A characterisation of Fuchsian groups of convergent type*

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Introduction

Let Γ be a Fuchsian group (I make no distinction here between Fuchsian and Frechsoid groups) acting on $U=\{z\in\mathbf{C}, |z|<1\}$ such that every point of ∂U is a limit point of Γ (i.e. Γ is of the first kind. The problem that will be examined does not arise for groups of the second kind). We say that Γ is of convergent type if:

$$\sum_{\gamma\in\Gamma}(1-|\gamma 0|)<+\infty$$

otherwise we say that it is of divergent type.

A group is of convergent type if and only if the corresponding Riemann surface is hyperbolic. It follows therefore that if Γ_0 is as above and is finitely generated then it is of divergent type. Indeed the Riemann surface U/Γ_0 can then be identified with $R\setminus\{r_1, r_2, \dots, r_k\}$ (for finitely many distinct $r_1, r_2, \dots, r_k\in R$) and R a compact surface cf. [11], [2].

In this paper I shall consider a subgroup $\Gamma\subset\Gamma_0$ where Γ_0 is finitely generated as above and I shall give a necessary and sufficient condition for Γ to be of convergent type. To state the theorem I shall need some algebraic preliminaries.

Let G be a discrete group generated by a finite number of generators $g_1, g_2, \dots, g_m\in G$. Let $H\subset G$ be a subgroup and let us fix $\xi_1, \dots, \xi_k\in G$ finitely many elements of G.

Let us also define $\mu_0, \mu_1, \dots, \mu_k\in\mathbf{P}(G)$ $k+1$ probability measures on G that are symmetric (i.e. $\mu_j(\{x\})=\mu_j(\{x^{-1}\})$) and which in addition satisfy:

(i) supp μ_0 is finite,

(ii) $\mu_0(g_j)>0$ $j=1, 2, \dots, m$,

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(iii) supp $\mu_j \subset Gp\{\xi_j)$ and if ord $\xi_j = +\infty$ (i.e. $\xi_j^N \neq e$=neutral element of $G, N \geqq 1$) then

$$C^{-1} \leqq n^2 \mu_j(\{\xi_j^n\}) \leqq C \quad n \in \mathbf{Z} \quad n \neq 0$$

for some $C > 0$, and all $j = 1, 2, \ldots, k$. The meaning of (iii) is that μ_j is essentially a Cauchy distribution on $\mathbf{Z}$ (if $Gp(\xi_j) \cong \mathbf{Z}$).

We shall now consider the symmetric random walk on G given by the transition matrix $P(g, h) = \mu(\{g^{-1}h\})$ $(g, h \in G)$ where $\mu = \sum_{j=0}^{k} \lambda_j \mu_j$ $(\lambda_j > 0; \sum \lambda_j = 1)$.

Definition. *I shall say that G is recurrent with respect to H and $\{\xi_1, \ldots, \xi_k\}$ as above if the random walk constructed above returns infinitely often to H. (I.e. if the induced random walk on the left cosset space G/H is recurrent cf.* [4], [5].*)*

It will of course be shown that the above definition is independent of the particulaı choice of the λ_j and of the μ_j $(j = 0, 1, \ldots, k)$, and of the particular choice of the generators $g_1, \ldots, g_m$. It will also be shown that changing the ξ_j's into conjugate ξ_j''s $(\xi_j' = \gamma_j \xi_j \gamma_j^{-1})$ will not affect the outcome.

Let now $\Gamma_0 \subset \mathrm{Aut}\,(U)$ be a finitely generated Fuchsian group (of the first kind) and let $\Gamma \subset \Gamma_0$ be a subgroup. Let Φ be the set of maximal cyclic paıabolic (i.e. each element is parabolic) subgroups of Γ_0. By a theorem of Heins Φ contains only finitely many conjugacy classes $\Phi = \Phi_1 \cup, \ldots, \cup \Phi_k$ and there is a natural one-one correspondance between these classes and the ideal boundary points $r_1, \ldots, r_k$ for which $U/\Gamma_0 \cong R \diagdown \{r_1, \ldots, r_k\}$ (with R compact as above) (cf. [3], [1]). This being said let $\xi_1, \ldots, \xi_k$ be generators of representatives of $\Phi_1, \ldots, \Phi_k$ respectively. We can then state

Theorem. *Let Γ_0 be a finitely generated Fuchsian group of the first kind and let $\Gamma \subset \Gamma_0$ be a subgroup of Γ_0. Let $F_1, F_2, \ldots, F_k \subset \Gamma_0$ be, inequivalent under conjugation, maximal cyclic parabolic subgroups that are a complete set of representatives in the corresponding conjugacy classes (cf. just above). Let ξ_j be a generator of F_j $(j = 1, \ldots, k)$.*

Then Γ is of divergence type if and only if Γ_0 is recurrent with respect to Γ and $\xi_1, \ldots, \xi_k$.

Observe that if Γ is of the first kind then, unless $[\Gamma_0 : \Gamma] < +\infty$, Γ will *never* be finitely generated (cf. [1] where it is implicit in the proof of 14.1).

Deciding in general, whether Γ_0 is recurrent with respect to some Γ and some $\xi_1, \ldots, \xi_k$, is not easy. But in specific situations we can *always* do it!

Here are some examples:

Let Γ_0 be transient (i.e. not recurrent) with respect to some Γ and $\xi_1, \ldots, \xi_k$, then it follows from the general theory [5] that it stays transient if we increase the number of ξ's i.e. it is transient for Γ and $\xi_1, \ldots, \xi_k, \xi_1^*, \ldots, \xi_s^*$ $(\xi_j^* \in \Gamma_0\ j = 1, \ldots, s)$. In particular as soon as the cosset space Γ_0/Γ is transient (i.e. we consider the previous definition with no ξ's and $\mu=\mu_0$) then Γ_0 is transient for Γ and any choice $\xi_1, \dots, \xi_k \in G$.

This remark becomes particularly significant when $\Gamma \lhd \Gamma_0$ is a normal subgroup. Indeed in that case the only known examples of recurrent groups G (that are finitely generated) are the finite extension of $\{e\}$, $\mathbf{Z}$ and $\mathbf{Z}^2$.

When $\Gamma \lhd \Gamma_0$ the problem in fact admits a simple answer:

(i) If Γ_0/Γ is cyclic by finite then Γ is of divergent type.

(ii) If Γ_0/Γ is not cyclic by finite and each parabolic element in Γ_0 generates a finite subgroup in Γ_0/Γ then Γ is of convergent type if and only if Γ_0/Γ is transient.

(iii) If Γ_0/Γ is not cyclic by finite and there exists one parabolic element in Γ_0 that geneiates an infinite subgroup of Γ_0/Γ then Γ is of convergent type.

The proof of these facts which depend on the main Theorem (and also on other things) will appear elsewhere.

I have tried as far as possible to make this paper independent of [5]. But to be honest I think it will be very difficult for the reader to embark on the proofs before looking at [5] first.

Special cases of the above theorem were considered in [5] and also in [10]. In [10] the authors treated the case $M=$ the sphere minus three points and $\Gamma=[\pi_1(M), \pi_1(M)]$ the commutator subgroup of $\pi_1=\Gamma_0$. Their methods are very different from ours.

§1. The Discrete Random Walk

Let G be a discrete group generated by finitely many elements $g_1, \dots, g_m \in G$ and let $H \subset G$ be a subgroup of G. We shall give on G and on the left coset space G/H its canonical left invariant metric $d=d_l$ (cf. [7], [5] §1) (on G we define for $x, y, a \in G$ $d(x, y)=|x^{-1}y|$; $|a|=\inf\{n;\ a=g_{s_1}^{\varepsilon_1}, \dots, g_{s_n}^{\varepsilon_n}, \varepsilon_j=\pm 1\}$ on G/H we define d as the quotient metric of the one defined on G).

Let us also denote by D the union of $k \geqq 1$ disjoint copies of the non-negative integers $\mathrm{N}=\{0, 1, 2, \dots\}$ with all their 0 points identified. We shall denote the points of D by $\{x_i^j;\ i=0, 1, \dots, j=1, \dots, k\}$ and we shall assume that

$$x_0 = x_0^1 = \dots = x_0^k.$$

Let us fix $\xi_1, \xi_2, \dots, \xi_k \in G$ k elements of G, and let us finally denote by $\Theta=D\times G$. We shall also denote by Θ the graph whose set of vertices is Θ and where the following pairs of points are joined by an edge:

(i) Two points $(x_0, g), (x_0, h)$ are joined if and only if $g^{-1}h=g_j^{\varepsilon}$; $\varepsilon=\pm 1$, $j=1, \dots, m$.

(ii) Two points of the form $(x_r^j, g), (x_{r+1}^j, h)$ for some $j=1, 2, \dots, k$ and $r=0, 1, \dots$ are joined if and only if $g^{-1}h=\xi_j^{\varepsilon}$ with $\varepsilon=0, \pm 1$.

No other points are joined by an edge.

The above graph is clearly connected and induces therefore a connected discrete (in the sense of §1 [5]) metric d^* on Θ. We simply set for $\theta_1, \theta_2 \in \Theta$.

(1.1) $d^*(\theta_1, \theta_2) = \inf\{n/\theta_1$ and θ_2 can be connected by n successive edges$\}$.

Observe that G acts on Θ in the obvious way $(g: (x, h) \to (x, gh)$ and the above graph structure is invariant by that action. The quotient space $\Theta/H = D \times (G/H)$ with the induced quotient graph structure and the induced quotient metric can then be defined. I shall denote that quotient metric on Θ/H again by d^*.

A random walk will now be defined on Θ by its transition matrix $P(\theta, \theta')$ $(\theta, \theta' \in \Theta)$. Together with P a measure λ $\{\lambda(\theta) > 0$ $\theta \in \Theta\}$ will be defined on Θ. The exact values of P and λ are irrelevant, what is essential is that they satisfy the following properties:

P_1: Both P and λ are G invariant i.e. $P(g\theta, g\theta') = P(\theta, \theta')$ and $\lambda(g\theta) = \lambda(\theta)$ $\theta, \theta' \in \Theta$ $g \in G$.

P_2: P is symmetric with respect to λ i.e. that $\Phi(\theta, \theta') = \lambda(\theta)P(\theta, \theta')$ is a symmetric function of $\theta, \theta' \in \Theta$.

P_3: P lives only on the graph Θ i.e. $P(\theta, \theta') = 0$ unless (θ, θ') is an edge of Θ, oı $\theta = \theta'$.

P_4: For every edge (θ, θ') of Θ we have $P(\theta, \theta') \geqq \varepsilon_0 > 0$ for some fixed ε_0 independent of (θ, θ').

P_5: λ is essentially constant over Θ i.e.

$$0 < \inf_\theta \lambda(\theta) \leqq \sup_\theta \lambda(\theta) < +\infty.$$

We shall infact impose the much stronger (but not essential) condition that there exist $A, B > 0$ such that $\lambda(x_i^j, g) = A$ or B according to whether $i = 0$ or $i \neq 0$ (respectively).

Let us now prove that the above conditions are compatible by constructing such a walk. Let:

$$N = \mathrm{Card}\{g_i^\varepsilon;\ i = 1, \ldots, m,\ \varepsilon = 0, \pm 1\},$$

$$n_j = \mathrm{Card}\{\xi_j^\varepsilon;\ \varepsilon = 0, \pm 1\};\quad j = 1, 2, \ldots, k,$$

and let us set:

$$P[(x_0, g), (x_0, gg_i^\varepsilon)] = \frac{1}{2N};\quad i = 1, 2, \ldots, m,\quad \varepsilon = \pm 1, 0; \tag{1.2}$$

$$P[(x_0, g), (x_1^j, g\xi_j^\varepsilon)] = \frac{1}{2n_j k};\quad j = 1, \ldots, k,\quad \varepsilon = \pm 1, 0; \tag{1.3}$$

$$P[(x_r, g), (x_{r\pm 1}, g\xi_j^\varepsilon)] = \frac{1}{2n_j};\quad r \geqq 1,\quad j = 1, \ldots, k,\quad \varepsilon = \pm 1, 0; \tag{1.4}$$

P is zero for all other pairs. This is clearly a Markovian matrix. P_1—P_5 are verified as soon as $A/k=B$. By the property P_1 we see that we can define a random walk on Θ/H by:

$$P_H(H\theta_1, H\theta_2) = \sum_{\theta\in H\theta_2} P(\theta_1, \theta)$$

and that this walk is symmetric with respect to the measure $\lambda_H(H\theta)=\lambda(\theta)$ on Θ/H and satisfies the analogues of P_3 and P_4 for the quotient graph.

The problem is to decide whether a walk P_H that satisfies P_2, P_3, P_4 is transient or recurrent.

The point here is that two such walks on G/H that satisfy P_2, P_3, P_4 and have the same symmetrising measure λ_H are both admissible for the metric d^* of Θ/H (in the sense of §4 [5]). The general theory tells us therefore (cf. [5] §4) that if one is transient so is the other. The transience becomes therefore an intrinsic property of Θ/H (and possibly also of the measure λ).

Let us now denote by:

$$\{\theta(n) = (x(n), g(n));\ n \geqq 0\};\ x(n)\in D,\ g(n)\in G$$

the path of our random walk on Θ and let us define successive stopping times: $T_1<T_2<\dots$ by

$$\begin{aligned} T_1 &= \inf\{n>0; \quad x(n)=x_0\}\\ T_2 &= \inf\{n>T_1; \quad x(n)=x_0\}\\ &\dots\\ T_p &= \inf\{n>T_{p-1};\ x(n)=x_0\}. \end{aligned}$$

If we only consider the path for which $x(0)=x_0$ we see that the paths:

$$[\theta(0),\ \theta(T_1),\ \theta(T_2)\ \dots]$$

now define a left translation invariant walk on G, which I shall call P^*.

P^* induces in turn a random walk on the coset space G/H which I shall call P_H^*. The following proposition is then clear:

Proposition. *Let P be a random walk on Θ that satisfies the properties P_1—P_5 then P is transient if and only if P_H^* is transient.*

(Alternatively P as above is transient if and only if P^* returns infinitely often to H.)

The transition probability $P^*(g, h)$ $(g, h\in G)$ is given by some measure $\mu\in\mathbf{P}(G)$ with $P^*(g, h)=\mu(\{g^{-1}h\})$ which can easily be computed.

To be specific let P be the walk given by (1.2), (1.3), (1.4), let us denote $\tilde{\mu}_0$ the measure on G that charges the point g_i^ε by $1/N$ $(\varepsilon=0, \pm 1, i=1, \dots, m)$. Let us also denote by $\tilde{\mu}_j$ the measure on G that charges the points ξ_j^ε $(\varepsilon=0, \pm 1)$ by $1/n_j$ (with the same notations as in (1.2), (1.3), (1.4)). Let us finally consider R the random walk on $\mathbf{N}=\{0, 1, \ldots\}$ given by:

$$R(0,0)=\tfrac{1}{2},\quad R(0,1)=\tfrac{1}{2},\quad R(n,n\pm 1)=\tfrac{1}{2}\quad (n\geqq 1)$$

(and $R(i,j)=0$ for all other values of $i,j\in\mathbf{N}$) and denote by $\{p_n;\, n=0,1,\ldots\}$ the distiibution of its first return time to 0. More explicitly:

$$P_n=\mathrm{Prob}_0\,[\text{first return occurs at time } t=n].$$

It is well known (cf. [8] or just about any other book of probability) that $p_n\sim n^{-3/2}$, Once we have all that, a moments reflexion gives us that the transition measure of P^* is:

$$\mu=p_0\tilde{\mu}_0+\sum_{j=1}^{\infty}p_j\left(\frac{\tilde{\mu}_1^j+\tilde{\mu}_2^j+\ldots+\tilde{\mu}_k^j}{k}\right).$$

Two facts have to be pointed out:

(F_1): μ in (1.5) is symmetric and can be written as

$$\mu=\sum_{j=0}^{k}\lambda_j\mu_j$$

where $\mu_0,\mu_1,\ldots,\mu_k;\ g_1,\ldots,g_m;\ \xi_1,\ldots,\xi_k;\ (\lambda_j>0,\sum\lambda_j=1)$ satisfy conditions (i), (ii), (iii) of the introduction.

The best way to see F_1 is *not* to compute but to use potential theory and estimate the hitting probability of two dimensional random walk on the real axis which is Cauchy. (Think of Brownian motion; if you do not like this then compute!)

(F_2): let $\mu'=\sum_{j=0}^{k}\lambda_j'\mu_j'$ be another measure that satisfies conditions (i), (ii) and (iii) but for different set of generators $g_1',\ldots,g_m'$, and new points

$$\xi_j'=\gamma_j\xi_j\gamma_j^{-1};\ \gamma_j\in G\quad j=1,\ldots,k$$

conjugate to the previous ones. Then the walk defined on G by μ is transient if and only if the walk defined on G by μ' is transient. The easiest way to see this is to observe that for some constant $C>0$ we have

$$\mu\leqq Ce^{-1}e^{\mu'};\ \mu'\leqq Ce^{-1}e^{\mu}$$

and to use the general theory developed in §4 [5].

§2. Metrics on Finite Riemann Surfaces

I shall say that M is a finite Riemann surface if it can be identified conformally with $\hat{M}\diagdown\{z_1,\ldots,z_k\}$ $(z_j\in\hat{M}\ j=1,\ldots,k)$ where $\hat{M}$ is a compact Riemann surface. The points z_j (that are then ideal boundary points of M) will be called the punctures of M. I shall also fix finitely many pairs (u_j,v_j) $j=1,\ldots,s$ $(u_j\in M,$

$v_j=1, 2, 3, \dots$). I shall consider now a class of smooth (say C^3) metrics on $M\setminus\{u_1, \dots, u_s\}=M'$ that I shall call almost flat ramified metrics (A.F.R.M. in short) with respect to $z_1, \dots, z_k$ and $u_1, \dots, u_s$. These will be the Riemannian metrics g on M' that are conformal and for which there exist $A_1, \dots, A_k$ and $B_1, \dots, B_s$ punctured Nhds. of $z_1, \dots, z_k$ and $u_1, \dots, u_s$ respectively (i.e. $A_j \cup z_j$ is a Nhd. of z_j in $\hat{M}$ and $B_j \cup u_j$ is a Nhd. of u_j in M) and for which the metric satisfies the following two conditions:

(i) The metric restricted to A_j $(j=1, \dots, k)$ reduces to the *flat* metric $ds^2= C\dfrac{|dz|^2}{|z|^2}$ of the punctured disc, for some C, and some local conformal coordinate z for which $A_j=\{0<|z|<1\}$.

(ii) The metric restricted to B_j $(j=1, \dots, s)$ in terms of some local conformal coordinate u $\big(B_j=(0<|u|<1)\big)$ takes the form

$$ds^2 = \varphi^2(u)|u|^{-2\left(1-\frac{1}{v_j}\right)}|du|^2 \tag{2.1}$$

for some function φ that extends to a smooth (C^3 say) positive function to $B_j \cup u_j$.

The points u_j $(j=1, \dots, s)$ will be called the ramification points of the metric g, and the v_j will be called their ramification index. The above metrics behave well under branched coverings.

Indeed let $\overline{M} \xrightarrow[p]{} M$ be a branched covering with ramification points only above $u_1, \dots, u_s$ the A.F.R.M. g on M' can then be pulled back to $\bar{g}$, a metric on $\overline{M}'=\overline{M}\setminus p^{-1}\{u_1, \dots, u_s\}$. Near each point $w_0 \in p^{-1}(u_j)$, in terms of a local coordinate w, the metric $\bar{g}$ looks like $\bar{\varphi}^2(w)w^{-2(1-1/\varrho_j)}|dw|^2$ with $\varrho_j=v_j/\mu$, where μ is the ramification index of p at w_0 and $\bar{\varphi}$ satisfies conditions as in (2.1).

It follows in particular that if $\mu=v_j$, for $w_0 \in p^{-1}(u_j)$ $(j=1, \dots, s)$, then the pull pack metric $\bar{g}$ extends to a smooth metric on $\overline{M}' \cup w_0$. An A.F.R.M., pulled back on a branched covering as above, will also be called an A.F.R.M. on $\overline{M}$.

To help your intuition, the way to think of an A.F.R.M. with *no* ramification points, is to take $\hat{M}$ the compact surface and glue infinite straight pipes (flat cylinders) at small circles around the points $z_1, \dots, z_k$. So that M ends up by looking like a sphere, with a number of handles and also a number of infinite straight pipes glued to it.

Let us consider now a general Riemannian manifold M (complete or not) and let $\Xi \subset M$ a discrete subset such that:

$$\begin{aligned} d(\xi, \zeta) &> 1/A; \quad \xi, \zeta \in \Xi, \quad \xi \neq \zeta \\ d(x, \Xi) &< A; \quad \forall x \in M \end{aligned} \tag{2.2}$$

for some large A. Such creatures, of course, exist in abundance, in any manifold. It suffices to choose a subset $\Xi \subset M$ that is maximal under (2.2). I shall call such a subset a grid. A grid Ξ as above, will always be considered as a discrete metric space (Ξ, d), with the metric d it inherits from the Riemannian metric of M.

Let (Δ, d) be a metric space that satisfies the following conditions:

1) for some $\alpha>0$ $d(x, y)\geqq\alpha>0$ $\forall x, y\in\Delta$, $x\neq y$,
2) for all $A>0$ there exist $B>0$ s.t.

$$\operatorname{card}\{y|d(x, y)<A\}<B;\ x\in\Delta.$$

3) there exists $C>0$ such that for all $x, y\in\Delta$ we can find a "chain" $x=x_0, x_1, \ldots, x_j=y\in\Delta$ such that $j\leqq Cd(x, y)$ and $d(x_i, x_{i+1})\leqq C$ $i=0, 1, \ldots, j-1$. We shall say that (Δ, d) is a discrete connected metric space. (Ξ, d) above satisfies 1) and 3) it satisfies 2) if M is "reasonable" cf. [5].

Two discrete connected metrics as above d_1, d_2, on the same space Δ, will be called equivalent if for every $a>0$ we can find some $b>0$ such that

$$d_1(x, y)\leqq a\Rightarrow d_2(x, y)\leqq b$$

$$d_2(x, y)\leqq a\Rightarrow d_1(x, y)\leqq b$$

for all $x, y\in\Delta$. This is equivalent to the existence of some $c>0$ such that:

$$\frac{1}{c}d_2(x, y)\leqq d_1(x, y)\leqq cd_2(x, y);\quad \forall x, y\in\Delta.$$

Let us go back now to M a finite Riemann surface endowed with an A.F.R.M. g (punctured at $z_1, \ldots, z_k$ and ramified at $(u_1, v_1), \ldots, (u_s, v_s)$ as above). A grid $\Delta\subset M$ can then be found in M' that can be identified with D the union of k distinct copies of $\mathbf{N}$ (with the zeros identified) (cf. §1) and the identification is a metric equivalence if we give on D the natural graph metric (cf. (1.1)) on D where the edges of that graph are just the pairs (x_i^j, x_{i+1}^j) $(j=1, \ldots, k;\ i=0, 1, \ldots)$.

The way to obtain such a grid is clear. We fix some point $m_0\in M'$ which we identify with $x_0\in D$, we then choose appropriately, sequences $(m_i^j)_{i=1}^{\infty}$ $(j=1, \ldots, k)$ such that $m_i^j\xrightarrow[i\to\infty]{} z_j$ (topologically in $\hat{M}$) and where we make the steps $d_M(m_i^j, m_{i+1}^j)$ $(1\leqq j\leqq k$ $i\geqq 0)$ all more or less equal and very large. d_M indicates the distance on M' induced by the Riemannian structure g. (Observe that for an A.F.R.M. every ramification point lies at a "finite distance".)

It will be important to pull back grids by $p\colon \overline{M}\to M$ a covering map. ($\overline{M}$ is a covering manifold and we endow $\overline{M}$ with the pull back metric of M.) Let $\Xi\subset M$ be a grid on M, and let us assume that there exists some $\alpha>0$ such that for every closed loop $\ell(t)\in M$ $\ell(0)=\ell(1)\in\Xi$ $(0\leqq t\leqq 1)$ in M, that is not homotopically zero, we have (length of $\ell(t)$ $0\leqq t\leqq 1)\geqq\alpha$. It is then clear that $p^{-1}(\Xi)\subset\overline{M}$ is a grid in $\overline{M}$.

The above condition is clearly verified for the grid $\Delta \cong D$ that we have constructed on a finite Riemann surface M, where M is endoved with an A.F.R.M. ramified at $u_1, \ldots, u_s \in M$, and where p is any branched covering of M ramified only above $u_1, \ldots, u_s$.

§3. Grids Associated to the Fuchsian Group

Let G be a finitely generated Fuchsian group of the first kind acting on the unit disc U. By [2], [11] it follows that the surface $U/G=M$ is a finite surface, say $\hat{M} \setminus \{z_1, \ldots, z_k\}$ ($\hat{M}$ compact), and that the coveiing $p_U\colon U \to M$ is ramified only over finitely many points (u_s, ϱ_s) $u_1, \ldots, u_s \in M$ $\varrho_1, \varrho_2, \ldots, \varrho_s \geqq 2$. Let then $H \subset G$ be a subgroup of G, let $\overline{M}=U/H$ be the intermediate surface, and let

$$U \xrightarrow[\bar{p}]{} \overline{M} \xrightarrow[p]{} M \quad p_U = p \cdot \bar{p}$$

be the factorisation of the universal covering. Observe that if $w \in p^{-1}(u_j)$ $(1 \leqq j \leqq k)$ and if $u \in U$ is such that $\bar{p}(u)=w$ then:

(3.1) (ramification index of $\bar{p}$ at u) $\times$ (ramification index of p at w)$=\varrho_j$.

Let us now fix some A.F.R.M. g on $M'=M \setminus \{u_1, \ldots, u_s\}$ with punctures at $z_1, \ldots, z_k$, ramification points at $u_1, \ldots, u_s$, and corresponding indices $\nu_1, \ldots, \nu_s \geqq 1$.

We shall then lift (as explained in §2) that metric to $\bar{g}$, an A.F.R.M. on $\overline{M}'=p^{-1}(M')$, and g_U, an A.F.R.M. on $U'=p_U^{-1}(M')$.

If we let $\Delta \subset M$ be the grid constructed in §2, then $p^{-1}(\Delta)$ and $p_U^{-1}(\Delta)$ are grids in $\overline{M}'$ and U' respectively.

Our aim will be to describe the above two grids in terms of G and H.

Towards that let us fix $\xi_1, \xi_2, \ldots, \xi_k \in G$ parabolic elements in G that generate maximal (in G) cyclic subgroups which correspond to the punctures $z_1, \ldots, z_k$ in the obvious way (e.g. cf. [1], [3]). Observe that two different ξ_i, ξ_j are not conjugate in G i.e. $\xi_i \neq \varphi^{-1}\xi_j\varphi$ $\varphi \in G$, and a diffeient choice $\xi_1', \xi_2', \ldots, \xi_k'$ of these ξ's would consist of conjugate elements). The exact choice of ξ_j among all its conjugates will be done later.

Using then $G, \xi_1, \ldots, \xi_k \in G$, and some fixed set of generators $g_1, \ldots, g_m \in G$, I shall construct the discrete metric space $\Theta=D \times G$ with its graph metiic d^* (a different choice of the generators g_j would give me an equivalent metric). I shall also consider the quotient space $\Theta/H=D \times (G/H)$ with the quotient metric d^*.

We then have:

Proposition. (i) *$p_U^{-1}(\Delta) \subset U'$ can be identified up to metric equivalence with Θ.* (ii) *$p^{-1}(\Delta) \subset \overline{M}'$ can be identified up to metric equivalence with Θ/H.*

It is enough to prove (i) since (ii) then follows by taking the quotient by H.

The strategy to prove (i) is the following:

(a) I shall find Q a Borel fundamental domain of $U' \to M'$ (which will in fact have a piecewise smooth boundary) and an identification $I\colon \Theta \to p_U^{-1}(\Delta)$ that commutes with the action of G and such that $I(D\times\{e\})=p_U^{-1}(\Delta)\cap Q$ (e is the neutral element of G). I shall even make sure (just to be on the safe side!) that

$$d\{[p_U^{-1}(\Delta)\cap Q],\ CQ\} > 0. \tag{3.2}$$

I shall then prove:

(b): $\exists A>0$ with the property: $\theta\in D\times\{e\}, \varphi\in\Theta$ $d^*(\theta,\varphi)\leqq 1$ implies $d_U(I(\theta), I(\varphi))\leqq A$: d_U denotes here the distance induced by g_U.

(c): $\forall B>0\, \exists C>0$ with the property: $X\in p_U^{-1}(\Delta)\cap Q$, $Y\in p_U^{-1}(\Delta)$ and $d_U(X, Y)\leqq B$ implies that $d^*(J(X), J(Y))\leqq C$ where $J=I^{-1}$.

If we use the group action G on U' (which now acts freely) and a moments reflection we see that the proposition follows from (a), (b) and (c).

Step (i). Let us start with $\hat{M}$ where $M=\hat{M}\setminus\{z_1, \ldots, z_k\}$ and let us introduce a *finite* number of cuts on $\hat{M}$ so as to make it simply connected. Each cut consists of simple smooth arcs of the form $l(t)$ $(0\leqq t\leqq 1)$ [possibly $l(0)=l(1)$ but no other double points]. We require that the points $z_1, \ldots, z_k, u_1, \ldots, u_s$ lie in the union of these cuts. We require further that each z_i lies in only *one* cut, say $l_i(t)$, with $l_i(0)\neq l_i(1)=z_i$, and l_i is a geodisc for the A.F.R.M. on M'; that cut, observe, will have to look eventually like a *generator* of the *flat* cylinder that is the Nhd. of the ideal point z_i in M.

The above can be done in many ways.

We obtain then Q by lifting (by monodromy) the set $M\setminus$\{all the cuts\} and adjoining to that open simply connected domain a number of its boundary curves. I shall also make sure that $d(\Delta$, union of cuts$)>0$. Having done that I shall identify D with Δ and Δ with $p_U^{-1}(\Delta)\cap Q$. Let $I\colon \{e\}\times D\to p_U^{-1}(\Delta)\cap Q$ be the composed identification. The group action of G on Θ and on $p_U^{-1}(\Delta)$ induces then an identification of Θ with $p_U^{-1}(\Delta)$ [by $I(g, x)=gI(e, x)\in U$].

Step (ii). It is here that a more specific choice of the ξ_j's (and of the metric) helps. We shall choose punctured Nhds. A_j of z_j in $\hat{M}$ and a local coordinate z in A_j such that $A_j=\{0<|z|<1\}$ and such that the metric g becomes $C\dfrac{|dz|^2}{|z|^2}$. We shall also choose some $\xi_j\in G$ that corresponds to z_j and is such that $\xi_j=\xi_j(1)$, where $\xi_j(t)$ is a one parameter group of parabolic transformations on U that induces the transformations $p[\xi_j(t)\tilde{p}(z)]=e^{2\pi i t}z$ for $z\in A_j$ and $\tilde{p}(z)=p^{-1}(z)\cap Q$.

To prove (b) observe first that we may as well assume that $\theta=(x_i^j, e)$ with i very large, φ can only be then $(x_{i\pm 1}^j, \xi_j^\varepsilon)\varepsilon=0, \pm 1$. By what has just been said the curve $\gamma=\{\xi_j(t)I(\theta);\ -1\leqq t\leqq 1\}$ passes through $I(\theta)$ and its length is clearly bounded by a constant.

The above specific choice of the ξ_j's is by no means essential, but it does clarify the picture. Another way to get a clear picture is to start with a fundamental domain Q_0 that is geometrically rigid (e.g. the Poincaré fundamental domain). In [9] §2 one can find a description of the above geometry.

Step (iii). Let $B>0$ be given. I distinguish two cases:

Case 1. $J(X)=(x_i^j, e)$ with $j=1, \ldots, k$ and $i\leqq i_0$ for some large i_0 to be specified later. Finitely many points have that property and so only finitely many choices of Y exist, by (3.2), that satisfy $d_U(X, Y)\leqq B$ [observe that away from the ramification points the volumes of balls of radius r for the A.F.R.M stay bounded above and below by positive numbers only depending on r]. From this it follows that some $C=C(B, i_0)$ can be found that satisfies condition (c).

Case 2. $J(X)=(x_i^j, e)$ with $j=1, \ldots, k$ and $i>i_0$ with i_0 as in Case 1: let $Y\in p_U^{-1}(\Delta)$ and let $J(Y)=(x_{i'}^{j'}, g)$ let us join X with Y in U' with a curve γ of g_U-length less then $2B$. The curve $p_U(\gamma)$ is then a curve in M' of length $\leqq 2B$ joining m_i^j to $m_{i'}^{j'}$ (points of the grid Δ).

If i_0 is large enough this forces $j=j'$ and $|i-i'|\leqq D$ (D depending only on B). Furthermore the curve $p_U(\gamma)$ will then lie entirely on the flat cylinder of M corresponding to the ideal boundary point z_j, and the number of times that this curve can go round that cylinder has to be bounded by a fixed multiple of B. This forces g to be ξ_j^k with k bounded by a fixed multiple of B.

The upshot is that Y can only be of the form $(x_{i'}^j, \xi_j^k)$ with

$$|i-i'|\leqq D \quad \text{and} \quad |k|\leqq D$$

(D only depending on B). The proof is complete.

§4. Proof of the Theorem

All the notations of the previous section will be preserved. I shall refer to formula (3.1) and consider first a special case. I shall assume namely that for every $j=1, 2, \ldots, s$ there exists some $\mu_j\geqq 1$ such that

(4.1) [ramification index of p at each $w\in p^{-1}(u_j)]=\mu_j$.

This is certainly the case if the converning p_U is unramified (i.e. if G already acts freely on U) or if H is normal in G (for then G/H acts transitively on $p^{-1}(u_j)$).

Proof of the special case. I shall specialise here $v_j=\mu_j$ $(j=1, \dots, k)$ (v_j being the remification indices of the A.F.R.M on M. considered in §3). By what has been said in §2 the metric $\bar{g}$ extends then to a smooth metric on $\overline{M}$. It is an easy matter to verify that this extended metric is complete and its curvature is bounded from above and below, and also that $p^{-1}(\Delta)\subset\overline{M}$ is also a grid for $(\overline{M}, \bar{g})$. [Cf. Appendix.] To be able to apply the general theory of [5] (especially §5) I shall need to make one final construction.

I shall decompose M into disjoint Borel subsets M_0, M_i^j $j=1, \dots, k$ $i=1, \dots$ such that with the identification of D with Δ we have for some appropriate $r, R>0$:

$$B_r(x_i^j) \subset M_i^{j\prime} \subset B_R(x_i^j); \; j = 1, \dots, k, \quad i = 0, 1, \dots, \tag{4.2}$$

where I denote $M_0^1=M_0^2=\dots=M_0^k=M_0$ and $B_\alpha(x)$ in M' is the metric ball centered at $x\in M'$ and radius α, $M_i^{j\prime}$ denotes $M_i^j\cap M'$.

It is clear that I can also make the choice of Δ and the decomposition such that:

(i) Each M_i^j $j=1, \dots, k$; $i\geqq 1$ lies entirely in the flat cylinder that corresponds to z_j.

(ii) $\mathrm{Vol}_M(M_i^j)=B$ $i\geqq 1$ $j=1, 2, \dots, k$ i.e. that the Riemannian volume is independent of i and j.

(iii) $\mathrm{Vol}\,(M_0')=A$ and $A/B=k$, $(M_0'=M'\cap M_0)$.

Only (ii) and (iii) are essential here. I only put down (i) to help you see how we get (ii).

I can then consider a decomposition of the fundamental domain $Q\subset U'$ into

$$Q_i^j = Q\cap p_{\bar{U}}^{-1}(M_i^{j\prime}) \quad j = 1, \dots, k; \; i = 0, 1, \dots \tag{4.3}$$

$(Q_0^1=Q_0^2=\dots=Q_0)$.

It is clear then that a decomposition of the whole of U' can be thus obtained by acting on the Q_i^j's by G. The sets of that decomposition are gQ_i^j $(i\geqq 0$, $1\leqq j\leqq k$, $g\in G)$. By identifying gQ_i^j with hgQ_i^j for $h\in H$ $g\in G$ $i\geqq 0, 1\leqq j\leqq k$ we obtain a disjoint Borel decomposition of $\overline{M}'$ into sets that I shall denote abusively by:

$$\dot{g}Q_i^j; \; \dot{g}\in G/H \quad i \geqq 0, \quad j = 1, 2, \dots, k. \tag{4.4}$$

It is clear that with the identification of $\Theta/H=D\times(G/H)$ with $p^{-1}(\Delta)$ we have:

$$B_r((x_j^i, \dot{g})) \subset \dot{g}Q_i^j \subset B_R((x_j^i, \dot{g})) \tag{4.5}$$

for $R, r>0$ as in (4.2) and all i, j and $\dot{g}$. It is just as clear that $\mathrm{Vol}_{\overline{M}}(\dot{g}Q_i^j)=B$, $i\geqq 0$; $\mathrm{Vol}_{\overline{M}}(\dot{g}Q_0)=A$ where A and B are as in (ii) and (iii). To obtain from this a decomposition of $\overline{M}$ we have to accommodate somewhere the points $p^{-1}\{u_1, \dots, u_s\}$.

But by condition (i), that now comes handy, we see that all the u_i can be made to lie in M_0. So we shall adjoin to $Q_0^1=Q_0^2=\ldots=Q_0^k$ the points $p^{-1}\{u_1,\ldots,u_s\}\cap Q$. The rest is done by the group action as before!

We finally see that (4.4) supplies us with a decomposition of $\overline{M}$ which together with the grid $p^{-1}(\Delta)$ satisfies all the conditions of the main theorem in [5] §5. [Cf. Appendix.]

That main theorem in [5] and the proposition of §1 in this paper completes the proof in the special case that we have considered.

The point where the above proof breaks down, when (4.1) fails, is that we cannot extend the Riemannian metric $\bar{g}$ from $\overline{M}\setminus p^{-1}\{u_1, \ldots, u_s\}$ to the whole of $\overline{M}$.

The decomposition of M_i^j that satisfies (4.1) (i), (ii), (iii) and the decompositions (4.3) and (4.4) can be made however as before. This time it will really be essential to have all the ramification points $u_1, \ldots, u_s$ well in the interior of M_0; their preimages on $\overline{M}$ will then be well in the interior of $\cup\{\dot{g}Q_0|\dot{g}\in G/H\}$. The only singularities of the metric $\bar{g}$ lies on these preimages, and this, for any choice of $v_1, v_2, \ldots, v_s\geqq 1$. I shall modify the metric $\bar{g}$ at small disjoint punctured neighbourhoods of the points of $p^{-1}\{u_1, \ldots, u_s\}$ so as to make the metric $\bar{g}$ smoothly extendable to $\overline{M}$.

To do that, we consider $(g_\alpha)_{\alpha\in A}$ $|A|=\varrho^s$ A.F.R.M.'s on M' obtained by the ϱ^s choices $1\leqq v_j\leqq\varrho=\max\varrho_j$ and make sure that all these metrics coincide outside very small disjoint Nhds. of the u_j's. It is clear then that with an appropriate partion of unity on $\overline{M}'$ we can obtain a metric $\bar{g}=\sum_{\alpha\in A}\lambda_\alpha\bar{g}_\alpha$ ($\bar{g}_\alpha$ is the pull back of g_α on $\overline{M}'$) that extends smoothly on $\overline{M}$ and that $\bar{g}$ coincides with each $\bar{g}_\alpha$ outside p^{-1} (union of small Nhds. of the u_j's).

It is an easy, if a little tedious, matter to verify that the λ's can be chosen so that the curvature of $\bar{g}$ is bounded. This simply involves the $\nabla^\alpha\lambda_\beta$ and $\mathrm{Hess}^\alpha(\lambda_\beta)$ (∇^α=covariant derivative w.r.t. $\bar{g}_\alpha$ and Hess^α=Hessian w.r.t. $\bar{g}_\alpha$).

It is also easy to see that the λ's can be chosen so that $p^{-1}(\Delta)$ remains a grid identifiable up to metric equivalence with Θ/H, and that (4.5) will still hold for some $R, r>0$. Indeed it is a matter of observing that in everything that we have done, when we considered the distance of two points $x, y\in\overline{M}'$, we could have considered the length of some curve that joined x to y and stayed away from the points $p^{-1}\{u_1, \ldots, u_s\}$.

The property $\mathrm{Vol}_{\bar{g}}(\dot{\gamma}Q_i^j)=B=\mathrm{Vol}_{\bar{g}_\alpha}(\dot{\gamma}Q_i^j)$ $i\geqq 1$ is clear by the definition, and what is also clear is that we can also guarantee that:

$$0<A_1=\inf_\gamma\,[\mathrm{Vol}_{\bar{g}}(\dot{\gamma}Q_0)]\leqq\sup_\gamma\,[\mathrm{Vol}_{\bar{g}}(\dot{\gamma}Q_0)]\leqq A_2<+\infty.$$

A further modification of $\bar{g}$ in each of the disjoined open sets $p^{-1}(\Omega)$ where $\Omega\subset M_0$ is a very small open set away from the previous Nhds. and from everything else will guarantee then that:

$$\mathrm{Vol}_{\bar{g}_{\text{modified}}}[\hat{\gamma}Q_0] = A.$$

By changing B if necessary we can also have $A/B=k$. [The way to change B the last minute is to give to $M_0 \subset M$ small pieces from the flat cylinders, and change the size of the slices M_i^j of the flat cylinders.]

After we have done all that we are back to a situation where the Theorem in §5 [5] applies.

The proof is completed as before.

Remark. Another way to finish this last proof is to go back to P_5 in §1 and relax the condition that $\lambda(\theta)$ takes only two values.

Appendix to §4

We preserve all the notations of §3 and §4.

For every $u_j \in M$ (ramification point of p_U with index ϱ_j) we can find B_j some Nhd. of u_j in M that admits a local coordinate u such that $u(u_j)=0$ $B_j=(|u|<1)$ and such that it has the following property:

Every connected component $\bar{B}$ of $p^{-1}(B_j) \subset \bar{M}$ intersects $p^{-1}(u_j)$ at only one point, $\bar{m}$ say, and there exists some local coordinate z of $\bar{B}$ such that $z(\bar{m})=0$ $\bar{B}=(|z|<1)$ and such that $u(p(\beta))=(z(\beta))^\mu$ $(\beta \in \bar{B})$ where μ is the ramification index of p at $\bar{m}$.

The above is clear in the special case $H=G, \bar{M}=U, p=p_U$. Indeed if we first fix some $\bar{m} \in p_U^{-1}(u_j)$ then it is clear that we can do the above with $B_j=p(\bar{B})$ where $\bar{B}$ is some appropriate Nhd. of $\bar{m}$. But then the group action of G on U allows us to go to any other point of $p_U^{-1}(u_j)$. From this special case the general case follows at once since for any $\bar{m} \in p_U^{-1}(u_j)$ $H \cap$(stabiliser of $\bar{m}$) is a finite cyclic group.

Let us assume that g is an A.F.R.M. on M such that $\bar{g}$, the pulled back metric on $\bar{B} \setminus \bar{m}$, extends smoothly to $\bar{m}$ (with the notations just above). The metric $\bar{g}$, with the above coordinates on $\bar{B}$, looks like $\varphi^2(z^\mu)|dz|^2$ with $\varphi=\varphi_j$ $(j=1, \dots, s)$ as in (2.1). The above φ only depends on j and *not* on $\bar{m}$. It follows that the curvature $K(\bar{M})$ and the injectivity radius $i(\bar{M})$ of $\bar{g}$ on the set $(|z|< 1/2) \subset \bar{B}$ are both controlled by constants that only depend on $j=1, \dots, s$ and on μ.

Let us observe also that *outside* the set $p^{-1}(\Omega)$ (where Ω is some fixed Nhd. of $u_1, \dots, u_s$ in M) we can control $K(\bar{M})$ and $i(\bar{M})$ by the curvature and the injectivity radius of g on $M \setminus \Omega$.

From the above two facts we finally conclude that when $\bar{g}$ extends smoothly on $\bar{M}$ a global uniform control of $K(\bar{M})$ and $i(\bar{M})$ can be obtained.

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