Convergence of complete spline interpolation for holomorphic functions

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1. Introduction

During the 2nd Edmonton conference on approximation theory in June 1982, I. J. Schoenberg stated a conjecture concerning convergence of complete spline interpolation.

Let $S_{2m-1}=S_{2m-1}\left(\frac{1}{n+1}, \dots, \frac{n}{n+1}\right)$ denote the space of spline functions of degree $2m-1$ with simple knots at n equidistant points $\frac{i}{n+1}$, $i=1, \dots, n$, in $(0, 1)$, $S_{2m-1}\subset C^{2m-2}(R)$ and any $S\in S_{2m-1}$ is a polynomial of degree $\leqq 2m-1$ between any two successive knots. The *complete spline interpolation problem* is to find $S(x)\in S_{2m-1}$, where

$$S\left(\frac{\nu}{n+1}\right)=f\left(\frac{\nu}{n+1}\right), \quad \nu = 1, 2, \dots, n \tag{1.1}$$

$$S^{(i)}(0)=f^{(i)}(0), \quad S^{(i)}(1)=f^{(i)}(1), \quad i=0, 1, \dots, m-1.$$

It is known that (1.1) has a unique solution (see [1]). Concerning this interpolation problem Schoenberg stated

Conjecture 1. *Assume that $f(x)$ is holomorphic in a neighborhood of the interval* $[0, 1]$. *Then there is a fixed value of n depending on f such that*

$$\lim_{m\to\infty} (S_{m,n}f)(x)=f(x) \tag{1.2}$$

uniformly on $[0, 1]$.

He first raised this conjecture in Budapest in 1968 and again in Oberwolfach in 1971. As a means to study this problem, he also formulated the weaker

Conjecture 2. *Let f again be holomorphic in a neighborhood of* [0, 1]. *Then there is a fixed value of n, depending on $f(x)$, such that*

$$\lim_{m\to\infty}\int_0^1 (S_{m,n}f)(x)\,dx = \int_0^1 f(x)\,dx. \tag{1.3}$$

Let $\Gamma_\varrho=\{z\colon |1-z^2|=\varrho\}$, $\varrho\geqq 1$ denote a lemniscate containing $[-1, 1]$ and let D_ϱ denote the interior of Γ_ϱ. Set

$$f_\zeta(x) = (\zeta-x)^{-1},\ \zeta\notin[-1, 1].$$

Let $A(D_\varrho)$ denote the class of functions holomorphic in the domain D_ϱ. Furthermore we denote the maximum norm on $[-1, 1]$ by $\|\cdot\|$. Suppose, more generally, that $S(x)=(S_{m,n}f)(x)$ is the complete spline interpolant to f on $[-1, 1]$ relative to any given knots $x_1<x_2<\ldots<x_n$ in $(-1, 1)$. In other words, $S(x)\in C^{2m-2}(\mathbf{R})$,

$$S(x_\nu) = f(x_\nu),\quad \nu = 1, 2, \ldots, n,$$
$$S^{(i)}(\pm 1) = f^{(i)}(\pm 1),\quad i = 0, 1, \ldots, m-1,$$

and $S(x)|_{(x_\nu, x_{\nu+1})}$ is a polynomial of degree $\leqq 2m-1$. In the sequel we shall denote this class of splines also by $S_{2m-1}(x_1, \ldots, x_n)$. With regard to the above conjectures we shall prove the following result:

Theorem 1. *Let $f\in A(D_\varrho)$ for some $\varrho>1$. Then for any n*

$$\lim_{m\to\infty}\|f-S_{m,n}f\|^{1/m} \leqq \varrho^{-1}. \tag{1.4}$$

Moreover, for any $\zeta\notin[-1, 1]$, we have

$$\lim_{m\to\infty}\frac{(1-\zeta^2)^m}{(1-x^2)^m}[f_\zeta(x)-(S_{m,n}f_\zeta)(x)] = \frac{B_n(x)}{B_n(\zeta)} f_\zeta(x) \tag{1.5}$$

uniformly for $x\in[-1, 1]$, where

$$B_n(z) = \prod_{i=1}^n \frac{z-x_i}{1-zx_i}. \tag{1.6}$$

In particular, this result shows that for each n, complete spline interpolation in $S_{2m-1}(x_1, \ldots, x_n)$ for the function $f_\zeta(x)$ diverges in $|x|<\sqrt{1-\sigma^2}$, if $\sigma^2=|1-\zeta^2|<1$, as $m\to\infty$. Of course, Theorem 1 also shows that Schoenberg's quadrature formula,

$$Q_{m,n}f = \int_{-1}^1 (S_{m,n}f)(x)dx \tag{1.7}$$

$$= \sum_{i=0}^{m-1}\frac{A_i^{(m)}}{i!}[f^{(i)}(-1)+(-1)^i f^{(i)}(1)]+\sum_{i=1}^n B_i^{(m)} f(x_i),$$

converges (diverges) to $\int_{-1}^1 f(x)\,dx$ under the same conditions on f as in Theorem 1. We will say more about this quadrature formula later (see also [4]).

2. Kernel $K_m(x, y)$

The proof of Theorem 1 is based on an analysis of the limit of the kernel

$$T_m(x, y) = K_m(x, y)/(1-x^2)K_m(x, x) \tag{2.1}$$

as $m\to\infty$. Here

$$K_m(x, y) = M(x|(-1)^m, y, 1^m) \tag{2.2}$$

is the B-spline of degree $2m-1$ with a simple knot at y and m fold knots ± 1, normalized to have integral one. In general, we write $M(x|x_0, \dots, x_n)$ for the B-spline of degree $n-1$ with knots at $x_0<x_1<\dots<x_n$ with integral one. This function is uniquely defined by the conditions that it belongs to $C^{n-2}(\mathbf{R})$, vanishes outside (x_0, x_n) and is a polynomial of degree $\leqq n-1$ on the interval $[x_i, x_{i+1}]$, $i=0, 1, \dots, n-1$. (See [1].)

To obtain an explicit expression for $K_m(x, y)$, we recall the identity (see [1])

(2.3)

$$M(x|x_0, \dots, x_n) = \frac{n}{n-1}\left\{\frac{x_n-x}{x_n-x_0} M(x|x_1, \dots, x_n)+\frac{x-x_0}{x_n-x_0} M(x|x_0, \dots, x_{n-1})\right\}.$$

Specializing this equation yields the following three relations:

$$K_m(x, y) = \frac{2m}{2m-1}\left\{\frac{1-x}{2} M(x|(-1)^{m-1}, y, 1^m)+\frac{1+x}{2} M(x|(-1)^m, y, 1^{m-1})\right\},$$

$$M(x|(-1)^{m-1}, y, 1^m) = \frac{2m-1}{2m-2}\left\{\frac{1-x}{1-y} M(x|(-1)^{m-1}, 1^m)+\frac{x-y}{1-y} K_{m-1}(x, y)\right\},$$

and

$$M(x|(-1)^m, y, 1^{m-1}) = \frac{2m-1}{2m-2}\left\{\frac{y-x}{1+y} K_{m-1}(x, y)+\frac{1+x}{1+y} M(x|(-1)^m, 1^{m-1})\right\}.$$

Combining these equations, we obtain

$$K_m(x, y) = \frac{m}{2(m-1)}\left\{\frac{(1-x)^2}{1-y} M(x|(-1)^{m-1}, 1^m)+\frac{(1+x)^2}{1+y} M(x|(-1)^m, 1^{m-1})\right.$$

$$\left.-\frac{2(x-y)^2}{1-y^2} K_{m-1}(x, y)\right\}.$$

Consequently, using the fact that

$$M(x|(-1)^r, 1^s) = r\binom{r+s-1}{s-1}\frac{(x+1)_+^{s-1}(1-x)_+^{r-1}}{2^{r+s-1}}, \tag{2.4}$$

we derive the recurrence

(2.5)
$$K_m(x, y) = \frac{m}{m-1}\left\{(m-1)\binom{2m-2}{m-1}\left(\frac{1-x^2}{4}\right)^{m-1} \frac{1-xy}{1-y^2} - \frac{(x-y)^2}{1-y^2} K_{m-1}(x, y)\right\}.$$

In particular,

$$K_m(x, x) = m\binom{2m-2}{m-1}\left(\frac{1-x^2}{4}\right)^{m-1}. \tag{2.6}$$

We shall now prove

Lemma 1. *We have the identity*

$$K_m(x, y) = m\left(\frac{x^2-1}{4\varrho}\right)^{m-1}\left[-\frac{|y-x|}{1-y^2} + \frac{1-xy}{1-y^2}\sum_{\nu=0}^{m-1}\binom{2\nu}{\nu}(-\varrho)^\nu\right], \tag{2.7}$$

where

$$\varrho = \frac{(1-x^2)(1-y^2)}{4(x-y)^2}. \tag{2.8}$$

Proof. Set

$$\hat{K}_m = \left(\frac{y^2-1}{(x-y)^2}\right)^{m-1} \frac{K_m(x, y)}{m}. \tag{2.9}$$

Then (2.5) gives

$$\hat{K}_m = \frac{1-xy}{1-y^2}\binom{2m-2}{m-1}(-\varrho)^{m-1} + \hat{K}_{m-1},$$

where ϱ is given by (2.8). Since

$$K_1(x, y) = \begin{cases} \dfrac{1+x}{1+y}, & -1 < x < y \\[2ex] \dfrac{1-x}{1-y}, & y < x < 1 \end{cases}$$

we obtain

$$\hat{K}_m(x, y) = \frac{1-xy}{1-y^2}\sum_{\nu=0}^{m-1}\binom{2\nu}{\nu}(-\varrho)^\nu + K_1 - \frac{1-xy}{1-y^2}$$

$$= -\frac{|y-x|}{1-y^2} + \frac{1-xy}{1-y^2}\sum_{\nu=0}^{m-1}\binom{2\nu}{\nu}(-\varrho)^\nu.$$

Using (2.9), we readily verify the desired formula (2.7).

Lemma 2. *The kernel*

$$G_m(x, y) = \frac{1}{(2m)!}(1-y^2)^m K_m(y, x) \tag{2.10}$$

is a symmetric and totally positive kernel for $(x, y) \subset (-1, 1)$.

Proof. From the definition of the B-spline and divided difference of f at $x_0, \ldots, x_n$, we have

$$[x_0, \ldots, x_n]f = \frac{1}{n!}\int_{-\infty}^{\infty} M(x|x_0, \ldots, x_n) f^{(n)}(x)\, dx.$$

Therefore for any function with $f^{(i)}(-1)=f^{(i)}(1)=0$, $(i=0, 1, \ldots, m-1)$, we have

$$\frac{f(y)}{(y^2-1)^m} = [\underbrace{-1, \ldots, -1}_{m}, y, \underbrace{1, \ldots, 1}_{m}]f = \frac{1}{(2m)!}\int_{-1}^{1} K_m(x, y) f^{(2m)}(x)\, dx.$$

In other words, we have shown that

$$f(x) = (-1)^m \int_{-1}^{1} G_m(x, y) f^{(2m)}(y)\, dy$$

and so $(-1)^m G_m(x, y)$ is the Green's function for the boundary-value problem:

$$f^{(2m)}(x) = g(x),$$
$$f^{(i)}(-1) = f^{(i)}(1) = 0, \quad i = 0, 1, \ldots, m-1.$$

Therefore $G_m(x, y)$ must be symmetric and its total positivity follows from the general theorem in [3]. Actually the symmetry of $G_m(x, y)$ can be seen also from (2.7) and (2.8) and the total positivity can be proved by Sylvester's determinant identity and the Schoenberg—Whitney theorem [1].

Let us also point out that since any $S(x)\in S_{2m-1}$ satisfying $S^{(i)}(-1)=S^{(i)}(1)=0$, $(i=0, 1, \ldots, m-1)$, can be written as a linear combination of the kernel $K_m(x, y)$, i.e.,

$$S(x) = \sum_{j=1}^{n} C_j K_m(x, x_j),$$

it follows from the uniqueness of complete spline interpolation that

$$K_m\begin{pmatrix} x_1, \ldots, x_n \\ x_1, \ldots, x_n \end{pmatrix} = \det_{i, j=1, \ldots, n} K_m(x_i, x_j) > 0.$$

More strict positivity properties of this sort appear in [2].

3. Kernel $T_m(x, y)$

We now introduce the Cauchy—Szegö kernel

$$T_\infty(z, \zeta) = \frac{1}{1-z\bar{\zeta}}, \quad |z|, |\zeta| < 1. \tag{3.1}$$

This kernel is known to be the reproducing kernel for the Hardy space on the unit disk (see [2]).

Lemma 3. *For* $x, y \in [-1, 1]$ *with* $xy \neq 1$, *we have*

$$|T_m(x, y) - T_\infty(x, y)| \leqq \frac{T_\infty(x, y)}{m-1}. \tag{3.2}$$

Proof. Using (2.6) and (2.7), we have

$$T_m(x, y) = \frac{(-1)^{m-1}}{\varrho^{m-1}\binom{2m-2}{m-1}(1-x^2)(1-y^2)}\left[-|y-x| + (1-xy)\sum_{v=0}^{m-1}\binom{2v}{v}(-\varrho)^v\right]$$

$$= \frac{(-1)^m(1-xy)}{\varrho^{m-1}\binom{2m-2}{m-1}(1-x^2)(1-y^2)}\left[\frac{1}{\sqrt{1-4\varrho}} - \sum_{v=0}^{m-1}\binom{2v}{v}(-\varrho)^v\right],$$

because $(1+4\varrho)^{-1/2} = |x-y|/(1-xy)$. By Taylor's formula with remainder, we get

$$(1+4\varrho)^{-1/2} - \sum_{v=0}^{m-1}\binom{2v}{v}(-\varrho)^v = m\binom{2m}{m}(-\varrho)^m \int_0^1 (1+4\varrho t)^{-m-1/2}(1-t)^{m-1}\, dt,$$

so that

$$T_m(x, y) = m\left(1 - \frac{1}{2m}\right)\frac{4\varrho(1-xy)}{(1-x^2)(1-y^2)}\int_0^1 (1+4\varrho t)^{-m-1/2}(1-t)^{m-1}\, dt.$$

By a suitable change of variable, we obtain

$$T_m(x, y) = m\left(1 - \frac{1}{2m}\right) T_\infty(x, y)(1+4\varrho)^{1/2}\int_0^1 v^{m-1}(1+4\varrho v)^{-1/2}\, dv,$$

which yields the following inequality

$$|T_m(x, y) - T_\infty(x, y)| \leqq T_\infty(x, y)\left\{m\int_0^1 v^{m-1}\left|\frac{(1+4\varrho)^{1/2}}{(1+4\varrho v)^{1/2}} - 1\right| dv\right.$$

$$\left. + \frac{(1+4\varrho)^{1/2}}{2}\int_0^1 v^{m-1}(1+4\varrho v)^{-1/2}\, dv\right\}.$$

Since the upper bound for $T_m - T_\infty$ as a function of ϱ is increasing, we see that

$$|T_m(x, y) - T_\infty(x, y)| \leqq T_\infty(x, y)\left\{m\int_0^1 v^{m-1}(v^{-1/2} - 1)\, dv + \tfrac{1}{2}\int_0^1 v^{m-3/2}\, dv\right\}$$

$$= \frac{2}{2m-1} T_\infty(x, y),$$

which proves the lemma.

For the convenience of the reader, we record the following fact which will be useful later.

Lemma 4. *The kernel*

$$T_\infty(x, y) = 1/(1-xy)$$

is extended totally positive on $(-1, 1)\times(-1, 1)$.

Proof. This a well-known result (see [2]) and follows from the Cauchy determinant identity

$$T_\infty\binom{x_1, \ldots, x_n}{y_1, \ldots, y_n} = \frac{\prod_{1\le k<j\le n}(x_k-x_j)(y_j-y_k)}{\prod_{j,k=1}^{n}(1-x_j y_k)}. \tag{3.3}$$

We shall denote by $R_n f$ the interpolant to f at $x_1, \ldots, x_n$ by the rational functions $T_\infty(x, x_1), \ldots, T_\infty(x, x_n)$. It is easy to see that

$$(R_n f)(x) = \sum_{i=1}^{n} f(x_i)\, l_i(x)$$

where the fundamental functions $l_i(x)$ are given by

$$l_i(x) = \frac{B_n(x)}{(x-x_i)\, B_n'(x_i)}, \quad B_n(x) = \prod_{i=1}^{n}\left(\frac{x-x_i}{1-xx_i}\right).$$

Lemma 5. *If* $F_m(x)=(1-x^2)^m f(x)$, *then for each* n *we have*

$$\lim_{m\to\infty}(1-x^2)^{-m}(S_{m,n}F_m)(x) = (R_n f)(x), \tag{3.4}$$

uniformly on $[-1, 1]$.

Proof. Since $F_m^{(i)}(\pm 1)=0$, $i=0, 1, \ldots, m-1$, we can write

$$(S_{m,n}F_m)(x) = \sum_{j=1}^{n} d_j^{(m)} M(x|(-1)^m, x_j, 1^m),$$

where

$$(1-x_r^2)^m f(x_r) = \sum_{j=1}^{n} d_j^{(m)} K_m(x_r, x_j).$$

Recalling the definition of $T_m(x)$ in (2.1), we obtain

$$\frac{(1-x_r^2)^{m-1} f(x_r)}{K_m(x_r, x_r)} = \sum_{j=1}^{n} d_j^{(m)}\, T_m(x_r, x_j).$$

By Stirling's formula, we can see that

$$4^{-m+1}\binom{2m-2}{m-1} \sim 1/\sqrt{\pi m},$$

so that by (2.6), we have

$$K_m(x, x) \sim \sqrt{\frac{m}{\pi}}\,(1-x^2)^{m-1}. \tag{3.5}$$

Therefore

$$\lim_{m\to\infty}\sqrt{\frac{m}{\pi}}\, d_j^{(m)} = d_j^{(\infty)}, \tag{3.6}$$

where $f(x_r)=\sum_{j=1}^{n} d_j^{(\infty)} T_\infty(x_r, x_j)$, $(r=1, 2, \dots, n)$. Hence

$$(1-x^2)^{-m}(S_{m,n}F_m)(x) = K_m(x, x)(1-x^2)^{-m+1} \sum_{j=1}^{n} d_j^{(m)} T_m(x, x_j).$$

From the above we get (3.4) by using equations (3.5) and (3.6).

Remark. When $f\in H^2$, i.e., Hardy—Szegö space on the unit disk, it is well-known [2] that $(R_n f)(x)$ is the minimal interpolant to f at $x_1, x_2, \dots, x_n$. In other words,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |(R_n f)(e^{i\theta})|^2\, d\theta \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(e^{i\theta})|^2\, d\theta$$

for all $g\in H^2$ such that $g(x_i)=f(x_i)$ $(i=1, 2, \dots, n)$. On the other hand, complete spline interpolation also has a similar extremal property [4], namely

$$\int_{-1}^{1} |(S_{m,n}f)^{(m)}(x)|^2\, dx \leqq \int_{-1}^{1} |g^{(m)}(x)|^2\, dx$$

for all g with $g^{(m)}\in L^2[-1, 1]$ and g interpolating f at $x_1, \dots, x_n$. Lemma 5 ties these two properties together.

4. Proof of Theorem 1

Let $(P_m f)(x)$ be the polynomial of degree $2m-1$ satisfying the conditions

$$(P_m f)^{(i)}(\pm 1) = f^{(i)}(\pm 1), \quad i = 0, 1, \dots, m-1.$$

Then

$$f(x)-(P_m f)(x) = \frac{1}{2\pi i}\int_\Gamma \frac{(1-x^2)^m}{(z-x)(1-z^2)^m} f(z)\, dz, \tag{4.1}$$

where Γ is any contour containing $D_1=\{z\colon |z^2-1|<1\}$. In particular, for any $\zeta\notin[-1, 1]$, we have

$$f_\zeta(x)-(P_m f_\zeta)(x) = \frac{(1-x^2)^m}{(\zeta-x)(1-\zeta^2)^m}.$$

Thus

$$f_\zeta(x)-(S_{m,n}f_\zeta)(x) = f_\zeta(x)-(P_m f_\zeta)(x) - \frac{1}{(1-\zeta^2)^m}(S_{m,n}F_{m,\zeta})(x),$$

where

$$F_{m,\zeta}(x) = \frac{(1-x^2)^m}{\zeta-x}.$$

By Lemma 5, we get

$$\lim_{m\to\infty}\frac{(1-\zeta^2)^m}{(1-x^2)^m}[f_\zeta(x)-(S_{m,n}f_\zeta)(x)]=f_\zeta(x)-(R_nf_\zeta)(x). \tag{4.2}$$

Since $(R_nf_\zeta)(x)$ interpolates $f_\zeta(x)$ at $x_1, \dots, x_n$, we conclude that

$$f_\zeta(x)-(R_nf_\zeta)(x)=\frac{B_n(x)}{(\zeta-x)B_n(\xi)}. \tag{4.3}$$

This proves (1.5).

In order to prove (1.4) we use (4.1) with Γ replaced by $\Gamma_\varrho=\{z\colon |1-z^2|=\varrho\}$, $\varrho>1$ and consequently we have

$$f(x)-(P_mf)(x)=a_f\,\varrho^{-m}. \tag{4.4}$$

Again

$$f(x)-(S_{m,n}f)(x)=f(x)-(P_mf)(x)-(S_{m,n}(f-P_m))(x)$$

$$=f(x)-(P_mf)(x)-\frac{1}{2\pi i}\int_{\Gamma_\varrho}\frac{f(z)}{(1-z^2)^m}(S_{m,n}F_{m,z})(x)\,dz,$$

where $F_{m,z}(x)=(1-x^2)^m/(z-x)$. By Lemma 4, $(S_{m,n}F_{m,z})(x)$ is clearly uniformly bounded for $x\in[-1,1]$ and $z\in\Gamma_\varrho$, $\varrho>1$. This together with (4.1) gives (1.4) and completes the proof of Theorem 1.

Remark. Combining Lemma 3 and (3.5) we have $M(x|-1^m, y, 1^m)\sim\sqrt{\frac{m}{\pi}}\cdot(1-x^2)^{m-1}/(1-xy)$, as $m\to\infty$. If we introduce the kernel $H_n(x|x_0, \dots, x_n)=(1-x^2)^n/\prod_{j=0}^n(1-xx_j)$, then it is easy to verify that

$$H(x|x_0,\dots,x_n)=\frac{x_n-x}{x_n-x_0}H(x|x_1,\dots,x_n)+\frac{x-x_0}{x_n-x_0}H(x|x_0,\dots,x_{n-1})$$

and so repeated appliciation of (2.3) gives

$$M(x|-1^m, x_0, \dots, x_n, 1^m)\sim\sqrt{\frac{m}{\pi}}(1-x^2)^nH(x|x_0,\dots,x_n).$$

5. Remarks on Schoenberg's quadrature formula

(a) **Asymptotics of the weights $B_i^{(m)}$.** Here we add some remarks on Schoenberg's quadrature formula

$$Q_{m,n}f=\int_{-1}^1(S_{m,n}f)(x)\,dx \tag{5.1}$$

$$=\sum_{i=0}^{m-1}\frac{A_i^{(m)}}{i!}\{f^{(i)}(-1)+(-1)^{(i)}f^{(i)}(1)\}+\sum_{i=1}^nB_i^{(m)}f(x_i).$$

Its extremal properties are discussed in [4]. Theorem 1 shows that for any n

$$\lim_{m\to\infty} Q_{m,n} f = \int_{-1}^{1} f(x)\,dx.$$

if f is regular in a neighborhood of $D_1 = \{z\colon |z^2-1| \leqq 1\}$ *and that moreover this* region is best possible for each n.

Let us also point out that since the quadrature formula (5.1) is equal to $\int_{-1}^{1} f(x)\,dx$ for $f \in S_{2m-1}(x_1, \ldots, x_n)$ it follows that the weights of the quadrature formula satisfy the following equations

$$\sum_{i=1}^{n} B_i^{(m)} K_m(x_i, x_j) = 1, \quad j = 1, 2, \ldots, n. \tag{5.2}$$

The matrix of this system of linear equations is non-singular by the remarks following Lemma 3. To obtain an asymptotic formula for the weights for large m, we set

$$C_i^{(m)} = B_i^{(m)} (1-x_i^2) K_m(x_i, x_i)$$

thereby getting

$$\sum_{i=1}^{n} C_i^{(m)} T_m(x_i, x_j) = 1, \quad j = 1, 2, \ldots, n.$$

Now, if we let $m\to\infty$, and use Lemma 4, we see that $\lim_{m\to\infty} C_i^{(m)} = C_i^{(\infty)}$ where

$$\sum_{i=1}^{n} C_i^{(\infty)} T_\infty(x_i, x_j) = 1, \quad j = 1, 2, \ldots, n.$$

Next we identify the constants C_i^∞ by observing that $T_\infty(0, x_j) = 1$, so that for any function f we have

$$\sum_{i=1}^{n} C_i^{(\infty)} f(x_i) = (R_n f)(0).$$

In particular, choosing $f(x) = f_\zeta(x)$ and using (4.3), we observe that

$$\frac{1}{\zeta} - \sum_{i=1}^{n} \frac{C_i^{(\infty)}}{\zeta - x_i} = \frac{1}{\zeta}\,\frac{B_n(0)}{B_n(\zeta)},$$

whence we easily see that

$$C_i^{(\infty)} = \frac{(-1)^{n-2}}{1-x_i^2} \prod_{j\neq i} x_j \prod_{j\neq i} \left(\frac{1-x_i x_j}{x_i - x_j}\right).$$

This shows that for m large enough, the $B_i^{(m)}$ are not of the same sign.

(b) **Numerical computation of** $A_i^{(m)}$ **and** $B_i^{(m)}$. We end this section with some remarks about the numerical computation of the $A_i^{(m)}$ and $B_i^{(m)}$. First the weights $B_i^{(m)}$ can be computed by solving the system (5.2). Then the coefficients $A_i^{(m)}$ can be evaluated from the following equations:

$$\sum_{j=0}^{m-1} A_j^{(m)} C_{ij}^{(m)} = E_i - \sum_{l=1}^{n} B_l^{(m)} f_i(x_l), \quad i = 0, 1, \ldots, m-1, \tag{5.3}$$

where we set

$$f_i(x) = (1+x)^{m-i-1}(1-x)^{m-i},$$

$$C_{ij}^{(m)} = \frac{1}{j!}\{f_i^{(j)}(-1)+(-1)^j f_i^{(j)}(1)\} = 2^{m-i}(-1)^{j-m+i}\binom{m-i-1}{j-m+i}$$

and

$$E_i = \int_{-1}^{1} f_i(x)\,dx = \frac{4^{m-i}}{(m-i)}\frac{1}{\binom{2m-2i}{m-i}}.$$

Note that $C_{ij}^{(m)}$ is non-zero only for $m-i \leqq j \leqq 2m-2i-1$ and so the linear system (5.3) can be solved by back substitution. Moreover the elements of the matrix $(C_{ij}^{(m)})_{i,j=0}^{m-1}$ can be easily computed from the recurrence

$$C_{i,j+1}^{(m)} = \tfrac{1}{2}(C_{i+1,j+1}^{(m)} - C_{i+1,j}^{(m)}). \tag{5.4}$$

6. An alternative to complete spline interpolation

It is interesting to point out that a suitable modification of complete spline interpolation allows us to give an affirmative answer to Conjecture 1. For example, let $\tilde{S}_{m,n}f$ satisfy the following requirements:

(i) $\quad (\tilde{S}_{m,n}f)^{(j)}\left(\frac{i}{n}\right) = f^{(j)}\left(\frac{i}{n}\right), \quad j = 0, 1, \dots, m-1; \quad i = 0, 1, \dots, n.$

(ii) $\quad (\tilde{S}_{m,n}f)(x) \in C^{m-1}(\mathbf{R}),$

(iii) On each interval $\left(\frac{j}{n}, \frac{j+1}{n}\right)$, $(\tilde{S}_{m,n}f)(x) \in \pi_{2m-1}$.

Then by (4.1), $\lim_{m\to\infty}(\tilde{S}_{m,n}f)(x) = f(x)$ uniformly on $[0, 1]$ when f is analytic in a neighborhood of the set D_n^* defined by

$$D_n^* = \bigcup_{i=0}^{n-1}\left\{z\colon \left|\left(z-\frac{i}{n}\right)\left(z-\frac{i+1}{n}\right)\right| \leqq \frac{1}{4n^2}\right\}.$$

Since $D_n^* \to [0, 1]$ as $n\to\infty$, because it is contained in the rectangle

$$\left[-\frac{\sqrt{2}-1}{2n}, 1+\frac{\sqrt{2}-1}{2n}\right]\times\left[-\frac{1}{4n}, \frac{1}{4n}\right],$$

it follows that given any function f analytic in a neighborhood of $[0, 1]$, there is an $n = n(f)$ such that

$$\lim_{n\to\infty}(\tilde{S}_{m,n}f)(x) = f(x),$$

uniformly on $[0, 1]$.

This simple observation suggests the following conjecture.

Conjecture. *Let $S_{m,n}f$ denote the complete spline interpolant to f on* [0, 1] *at the knots* $\frac{i}{n+1}$ $(i=1, \ldots, n)$. *If f is holomorphic in a neighborhood of* [0, 1], *we conjecture that there exists an n such that*

$$\lim_{m\to\infty} S_{m,mn}f = f$$

uniformly on [0, 1].

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References

1. DE BOOR, CARL, *Practical guide to splines*, Springer-Verlag, New York 1978.
2. FISHER, S. D. and MICHELLI, C. A., Optimal sampling for holomorphic functions, *Amer. J. Math.* (1984), 593—609.
3. KARLIN, S., Total positivity, interpolation by splines and Green's functions of differential operators, *J. Approx. Theory* **4**, (1971), 91—112.
4. SCHOENBERG, I. J., On monosplines of least deviation and best quadrature formulae, *SIAM J. Numer. Anal.* **2**, (1965).

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