H^∞+BUC does not have the best approximation property

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§ 1. Introduction

Let L^∞ denote the usual Lebesgue space of functions on the unit circle $[|z|=1]$ and let H^∞ denote the bounded analytic functions on the unit disc $[|z|<1]$. By identifying functions in H^∞ with their boundary values we may regard H^∞ as a closed subalgebra of L^∞. The closed algebras between H^∞ and L^∞ are called *Douglas algebras* and have been studied extensively ([3], [4], [5], [9], [11], [14], [15]). For background and general information on Douglas algebras see [6] and [13].

Let C denote the space of continuous functions on the unit circle. It was shown by Sarason [10] that the linear span $H^\infty+C$ is a Douglas algebra. In fact it is the smallest such algebra properly containing H^∞; see [7]. In [12], Sarason asked whether $H^\infty+C$ has the *best approximation property,* i.e. whether given any $f\in L^\infty$ there existed a $g\in H^\infty+C$ such that

$$\|f-g\|_\infty = \mathrm{d}(f, H^\infty+C) \underset{\text{def}}{=} \inf\{\|f-g\|_\infty\colon\ g\in H^\infty+C\}.$$

This question was answered affirmatively by Axler, Berg, Jewell, and Shields [1], who then raised the question of whether all Douglas algebras possess this property.

A subsequent paper of Luecking [8] provided a simpler proof of the $H^\infty+C$ case using the theory of M-ideals. In an unpublished manuscript, Marshall and Zame give a very simple proof of this case and also give many interesting examples of Douglas algebras possessing the best approximation property. Another such example is given by Younis in [16].

In this paper we answer the question for general Douglas algebras negatively, our counterexample being a certain "natural" Douglas algebra. In order to describe and work with this algebra it is convenient to move over to the real line $\mathbf{R}$ and the upper half plane $\Delta=\{z=x+iy\colon x, y\in\mathbf{R}, y>0\}$. Henceforth in this paper L^∞ and H^∞ will refer to the corresponding function spaces on $\mathbf{R}$ and Δ. Let BUC denote the space of bounded uniformly continuous functions on $\mathbf{R}$. It is shown by Sarason [11] that $H^\infty+\mathrm{BUC}$ is a Douglas algebra, and this is the algebra which we will show fails the best approximation property.

The following definitions and notations will be used. For $f\in L^\infty$ and $z=x+iy\in\Delta$ we define the Poisson integral of f at z by

$$P[f](z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-t)^2+y^2}f(t)\,dt;$$

then $P[f]$ is harmonic in Δ with boundary value f, and if $f\in H^\infty$ then $P[f](z)=f(z)$. For $z, w\in\Delta$ we define the pseudo-hyperbolic distance between z and w by $\varrho(z-w)=\left|\frac{z-w}{z-\bar{w}}\right|$. For an interval $I\subset\mathbf{R}$ and a function f on $\mathbf{R}$ we define $\mathrm{Var}_I(f)=\sup_{x_1,x_2\in I}|f(x_1)-f(x_2)|$ and $\|f\|_I=\sup_{x\in I}|f(x)|$. We denote the length of I by $|I|$. Finally we will use the following facts, the first of which is shown by Sarason in [11] and the second of which is an easy exercise with the Poisson integral formula: if $f, g\in H^\infty+\mathrm{BUC}$ then

$$\sup_{x\in\mathbf{R}}|P[fg](x+iy)-P[f](x+iy)P[g](x+iy)|\to 0 \quad \text{as} \quad y\to 0;$$

and if $f\in\mathrm{BUC}$ and $0<\varkappa<1$ then

$$\sup\{|P[f](w)-P[f](z)|\colon \varrho(z,w)\leqq\varkappa\}\to 0 \quad \text{as} \quad \operatorname{Im} z\to 0.$$

Other information about $H^\infty+\mathrm{BUC}$ is developed in [11] and in Exercise 8, Chapter IX of [6].

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§ 2. **Theorem.** $H^\infty+\mathrm{BUC}$ *does not have the best approximation property.*

Proof. First, a bit of motivation for the construction. Returning to the unit circle for a moment, Marshall and Zame pointed out that $(H^\infty+C)/H^\infty$ has *continuous* best approximations, i.e. given $f\in L^\infty$ such that $\mathrm{d}(f, H^\infty)\leqq 1+\varepsilon$ and $\mathrm{d}(f, H^\infty+C)=1$, there exists $h\in C$ such that $\mathrm{d}(f-h, H^\infty)=1$ and $\|h\|_\infty\leqq\delta(\varepsilon)$, where $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$. We now will do a preliminary construction whose essential point is that this property fails in $H^\infty+\mathrm{BUC}$, and the theorem will then follow easily.

Let $\varepsilon, R>8, \eta$ be given positive numbers — we are thinking of ε and η as being small and R as being large. For $k=1, 2, \ldots$ we pick widely spaced intervals $I_k\subset\mathbf{R}$, all of length η. Let $I_k\subset\tilde{I}_k\subset\tilde{\tilde{I}}_k$ where I_k, $\tilde{I}_k$, and $\tilde{\tilde{I}}_k$ have the same midpoint, $|\tilde{I}_k|/|I_k|\to\infty$ rapidly as $k\to\infty$, and $|\tilde{\tilde{I}}_k|-|\tilde{I}_k|$ is constant. We choose all these intervals so that the $\tilde{\tilde{I}}_k$'s are disjoint. Denote the midpoint of I_k by x_k.

Pick $\varkappa$ very close to 1, $0<\varkappa<1$, pick $\delta_k>0$ to be small numbers such that $\delta_k\to 0$ as $k\to\infty$, and define $l_k=\{x+i\delta_k: x\in I_k\}$. Let $\{z_{kn}\}_{n=-N_k,\dots,N_k}$ be a maximal set of points on the line l_k having pseudohyperbolic separation of adjacent points being equal to $\varkappa$, and such that $z_{k0}=x_k+i\delta_k$. Thus the points $\{z_{kn}\}$ are distributed symmetrically with respect to the line $\operatorname{Re} z=x_k$. Define the finite Blaschke product b_k with these points as zeros: $b_k(z)=\prod_{n=-N_k}^{N_k}\frac{z-z_{kn}}{z-\bar{z}_{kn}}$. Then from the symmetry of the $\{z_{kn}\}$ it follows easily that $b_k(x_k+iy)>0$ for all $y>0$. Define $w_k=x_k+i\frac{\Pi}{320R}\eta$.

Now set $b=\prod_{k=1}^{\infty}b_k$, $S(z)=\prod_{k=1}^{\infty}\frac{z-w_k}{z-\bar{w}_k}$. Standard methods easily show that both products converge uniformly on compact subsets of $\Delta\cup\mathbf{R}$ and define Blaschke products if the intervals are widely enough dispersed. Clearly $S\in$BUC. It is also easy to check that if the I_k's are widely dispersed then $|1-b(w_k)|<1$ $\forall k$, and if in addition $|\tilde{I}_k|/|I_k|\to\infty$ fast enough (where "fast enough" will depend on the choice of the δ_k's), then $|1-b(x)|<1/2\,\varepsilon$ for $x\notin\cup\tilde{I}_k$.

If $\varkappa$ is chosen close enough to 1, the δ_k's are all small enough, and the I_k's are widely enough dispersed, then the set $\{w_k\}\cup\{z_{kn}\}$ will be an interpolating sequence with interpolation constant close to 1 (see [2] and [6], Chapter VII); what this means for us is that if complex numbers α_k, β_{kn} are given for $k\geqq 1$ and $-N_k\leqq n\leqq N_k$ and $|\alpha_k|\leqq 1$, $|\beta_{kn}|\leqq 1$, then there exists $\varphi\in H^\infty$ such that $\|\varphi\|_\infty\leqq 1+\varepsilon/2$ and $\varphi(w_k)=\alpha_k$, $\varphi(z_{kn})=\beta_{kn}$.

Because of the conditions imposed on the lengths of the intervals I_k, $\tilde{I}_k$, $\tilde{\tilde{I}}_k$, we can find $\chi\in$BUC such that $0\leqq\chi\leqq 1$, $\chi\equiv 1$ on $\bigcap_k\tilde{I}_k$, and $\chi\equiv 0$ off $\bigcap_k\tilde{\tilde{I}}_k$. Then

$$\begin{aligned}\mathrm{d}(\chi\bar{S}\bar{b}-\chi\bar{S}, H^\infty) &= \mathrm{d}(\chi-\chi b, SbH^\infty)\\ &\leqq \|(1-\chi)(1-b)\|_\infty+\mathrm{d}(1-b, SbH^\infty).\end{aligned}$$

The first term is bounded by $1/2\varepsilon$ since if $x\in\mathbf{R}$, $1-\chi(x)\neq 0$, then $x\notin\bigcap_k\tilde{I}_k$, hence $|1-b(x)|<\varepsilon/2$. To estimate the second term we write

$$\mathrm{d}(1-b, SbH^\infty) = \inf_{g\in H^\infty}\|1-b-Sbg\|_\infty$$

$$= \inf\{\|\varphi\|_\infty\colon \varphi(w_k) = 1-b(w_k),\ \varphi(z_{kn}) = 1-b(z_{kn}) = 1\}\leqq 1+\frac{\varepsilon}{2}$$

by the above comments and the fact that $|1-b(w_k)|<1$. Hence $\mathrm{d}(\chi\bar{S}\bar{b}-\chi\bar{S}, H^\infty)<1+\varepsilon$.

What we have done so far has been to start with a function having distance 1 from H^∞, namely $\chi\bar{S}\bar{b}$, and then to change it by the large BUC function $\chi\bar{S}$ to get a function whose distance from H^∞ is only slightly greater than 1. The point of what we will do next is that it is impossible to get back to a function having distance 1 from H^∞ by adding a small BUC function. Actually we need a local version of this fact.

Assume, to get a contradiction, that there is a function $h\in\text{BUC}$ and a $g\in H^\infty$ such that $\|h\|_\infty\leqq R$, $\|g\|_\infty\leqq R$, $\text{Var}_{I_k}(h)<1/2$ for all k, and $\|\chi\bar S\bar b-\chi\bar S-h-g\|_{I_k}\leqq 1$ for all k. Define $h_k=h-h(x_k)$, $g_k=g+h(x_k)$. Then $\|h_k\|_\infty\leqq 2R$, $\|g_k\|_\infty\leqq 2R$, $\|h_k\|_{I_k}\leqq 1/2$, and

$$\|\chi\bar S\bar b-\chi\bar S-h-g\|_{\tilde I_k}=\|\bar S\bar b-\bar S-h_k-g_k\|_{\tilde I_k}=\|1-b-Sbh_k-Sbg_k\|_{\tilde I_k},$$

so that $\|1-b-Sbh_k-Sbg_k\|_{\tilde I_k}\leqq 1$. Fixing attention on a point z_{kn} we write

$$1-b-Sbh_k-Sbg_k=1-b-P[h_k](z_{kn})Sb-g_kSb-[h_k-P[h_k](z_{kn})]Sb.$$

Now $h_k-P[h_k](z_{kn})=h-P[h](z_{kn})$, and since $h\in\text{BUC}$ and $\operatorname{Im} z_{kn}=\delta_k\to 0$ as $k\to\infty$ we have for z satisfying $\varrho(z,z_{kn})\leqq\varkappa$ that

$$\begin{aligned}&\left|P\left[\left(h_k-P[h_k](z_{kn})\right)Sb\right](z)\right|=\left|P\left[\left(h-P[h](z_{kn})\right)Sb\right](z)\right|\\&=|P[hSb](z)-P[h](z_{kn})S(z)b(z)|\leqq|P[hSb](z)-P[h](z)P[Sb](z)|\\&+|P[h](z)-P[h](z_{kn})|\,|S(z)|\,|b(z)|<\lambda_k\end{aligned}$$

where $\lambda_k\to 0$ as $k\to\infty$.

Also since $\delta_k\to 0$ and $|\tilde I_k|/|I_k|\to\infty$ as $k\to\infty$, we have by the Poisson integral formula that $\|1-b-Sbh_k-Sbg_k\|_{\tilde I_k}\leqq 1$ implies that

$$\sup\{|P[1-b-Sbh_k-Sbg_k](z)|:\ \varrho(z,z_{kn})\leqq\varkappa\}\leqq 1+\lambda_k'$$

where $\lambda_k'\to 0$ as $k\to\infty$. Hence

$$\begin{aligned}&\sup\{|1-b(z)-P[h_k](z_{kn})S(z)b(z)-g_k(z)S(z)b(z)|:\ \varrho(z,z_{kn})\leqq\varkappa\}\\&=\sup\{|P[1-b-P[h_k](z_{kn})Sb-g_kSb](z)|:\ \varrho(z,z_{kn})\leqq\varkappa\}\leqq 1+\lambda_k+\lambda_k'\to 1.\end{aligned}$$

Writing $x_{kn}=\operatorname{Re} z_{kn}$, define

$$B_{kn}(z)=b(z\delta_k+x_{kn}),\ G_{kn}(z)=1+P[h_k](z_{kn})S(z\delta_k+x_{kn})-g_k(z\delta_k+x_{kn})S(z\delta_k+x_{kn}).$$

We then have that B_{kn} is a Blaschke product for which $|B_{kn}'(i)|=|b'(z_{kn})|\delta_k$ is bounded below by some positive constant not depending on k,n (since $\{z_{kn}\}$ is an interpolating sequence, see Chapter VII of [6]), $G_{kn}\in H^\infty$, $\|G_{kn}\|_\infty\leqq 1+4R$, and

$$\sup\{|1-B_{kn}(z)G_{kn}(z)|:\ \varrho(z,i)\leqq\varkappa\}\leqq 1+\lambda_k+\lambda_k'.$$

A simple argument based on normal families and the open mapping theorem now yields $\lambda_k''\to 0$ such that $\sup\{|G_{kn}(z)|:\varrho(z,i)\leqq\varkappa\}\leqq\lambda_k''$. Hence

$$|1+P[h_k](z_{kn})S(z)-g_k(z)S(z)|<\lambda_k''\quad\text{if}\quad\varrho(z,z_{kn})<\varkappa.$$

Then for such z,

$$\begin{aligned}P[1+h_kS+g_kS](z)&\leqq|1+P[h_k](z_{kn})S(z)+g_k(z)S(z)|\\&+|P[h_kS](z)-P[h_k](z)S(z)|+|P[h_k](z)-P[h_k](z_{kn})|\,|S(z)|.\end{aligned}$$

The first term is bounded by λ_k'', and the arguments we have used show that the second and third terms are bounded by numbers $\lambda_k''', \lambda_k''''$ which go to 0 as $k\to\infty$, since $h\in$BUC. Hence for k large enough, $|P[1+h_k S+g_k S](z)|<1/16$ for $z\in l_k$. The Poisson integral formula (on the line $\operatorname{Im} z=\delta_k$) together with the choice of w_k and the facts that $\|1+h_k S+g_k S\|_\infty<5R$, $\|h_k\|_\infty<2R$, and $\|h_k\|_{\tilde{I}_k}\leqq 1/2$ now implies that

$$|P[1+h_k S+g_k S](w_k)|<\frac{1}{8}\text{ and }\quad |P[h_k S](w_k)|<\frac{9}{16}.$$

This leads to a contradiction since

$$P[1+h_k S+g_k S](w_k)=1+P[h_k S](w_k).$$

(It is of interest to note the similarity at this point to the example at the end of Section 3 of [15].)

We have thus shown that if $g\in H^\infty$, $h\in$BUC, $\|g\|_\infty\leqq R$, $\|h\|_\infty\leqq R$, and

$$\|\chi\bar{S}\bar{b}-\chi\bar{S}-h-g\|_{I_k}\leqq 1 \quad \text{for all } k,$$

then

$$\operatorname{Var}_{I_k}(h)\geqq \tfrac{1}{2} \quad \text{for some } k.$$

Now find $g\in H^\infty$ such that $\|\chi\bar{S}\bar{b}-\chi\bar{S}-g\|_\infty\leqq 1+\varepsilon$ and define $f=\chi^2\bar{S}\bar{b}-\chi^2\bar{S}-\chi g$, so that $\|f\|_\infty\leqq 1+\varepsilon$. Clearly $\|g\|_\infty\leqq 4<1/2\,R$. Then f is supported in $\bigcup_k \tilde{\tilde{I}}_k$ and if $F=-\chi^2\bar{S}-\chi g$ we have that $F\in H^\infty$+BUC and $\|f-F\|_\infty=1$. If, however, $\varphi\in H^\infty$, $h\in$BUC with $\|\varphi\|_\infty<1/2\,R$, $\|h\|_\infty<R$, and $\|f-(\varphi+h)\|_{I_k}\leqq 1$ foı all k, then

$$1\geqq\|\chi^2\bar{S}\bar{b}-\chi^2\bar{S}-\chi g-\varphi-h\|_{I_k}=\|\bar{S}\bar{b}-\bar{S}-h-(g+\varphi)\|_{I_k} \quad \text{for all } k,$$

so $\operatorname{Var}_{I_k} h\geqq 1/2$ for some k since $g+\varphi\in H^\infty$ and $\|g+\varphi\|_\infty\leqq R$.

It is now easy to find a function with no best approximant in H^∞+BUC. For $j=1,2,\dots$ pick positive $\varepsilon_j, R_j>8, \eta_j$ such that $\varepsilon_j\to 0$, $R_j\to\infty$, $\eta_j\to 0$ as $j\to\infty$. Carry out the above construction to get intervals $I_k^j\subset\tilde{I}_k^j\subset\tilde{\tilde{I}}_k^j$ such that $|I_k^j|=\eta_j$, functions f_j supported in $\bigcup_k\tilde{\tilde{I}}_k^j$ with $\|f_j\|_\infty\leqq 1+\varepsilon_j$, and functions F_j supported in $\bigcup_k\tilde{\tilde{I}}_k^j$ such that $F_j\in H^\infty$+BUC and $\|f_j-F_j\|_\infty=1$, and such that if $\varphi\in H^\infty$, $h\in$BUC with $\|\varphi\|_\infty\leqq 1/2\,R_j$, $\|h\|_\infty\leqq R_j$ and $\|f_j-(\varphi+h)\|_{I_k^j}\leqq 1$ for all k then $\operatorname{Var}_{I_k^j}(h)\geqq 1/2$ for some k. This can be done so that $\bigcup_k\tilde{\tilde{I}}_k^{j_1}\cap\bigcup_k\tilde{\tilde{I}}_k^{j_2}=\emptyset$ if $j_1\neq j_2$, so that the supports of the various f_j's are disjoint. Let $f=\sum_{j=1}^\infty f_j$. Since $\sum_{j=1}^N F_j\in H^\infty$+BUC and

$$\Big\|f-\sum_{j=1}^N F_j\Big\|_\infty=\Big\|\sum_{j=1}^N(f_j-F_j)+\sum_{j=N+1}^\infty f_j\Big\|_\infty$$

$$\leqq\sup_{j\geqq N+1} 1+\varepsilon_j\to 1, \quad \text{we have that}$$

$d(f, H^\infty+\mathrm{BUC})\leqq 1$. However say we could find $\varphi\in H^\infty, h\in\mathrm{BUC}$ such that $\|f-(\varphi+h)\|_\infty\leqq 1$. If j is high enough then $\|\varphi\|_\infty<1/2\,R_j$ and $\|h\|_\infty<R_j$. Then $\|f-(\varphi+h)\|_\infty\leqq 1$ implies that $\|f_j-(\varphi+h)\|_{I_k^j}\leqq 1$ for all k, which then implies that $\mathrm{Var}_{I_k^j}(h)\geqq 1/2$ for some k. Since $|\tilde{I}_k^j|=\eta_j\to 0$ this would violate the uniform continuity of h. This completes our proof.

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