An interpolation theorem

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Given an interpolation couple (A_0, A_1), the approximation functional is defined by:

(1) $$E(t, a; A_0, A_1) = \inf \{|a-a_0|_{A_1} / |a_0|_{A_0} \leqq t\}.$$

An operator $T\colon A_0+A_1 \to B_0+B_1$ is E-quasi-linear (see [4]) iff

(2) $$E(t_0+t_1, T(a_0+a_1); B_0, B_1) \leqq C\{E(dt_0, Ta_0; B_0, B_1) + E(dt_1, Ta_1; B_0, B_1)\}.$$

The following interpolation theorem is proved in [4]:

Theorem 1. *If* $T\colon A_0+A_1 \to B_0+B_1$ *is E-quasi-linear and*

(3) $$\frac{1}{t}\int_0^t E(s, Ta; B_0, B_1)\, ds - E(t, Ta; B_0, B_1) \leqq C_1 |a|_{A_1},$$

(4) $$E(t, Ta; B_0, B_1) \leqq C_0 t^{-\beta} |a|_{A_0}, \quad 0 < \beta < \infty.$$

Then

$$|Ta|_{\beta(1-\theta), q; E} \leqq C\, |a|_{\theta, q; K}, \quad 0 < \theta < 1.$$

Condition (3) is interesting: it gives an abstract definition of T being of weak type. This has yielded in [4] a significant generalization of a theorem of J. Gilbert on interpolation with change of measure [2], and an extension of a theorem of Bennett—DeVore—Sharpley [1].

The proof of Theorem 1 in [4] is direct, and this entails a shortcoming: it makes it harder to apply interpolation theory to the new results. In this paper we intend to prove Theorem 1 again, within the framework of interpolation theory. Using this approach we are indeed able to strengthen the theorem: condition (4) which is $T\colon A_0 \to (B_0, B_1)_{\beta, \infty; E}$ is replaced by $T\colon A_0 \to B_0$.

Definition 2. Let f be integrable on $(0, t)$, all t. We define

(5) $$f_{\#}(t) = \frac{1}{t}\int_0^t f(u)\,du - f(t),$$

(6) $$|f|_W = \operatorname*{ess\,sup}_{0<t} |f_{\#}(t)|.$$

If we identify functions differing by a constant, $|\ |_W$ serves as a norm on the space of equivalence classes. Denote this space by W. Condition (3) is therefore $|E(s, Ta; B_0, B_1)|_W \leqq c_1 |a|_{A_1}$. Our space W is not the class W of [1]. If we denote by $W(A_0, A_1)$ the class of elements of A_0+A_1 which satisfy $|E(s, a; A_0, A_1)|_W < \infty$, the class W of [1] is $W(L_0, L_\infty)$. Note that $|Ta|_{B_1} \leqq c_1 |a|_{A_1}$, which is the usual hypothesis in the interpolation theorem, implies $E(s, Ta; B_0, B_1) \leqq c_1 |a|_{A_1}$. Our generalization consists therefore in replacing L_∞ estimates on E, by a W estimate.

The identification of $W(A_0, A_1)$ for given interpolation couples (A_0, A_1) will yield the conclusions of known interpolation theorems from weaker hypotheses, much in the same way as was done in [4] for interpolation with change of measure. We shall return to these and related problems in subsequent papers.

We are going to interpolate between W and L_p spaces. For the application of Wolff's theorem we need the following theorem:

Theorem 3. *W is complete.*

Proof. $\{\tilde{f}_n\}$ is a Cauchy sequence in W. $f_n \in \tilde{f}_n$ are chosen, so that:

(7) $$\int_0^1 f_n = 0.$$

$\{(f_n)_{\#}\}$ is Cauchy in L_∞, and so there exists $h \in L_\infty$ so that $f_{n\#} \to h\ (L_\infty)$. On the other hand, using

(8) $$\frac{1}{t}\int_0^t f - \frac{1}{s}\int_0^s f = \int_t^s f_{\#}(u)\frac{du}{u}$$

we have

(9) $$\frac{1}{t}\int_0^t f_n = \int_t^1 (f_n)_{\#}(u)\frac{du}{u}$$

so that

(10) $$\left|\int_0^t (f_n - f_m)\right| \leqq t\,|\log t|\,|f_n - f_m|_W$$

and $\int_0^t f_n \to g(t)$, all t. Since also

(11) $$|f_n(t) - f_m(t)| \leqq |(f_n - f_m)_{\#}(t)| + \frac{1}{t}\left|\int_0^t (f_n - f_m)\right|.$$

We also have

(12) $$f_n(t) \to f(t) \quad (L_1(0, M), \text{ for any } M), \quad \text{and for a.e.} \quad t > 0.$$

Therefore f is integrable on $(0, M)$ for any M and $\int_0^t f = g$. Finally

$$(f_n)_\#(t) = \frac{1}{t}\int_0^t f_n - f_n(t) \to \frac{1}{t}\int_0^t f - f(t) = f_\#(t) \tag{13}$$

so that $f_\# = h$ and thus $|f_{n\#} - f_\#|_\infty \to 0$ and $f_n \to f$ (W). The proof is complete.

Theorem 4. $(L_0, W)_{1/p, q; E} = L(p, q)$, $0<p<\infty$; $0<q\leqq\infty$. *All function spaces here are taken on* $(0, \infty)$.

Proof. We first interpolate W with L_1 and then, using a theorem of Wolff we get the full result.

If $f_\# \in L(p, q)$, $1<p<\infty$, then there exists a constant c so that $f+c \in L(p, q)$. For: if $f_\# \in L(p, q)$ then $\int_t^\infty |f_\#(u)|\, du/u < \infty$. From (8) then we have:

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s f(u)\, du = c$$

exists. Let $g = f - c$. Of course $g_\# = f_\#$ so that from (8) again we have

$$\frac{1}{t}\int_0^t g = \int_t^\infty g_\#(u)\frac{du}{u}. \tag{14}$$

From $g_\# \in L(p, q)$, using Hardy's inequality we get $\frac{1}{t}\int_0^t g \in L(p, q)$. Since

$$g(t) \leqq \left|\frac{1}{t}\int_0^t g\right| + |g_\#(t)| \tag{15}$$

we get $g \in L(p, q)$.

As elements of W, $f \equiv g$ and in the sequel we shall therefore write for $f \in W$: $f_\# \in L(p, q) \Rightarrow f \in L(p, q)$. This amounts to taking the element in the equivalence class of f for which $\lim_{s\to\infty} \frac{1}{s}\int_0^s f = 0$.

On the other hand, $f \in L(p, q)$, $1<p<\infty$, implies $f_\# \in L(p, q)$. To see this, consider the linear operator $\# : f \to f_\#$. Obviously:

$$\begin{aligned} \# &: L_1 \to L(1, \infty), \\ \# &: L_\infty \to L_\infty. \end{aligned} \tag{16}$$

Interpolating we get

$$\# : L(p, q) \to L(p, q), \quad 1 < p < \infty, \quad 0 < q \leqq \infty. \tag{17}$$

Now for the identification

$$(L_1, W)_{\theta, q; K} = L(p, q), \quad \frac{1}{p} = 1 - \theta, \quad 0 < q \leqq \infty. \tag{18}$$

Since $L_\infty \subset W$, we have $L(p,q) \subset (L_1, W)_{\theta,q;K}$. For the converse note that $\#$ actually maps W to L_∞ so that

(19) $$\#: (L_1, W)_{\theta,q;K} \to L(p,q).$$

Therefore

(20) $$|f_\#|_{p,q} \leqq C_p |f|_{(L_1,W)_{\theta,q;K}}.$$

Since however $|f|_{p,q} \leqq C_p |f_\#|_{p,q}$, we get (18). To get the full theorem we apply Wolff's theorem. We restate it in a form more convenient for our application.

A_1, A_2, A_3, A_4 are quasi-Banach Abelian groups and $A_1 \cap A_4 \subset A_2 \cap A_3$. Assume

(21) $$(A_1, A_3)_{\beta,q;E} = A_2, \quad 0 < \beta < \infty, \quad 0 < q \leqq \infty,$$

(22) $$(A_2, A_4)_{\psi,r;K} = A_3, \quad 0 < \psi < 1, \quad 0 < r \leqq \infty.$$

Then

(23) $$(A_1, A_4)_{\alpha_2,q;E} = A_2, \quad \alpha_2 = \beta/\psi,$$

(24) $$(A_1, A_4)_{\alpha_3,r;E} = A_3, \quad \alpha_3 = \beta\frac{1-\psi}{\psi}.$$

In [5] the statement of the theorem is for quasi-Banach spaces, i.e., $|rx|_A = |r|\,|x|_A$ for scalars r is required. This, in fact, is not used in the proof. The added generality is needed here for L_0 defined by

(25) $$|f|_{L_0} = \int_{\{x \,|\, |f(x)| > 0\}} d\mu(x)$$

does not have the homogeneity property. It is easy to see that for $0 < p \leqq \infty$

(26) $$E(t, f; L_0, L_p) = \left(\int_t^\infty [f^*(s)]^p\, ds\right)^{1/p}$$

and, applying an extension of Hardy's inequality applicable to decreasing functions, see [3], we get from (26) for $1 < p$:

(27) $$L_1 = (L_0, L_p)_{1/p',1;E}; \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Since, by (18), we have

(28) $$(L_1, W)_{1/p',p;K} = L_p$$

we get, using (23) and (24),

(29) $$(L_0, W)_{1,1;E} = L_1,$$

(30) $$(L_0, W)_{1/p,p;E} = L_p; \quad 1 < p < \infty.$$

To get the result for the full range we shall need a version of the reiteration theorem:

$$(A_0, (A_0, A_1)_{\alpha, r; E})_{\beta, q; E} = (A_0, A_1)_{\alpha+\beta, q; E}. \tag{31}$$

Fix $0<r<1$ and write

$$L_r = (L_0, L_1)_{1/r-1, r; E} = (L_0, (L_0, W)_{1,1;E})_{1/r-1, r; E} = (L_0, W)_{1/r, r; E}.$$

Finally, another application of the reiteration theorem gives the result in full generality. The theorem is proved.

Theorem 5. *T: $A_0+A_1 \to B_0+B_1$ is E-quasi-linear and*

$$\frac{1}{t}\int_0^t E(s, Ta; B_0, B_1)\, ds - E(t, Ta; B_0, B_1) \leqq C_1 |a|_{A_1}. \tag{32}$$

$$|Ta|_{B_0} \leqq C_0 |a|_{A_0}. \tag{33}$$

Then, for $0<\alpha<\infty$, $0<q\leqq\infty$:

$$|Ta|_{(B_0, B_1)_{\alpha, q; E}} \leqq C |a|_{(A_0, A_1)_{\alpha, q; E}}. \tag{34}$$

Proof. Consider E_T: $A_0+A_1 \to L_0+W$ defined by

$$E_T(a)(s) = E(s, Ta; B_0, B_1). \tag{35}$$

Conditions (32), (33) give

$$E_T\colon A_1 \to W, \tag{36}$$

$$E_T\colon A_0 \to L_0, \tag{37}$$

while from the E-quasi-linearity of T we have, for each $0<\alpha<\infty$, $0<q\leqq\infty$: $|Ta|_{(B_0, B_1)_{\alpha, q; E}}$ is a semi-quasi-norm on $A_0 \cap A_1$, satisfying:

$$\begin{aligned} |Ta|_{(B_0, B_1)_{\alpha, q; E}} &\sim |E_T(a)|_{(L_0, W)_{\alpha, q; E}} \\ &\leqq |E_T(a)|_{L_0}^{\alpha} |E_T(a)|_W \\ &\leqq C_0^{\alpha} C_1 |a|_{A_0}^{\alpha} |a|_{A_1}. \end{aligned}$$

Reiteration between different values of α now yields for $0<\alpha<\infty$, $0<q\leqq\infty$:

$$T\colon (A_0, A_1)_{\alpha, q; E} \to (B_0, B_1)_{\alpha, q; E} \tag{38}$$

and the theorem is proved.

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