t)=(1-t)_+^{\mu-1}$ we obtain

$$\|M_{m_\mu\circ\varrho}f\|_p\leqq C_\mu\|f\|_p \tag{2.5}$$

for $p=2$, $\operatorname{Re}\mu-1>\nu>0$ and $p=\infty$, $\operatorname{Re}\mu-1>\nu>(n-1)/2$, where C_μ is only of polynomial growth in $|\mu|$. An application of the interpolation theorem for analytic families of operators along the lines of Stein and Weiss [22; p. 280] gives that (2.5) holds for $\mu>(n-1)\Big(\dfrac{1}{2}-\dfrac{1}{p}\Big)+1$, $2\leqq p\leqq\infty$. Now we have for $f\in S$ (cf. Gasper and Trebels [8] or Trebels [23])

$$\begin{aligned}(2.6)\qquad &|F^{-1}[m(\varrho(\xi)/t)f^\wedge](x)|\\ &=C\Big|\int_{\mathbf{R}^n}\int_0^\infty (1-\varrho(\xi)/ts)_+^{\mu-1}s^{\mu-1}m^{(\mu)}(s)\,ds\,f^\wedge(\xi)e^{ix\cdot\xi}\,d\xi\Big|\\ &\leqq C\int_0^\infty s^{\mu-1}|m^{(\mu)}(s)|\,|F^{-1}[m(\varrho(\xi)/ts)f^\wedge](x)|\,ds\\ &\leqq C\int_0^\infty s^{\mu-1}|m^{(\mu)}(s)|\,ds\,M_{m_\mu\circ\varrho}f(x)\quad\text{a.e.}\end{aligned}$$

uniformly in $t>0$ which, in combination with (2.5), gives the assertion.

The proof of part a) relies heavily upon the following estimate for square functions with respect to Riesz kernels, thus underlining the basic nature of Littlewood—Paley functions. Set for $f\in S$

$$S_t^\lambda(f;x)=F^{-1}[(1-\varrho(\xi)/t)_+^\lambda f^\wedge](x)$$

and

$$g_\lambda(f;x)=\Big(\int_0^\infty |S_t^\lambda(f;x)-S_t^{\lambda-1}(f;x)|^2\frac{dt}{t}\Big)^{1/2} \tag{2.7}$$

Theorem 2. *Let* $\varrho\in C^{[n/2+1]}(\mathbf{R}_0^n)$ *and the operator* g_λ *be defined by* (2.7). *Then* g_λ *is of weak type* (1, 1) *and strong type* (p,p), $1<p<\infty$, *provided that* $\lambda>n\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}$.

A proof of Theorem 2 in the case $\varrho(\xi)=|\xi|^2$ can be found in Igari and Kuratsubo [12]. An essential improvement in two dimensions for $\varrho(\xi)=|\xi|$ is due to Carbery [3]:

$$\int_{\mathbf{R}^2}|g_\lambda(f;\ x)|^4\,dx \leqq C\int_{\mathbf{R}^2}|f(x)|^4\,dx,\quad \lambda>1/2. \tag{2.8}$$

Suppose for the moment that Theorem 2 holds. Then, by (2.6),

$$\begin{aligned}&\left|F^{-1}[m(\varrho(\xi)/t)f^{\wedge}](x)\right|\\ &\leqq C\sum_{j=0}^{k}\int_0^\infty s^\lambda|m^{(\lambda)}(s)|\,|S_{st}^{\lambda-1+j}(f;\ x)-S_{st}^{\lambda+j}(f;\ x)|\frac{ds}{s}\\ &\quad+C\int_0^\infty s^{\lambda-1}|m^{(\lambda)}(s)|\,|S_{st}^{\lambda+k}(f;\ x)|\,ds\\ &\leqq C\left(\int_0^\infty |s^\lambda m^{(\lambda)}(s)|^2\frac{ds}{s}\right)^{1/2}\sum_{j=0}^{k}g_{\lambda+j}(f;\ x)\\ &\quad+C\int_0^\infty s^{\lambda-1}|m^{(\lambda)}(s)|\,ds\sup_{t>0}|S_t^{\lambda+k}(f;\ x)| \text{ a.e.,}\end{aligned}$$

since k can be chosen sufficiently large so that $F^{-1}[(1-\varrho(\xi))_+^{\lambda+k}]$ is integrable. Thus, $\sup_{t>0}|S_t^{\lambda+k}(f;x)|$ is of strong type (∞,∞) and the assertion follows by Theorem 3 and the interpolation theorem of Marcinkiewicz.

Remarks. i) If one follows Carbery's [3] approach, starting with

$$\frac{m(t)}{t}=C_\lambda\int_t^\infty (s-t)^{\lambda-1}(m(s)/s)^{(\lambda)}\,ds,$$

one is, analogously to the above, lead to

$$M_{m\circ\varrho}f(x)\leqq Cg_\lambda(f;\ x)\left(\int_0^\infty \left|s^{\lambda+1}\left(\frac{m(s)}{s}\right)^{(\lambda)}\right|^2\frac{ds}{s}\right)^{1/2},\quad \lambda>(n+1)/2,$$

for $\varrho\in C^{[n/2+1]}(\mathbf{R}_0^n)$.

ii) If $\sum_\varrho=\{\xi:\ \varrho(\xi)=1\}$ is strictly convex, $\varrho\in C^{[3n/2+5]}(\mathbf{R}_0^n)$, and m satisfies only (2.3) for $\mu>(n-1)\left(\frac{1}{p}-\frac{1}{2}\right)+1$, then $M_{m\circ\varrho}$ is of weak type (1, 1) and of type (p,p), $1<p\leqq 2$. To realize this observe

$$\left|F^{-1}[m(\varrho(\xi)/t)f^{\wedge}](x)\right| \leqq C\int_0^\infty s^{\lambda-1}|m^{(\lambda)}(s)|\,ds \sup_{t>0}|S_t^\lambda(f;\,x)|.$$

The result now follows from the Remark to Theorem 3 since, by methods of Herz [10] (see also Ashurov [2]), the hypothesis $\varrho\in C^{[3n/2+5]}(\mathbf{R}_0^n)$ allows to dominate $F^{-1}[(1-\varrho(\xi))_+^\lambda]$ by a radial decreasing integrable function. For the same reason, $M_{m\circ\varrho}$ is also of type (p,p) for $p>1$, p near 1, and of type $(2,2)$ by Theorem 1b; thus the interpolation for analytic families of operators again gives the assertion.

Examples. 1) For $m_\beta(t)=t^{1-\beta-n/2}J_{n/2+\beta-1}(t)$ it is not hard to check that $m_\beta^{(\lambda)}(t)=O(1)$ for $t\to 0$, $\lambda>0$ and $O(t^{1/2-\mathrm{Re}\,\beta-n/2})$ for $t\to\infty$. Hence, by Theorem 1, we have for all $f\in S$

$$\|M_{m_\beta\circ\varrho}f\|_p \leqq C\|f\|_p, \quad 1<p\leqq 2, \quad \mathrm{Re}\,\beta>1-n+n/p \tag{2.9}$$

which for $\varrho(\xi)=|\xi|$ is Theorem 14a of Stein and Wainger [21]. The case $\beta=1$, $\varrho(\xi)=|\xi|$, corresponds to the classical Hardy—Littlewood maximal function; the restriction $n>n/p$ shows that Theorem 1 gives only the "right" mapping properties for $p>1$, i.e., our procedure is just not sharp enough to regain the crucial case $p=1$, $\varrho(\xi)=|\xi|$ for Mf. For a further discussion of the case $\varrho(\xi)=|\xi|$ see Carbery [3].

2) The solution of the Schrödinger initial value problem

$$u_t=-i\Delta u, \quad u(x,0)=f(x)$$

is given by

$$G(t)f=F^{-1}[e^{it|\xi|^2}]*f, \quad f\in S,$$

and can be extended to all of L^p only if $p=2$ (see Hörmander [11]). Sjöstrand [17] examines in particular the question when the β-th Riesz means of $G(t)$

$$\frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1}G(s)\,ds$$

are bounded in L^p. The associated Fourier multiplier is

$$m_\beta(t|\xi|^2)=\frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1}e^{is|\xi|^2}\,ds$$

$$=\frac{\beta}{(t|\xi|^2)^\beta}\int_0^{t|\xi|^2}(t|\xi|^2-s)^{\beta-1}e^{is}\,ds.$$

Thus, one is lead to examine the C^∞-function

$$m_\beta(t) = \frac{\beta}{t^\beta}\int_0^t (t-s)^{\beta-1} e^{is}\,ds$$
$$= \beta\int_0^1 (1-s)^{\beta-1} e^{ist}\,ds = w_\beta(t)+u_\beta(t),$$

where $w_\beta(t)=Ct^{-\beta}e^{it}\varphi(t)$, $\varphi\in C^\infty(\mathbf{R})$ vanishes for $t\leqq 1$ and equals 1 for $t\geqq 2$, and u_β is a nice function in the sense that it satisfies (2.3) and (2.4) for all $\lambda>0$. An examination of w_β shows that (2.3) and (2.4) are fulfilled for $\operatorname{Re}\beta>n\left|\frac{1}{p}-\frac{1}{2}\right|+\frac{1}{2}$, i.e., in this instance

$$\|\sup_{t>0}|F^{-1}[m_\beta(t|\xi|^2)]*f(x)|\|_p \leqq C\|f\|_p,\quad 1<p\leqq\infty. \tag{2.10}$$

Note that using (2.8) in Theorem 1b, Theorem 1b for $p=2$, for $p=\infty$ $w_\beta(|\xi|^2)\in[L^1(\mathbf{R}^n)]^\wedge$ when $\beta>1$, and Stein's interpolation theorem for analytic families of operators yields that (2.10) is valid for $\beta>\max\left\{2\left(\frac{1}{2}-\frac{1}{p}\right),\frac{1}{2}\right\}$ when $n=2$ and $2\leqq p\leqq\infty$.

This has to be seen on the background that $m_\beta(|\xi|^2)\in M_p$ implies $\beta\geqq n\left|\frac{1}{p}-\frac{1}{2}\right|$ (see Sjöstrand [17]).

3) e^{-t} and $(1-e^{-t})/t$ trivially satisfy (2.3), (2.4) for all $\lambda>0$ and hence in particular for $b>0$

$$\|\sup_{t>0}|F^{-1}[e^{-t|\xi|^b}]*f(x)|\|_p \leqq C\|f\|_p,\quad 1<p\leqq\infty,$$

which for $b=1$ coincides with the maximal function for the Poisson integral, and

$$\|\sup_{t>0}|F^{-1}[(t|\xi|)^{-b}(e^{-t|\xi|^b}-1)]*f(x)|\|_p \leqq C\|f\|_p,\quad 1<p\leqq\infty,$$

which may be useful to prove pointwise convergence theorems of Voronovskaja type for the generalized Weierstrass means (cf. Görlich and Stark [9]).

4) For $m_\beta(t)=(1-t)_+^\beta$ one has $m_\beta^{(\lambda)}(t)=C(1-t)_+^{\beta-\lambda}$, $\lambda<\beta+1$. Hence Theorem 1 yields

$$\left\|\sup_{t>0}\left|F^{-1}\left[\left(1-\frac{\varrho(\xi)}{t}\right)_+^\beta\right]*f(x)\right|\right\|_p \leqq C\|f\|_p$$

for $\beta>n\left(\frac{1}{p}-\frac{1}{2}\right)$, $1<p\leqq 2$ and $\beta>(n-1)\left(\frac{1}{2}-\frac{1}{p}\right)$, $2\leqq p\leqq\infty$.

5) By a result of Kenig and Tomas [13] all the above results also hold on the n-dimensional torus if $1<p\leqq\infty$. In particular, the transference of Example 4 for $n=2$ to the 2-dimensional torus improves and supplements results of Podkorytov

[16]. Podkorytov assumes $\beta=1, p=\infty$ and $\sum_\varrho$ convex and obtains $(1-\varrho(\xi))_+^\beta \in M_\infty$, whereas here $\beta>1/2$, $p=\infty$ and $\varrho\in C^2(\mathbf{R}_0^2)$ (which may be relaxed to the condition that ϱ satisfies in $L^\infty(\mathbf{R}^2)$ a Lipschitz condition of order $1+\varepsilon$, $\varepsilon>0$), $\sum_\varrho$ not necessarily convex.

3. Proof of Theorem 2

The case $p=2$ follows easily by the Parseval formula and repeated changes of the integration order:

$$\|g_\lambda(f)\|_2^2 = \int_{\mathbf{R}^n} |f^\wedge(\xi)|^2 \int_0^\infty \left(\frac{\varrho(\xi)}{t}\right)^2 \left(1-\frac{\varrho(\xi)}{t}\right)_+^{2\operatorname{Re}\lambda-2} \frac{dt}{t}\, d\xi$$
$$\leqq C\|f^\wedge\|_2^2 = C'\|f\|_2^2, \quad \operatorname{Re}\lambda > 1/2.$$

Now proceeding as in Igari and Kuratsubo [12] one can use Stein's interpolation theorem for analytic families of operators living on Hilbert-valued L^p-spaces once it is shown that g_λ is of weak type (1, 1) for $\operatorname{Re}\lambda>(n+1)/2$. To this end it is sufficient (see Stein [18; p. 46]) to establish that

$$\int_{r(x)\geqq C} \left(\int_0^\infty |K_t(x-y)-K_t(x)|^2 \frac{dt}{t}\right)^{1/2} dx \leqq B, \quad y\in B_r \tag{3.1}$$

for a sufficiently large constant C; here $B_r=\{x\colon r(x)\leqq 1\}$, $K_t(x)=t^\nu K(A_t'x)$, A_t' is the adjoint of A_t, $r(x)$ a positive, A_t'-homogeneous distance function and $K=F^{-1}[\varrho(\xi)(1-\varrho(\xi))_+^\lambda]$. We first show that, without loss of generality, we may assume $\varrho\in C^{[n/2+1]}(\mathbf{R}^n)$. Observe that there is a $k\in\mathbf{N}$ such that $\varrho(\xi)^k\in C^{[n/2+1]}(\mathbf{R}^n)$; setting $F_\lambda(t)=t^{1/k-1}(1-t^{1/k})_+^{\lambda-1}$ we obtain for $(n-1)/2<\mu<\operatorname{Re}\lambda-1$

$$\varrho(\xi)(1-\varrho(\xi))_+^{\lambda-1} = C\int_0^\infty s^{\mu+1}F_\lambda^{(\mu+1)}(s)[R_s^{\mu,k}]^\wedge(\xi)\, ds,$$

where

$$[R_s^{\mu,k}]^\wedge(\xi) = (1-\varrho(\xi)^k/s)_+^\mu \varrho(\xi)^k/s,$$
$$\int_0^\infty s^{\mu+1}|F_\lambda^{(\mu+1)}(s)|\, ds = O(1+|\lambda|^{[\mu+2]})$$

(see Trebels [23]). Since $R_1^{\mu,k}\in L^1$ for $\mu>(n-1)/2$ (see Dappa [5]) it follows that

$$K(x) = C\int_0^\infty s^{\mu+1}F_\lambda^{(\mu+1)}(s)R_s^{\mu,k}(x)\, ds \quad \text{a.e.}. \tag{3.2}$$

Thus we can estimate the left side of (3.1) with the aid of the integral Minkowski inequality by

$$C\int_0^\infty s^{\mu+1}|F_\lambda^{(\mu+1)}(s)| \int_{r(x)\geqq C}\left(\int_0^\infty |R_{st^k}^{\mu,k}(x-y)-R_{st^k}^{\mu,k}(x)|^2\frac{dt}{t}\right)^{1/2} dx\, ds$$
$$\leqq C(1+|\lambda|^{[\mu+2]})\int_{r(x)\geqq C}\left(\int_0^\infty |R_t^{\mu,k}(x-y)-R_t^{\mu,k}(x)|^2\frac{dt}{t}\right)^{1/2} dx.$$

Thus (3.1) is satisfied if we can show the corresponding estimate for

$$H_\mu = H = F^{-1}[\varrho(\xi)(1-\varrho(\xi))_+^\mu], \quad (n-1)/2 < \mu < \operatorname{Re}\lambda - 1,$$

with $\varrho \in C^{[n/2+1]}(\mathbf{R}^n)$. Now decompose

$$\int_{r(x)\geqq C}\Big(\int_0^\infty |H_t(x-y)-H_t(x)|^2\,\frac{dt}{t}\Big)^{1/2} dx$$

$$\leqq \int_{r(x)\geqq C}\Big(\int_0^1 \dots\Big)^{1/2} dx + \int_{r(x)\geqq C}\Big(\int_1^\infty \dots\Big)^{1/2} dx = I_1 + I_2.$$

Concerning I_1 observe that

$$|H_t(x-y)-H_t(x)| \leqq |A_t' y| \int_0^1 |(\nabla H)(A_t' x - sA_t' y)|\, ds,$$

that

$$|A_t' y| \leqq \|A_t'\|\,|y| \leqq C\|A_t'\|, \quad y\in B_r,$$

and that the triangle inequality $r(x+y)\leqq b(r(x)+r(y))$ and $r(x)\geqq C$ for sufficiently large C, say $C=2b$, imply $r(x-sy)\geqq 1$ for all s, $0\leqq s\leqq 1$. Hence, by the integral Minkowski inequality,

$$I_1 \leqq C\int_0^1\int_{r(x-sy)\geqq 1}\Big(\int_0^1 \|A_t'\|^2 |t^\nu(\nabla H)(A_t'(x-sy))|^2\frac{dt}{t}\Big)^{1/2} dx\, ds$$

$$\leqq C\sum_{k=1}^\infty \int_{r(x)\geqq 1}\Big(\int_{2^{-k}}^{2^{1-k}} \|A_t'\|^2 |t^\nu(\nabla H)(A_t' x)|^2\frac{dt}{t}\Big)^{1/2} dx = C\sum J_k.$$

Noting that (2.2) for $r(x)$ instead of $\varrho(\xi)$ gives $|A_t' x|\leqq C(r(x)t)^{a-\varepsilon}$, $|x|=1$, thus $\|A_t'\|\leqq Ct^{a-\varepsilon}$ for $0<t\leqq 1$, that the semi-group property of A_t' yields

$$|A_{2^{-k}}' x| \leqq \|A_{2^{-k}/t}'\|\,|A_t' x| \leqq C|A_t' x|, \quad 2^{-k}\leqq t\leqq 2^{1-k},$$

and that by (2.2) $|x|\leqq C2^{-k(A+\varepsilon)}$ if $r(x)\geqq 2^{-k}$, we obtain by the Schwarz inequality for $\beta > n/2$

$$J_k \leqq \Big(\int_{r(x)\geqq 1} |A_{2^{-k}}' x|^{-2\beta} 2^{-k\nu}\, dx\Big)^{1/2}$$

$$\cdot\Big(\int_{\mathbf{R}^n}\int_{2^{-k}}^{2^{1-k}} |A_{2^{-k}}' x|^{2\beta} 2^{-k\nu} \|A_t'\|^2 |t^\nu(\nabla H)(A_t' x)|^2 \frac{dt}{t}\, dx\Big)^{1/2}$$

$$\leqq C2^{k(A+\varepsilon)(\beta-n/2)}\Big(\int_{2^{-k}}^{2^{1-k}}\Big(\int_{\mathbf{R}^n} |A_t' x|^{2\beta} |(\nabla H)(A_t' x)|^2 t^\nu\, dx\Big)\|A_t'\|^2\frac{dt}{t}\Big)^{1/2}$$

$$\leqq C2^{k(A+\varepsilon)(\beta-n/2)-k(a-\varepsilon)}\Big(\int_{\mathbf{R}^n} |x|^{2\beta} |\nabla H(x)|^2\, dx\Big)^{1/2}$$

for every $\varepsilon>0$. Hence $\sum J_k$ converges if $n/2<\beta<n/2+a/A$ provided that the last integral is finite. Analogously,

$$I_2 \leqq C\sum_{k=1}^\infty 2^{-k(a-\varepsilon)(\beta-n/2)}\Big(\int_{\mathbf{R}^n} |x|^{2\beta}|H(x)|^2\, dx\Big)^{1/2}.$$

Since it is shown in Dappa [5] that for $\mu > \beta - 1/2$ there holds

$$\| |\cdot|^{\beta} H_\mu \|_2 + \| |\cdot|^{\beta} \nabla H_\mu \|_2 = O(1),$$

Theorem 2 is established.

4. The case weak (1, 1) revisited

Here we want to deduce an improvement of Theorem 1a based on the following version of a criterion due to Zo [24] (see also Stein and Wainger [21; p. 1277]).

Theorem A. *Let* K, $K_t(x) = t^\nu K(A_t' x)$, *be such that* $K^\wedge \in L^\infty$ *and*

$$K^* f(x) = \sup_{t>0} |K_t * f(x)|, \quad f \in S,$$

be of strong type (p, p) *for some* p, $1 < p \leqq \infty$. *If*

$$\int_{r(x) \geqq 2bt} \sup_{s>0} |K_s(x-y) - K_s(x)|\, dx \leqq B, \quad r(y) \leqq t, \tag{4.1}$$

then K^* *is of weak type* $(1, 1)$.

For the proof of this theorem, one may employ the following version of the

Decomposition lemma. *Let* $f \in S(\mathbf{R}^n)$ *and* $s > 0$ *be arbitrary. Then* $f = g + \sum_{j \in N} b_j$, $N \subset \mathbf{N}$, *where*

(i) $g \in S(\mathbf{R}^n)'$ *and* $\|g\|_2^2 \leqq Cs \|f\|_1$;

(ii) *for each* $j \in N$ *there holds* $b_j \in C^\infty(\mathbf{R}^n)$;

(iii) $\operatorname{supp} b_j \subset I_j = \{x\colon r(x - x_j) < c_j\}$ *for some* $x_j \in \mathbf{R}^n$ *and* $c_j > 0$;

(iv) $\sum_{j \in N} |I_j| \leqq \frac{C}{s} \|f\|_1$ *and there exists an* $n_0 \in \mathbf{N}$ *such that at most* n_0 *of the* I_j*'s intersect;*

(v) $\int b_j = 0$, $\int |b_j| \leqq Cs |I_j|$.

The proof of this lemma consists in a simple combination of a partition of the unity as given e.g. in Stein [18; §6, 1.3] and the decomposition lemma in Madych [14].

Theorem 3. *Let* $\varrho \in C^{[n/2+1]}(\mathbf{R}_0^n)$ *be a positive,* A_t*-homogeneous distance function and* $M_{m \circ \varrho}$ *of strong type* (p, p) *for some* p, $1 < p \leqq \infty$. *If* m *defined on* $(0, \infty)$ *satisfies*

$$\|m\|_{2,\lambda} = \|m\|_\infty + \sup_{k \in \mathbf{Z}} \left(\int_{2^k}^{2^{k+1}} |t^\lambda m^{(\lambda)}(t)|^2 \frac{dt}{t} \right)^{1/2} \leqq B \tag{4.2}$$

for $\lambda > (n+1)/2$, *then* $M_{m \circ \varrho}$ *is of weak type* $(1, 1)$ *and the (weak) operator norm is bounded by* $\|m\|_{2,\lambda}$ *and the* (p, p)*-operator norm of* $M_{m \circ \varrho}$.

Remark. Again, if $\sum_\varrho$ is strictly convex and $\varrho\in C^{[\lambda+n+4]}(\mathbf{R}^n_0)$, then

$$\|m\|_\infty+\sup_{k\in\mathbf{Z}}\int_{2^k}^{2^{k+1}} t^{\lambda-1}|m^{(\lambda)}(t)|\,dt \leqq B, \quad \lambda>(n+1)/2,$$

and the L^p-boundedness of $M_{m\circ\varrho}$ is sufficient for the above assertion.

Proof. Let $\chi_h(s)=1$, $h=(h_1,h_2)$ for $h_1\leqq s\leqq h_2$ and 0 for $0\leqq s\leqq h_1/2$ and $s\geqq 2h_2$ be a positive C^∞-function. Then we have by Gasper and Trebels [8] for every $t>0$

$$K_{t,h}^{\wedge}(\xi)=(m\chi_h)(\varrho(\xi)/t)=C\int_0^\infty s^{\lambda-1}(m\chi_h)^{(\lambda)}(s)(1-\varrho(\xi)/ts)_+^{\lambda-1}\,ds$$

which is the Fourier transform of an L^1-function, because $R_{\lambda-1}=F^{-1}[(1-\varrho(\xi))_+^{\lambda-1}]$ is integrable. Therefore, we may take the inverse to obtain by the Schwarz inequality

$$|K_{t,h}(x-y)-K_{t,h}(x)|$$

$$\leqq\sum_{j=-\infty}^{\infty}\Big(\int_{2^j/t}^{2^{j+1}/t}|s^\lambda(m\chi_h)^{(\lambda)}(s)|^2\frac{ds}{s}\Big)^{1/2}\Big(\int_{2^j/t}^{2^{j+1}/t}|R_{\lambda-1,ts}(x-y)-R_{\lambda-1,ts}(x)|^2\frac{ds}{s}\Big)^{1/2}$$

$$\leqq\|m\chi_h\|_{2,\lambda}\sum_j\Big(\int_{2^j}^{2^{j+1}}|R_{\lambda-1,s}(x-y)-R_{\lambda-1,s}(x)|^2\frac{ds}{s}\Big)^{1/2}$$

$$\leqq C\|m\|_{2,\lambda}\sum_j I_j(x,y)$$

uniformly in $0<h_1, h_2^{-1}<1$ and $t>0$ by Gasper and Trebels [8]. Thus the left side of (4.1) (with K_t replaced by $K_{t,h}$) can be estimated by $C\|m\|_{2,\lambda}$ times

$$\sum_j\int_{r(x)\geqq 2bt}I_j(x,y)\,dx=\sum_j H_j.$$

As in the proof of Theorem 2 it follows for $2^jt\leqq 1$, $\mu>n/2$ that

$$H_j\leqq C(2^jt)^{a-\varepsilon}\int_{\mathbf{R}^n}\Big(\int_{2^j}^{2^{j+1}}|s^\nu(\nabla R_{\lambda-1})(A_s'x)|^2\frac{ds}{s}\Big)^{1/2}dx$$

$$\leqq C(2^jt)^{a-\varepsilon}\Big(\int_{\mathbf{R}^n}(1+|A_{2^j}'x|)^{-2\mu}2^{j\nu}\,dx\Big)^{1/2}\cdot$$

$$\cdot\Big(\int_{\mathbf{R}^n}\int_{2^j}^{2^{j+1}}(1+|A_s'x|)^{2\mu}s^\nu|\nabla R_{\lambda-1}(A_s'x)|^2\frac{ds}{s}\,dx\Big)^{1/2}$$

$$\leqq C(2^jt)^{a-\varepsilon}\Big(\int_{\mathbf{R}^n}(1+|x|)^{2\mu}|\nabla R_{\lambda-1}(x)|^2\,dx\Big)^{1/2}.$$

Here the last integral is finite if $n/2<\mu<\lambda-1/2$ by Dappa [5]. For $2^jt>1$ note that analogously

$$H_j \leqq 2\int_{r(x)\geqq t}\Big(\int_{2^j}^{2^{j+1}} |R_{\lambda-1,s}(x)|^2\frac{ds}{s}\Big)^{1/2} dx$$

$$\leqq C\Big(\int_{r(x)\geqq t} |A'_{2^j}x|^{-2\mu}2^{j\nu}\,dx\Big)^{1/2}\cdot$$

$$\cdot\Big(\int_{\mathbf{R}^n}\int_{2^j}^{2^{j+1}} |A'_s x|^{2\mu}s^{\nu}|R_{\lambda-1}(A'_s x)|^2\frac{ds}{s}\,dx\Big)^{1/2}$$

$$\leqq C(2^j t)^{-(\mu-n/2)(a-\varepsilon)}\Big(\int_{\mathbf{R}^n} |x|^{2\mu}|R_{\lambda-1}(x)|^2\,dx\Big)^{1/2}$$

where the last integral is again finite for $n/2<\mu<\lambda-1/2$ by Dappa [5]. Thus

$$\int_{r(x)\geqq 2bt}\sup_{0<h_1,h_2^{-1}<1}\sup_{s>0}|K_{s,h}(x-y)-K_{s,h}(x)|\,dx \tag{4.3}$$

$$\leqq C\big(\textstyle\sum_{2^j t\leqq 1}\cdots+\sum_{2^j t>1}\cdots\big)\leqq B$$

uniformly in y, $r(y)\leqq t$.

Now choose $\chi_h(s)=G(s/h_2)-G(2s/h_1)$, where $G(s)=1$ for $s\leqq 1$, 0 for $s\geqq 2$ and $G\in C^\infty$. With $h_2=t/\delta$, $h_1=\delta/t$ it follows that

$$(m\chi_h\cdot)(\varrho(\xi)/t)=m(\varrho(\xi)/t)\{G(\delta\varrho(\xi))-G(2\varrho(\xi)/\delta)\}.$$

Clearly $(m\chi_h)(\varrho(\xi)/t)$ tends to $m(\varrho(\xi)/t)$ in S' for $\delta\to 0+$ and therefore

$$|F^{-1}[m(\varrho(\xi)/t)]*f(x)|=\liminf_{\delta\to 0}|F^{-1}[(m\chi_h)(\varrho(\xi)/t)]*f(x)|$$

$$\leqq \liminf_{\delta\to 0}\sup_{t>0}|F^{-1}[(m\chi_h)(\varrho(\xi)/t)]*f(x)|=M_{m\chi\circ\varrho}f(x)$$

uniformly in $t>0$, i.e.,

$$M_{m\circ\varrho}f(x)\leqq M_{m\chi\circ\varrho}f(x),\quad f\in S. \tag{4.4}$$

Furthermore, by Fatou's lemma,

$$\Big\|\liminf_{\delta\to 0+}\sup_{t>0}|F^{-1}[(m\chi_h)(\varrho(\xi)/t)]*f(x)|\Big\|_p \tag{4.5}$$

$$\leqq \liminf_{\delta\to 0}\Big\|\sup_{t>0}|F^{-1}[m(\varrho(\xi)/t)(G(\delta\varrho(\xi))-G(2\varrho(\xi)/\delta)f^\wedge](x)|\Big\|_p$$

$$\leqq C\liminf_{\delta\to 0}\|F^{-1}[(G(\delta\varrho(\xi))-G(2\varrho(\xi)/\delta)]*f\|_p\leqq C'\|f\|_p$$

where the second inequality follows from the hypothesis and the third, since $G(\varrho(\xi))\in[L^1]^\wedge$. Now relations (4.3) and (4.5) imply that $M_{m\chi\circ\varrho}$ is of weak type $(1,1)$ and therefore, by (4.4), also $M_{m\circ\varrho}$ is of weak type $(1,1)$.

There is the open question if already a condition of type (4.2) yields that $M_{m\circ\varrho}$ is of type $(2,2)$. We note that an m on $(0,\infty)$, satisfying (4.2) for certain parameters λ, yields $m\circ\varrho\in M_p$; thus there is the question in how far conditions of type (4.2) (if necessary with increased λ) also guarantee that $M_{m\circ\varrho}$ is of type (p,p).

5. The case of arbitrary Fourier multipliers

Let us recall the definition of the maximal function associated to $m \in L^\infty(\mathbf{R}^n)$:

$$M_m f(x) = \sup_{t>0} |F^{-1}[m(A_t\xi)f^\wedge(\xi)](x)|, \quad f \in S,$$

with A_t the previously introduced dilation matrix. Let $\varrho \in C^\infty(\mathbf{R}_0^n)$ be the associated distance function as described in Stein and Wainger [21; p. 1255] (denote by $r(x)$ the adjoint positive, A_t'-homogeneous $C^\infty(\mathbf{R}_0^n)$-distance function). We first deal with the boundedness of M_m on $L^2(\mathbf{R}^n)$.

Theorem 4. *Let* $m \in L^\infty(\mathbf{R}^n)$ *satisfy*

$$\sup_{\varrho(\xi)=1} \int_0^\infty |m(A_t\xi) - \varphi(A_t\xi)|^2 \frac{dt}{t} \leqq B^2 < \infty \tag{5.1}$$

for some $\varphi \in S$ *and, for* $1/2 < \lambda < 1$,

$$\sup_{\varrho(\xi)=1} \sum_{k\in\mathbf{Z}} \int_{2^{k-2}}^{2^{k+2}} \int_{2^{k-2}}^{2^{k+2}} 2^{k(1+2\lambda)} \frac{|m(A_t\xi) - m(A_s\xi)|^2}{|t-s|^{1+2\lambda}} \frac{ds}{s} \frac{dt}{t} \leqq B^2. \tag{5.2}$$

Then there holds

$$\|M_m f\|_2 \leqq CB\|f\|_2, \quad f \in S.$$

Remark. Condition (5.1) corresponds to (3.15) in Stein and Wainger [21]. Sufficient for (5.2) is $m \in C^1(\mathbf{R}_0^n)$ with

$$|\nabla m(\xi)| = O(\varrho(\xi)^{-a'}), \quad |\xi| \to 0 \tag{5.3}$$

$$|\nabla m(\xi)| = O(\varrho(\xi)^{-A'}), \quad |\xi| \to \infty \tag{5.4}$$

where $a' < a = \min \operatorname{Re} \alpha_j$ and $A' > A = \max \operatorname{Re} \alpha_j$. To realize this we note that

$$\|A_t - A_s\| \leqq \|A_s\| \, \|A_{t/s} - I\| \leqq C_\varepsilon s^{a-\delta-\varepsilon} |t-s|^\delta \tag{5.5}$$

for $0 \leqq \delta \leqq 1$, $0 < s, t \leqq 1$ and $\varepsilon > 0$ arbitrarily small. Then, for negative k, we can estimate the double integral in (5.2) by

$$2^{k(1+2\lambda)} \int_{2^{k-2}}^{2^{k+2}} \int_{2^{k-2}}^{2^{k+2}} |t-s|^{-1-2\lambda} |(A_t - A_s)\xi|^2 \int_0^1 |(\nabla m)(A_s\xi + u(A_t\xi - A_s\xi))|^2 du \frac{ds}{s} \frac{dt}{t}$$

$$\leqq C 2^{k(1+2\lambda-2+2a-2\delta-2\varepsilon-2a')} \int_{2^{k-2}}^{2^{k+2}} \int_{2^{k-2}}^{2^{k+2}} |s-t|^{-1-2\lambda+2\delta} \, ds \, dt$$

by (5.5) and (5.3) since, on account of $\varrho(u\xi) \leqq C$ for $0 \leqq u \leqq 1$ and $\varrho(\xi) = 1$, the triangle inequality gives

$$\varrho(A_s\xi + u(A_t\xi - A_s\xi)) \leqq b(\varrho(A_s\xi) + b(\varrho(A_t u\xi) + \varrho(A_s u\xi))) = O(s)$$

(note that s and t are comparable). For $\delta > \lambda$ the last double integral behaves like $O(2^{k(1+2\delta-2\lambda)})$ so that the summation in (5.2) over negative k converges if $a-\varepsilon-$

$a'>0$ which holds by hypothesis because $\varepsilon>0$ may be chosen arbitrarily small. Analogously one uses (5.4) and $\|A_t-A_s\|\leqq C_\varepsilon s^{A+\varepsilon-1}|t-s|$ to prove that the summation in (5.2) over positive k also converges when $A'>A$.

Proof of Theorem 4. Clearly $m(A_s\xi)$ is measurable in (s, ξ) and therefore, by the Fubini Theorem, (5.2) is well defined. Consider

$$F(x, t) = F^{-1}[m(A_t\cdot)f^\wedge](x),$$

which is a C^∞-function on $\{(x, t)\colon x\in\mathbf{R}^n,\ t>0\}$, and $\chi\in C^\infty(\mathbf{R})$ with $\chi(t)=1$ on $[3/4, 3/2]$ and $\operatorname{supp}\chi\subset[1/2, 2]$. Then

$$I_k(x) = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|F(x, 2^k t)\chi(t)-F(x, 2^k s)\chi(s)|^2}{|t-s|^{1+2\lambda}}\,ds\,dt\right)^{1/2} \tag{5.6}$$

is finite for each $x\in\mathbf{R}^n$ when $1/2<\lambda<1$. Hence

$$\sup_{t>0}\left|F(x, t)\chi\left(\frac{t}{2^k}\right)\right| \leqq CI_k(x), \tag{5.7}$$

as can be seen by the following argument of E. M. Stein: Denote by

$$g(\tau, x) = \int_{-\infty}^{\infty} F(x, 2^k t)\chi(t)e^{it\tau}\,dt$$

the Fourier transform of $F(x, 2^k t)\chi(t)$ with respect to t; then

$$\begin{aligned}
|F(x, 2^k t)\chi(t)| &= C\left|\left(\int_{|t\tau|\leqq 1}+\int_{|t\tau|\geqq 1}\right)(e^{it\tau}-1)\,g(\tau, x)\,d\tau\right| \\
&\leqq C|t|\left(\int_{|t\tau|\leqq 1}|\tau|^{2\lambda}|g(\tau, x)|^2\,d\tau\right)^{1/2}\left(\int_{|t\tau|\leqq 1}|\tau|^{2-2\lambda}\,d\tau\right)^{1/2} \\
&\quad +C\left(\int_{|t\tau|\geqq 1}|\tau|^{2\lambda}|g(\tau, x)|^2\,d\tau\right)^{1/2}\left(\int_{|t\tau|\geqq 1}|\tau|^{-2\lambda}\,d\tau\right)^{1/2} \\
&\leqq C|t|^{\lambda-1/2}\left(\int_{-\infty}^{\infty}\left||\tau|^{\lambda}g(\tau, x)\right|^2\,d\tau\right)^{1/2} \\
&\leqq C\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|t-s|^{-1-2\lambda}|F(x, 2^k t)\chi(t)-F(x, 2^k s)|^2\,ds\,dt\right)^{1/2}
\end{aligned}$$

where the last inequality follows by observing that $\operatorname{supp}\chi\subset[1/2, 2]$ and by a characterization in Stein [18; p. 139]. Thus (5.7) is established. The idea to use Fourier methods with respect to the parameter space is roughly sketched in Cowling [4]. Furthermore,

$$\sup_{t>0}|F(x, t)| \leqq \Big(\sum\nolimits_{k\in\mathbf{Z}}\sup_{t>0}|F(x, t)\chi(2^{-k}t)|^2\Big)^{1/2},$$

thus

$$\Big\|\sup_{t>0}|F(\cdot, t)|\Big\|_2 \leqq C\,\Big\|\Big(\sum\nolimits_{k\in\mathbf{Z}}|I_k(\cdot)|^2\Big)^{1/2}\Big\|_2,$$

so that there only remains to estimate the right side by $CB\|f\|_2$. Splitting up the integration domain in (5.6) into a strip (in which $|t-s|\leqq 1$ holds) and the complement leads, after an elementary calculation, to

$$I_k(x) \leqq C\left(\int_{2^{k-2}}^{2^{k+2}} |F(x,t)|^2 \frac{dt}{t}\right)^{1/2}$$
$$+C\left(\int_{2^{k-2}}^{2^{k+2}}\int_{2^{k-2}}^{2^{k+2}} 2^{k(1+2\lambda)} \frac{|F(x,t)-F(x,s)|^2}{|t-s|^{1+2\lambda}} \frac{dt}{t}\frac{ds}{s}\right)^{1/2}$$

and, therefore,

$$\left(\sum_{k\in\mathbf{Z}} |I_k(x)|^2\right)^{1/2} \leqq C\left(\int_0^\infty |F(x,t)|^2 \frac{dt}{t}\right)^{1/2} + C\left(\sum_{k\in\mathbf{Z}} \int_{2^{k-2}}^{2^{k+2}}\int_{2^{k-2}}^{2^{k+2}} \dots\right)^{1/2}$$
$$= J_1(x)+J_2(x).$$

By the Parseval formula we obtain

$$\|J_1\|_2^2 = C\int_{\mathbf{R}^n}\int_0^\infty |m(A_t\xi)|^2 \frac{dt}{t} |f\hat{\ }(\xi)|^2\, d\xi \leqq CB^2\|f\|_2^2$$

and analogously $\|J_2\|_2 \leqq CB\|f\|_2$. Finally, if

$$\sup_{\varrho(\xi)=1} \int_0^\infty |m(A_t\xi)|^2 \frac{dt}{t}$$

is not finite, one can work with $\tilde{m}(\xi)=m(\xi)-\varphi(\xi)$.

Next we want to discuss the analogue of Theorem 3. To this end we need

Lemma 1. *Let* $\varphi_0\in C^\infty(\mathbf{R})$ *be a nonnegative function with* $\operatorname{supp}\varphi_0\subset[1/2, 2]$ *and* $\sum_{k\in\mathbf{Z}} \varphi_0^2(2^{-k}t)=1$ *for* $t>0$; *set* $\phi\hat{\ }(\xi)=\varphi_0(\varrho(\xi))$ *and let* $m\in L^\infty(\mathbf{R}^n)$ *satisfy*

$$B_1^{n+\delta}(m) = \sup_{s>0} \left\|F[(1+|x|^2)^{(n+\delta)/2} F^{-1}[m_{s,k}](x)]\right\|_1 < \infty$$

for some $\delta>0$, *where* $m_{s,k}(\xi)=m(A_{s2^k}\xi)\phi\hat{\ }(\xi)$. *Then, for* $r(y)\leqq t$,

(5.8)
$$\sum_{k\in\mathbf{Z}} \int_{r(x)\geqq 2bt} 2^{kv} \sup_{s>0} \left|F^{-1}[m_{s,k}\phi\hat{\ }](A'_{2^k}(x-y)) - F^{-1}[m_{s,k}\phi\hat{\ }](A'_{2^k}x)\right| dx \leqq CB_1^{n+\delta}(m).$$

Proof. $F^{-1}[m_{s,k}]$ is well-defined as a C^∞-function and

$$\left|F^{-1}[m_{s,k}\phi\hat{\ }](A'_{2^k}(x-y)) - F^{-1}[m_{s,k}\phi\hat{\ }](A'_{2^k}x)\right|$$
$$\leqq \int_{\mathbf{R}^n} \left|\phi(A'_{2^k}(x-y)-z)-\phi(A'_{2^k}x-z)\right| |F^{-1}[m_{s,k}](z)|\, dz$$
$$\leqq B_1^{n+\delta}(m)\int_{\mathbf{R}^n} \left|\phi(A'_{2^k}(x-y)-z)-\phi(A'_{2^k}x-z)\right| (1+|z|^2)^{-(n+\delta)/2}\, dz,$$

where we used

$$\left|(1+|z|^2)^{(n+\delta)/2} F^{-1}[m(A_{s2^k}\cdot)\phi\hat{\ }](z)\right| \leqq B_1^{n+\delta}(m).$$

Since $\varphi \in S$ we have

$$|\phi(A'_{2^k}(x-y)-z)-\varphi(A_{2^k}x-z)|$$

$$\leqq C\int_0^1 \frac{|A'_{2^k}y|}{(1+r(A'_{2^k}x-z-uA'_{2^k}y))^{\nu+1}}\,du.$$

Observe that $r(y)\leqq t$, hence $|A'_{1/t}y|\leqq C$ and, therefore, as in the proof of Theorem 2

$$|A'_{2^k}y| \leqq \|A'_{t2^k}\|\,|A'_{1/t}y| \leqq C(t2^k)^{a-\varepsilon}.$$

Thus, after an interchange of integration,

$$J=\int_{r(x)\geqq 2bt} 2^{k\nu}\sup_{s>0}|F^{-1}[m_{s,k}](A'_{2^k}(x-y))-F^{-1}[m_{s,k}](A'_{2^k}x)|\,dx$$

$$\leqq CB_1^{n+\delta}(m)(2^k t)^{a-\varepsilon}\int_0^1\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(1+r(A'_{2^k}x-z-uA'_{2^k}y))^{-\nu-1}2^{k\nu}\,dx\,(1+|z|)^{-n-\delta}\,dz\,du$$

$$\leqq CB_1^{n+\delta}(m)(2^k t)^{a-\varepsilon}$$

which implies the convergence of the $\sum_{2^k t\leqq 1}$-part of the sum in (5.8). So it remains to consider the case $2^k t>1$.

$$\int_{\mathbf{R}^n}|\phi(A'_{2^k}(x-y)-z)-\phi(A'_{2^k}x-z)|(1+|z|)^{-n-\delta}\,dz$$

$$\leqq |\phi|*(1+|\cdot|)^{-n-\delta}(A'_{2^k}(x-y))+|\phi|*(1+|\cdot|)^{-n-\delta}(A'_{2^k}x)$$

$$\leqq C(1+|A'_{2^k}(x-y)|)^{-n-\delta}+C(1+|A'_{2^k}x|)^{-n-\delta},$$

since $\phi\in S$ implies $|\phi|*(1+|\cdot|)^{-n-\delta}(x)\leqq C(1+|x|)^{-n-\delta}$. Hence

$$J\leqq CB_1^{n+\delta}(m)\Big(\int_{r(x)\geqq 2bt}(1+|A'_{2^k}x-A'_{2^k}y|)^{-n-\delta}2^{k\nu}\,dx$$

$$+\int_{r(x)\geqq 2bt}(1+|A'_{2^k}x|)^{-n-\delta}2^{k\nu}\,dx\Big)$$

$$=CB_1^{n+\delta}(m)\Big(\int_{r(x+A_{2^k}y)\geqq 2^{k+1}bt}|x|^{-n-\delta}\,dx+\int_{r(x)\geqq 2^{k+1}bt}|x|^{-n-\delta}\,dx\Big).$$

Now $r(x)\geqq 2^{k+1}bt>1$ implies by (2.2) that $|x|\geqq C(2^k t)^{a-\varepsilon}$, and the triangle inequality and $r(y)\leqq t$ show

$$r(x)\geqq \frac{1}{b}r(x+A'_{2^k}y)-r(A'_{2^k}y)\geqq 2^{k+1}t-2^k t=2^k t$$

so that

$$J\leqq CB_1^{n+\delta}(m)\int_{|x|\geqq c(2^k t)^{a-\varepsilon}}|x|^{-n-\delta}\,dx\leqq C'B_1^{n+\delta}(m)(2^k t)^{-\delta(a-\varepsilon)}.$$

Thus the $\sum_{2^k t>1}$-part of the sum in (5.8) also converges and the Lemma is established.

Theorem 5. *Let $m\in L^\infty$ satisfy $B_1^{n+\delta}(m)<\infty$ for some $\delta>0$; let M_m be of strong type (p,p) for some p, $1<p<\infty$. Then M_m is of weak type $(1, 1)$ and strong type (q, q), $1<q\leqq p$.*

Proof. In S' we have

$$K_s(x) = \sum_{k\in \mathbf{Z}} F^{-1}[m(A_s\xi)(\phi\hat{}(A_{2^{-k}}\xi))^2](x)$$

$$= \sum_{k\in \mathbf{Z}} 2^{kv} F^{-1}[m(A_{2^k t}\xi)(\phi\hat{}(\xi))^2](A'_{2^k}x).$$

Now Lemma 1 shows $K_s\in L^1_{\mathrm{loc}}(\mathbf{R}^n_0)$ and, furthermore, that condition (4.1) is satisfied by K_s. Hence Theorem A gives that M_m is of weak type $(1, 1)$ and the interpolation theorem of Marcinkiewicz gives the rest of the assertion.

As an application, consider the hyperbolic Riesz means as introduced by El Kohen [7] who proved

$$m_\lambda(\xi) = (1-\xi_1^2\xi_2^2)_+^\lambda \in M_p(\mathbf{R}^2),\quad \lambda > 2\left|\frac{1}{p}-\frac{1}{2}\right|,\quad 1<p<\infty. \tag{5.9}$$

That (5.9) is true may be seen quite easily: $\|m_\lambda\|_{M_2}=O(1)$ for $\operatorname{Re}\lambda\geqq 0$ and the Marcinkiewicz multiplier theorem (see e.g. Stein [18; p. 109]) shows $\|m_\lambda\|_{M_p}= O(|\lambda|^2)$ for $\operatorname{Re}\lambda>1$ and $1<p<\infty$; thus, by the interpolation theorem for analytic families of operators, (5.9) follows. In order to apply Theorem 5, choose $A_t=\operatorname{diag}(t^{1/4}, t^{1/4})$, $\delta=1$; then it is not hard to check that $B_1^3(m_\lambda)<\infty$ for $\lambda>2$. But we cannot prove that M_{m_λ} is of strong type (p,p) for some $p>1$. (It is perhaps interesting to note that using only appropriate square functions one can show

$$\Big\|\sup_{1/M\leqq t\leqq N} |F^{-1}[m_\lambda(t^{1/4}\cdot)f\hat{}](x)|\Big\|_2 \leqq C\log MN\|f\|_2,\quad f\in S,$$

for $\lambda>0$). However, setting

$$F_\lambda(x, t) = F^{-1}[m_\lambda(t^{1/4}\cdot)f\hat{}](x),\quad f\in S,$$

one obtains

$$\Big\|\sup_{0<t\leqq 1} |F_\lambda(x, t)|\Big\|_2 \leqq C\|f\|_{L^2_\beta},\quad \lambda>1/2,\quad \beta>0. \tag{5.10}$$

To see this, observe that

$$F_\lambda(x, t) = -\int_t^1 \frac{d}{ds} F_\lambda(x, s)\,ds + F_\lambda(x, 1)$$

and, therefore,

$$\sup_{0<t\leqq 1} |F_\lambda(x, t)| \leqq \int_0^1 \left|\frac{d}{ds} F_\lambda(x, s)\right| ds + |F_\lambda(x, 1)|$$

$$\leqq C_\varepsilon\left(\int_0^1 s^{1-\varepsilon}\left|\frac{d}{ds} F_\lambda(x, s)\right|^2 ds\right)^{1/2} + |F_\lambda(x, 1)| = I_1(x)+I_2(x).$$

Taking L^2-norms leads to

$$\|I_2\|_2 \leqq C\|f\|_2 \leqq C\|f\|_{L^2_\beta}, \quad \lambda, \beta > 0,$$

$$\|I_1\|_2 \leqq C_\varepsilon \Big(\int_{\mathbf{R}^2}\int_0^1 |f^\wedge(\xi)|^2 |\xi_1^2\xi_2^2|^2 s^{1-\varepsilon}(1-s\xi_1^2\xi_2^2)_+^{2(\lambda-1)}\, ds\, d\xi\Big)^{1/2}$$

$$\leqq C_\varepsilon \Big(\int_{\mathbf{R}^2} |f^\wedge(\xi)|^2 |\xi_1^2\xi_2^2|^\varepsilon \int_0^\infty s^{1-\varepsilon}(1-s)_+^{2(\lambda-1)}\, ds\, d\xi\Big)^{1/2}$$

$$\leqq C_\varepsilon \||\xi|^{2\varepsilon} f^\wedge\|_2 \leqq C'_\varepsilon \|f\|_{L^2_\beta}, \quad \beta = 2\varepsilon, \ \lambda > 1/2.$$

On the other hand, if we consider instead of $F^{-1}[m_\lambda(t^{1/4}\cdot)]$ the smoother kernel $F^{-1}[m_\lambda(t^{1/4}\cdot)G_\beta^\wedge]$, $G_\beta^\wedge(\xi)=(1+|\xi|^2)^{-\beta/2}$, then the proofs of Theorem 5 and Lemma 1 show that it is sufficient to establish for the same $\varphi^\wedge$ as in Lemma 1 the boundedness of

$$\text{(5.11)} \qquad \sum_{i=1}^2 \sup_{s>0} \Big\|\Big(1+\Big(\frac{\partial}{\partial\xi_i}\Big)^3\Big)\big(m_\lambda((s2^k)^{1/4}\xi)G_\beta^\wedge(2^{k/4}\xi)\big)\Big\|_1$$
$$\leqq C\sum_{j=0}^3\sum_{i=1}^2 \sup_{s>0}\Big\|\Big(\frac{\partial}{\partial\xi_i}\Big)^j m_\lambda((s2^k)^{1/4}\xi)\Big\|_1,$$

where the latter inequality follows by Leibniz' formula since

$$\operatorname*{ess\,sup}_{1/2\leqq|\xi|^4\leqq 2} \Big|\Big(\frac{\partial}{\partial\xi_i}\Big)^j(1+|2^{k/4}\xi|^2)^{-\beta/2}\Big| = O(1), \quad 0\leqq j\leqq 3, \quad i=1,2.$$

The norms in (5.11) are bounded for $\lambda>2$ so that, on account of the previous L^2-estimate, Theorem A yields

$$|\{x\in\mathbf{R}^2\colon M_{m_{\lambda,\beta}}g(x) > s\}| \leqq \frac{C_\beta}{s}\|g\|_1, \quad g\in S;$$

here

$$M_{m_{\lambda,\beta}}g(x) = \sup_{0<t\leqq 1}|F^{-1}[m_\lambda(t^{1/4}\cdot)G_\beta^\wedge g^\wedge](x)|.$$

Now the Marczinkiewicz interpolation theorem gives

$$\Big\|\sup_{0<t\leqq 1}|F^{-1}[(1-t\xi_1^2\xi_2)_+^\lambda]*f(x)|\Big\|_p \leqq C_\beta\|f\|_{L^p_\beta}$$

for $1<p\leqq 2$, $\beta>0$, $\lambda>2$, when we use the interpretation $f=G_\beta * g$.

References

1. AGUILERA, N. E., Maximal operators associated to radial functions in $L^2(\mathbf{R}^2)$, *Proc. Amer. Math. Soc.*, **80** (1980), 283—286.
2. ASHUROV, R., Conditions for Riesz means of spectral resolutions to be a basis in $L^p(\mathbf{R}^n)$, *Math. Notes*, **29** (1981), 342—348.
3. CARBERY, A., Radial Fourier multipliers and associated maximal functions, preprint.

4. Cowling, M. G., On Littlewood—Paley—Stein theory, *Rend. Circ. Mat. Palermo,* **2** (1981), 21—55.
5. Dappa, H., Quasiradiale Fouriermultiplikatoren, Dissertation, TH Darmstadt 1982.
6. Dappa, H. and Trebels, W., On hypersingular integrals and anisotropic Bessel potential spaces, *Trans. Amer. Math. Soc.,* **286** (1984), 419—429.
7. El Kohen, A., On the hyperbolic Riesz means, *Proc. Amer. Math. Soc.,* **89** (1983), 113—116.
8. Gasper, G. and Trebels, W., A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers, *Studia Math.,* **65** (1979), 243—278.
9. Görlich, E. and Stark, E. L., Über beste Konstanten und asymptotische Entwicklungen positiver Faltungsintegrale und deren Zusammenhang mit dem Saturationsproblem, *Jahresber. Deutsch. Math.-Verein.,* **72** (1970), 18—61.
10. Herz, C., Fourier transforms related to convex sets, *Ann. of Math.,* **75** (1962), 81—92.
11. Hörmander, L., Estimates for translation invariant operators in L^p-spaces, *Acta Math.,* **104** (1960), 93—139.
12. Igari, S. and Kuratsubo, S., A sufficient condition for L^p-multipliers, *Pacific J. Math.,* **38** (1971), 85—88.
13. Kenig, C. E. and Tomas, P. A., Maximal operators defined by Fourier multipliers, *Studia Math.,* **68** (1980), 79—83.
14. Madych, W., On Littlewood—Paley functions, *Studia Math.,* **50** (1974), 43—63.
15. Peetre, J., Applications de la théorie des espaces d'interpolation dans l'analyse harmonique, *Ricerche Mat.,* **15** (1966), 3—36.
16. Podkorytov, A. N., Fejér means in the two-dimensional case, *Vestnik Leningrad Univ. Math.,* **11** (1983), 197—205.
17. Sjöstrand, S., On the Riesz means of the solutions of the Schrödinger equation, *Ann. Scuola Norm. Sup. Pisa,* **24** (1970), 331—348.
18. Stein, E. M., *Singular Integrals and Differentiability Properties of Functions,* Princeton Univ. Press, Princeton, N.J. 1970.
19. Stein, E. M., *Topics in Harmonic Analysis Related to the Littlewood—Paley Theory,* Princeton Univ. Press, Princeton, N.J. 1970.
20. Stein, E. M., Maximal functions: Spherical means, *Proc. Nat. Acad. Sci. U.S.A.,* **73** (1976), 2174—2175.
21. Stein, E. M. and Wainger, S., Problems in harmonic analysis related to curvature, *Bull. Amer. Math. Soc.,* **84** (1978), 1239—1295.
22. Stein, E. M. and Weiss, G., *Introduction to Fourier Analysis on Euclidean Spaces,* Princeton Univ. Press, Princeton, N.J. 1971.
23. Trebels, W., Some Fourier multiplier criteria and the spherical Bochner—Riesz kernel, *Rev. Roumaine Math. Pures Appl.,* **20** (1975), 1173—1185.
24. Zo, F., A note on approximation of the identity, *Studia Math.,* **55** (1976), 111—122.

Received April 13, 1984

H. Dappa and W. Trebels
FB Mathematik
TH Darmstadt
D-6100 Darmstadt, FRG