Riemann's zeta-function and the divisor problem

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1. Introduction and statement of the results

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It was found by Atkinson (see [1], [2]) that the mean value problem for $\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$ has "more than superficial affinities" with the classical Dirichlet divisor problem.

Let

$$I(T) = \int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt,$$

$$D(x) = \sum_{n\leq x} d(n),$$

where $d(n)$ denotes the number of positive divisors of n. In [1] Atkinson reproved Ingham's result, viz.

$$I(T) = T\log(T/2\pi)+(2\gamma-1)T+E(T) \tag{1.1}$$

with

$$E(T) \ll T^{1/2+\varepsilon}, \tag{1.2}$$

via the formula

$$I(T) = 2\pi D(T/2\pi)+O(T^{1/2+\varepsilon}).$$

(Atkinson's proof is also presented in Titchmarsh [11], pp. 120—122.) An application of Dirichlet's formula

$$D(x) = x\log x+(2\gamma-1)x+\Delta(x) \tag{1.3}$$

with the elementary estimate $\Delta(x) \ll x^{1/2}$ yields now (1.1) — (1.2). In (1.1) and (1.3) γ denotes Euler's constant.

The error term $\Delta(x)$ in (1.3) can be given by a formula of Voronoi, which in a truncated form reads as follows (see Titchmarsh [11], eq. (12.4.4)): for $1 \leqq N \ll x$

$$(1.4) \quad \Delta(x) = (\pi\sqrt{2})^{-1} x^{1/4} \sum_{n \leqq N} d(n) n^{-3/4} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon} N^{-1/2}).$$

In [2] Atkinson proved an analogous result for $E(T)$. Let

$$(1.5) \qquad e(T, n) = \left(1 + \frac{\pi n}{2T}\right)^{-1/4} \left\{\left(\frac{2T}{\pi n}\right)^{1/2} \operatorname{ar\,sinh}\left(\left(\frac{\pi n}{2T}\right)^{1/2}\right)\right\}^{-1},$$

$$(1.6) \qquad f(T, n) = 2T \operatorname{ar\,sinh}\left(\left(\frac{\pi n}{2T}\right)^{1/2}\right) + (\pi^2 n^2 + 2\pi n T)^{1/2} - \pi/4,$$

$$(1.7) \qquad g(T, n) = T \log\left(\frac{T}{2\pi n}\right) - T + \pi/4.$$

Atkinson's formula states that

$$(1.8) \qquad E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T),$$

where

$$(1.9) \qquad \Sigma_1(T) = \sqrt{2}(T/2\pi)^{1/4} \sum_{n \leqq N} (-1)^n d(n) n^{-3/4} e(T, n) \cos(f(T, n)),$$

$$(1.10) \qquad \Sigma_2(T) = -2 \sum_{n \leqq N'} d(n) n^{-1/2} \left(\log\left(\frac{T}{2\pi n}\right)\right)^{-1} \cos(g(T, n)),$$

$$(1.11) \qquad AT \leqq N \leqq A'T,$$

$$(1.12) \qquad N' = N'(T, N) = T/2\pi + N/2 - (N^2/4 + NT/2\pi)^{1/2};$$

in (1.11) A and A' are any fixed positive constants with $A < A'$. (In the formulation of the main theorem in [2] the sign in front of the second sum should be $+$, as can be seen by (4.4), (9.1) and Lemma 3.)

By (1.5) and (1.6) we have

$$(1.13) \qquad e(T, n) = 1 + O(nT^{-1}),$$

$$(1.14) \qquad f(T, n) = -\pi/4 + 4\pi(nT/2\pi)^{1/2} + O(n^{3/2} T^{-1/2})$$

for $n \leqq N$. Now a comparison of (1.9) and (1.4) reveals a similarity between $\Sigma_1(T)$ and $2\pi\Delta(T/2\pi)$, apart from the oscillating sign $(-1)^n$ in (1.9), for the first $o(T^{1/3})$ terms in the respective sums are asymptotically equal in absolute value. In applications the effect of the sum $\Sigma_2(T)$ can usually be eliminated by a smoothing process, so that in a certain sense there is also an analogy between $E(T)$ and $2\pi\Delta(T/2\pi)$.

It is somewhat surprising that Atkinson's formula has found concrete applications only recently, in two interesting papers of Heath-Brown ([7], [8]). However, in future this formula will probably play a more central rôle in the theory of the zeta-function. Therefore it would be desireable to have a more flexible version at disposal, with the condition (1.11) replaced e.g. by the condition $1 \leqq N \ll T^2$, in analogy with the approximate functional equation for $\zeta^2(s)$. This generalization seems to be possible, if the error term in (1.8) is allowed to depend on N.

The estimate (1.2) for $E(T)$ was sharpened by Titchmarsh, who was able to replace $\frac{1}{2}$ by $\frac{5}{12}$ in the exponent. But in view of the above mentioned analogy, one would expect that if θ_1, θ_2 are the least constants such that for all $x \geqq 2$, $T \geqq 2$ and any fixed $\varepsilon > 0$.

$$\Delta(x) \ll x^{\theta_1+\varepsilon}, \tag{1.15}$$

$$E(T) \ll T^{\theta_2+\varepsilon}, \tag{1.16}$$

then $\theta_1 = \theta_2$; also there are reasons to conjecture that $\theta_1 = \theta_2 = 1/4$. It is not known whether or not $\theta_1 = \theta_2$, but anyway the best known estimates for these constants are equal. This result was first obtained by Balasubramanian [3], by using the Riemann—Siegel formula and very complicated techniques. Another proof was given by Heath-Brown [6]. But it seems to have remained unobserved that this result can be deduced fairly easily from Atkinson's formula. We shall sketch the argument in the end of the paper.

The best known estimate for θ_1 and θ_2 is $346/1067 = 0.3242 \ldots$, due to Kolesnik [9]. It follows from (1.16) that

$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \ll t^{\theta_2/2+\varepsilon} \quad \text{for} \quad t \geqq 2, \tag{1.17}$$

for by Lemma 3 of Heath-Brown [7] we have

$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 \ll (\log t)\left(1+\int_{-\log^2 t}^{\log^2 t} e^{-u}\left|\zeta\left(\tfrac{1}{2}+i(t+u)\right)\right|^2 du\right). \tag{1.18}$$

For the time being nothing better than (1.17) is known. This argument has, however, a theoretical limit, for it is known that $\theta_2 \geqq 1/4$ (see Good [5], or Heath-Brown [8]); this is an analogue of Hardy's theorem $\theta_1 \geqq 1/4$.

In attempts to analyze or utilize the analogy between $\Sigma_1(T)$ and $2\pi\Delta(T/2\pi)$, one faces two difficulties: firstly, there are the extra factors $(-1)^n$ in (1.9), and secondly, the terms with $n \gg T^{1/3}$ in the respective sums are no more comparable in absolute value.

The first obstacle can be removed by going over from $D(x)$ to the function

$$D^*(x) = -D(x) + 2D(2x) - \tfrac{1}{2}D(4x), \tag{1.19}$$

for which we have by (1.3) the asymptotic formula

$$D^*(x) = x\log x + (2\gamma - 1)x + \Delta^*(x), \tag{1.20}$$

$$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \tfrac{1}{2}\Delta(4x). \tag{1.21}$$

It turns out that $2\pi\Delta^*(T/2\pi)$ is the "right" analogue of $\Sigma_1(T)$, for with $1 \leqq N \ll x$ we have

(1.22)
$$\Delta^*(x) = (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n\leqq N}(-1)^n d(n)n^{-3/4}\cos(4\pi\sqrt{nx}-\pi/4)+O(x^{1/2+\varepsilon}N^{-1/2}).$$

It may be of interest to point out that the function $D^*(x)$ can also be interpreted as the sum function of a certain arithmetic function $d^*(n)$. Namely, by elementary considerations it is easy to verify that

(1.23)
$$D^*(x) = \sum_{n\leqq 4x} d^*(n),$$
where

(1.24)
$$d^*(n) = \sum_{4hk=n} 1 - \tfrac{1}{2}\sum_{hk=n}(-1)^{h+k}$$

(see the remark in the end of section 2). Observe that $d^*(n)$ takes both positive and negative values.

The other difficulty mentioned above was how to treat the tails of the sums in (1.9) and (1.22). By a smoothing process, similar to that in Heath-Brown [7], we first truncate these sums, and take then mean squares of the truncated sums. By a lemma of Gallagher, the mean squares are expressed in terms of coefficient sums over short intervals. The absolute values of these short sums are still comparable with each other, though a termwise comparison is no more useful. This argument leads to a relation between integrals of $\left|\zeta\left(\frac{1}{2}+it\right)\right|^2$ over short intervals, and sums of $d^*(n)$ (or $d(n)$) over other short intervals. This correspondence with its consequences is the main topic of this paper.

Because of the square roots in (1.6) and (1.22), it is convenient to work with the functions

(1.25)
$$\Delta_0^*(y) = 2\pi\Delta^*(y^2/2\pi),$$

(1.26)
$$E_0(y) = E(y^2).$$

We introduce some notations. Let T be a large positive number, $\tau = T^{1/2}$, $L = \log T$. Let G be a parameter satisfying, for some constants $1/2 > a > b > 0$, the inequality

(1.27)
$$T^{-a} \leqq G \leqq T^{-b}$$

and write $H = GL$. The main result will be stated in terms of the smoothed functions

(1.28)
$$\Delta_1^*(x) = G^{-1}\int_{-H}^{H}\Delta_0^*(x+u)e^{-(u/G)^2}du,$$

(1.29)
$$E_1(x) = G^{-1}\int_{-H}^{H}E_0(x+u)e^{-(u/G)^2}du,$$

and the corollaries are formulated for "natural" functions $E(t)$, $\Delta^*(x)$, and $\left|\zeta\left(\frac{1}{2}+it\right)\right|$, which we are in the first place interested in.

Theorem. *Suppose that G satisfies* (1.27) with $b=1/6$, *and write* $K=T^{-1/2}G^{-2}$. *Then for* $0<\xi\ll G$ *and any fixed* $\varepsilon>0$ *we have*

(1.30)
$$\int_{-K}^{K}(E_1(\tau+y+\xi)-E_1(\tau+y))^2dy$$
$$\ll T^{\varepsilon}\int_{-\tau/3}^{\tau/3}(\Delta_1^*(\tau+y+\xi)-\Delta_1^*(\tau+y))^2\frac{K^2}{K^2+y^2}dy+T^{\varepsilon}(T^{-5}G^{-17}+T^{-1/2}G^{-2}).$$

Corollary 1. *Let* $T^{1/20}\ll X\ll T^{1/3}$, $Y=TX^{-2}$, *and let* $A_1, \ldots, A_R$ *be non-overlapping subintervals of lenght* $A\gg X$ *of the interval* $[T-Y,\ T+Y]$ *such that*

(1.31)
$$\sup_{t_1,t_2\in A_i}|E(t_1)-E(t_2)|\geqq U\gg X,\quad i=1,\ldots,R.$$

Then

(1.32)
$$R\ll AU^{-2}T^{2+\varepsilon}X^{-6}\sup_{0<\eta\leqq X}\ \sup_{Y\leqq Z\leqq T/8}Z^{-2}\int_{T/2\pi-Z}^{T/2\pi+Z}(\Delta^*(x+\eta)-\Delta^*(x))^2dx$$
$$+AU^{-2}T^{\varepsilon}(T^4X^{-19}+TX^{-4}).$$

Note that for $t_1<t_2$ the difference $|E(t_1)-E(t_2)|$ indicates how much the integral

$$\int_{t_1}^{t_2}\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2dt$$

deviates from its expected value. Hence Corollary 1 gives information on both large and small values of such integrals.

Let us consider large values of $\left|\zeta\left(\frac{1}{2}+it\right)\right|$. If for some $t\in\left[\frac{1}{2}T,\ 2T\right]$

(1.33)
$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right|\geqq V\gg T^{\varepsilon},$$

then by (1.18) and (1.1)

(1.34)
$$E(t+L^2)-E(t-L^2)\gg V^2L^{-1}.$$

An application of Corollary 1 gives the following result.

Corollary 2. *Suppose that* $V\gg T^{1/20}$, *and let* $t_1, \ldots, t_R$ *be a set of points in the interval* $[T-TV^{-4},\ T+TV^{-4}]$ *such that* (1.33) *holds for each* t_i, *and* $|t_i-t_j|\geqq 1$ *for* $i\neq j$. *Then*

(1.35)
$$R\ll T^{2+\varepsilon}V^{-14}\sup_{0<\eta\leqq V^2}\ \sup_{TV^{-4}\leqq Z\leqq T/8}Z^{-2}\int_{T/2\pi-Z}^{T/2\pi+Z}(\Delta^*(x+\eta)-\Delta^*(x))^2dx+T^{4+\varepsilon}V^{-40}.$$

The truth of the famous conjecture

$$\Delta(x) \ll x^{1/4+\varepsilon} \tag{1.36}$$

would imply the same estimate for $\Delta^*(x)$ by (1.21). On this conjecture, the first term on the right of (1.35) is $\ll T^{3/2+\varepsilon}V^{-10}$. Then for $V=T^{3/20+\varepsilon}$ the inequality (1.35) is impossible even for $R=1$. Thus we obtain a conditional improvement of the estimate of $\left|\zeta\left(\frac{1}{2}+it\right)\right|$. The hypothesis (1.36) would also imply a new estimate for $E(t)$.

Corollary 3. *If the conjecture* (1.36) *is true, then for all* $t \geqq 2$

$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \ll t^{3/20+\varepsilon}, \tag{1.37}$$

$$|E(t)| \ll t^{5/16+\varepsilon}. \tag{1.38}$$

For comparison, the respective best known exponents are $173/1067=0.1621\,\ldots$, $346/1067=0.3242\,\ldots$, as was mentioned earlier, while $3/20=0.15$, $5/16=0.3125$. Thus the improvements would not be very striking, but it is of some interest in principle that such implications exist. Also it should be noted that we do not need the full power of the conjecture (1.36) at all, for (1.37) follows already from the local estimate

$$\Delta(x+y)-\Delta(x) \ll x^{1/4+\varepsilon}, \quad 0<y\ll x^{3/10}, \tag{1.39}$$

and the consequence of this is, beyond (1.37), that

$$\int_T^{T+X} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt \sim X \log T \text{ for } T^{3/20+\varepsilon} \leqq X \leqq T.$$

This does not follow e.g. from the Lindelöf hypothesis.

Though the hypothesis (1.39) may seem to be essentially easier than (1.36), both problems are nevertheless of equal difficulty at the present state of knowledge, for we cannot estimate non-trivially the change of $\Delta(x)$ in an interval $[x, x+x_0]$ which is shorter than the best known estimate of $\Delta(x)$. However, the mean value estimate

$$\int_1^X (\Delta(x+y)-\Delta(x))^2 dx \ll Xy \log^3 X, \quad 1 \leqq y \leqq X^{1/2}, \tag{1.40}$$

suggests that perhaps

$$\Delta(x+y)-\Delta(x) \ll y^{1/2}x^{\varepsilon}.$$

This would give the exponent $1/8+\varepsilon$ in (1.37). The estimate (1.40) is an analogue of a result of Good [5] concerning differences of the function $E(t)$. In fact Good has an asymptotic formula with an error term, and a similar refinement is possible also in (1.40). The proof can be based on Voronoi's formula (1.4).

In conclusion we point out that if $E(t)$ changes rapidly near T, e.g. because $\left|\zeta\left(\frac{1}{2}+iT\right)\right|$ is very large, then $\Delta^*(x)$ also necessarily changes rapidly near $T/2\pi$. This means that $d(n)$ behaves in an anomalous way near at least one of the numbers $T/2\pi$, T/π and $2T/\pi$. We do not formulate a quantitative result in this direction. Qualitatively it can be said that if $\left|\zeta\left(\frac{1}{2}+iT\right)\right|$ is very large, i.e. of the order T^α with α near 1/6, then there exists an interval $[T/2\pi-T^\beta,\ T/2\pi+T^\beta]$ with β slightly larger than 1/3, where the oscillation of $\Delta^*(x)$ exceeds T^γ, with γ slightly less than 1/3.

2. The analogue of Voronoi's formula for $\Delta^*(x)$

We are going to prove the expression (1.22) for $\Delta^*(x)$ by modifying suitably the proof of Voronoi's formula (1.4) in Titchmarsh [11], § 12.4.

Let $N \ll x$ be a positive integer, $N+\frac{1}{2}=T^2(4\pi^2x)^{-1}$, $\alpha=1+1/\log x$. We shall evaluate the integral

$$I=\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}(1-2^{s-1})^2\zeta^2(s)x^s s^{-1}ds$$

in two ways.

First, by Perron's formula,

$$I=\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\zeta^2(s)\,s^{-1}\left(x^s-(2x)^s+\frac{1}{4}(4x)^s\right)ds \tag{2.1}$$

$$=D(x)-D(2x)+\frac{1}{4}D(4x)+O(x^{1+\varepsilon}T^{-1}).$$

The error term can be written also as $O(x^{1/2+\varepsilon}N^{-1/2})$.

Secondly, we apply the theorem of residues to the rectangle with the vertices $\alpha\pm iT$, $\beta\pm iT$, where $\beta=-1/\log x$. Note that the integrand is regular at $s=1$, and that its residue at $s=0$ is $O(1)$. The integrals over the horizontal sides can be estimated by the error term in (2.1). Thus we obtain

$$I=\frac{1}{2\pi i}\int_{\beta-iT}^{\beta+iT}(1-2^{s-1})^2\zeta^2(s)x^s s^{-1}ds+O(x^{1/2+\varepsilon}N^{-1/2}).$$

Next we use the functional equation $\zeta(s)=\chi(s)\zeta(1-s)$, and observe that for $\mathrm{Re}\,s<0$

$$(1-2^{s-1})^2\zeta^2(1-s)=\sum_{\substack{n=1\\2\nmid n}}^{\infty} d(n)n^{s-1},$$

by considering the Euler product of the function on the left. This is substituted in the preceding integral, and the series is integrated termwise. The situation is now the same as in the proof of Voronoi's formula, except that we have the extra condition $2\nmid n$. Hence we get, in analogue with (1.4),

$$I = (\pi\sqrt{2})^{-1} \sum_{\substack{n\le N \\ 2\nmid n}} d(n) n^{-3/4} \cos(4\pi\sqrt{nx}-\pi/4) + O(x^{1/2+\varepsilon} N^{-1/2}).$$

This and (1.4) show together that $\Delta(x)-2I$ equals the right hand side of (1.22). On the other hand, by (2.1), (1.3) and (1.21)

$$\Delta(x)-2I = \Delta^*(x) + O(x^{1/2+\varepsilon} N^{-1/2}).$$

The formula (1.22) now follows by a comparison of these expressions for $\Delta(x)-2I$.

Remark. The equation (1.23) can be verified as follows. Defining $d^*(n)$ by (1.24), we have

$$\sum_{n\le 4x} d^*(n) = \sum_{4hk\le 4x} 1 - \tfrac{1}{2} \sum_{hk\le 4x} (-1)^{h+k}$$

$$= D(x) - \tfrac{1}{2} \Big(\sum_{\substack{hk\le 4x \\ 2|h,\, 2|k}} 1 - 2 \sum_{\substack{hk\le 4x \\ 2|h,\, 2\nmid k}} 1 + \sum_{\substack{hk\le 4x \\ 2\nmid hk}} 1 \Big)$$

$$= D(x) - \tfrac{1}{2} \Big\{ D(x) - 2\sum_{hk\le 2x} 1 + 2 \sum_{\substack{hk\le 2x \\ 2|k}} 1 + \sum_{hk\le 4x} 1 - 2 \sum_{\substack{hk\le 4x \\ 2|h}} 1 + \sum_{\substack{hk\le 4x \\ 2|h,\, 2|k}} 1 \Big\}$$

$$= D(x) - \tfrac{1}{2} \{ D(x) - 2D(2x) + 2D(x) + D(4x) - 2D(2x) + D(x) \}$$

$$= -D(x) + 2D(2x) - \tfrac{1}{2} D(4x) = D^*(x).$$

3. Gallagher's lemma

This lemma (see Gallagher [4], or Montgomery [10], Lemma 1.9) relates the mean square of the absolutely convergent trigonometric series

$$S(y) = \sum_\nu c(\nu) e(\nu y) \quad (e(\alpha) = e^{2\pi i \alpha}) \tag{3.1}$$

over the interval $[-Y, Y]$ to short sums of its coefficients. Let $\delta = \dfrac{1}{2} Y^{-1}$,

$$C(x) = \delta^{-1} \sum_{|\nu - x| < \frac{1}{2}\delta} c(\nu).$$

Gallagher's lemma is usually stated and applied in the form of the inequality

$$\int_{-Y}^{Y} |S(y)|^2 \, dy \ll \int_{-\infty}^{\infty} |C(x)|^2 \, dx. \tag{3.2}$$

A precise form of the result is, however, the identity

$$\int_{-\infty}^{\infty} |S(y)|^2 \left(\frac{\sin \pi\delta y}{\pi\delta y} \right)^2 dy = \int_{-\infty}^{\infty} |C(x)|^2 \, dx, \tag{3.3}$$

which obviously implies (3.2); this was Gallagher's proof of (3.2). But (3.3) can be applied to bound the right hand side in terms of $S(y)$. We shall apply this argument in the proof of the theorem, except that in place of $C(x)$ we shall have a somewhat modified sum.

For future reference we prove the following lemma.

Lemma 1. *Suppose that the trigonometric series* (3.1) *is absolutely convergent. Let* $Y>0$, $\delta=\frac{1}{2}Y^{-1}$, *and suppose that* $\alpha(x)$ *is a continuously differentiable function such that for all real* x

$$|\alpha(x)| \leqq B_1, \tag{3.4}$$

$$\int_x^{x+\delta} |\alpha'(u)|^2\,du \leqq B_2\delta^{-1}. \tag{3.5}$$

Let $B_3=\max(B_1^2, B_2)$. *Then we have*

$$\int_{-\infty}^{\infty} |\textstyle\sum_{|v-x|<\frac{1}{2}\delta} c(v)\alpha(v)|^2\,dx \ll B_3\displaystyle\int_{-\infty}^{\infty}\frac{|S(y)|^2}{Y^2+y^2}\,dy. \tag{3.6}$$

If $c(v)\neq 0$ *only for* v *in a certain finite interval* $[A, B]$, *then it is enough to suppose the validity of* (3.4) *and* (3.5) *for* $A-\delta\leqq x\leqq B+\delta$.

Proof. Let

$$C(x, u) = \textstyle\sum_{x-\frac{1}{2}\delta<v<x-\frac{1}{2}\delta+u} c(v).$$

By partial summation,

$$\textstyle\sum_{|v-x|<\frac{1}{2}\delta} c(v)\alpha(v) = C(x, \delta)\alpha\left(x+\frac{1}{2}\delta\right)-\displaystyle\int_0^\delta C(x, u)\alpha'\left(x-\tfrac{1}{2}\delta+u\right)du.$$

Hence by (3.4), Cauchy's inequality, and (3.5), the integrand on the left of (3.6) is

$$\ll B_1^2|C(x, \delta)|^2+B_2\delta^{-1}\int_0^\delta |C(x, u)|^2\,du.$$

The integral of this with respect to x is by (3.3)

$$\ll B_1^2Y^{-2}\int_{-\infty}^{\infty} |S(y)|^2\left(\frac{\sin\pi\delta y}{\pi\delta y}\right)^2 dy+B_2\delta^{-1}\int_0^\delta u^2\,du\int_{-\infty}^{\infty} |S(y)|^2\left(\frac{\sin\pi u y}{\pi u y}\right)^2 dy$$

$$\ll B_1^2\int_{-\infty}^{\infty}\frac{|S(y)|^2}{Y^2+y^2}\,dy+B_2\delta^{-1}\int_{-\infty}^{\infty} |S(y)|^2\left(\int_0^\delta \frac{du}{y^2+u^{-2}}\right) dy$$

$$\ll B_3\int_{-\infty}^{\infty}\frac{|S(y)|^2}{Y^2+y^2}\,dy.$$

This proves (3.6), and the last assertion is obvious.

4. Formulae for $\Delta_1^*(x)$ and $E_1(x)$

In the next lemma these functions, which were defined in (1.28) and (1.29), are expressed by truncated sums of Voronoi and Atkinson type. The method of proof is due to Heath-Brown [7], but for completeness we give the details also here. We retain the notations and assumptions of the introduction. For convenience we shall denote by ε generally a small positive number, not necessarily the same at each occurence.

Lemma 2. *For* $|x-\tau|\leqq\frac{1}{2}\tau$ *we have*

(4.1)
$$\Delta_1^*(x) = (2\pi x^2)^{1/4} \sum_{n\leqq M} (-1)^n d(n) n^{-3/4} \exp(-2\pi n G^2) \cos(2\sqrt{2\pi n}x - \pi/4) + O(T^\varepsilon).$$

(4.2)
$$E_1(x) = (2\pi x^2)^{1/4} \sum_{n\leqq M} (-1)^n d(n) n^{-3/4} e(x^2, n) r(x, n) \cos(f(x^2, n)) + O(L^2),$$

where $M=G^{-2}L^2$, *and*

$$r(x, n) = \exp\{-4G^2(x \operatorname{ar\,sinh}((\pi n/2)^{1/2}x^{-1}))^2\}. \tag{4.3}$$

Proof. We choose $N=T$ in the formula (1.22) for $\Delta^*(x)$. Then by (1.25) and (1.28)

$$\Delta_1^*(x) = (2/\pi)^{1/4} G^{-1} \int_{-H}^{H} (x+u)^{1/2} \sum_{n\leqq T} (-1)^n d(n) n^{-3/4}$$
$$\cdot \cos(2\sqrt{2\pi n}(x+u) - \pi/4) e^{-(u/G)^2} du + O(T^\varepsilon).$$

Here $(x+u)^{1/2}$ can be replaced by $x^{1/2}$ with an error $\ll 1$. The integration can be extended over the whole real line with a negligible error. The sum is integrated termwise, and the integrals are evaluated by the formula

$$\int_{-\infty}^{\infty} e^{\alpha u - \beta u^2} du = (\pi/\beta)^{1/2} \exp(\alpha^2/4\beta) \quad (\operatorname{Re}\beta > 0). \tag{4.4}$$

The contribution of the terms with $n>\mathrm{M}$ is negligible, and (4.1) follows.

Turning to the proof of (4.2), we express $E(t)$ by (1.8), choosing $N=T$ in the definition of $\sum_1(t)$. By (1.29) and (1.26) we have

$$E_1(x) = \sum_{j=1}^{2} G^{-1} \int_{-H}^{H} \Sigma_j((x+u)^2) e^{-(u/G)^2} du + O(L^2). \tag{4.5}$$

Consider first the term with $j=1$. By (1.9) this is

$$(2\pi)^{1/4} G^{-1} \int_{-H}^{H} (x+u)^{1/2} \sum_{n\leqq T} (-1)^n d(n) n^{-3/4} e((x+u)^2, n) \tag{4.6}$$
$$\cdot \cos(f((x+u)^2, n)) e^{-(u/G)^2} du.$$

As above, we may replace $(x+u)^{1/2}$ by $x^{1/2}$, and likewise $e((x+u)^2, n)$ may be replaced by $e(x^2, n)$. Further, by (1.6)

$$f'(t, n) = 2 \operatorname{ar\,sinh}\left(\left(\frac{\pi n}{2t}\right)^{1/2}\right), \tag{4.7}$$

whence

$$\frac{\partial^2}{\partial u^2} f((x+u)^2, n) \ll n^{3/2} T^{-3/2},$$

$$\frac{\partial^3}{\partial u^3} f((x+u)^2, n) \ll n^{3/2} T^{-2}.$$

Hence we have the approximation

$$f((x+u)^2, n) = f(x^2, n) + 4x \operatorname{ar\,sinh}\left((\pi n/2)^{1/2} x^{-1}\right) u + A(n, x) u^2 + O(T^{-1/2} G^3 L^3),$$

where

$$A(n, x) \ll (n/T)^{3/2}.$$

We substitute this in (4.6), omitting the error term. This effects an error $\ll G^3 L^4 \ll 1$. The sum in (4.6) is integrated termwise, and the integrals are again extended over the real line. The new integrals are evaluated by (4.4), and as before, those with $n > M$ are very small. Hence (4.6) can be rewritten as follows:

$$(2/\pi)^{1/4} x^{1/2} G^{-1} \sum_{n \leq M} (-1)^n d(n) n^{-3/4} e(x^2, n) \tag{4.8}$$

$$\times \operatorname{Re}\left\{ e^{if(x^2, n)} \left(\frac{\pi}{\beta(n, x)}\right)^{1/2} \exp\left(-\frac{4\left(x \operatorname{ar\,sinh}\left((\pi n/2)^{1/2} x^{-1}\right)\right)^2}{\beta(n, x)}\right)\right\} + O(1),$$

where

$$\beta(n, x) = G^{-2} - A(n, x) i.$$

We replace everywhere $\beta(n, x)$ by G^{-2}, and the total error is

$$\ll T^{-5/4} G^{-3/2} L^5 \ll 1.$$

Hence (4.8) equals the right hand side of (4.2).

To complete the proof of the lemma, it suffices to show that the term with $j=2$ in (4.5) is $\ll 1$.

Note that since we fixed $N = T$ in the definition of the sum $\Sigma_1((x+u)^2)$, the number N' in the definition (1.10) of the sum $\Sigma_2((x+u)^2)$ depends on $(x+u)^2$ according to (1.12). However, it is convenient to replace $N'((x+u)^2, T)$ by $N'(x^2, T)$. By (1.12)

$$N'(x^2, T) - N'((x+u)^2, T) \ll T^{1/2} GL.$$

For $n \leqq N'((x+H)^2, T)$ we have

$$\left(\log\left(\frac{(x+u)^2}{2\pi n}\right)\right)^{-1} \ll 1,$$

$$\left(\log\left(\frac{(x+u)^2}{2\pi n}\right)\right)^{-1} = \left(\log\left(\frac{x^2}{2\pi n}\right)\right)^{-1} + O(T^{-1/2}GL).$$

Hence by (1.10), for $|u| \leqq H$,

$$\Sigma_2((x+u)^2) = -2 \sum_{n \leqq N'(x^2, T)} d(n) n^{-1/2} \left(\log \frac{x^2}{2\pi n}\right)^{-1} \cos\left(g((x+u)^2, n)\right) + O(T^\varepsilon G).$$

By (1.7) we have

$$g((x+u)^2, n)) = g(x^2, n) + 2x \log\left(\frac{x^2}{2\pi n}\right) u + \left(\log\left(\frac{x^2}{2\pi n}\right) + 2\right) u^2 + O(|u|^3 T^{-1/2}).$$

Substituting this in the previous formula and arguing as the case $j=1$, it is easily seen that the term with $j=2$ in (4.5) is indeed $\ll 1$. This completes the proof of Lemma 2.

Lemma 3. *Let $0 < \xi \ll G$. Then with the assumptions and notations of the preceding lemma, we have for $|y| \leqq \tau/3$*

(4.9)

$$\Delta_1^*(\tau+y+\xi) - \Delta_1^*(\tau+y) = (2\pi)^{1/4} (\tau+y)^{1/2} \sum_{n \leqq M} (-1)^n d(n) n^{-3/4} \exp(-2\pi n G^2)$$
$$\times \operatorname{Re}\left\{e^{i(2\sqrt{2\pi n}(\tau+y) - \pi/4)} \left(e^{2i\sqrt{2\pi n}\,\xi} - 1\right)\right\} + O(T^\varepsilon),$$

and for $|y| \leqq K$

(4.10)

$$E_1(\tau+y+\xi) - E_1(\tau+y) = (2\pi)^{1/4} (\tau+y)^{1/2} \sum_{n \leqq M} (-1)^n d(n) n^{-3/4} e(T, n) r(\tau, n)$$
$$\times \operatorname{Re}\left\{e^{i(f(T,n) + h(T,n)y)} \left(e^{2i\sqrt{2\pi n}\,\xi} - 1\right)\right\} + O(T^{-9/4+\varepsilon} G^{-15/2}) + O(T^\varepsilon),$$

where

$$h(T, n) = 4T^{1/2} \operatorname{ar\,sinh}\left((\pi n/2T)^{1/2}\right). \tag{4.11}$$

Proof. We begin by writing the left hand sides of (4.9) and (4.10) by means of (4.1) and (4.2). In $\Delta_1^*(\tau+y+\xi)$ and $E_1(\tau+y+\xi)$ we may replace $(\tau+y+\xi)^{1/2}$ by $(\tau+y)^{1/2}$ with an error $\ll 1$. Then (4.9) follows immediately from (4.1).

Turning to the proof of (4.10), we make a further simplification on replacing $e(x^2, n)$ (resp. $r(x, n)$) with $x = \tau+y+\xi$ or $\tau+y$ by $e(T, n)$ (resp. $r(\tau, n)$). To estimate the error, note that by (1.5) we have, if nT^{-1} is sufficiently small,

$$e(T, n) = 1 + P(nT^{-1}), \tag{4.12}$$

where P is a power series with vanishing constant term. Hence

$$\frac{d}{dx} e(x^2, n) \ll nx^{-3} \ll nT^{-3/2}.$$

Also, by (4.3)

$$r'(x, n) \ll G^2 n^2 x^{-3} \ll nT^{-3/2} L^2.$$

Hence the error to be estimated is

$$\ll T^{-7/4+\varepsilon} G^{-9/2},$$

which can be absorbed into the error terms in (4.10).

Writing

$$\varphi(x, n) = f(x^2, n),$$

we now have

(4.13)

$$E_1(\tau+y+\xi)-E_1(\tau+y) = (2\pi)^{1/4}(\tau+y)^{1/2} \sum_{n\leqq M} (-1)^n d(n) n^{-3/4} e(T, n) r(\tau, n)$$
$$\times \operatorname{Re}\{e^{i\varphi(\tau+y+\xi, n)} - e^{i\varphi(\tau+y, n)}\} + O(T^{-9/4+\varepsilon} G^{-15/2} + O(T^{\varepsilon}).$$

Next we replace here the function φ by a linear approximation. For that we need its two first derivatives. It follows from (1.6) that if nT^{-1} is sufficiently small, then

(4.14) $$f(T, n) = -\pi/4 + 2(2\pi nT)^{1/2}(1+Q(nT^{-1})),$$

where Q is a power series with vanishing constant term; this is a more precise form of (1.14). Now by (4.7) and (4.11)

(4.15)

$$\varphi'(\tau, n) = 4\tau \operatorname{ar\,sinh}((\pi n/2)^{1/2}\tau^{-1}) = h(T, n) = 2(2\pi n)^{1/2} + O(T^{-1} n^{3/2}),$$

and by (4.14)

$$\varphi''(x, n) \ll n^{3/2} T^{-3/2}$$

for $\tau \ll x \ll \tau$, $n \leqq M$. Hence for $y \ll K$

$$\varphi(\tau+y, n) = f(T, n) + h(T, n) y + O(n^{3/2} T^{-3/2} K^2).$$

When this is substituted in (4.13), the error term can be omitted at the cost of an error $\ll T^{-9/4+\varepsilon} G^{-15/2}$. We get (4.10), except that in place of

$$e^{2i\sqrt{2\pi n}\,\xi} - 1$$

we have

$$e^{ih(T, n)\xi} - 1;$$

by the last equation in (4.15) these factors can be interchanged with an error $\ll T^{-3/4+\varepsilon} G^{-5/2}$, which is admissible.

5. Proof of the theorem

We want to estimate the integral

$$J=\int_{-K}^{K}(E_1(\tau+y+\xi)-E_1(\tau+y))^2\,dy$$

in terms of the function Δ_1^*. Denote the error term on the right of (1.30) by R. By (4.10) we have

$$J\ll T^{1/2}\int_{-K}^{K}\left|\sum_\lambda a(\lambda)e(\lambda y)\right|^2 dy+R, \tag{5.1}$$

where λ runs over the sequence

$$\lambda=\lambda(n)=(2\pi)^{-1}h(T,n),\quad 1\leqq n\leqq M, \tag{5.2}$$

as well as the negative numbers $-\lambda(n)$, and

$$a(\lambda)=a(\lambda(n)) \tag{5.3}$$
$$=(-1)^n d(n)n^{-3/4}e(T,n)r(\tau,n)e^{if(T,n)}(e^{2i\sqrt{2\pi n}\,\xi}-1),\quad \lambda>0,$$

$$a(\lambda)=\overline{a(-\lambda)},\quad \lambda<0. \tag{5.4}$$

The trigonometric sum on the right of (4.9) can be written, up to a factor $\frac{1}{2}$, as

$$\sum_\nu c(\nu)e(\nu y), \tag{5.5}$$

where ν runs over the numbers

$$\nu=\nu(n)=(2n/\pi)^{1/2},\quad 1\leqq n\leqq M, \tag{5.6}$$

as well as the negative numbers $-\nu(n)$, and

$$c(\nu)=c(\nu(n)) \tag{5.7}$$
$$=(-1)^n d(n)n^{-3/4}\exp(-2\pi nG^2)e^{i(\sqrt{2\pi n}\,\tau-\pi/4)}(e^{2i\sqrt{2\pi n}\,\xi}-1),\quad \nu>0,$$

$$c(\nu)=\overline{c(-\nu)},\quad \nu<0. \tag{5.8}$$

We may now apply Gallagher's lemma. By (3.2) it follows from (5.1) that

$$J\ll T^{1/2}K^2\int_{-\infty}^{\infty}\left|\sum_{|\lambda-x|<\frac{1}{2}\delta}a(\lambda)\right|^2dx+R, \tag{5.9}$$

where

$$\delta=\tfrac{1}{2}K^{-1}=\tfrac{1}{2}T^{1/2}G^2. \tag{5.10}$$

There is a one-to-one order preserving correspondence between the sequences λ and ν, defined by the mapping $\lambda=\lambda(n)\to\nu(n)$ for $\lambda>0$, $\lambda=-\lambda(n)\to-\nu(n)$ for $\lambda<0$. Hence we may write $a(\lambda)=b(\nu)$, and the condition of summation for λ in (5.9) can be restated in terms of ν. We shall do this explicitely below.

By (5.2), (5.6) and (4.11) we have

$$\nu(n) = 2\pi^{-1}\tau \sinh\left(\tfrac{1}{2}\pi\lambda(n)\tau^{-1}\right).$$

In the variable

$$u = u(x) = 2\pi^{-1}\tau \sinh\left(\tfrac{1}{2}\pi x\tau^{-1}\right), \tag{5.11}$$

the condition of summation in (5.9) may be written as

$$u\left(x-\tfrac{1}{2}\delta\right) < \nu < u\left(x+\tfrac{1}{2}\delta\right). \tag{5.12}$$

But in place of this, we want to have the simpler condition

$$|u(x)-\nu| < \tfrac{1}{2}\delta. \tag{5.13}$$

The conditions (5.12) and (5.13) are actually almost equivalent, for the respective sums are identical up to at most two terms. To see this, note first that by (5.11)

$$u'(x) = 1+O(x^2T^{-1}),$$

whence

$$\left|u\left(x\pm\tfrac{1}{2}\delta\right)-\left(u(x)\pm\tfrac{1}{2}\delta\right)\right| \ll \delta(|x|+\delta)^2T^{-1}.$$

On the other hand, the difference between consecutive numbers ν near $x\pm\dfrac{1}{2}\delta$ is by (5.6) at least $\gg M^{-1/2}$. The assertion now follows from the estimations

$$M^{1/2}\delta(|x|+\delta)^2T^{-1} \ll K^{-1}M^{3/2}T^{-1} \ll T^{-1/2}G^{-1}L^3 \ll T^{-\varepsilon}.$$

Observing also that by (5.3)

$$a(\lambda(n)) \ll d(n)n^{-1/4}G,$$

we have

$$\left|\sum\nolimits_{|\lambda-x|<\frac{1}{2}\delta} a(\lambda)\right|^2 \ll \left|\sum\nolimits_{|\nu-u(x)|<\frac{1}{2}\delta} b(\nu)\right|^2$$
$$+T^\varepsilon G^2\left\{\min\left(\left|x-\tfrac{1}{2}\delta\right|^{-1}, 1\right)+\min\left(\left|x+\tfrac{1}{2}\delta\right|^{-1}, 1\right)\right\}.$$

Hence (5.9) now takes the form

$$J \ll T^{1/2}K^{-2}\int_{-\infty}^{\infty}\left|\sum\nolimits_{|\nu-u|<\frac{1}{2}\delta} b(\nu)\right|^2du+R; \tag{5.14}$$

note that

$$T^{1/2+\varepsilon}K^2G^2 \ll T^{-1/2+\varepsilon}G^{-2} \ll R.$$

Next we compare $b(\nu)$ with $c(\nu)$. Suppose first that $\nu=\nu(n)>0$. Then by (5.6)

$$n = \tfrac{1}{2}\pi\nu^2,$$

and comparing (5.3) with (5.7) we find that

$$b(\nu) = c(\nu)\alpha(\nu), \tag{5.15}$$

where

(5.16)
$$\alpha(v) = e\left(T, \tfrac{1}{2}\pi v^2\right) r\left(\tau, \tfrac{1}{2}\pi v^2\right) \exp\left\{\pi^2 G^2 v^2 + i\left(\pi/4 + f\left(T, \tfrac{1}{2}\pi v^2\right) - 2v\pi\tau\right)\right\}.$$

Next we define for $x<0$
$$\alpha(x) = \overline{\alpha(-x)};$$
then (5.15) is valid for all v. Note that by (4.14) we may now write for $|x| \ll M^{1/2}$

$$\alpha(x) = e\left(T, \tfrac{1}{2}\pi x^2\right) r\left(\tau, \tfrac{1}{2}\pi x^2\right) \exp\left\{\pi^2 G^2 x^2 + 2\pi i x T^{1/2} Q\left(\tfrac{1}{2}\pi x^2 T^{-1}\right)\right\}. \tag{5.17}$$

Hence, in particular, $\alpha(x)$ is continuously differentiable in the interval $|x| \leqq (2\pi^{-1}M)^{1/2}$ where v varies.

With an application of Lemma 1 in mind, we find estimates for $\alpha(x)$ and $\alpha'(x)$. The function $\alpha(x)$ is bounded, for by (4.3)

(5.18)
$$r\left(\tau, \tfrac{1}{2}\pi x^2\right) e^{\pi^2 G^2 x^2} = \exp\left\{-G^2 x^4 T^{-1}\left(c_0 + c_1(x^2 T^{-1}) + c_2 (x^2 T^{-1})^2 + \dots\right)\right\},$$

where the exponent is bounded for the relevant values of x. Also, by (5.17), (5.18), (4.12) and (5.10), for $|x| \ll M^{1/2}$,
$$\alpha'(x) \ll G^{-1} L^3 T^{-1} + G^{-2} T^{-1/2} L^2 \ll \delta^{-1} L^2.$$
Consequently we may apply Lemma 1 with $B_1 \ll 1$, $B_2 \ll L^4$. It follows from (5.14), (5.15) and (3.6) that
$$J \ll T^{1/2+\varepsilon} K^2 \int_{-\infty}^{\infty} \left|\sum_v c(v) e(vy)\right|^2 (K^2+y^2)^{-1}\, dy + R.$$

But the trigonometric sum in the integrand is the same as in (4.9), so that we get further

$$J \ll T^{\varepsilon} \int_{-\tau/3}^{\tau/3} \left(\Delta_1^*(\tau+y+\xi) - \Delta_1^*(\tau+y)\right)^2 \frac{K^2}{K^2+y^2}\, dy \tag{5.19}$$
$$+ T^{1/2+\varepsilon} K^2 \int_{|y| \geqq \tau/3} \left|\sum_v c(v) e(vy)\right|^2 y^{-2}\, dy + R.$$

To estimate the integral over $|y| \geqq \tau/3$, note that for $Y \gg \tau$ we have by (3.2) and (5.7)—(5.8)
$$\int_{-Y}^{Y} \left|\sum_v c(v) e(vy)\right|^2 dy \ll Y \sum_v |c(v)|^2 \ll Y T^{\varepsilon} G,$$

since the sums $C(x)$ in (3.2) are in this case either empty or contain only one term. Hence the second term on the right of (5.19) is
$$\ll T^{\varepsilon} K^2 G = T^{-1+\varepsilon} G^{-3} \ll T^{-1/2} G^{-2} \ll R,$$

and the proof of the theorem is complete.

6. A modified version of the theorem

In the next lemma we give an inequality of the type (1.30) in terms of the function $E(t)$ and $\Delta^*(x)$. The proofs of the corollaries will be based on this lemma.

Lemma 4. *Suppose that* G *satisfies* (1.27) *with* $b=1/6$. *Let* $G_1=T^{1/2}G$, $H_1=G_1L$, $K_1=T^{1/2}K=TG_1^{-2}$. *Then we have*

(6.1)
$$G_1^{-3}\int_0^{G_1} d\eta \int_{T-K_1}^{T+K_1}\Big\{\int_{-H_1}^{H_1}\big(E(t+v+\eta)-E(t+v)\big)e^{-(v/2G_1)^2}\,dv\Big\}^2 dt$$
$$\ll T^{\varepsilon}\sup_{0<\eta\le G_1}\ \sup_{K_1\le Z\le T/8} K_1^2Z^{-2}\int_{T/2\pi-Z}^{T/2\pi+Z}(\Delta^*(x+\eta)-\Delta^*(x))^2\,dx+T^{\varepsilon}(T^4G_1^{-17}+TG_1^{-2}).$$

Proof. We average the inequality (1.30) of the theorem with respect to the parameter ξ over the interval $[0, G]$. The inequality (6.1) will follow when the left hand side of the averaged inequality is estimated from below, and the right hand side is estimated from above.

Consider first the left side. In terms of $E(t)$ it reads as follows:

(6.2)
$$G^{-3}\int_0^G d\xi\int_{-K}^{K}\Big\{\int_{-H}^{H}\big(E((\tau+y+\xi+u)^2)-E((\tau+y+u)^2)\big)e^{-(u/G)^2}\,du\Big\}^2 dy.$$

In place of y and ξ we introduce the new variables

$$t=(\tau+y)^2,\quad \eta=2(\tau+y)\xi,$$

and in the innermost integral we integrate with respect to the variable $v=2t^{1/2}u$. The range of integration in the t,η-plane contains the rectangle $[T-K_1, T+K_1]\times[0, G_1]$. The Jacobian determinant of the change of variables $(y,\xi)\to(t,\eta)$ is of the order T^{-1}.

In the new variables,

$$E\big((\tau+y+\xi+u)^2\big)-E\big((\tau+y+u)^2\big)$$
$$=E\big(t+v+\eta+0(G^2L^2)\big)-E\big(t+v+O(G^2L^2)\big)=E(t+v+\eta)-E(t+v)+O(L^2);$$

in the last step we used the assumption $G\ll T^{-1/6}$ and a standard estimate of $\left|\zeta\left(\frac{1}{2}+it\right)\right|$. Denoting by I the expression in (6.2), we now have

(6.3)
$$T^{-1/2}G_1^{-3}\int_0^{G_1} d\eta\int_{T-K_1}^{T+K_1}\Big\{\int_{-H_1}^{H_1}\big(E(t+v+\eta)-E(t+v)\big)e^{-(v/2G_1)^2(T/t)}\,dv\Big\}^2 dt$$
$$\ll I+T^{1/2+\varepsilon}G_1^{-2}.$$

Consider next the right hand side of the basic inequality. By (1.28), (1.25) and Cauchy's inequality

$$\left(\Delta_1^*(\tau+y+\xi)-\Delta_1^*(\tau+y)\right)^2$$
$$\ll G^{-1}\int_{-H}^{H}\left(\Delta^*((\tau+y+\xi+u)^2/2\pi)-\Delta^*((\tau+y+u)^2/2\pi)\right)^2\,du.$$

The expression to be estimated is hence

(6.4)
$$\ll G^{-1}T^{\varepsilon}\int_0^G d\xi\int_{-\tau/3-H}^{\tau/3+H}\left(\Delta^*((\tau+y+\xi)^2/2\pi)-\Delta^*((\tau+y)^2/2\pi)\right)^2\frac{K^2}{K^2+y^2}\,dy+R,$$

where R again denotes the error term in (1.30).

We introduce here the new variables

$$x=(\tau+y)^2/2\pi,$$
$$\eta=(\tau+y+\xi)^2/2\pi-(\tau+y)^2/2\pi.$$

Then the range of integration in the x, η-plane is contained in the rectangle

$$|T/2\pi-x|\leqq T/8,\quad 0\leqq\eta\leqq G_1,$$

and the Jacobian determinant is $\ll T^{-1}$. The x-integral is estimated by considering separately the ranges $|T/2\pi-x|\leqq K_1$, and $2^jK_1\leqq|T/2\pi-x|\leqq 2^{j+1}K_1$, $j=0,1,\ldots$. The η-integral is estimated trivially by taking the supremum of the integrand. The expression (6.4) is found to be at most of the order of the right hand side of (6.1) multiplied by $T^{-1/2}$. Taking also into account (6.3), we obtain an inequality, which gives a slightly modified form of (6.1) if its both sides are multiplied by $T^{1/2}$. In order to get (6.1) precisely, we have to remove the extra factor T/t in the exponent in (6.3). Since $T/t=1+O(G_1^{-2})$ and $E(T)\ll T^{1/3}$, this change can be made if the term $T^{5/3+\varepsilon}G_1^{-6}$ is added to the right side of (6.1). But this term can be absorbed into the error terms in (6.1).

7. Proof of Corollary 1

The function $E(t)$ may increase rapidly, but it follows immediately from (1.1) that it can decrease only relatively slowly, more exactly,

(7.1) $$E(t+x)-E(t)\geqq -cx\log t\quad\text{for}\quad 2\leqq t\leqq t+x\leqq 2t.$$

It follows that if $E(t_2)-E(t_1)$ is large and positive for $t_1<t_2$, then $E(t_2')-E(t_1')$ is also large if the numbers $t_2'\geqq t_2$ and $t_1'\leqq t_1$ lie in certain intervals.

We may suppose that the distance between two adjacent intervals A_i is at least A, by going over to a suitable subset of cardinality $\gg R$ if necessary.

We consider in detail those intervals A_i for which there exists a pair of points $t_1, t_2 \in A_i$ such that

$$t_1 < t_2, \quad E(t_2)-E(t_1) \geqq U \gg X. \tag{7.2}$$

The remaining intervals can be dealt with analogously.

By (7.1) and (7.2) we have

$$E(t_2')-E(t_1') \geqq \tfrac{1}{2}U$$

for

$$t_1 - XL^{-2} \leqq t_1' \leqq t_1, \quad t_2 \leqq t_2' \leqq t_2+XL^{-2}.$$

Hence, defining

$$G = T^{-1/2}XL^{-4}, \quad \text{i.e.} \quad G_1 = XL^{-4}, \quad H_1 = XL^{-3},$$

we have

$$G_1^{-1}\int_{-H_1}^{H_1}\big(E(t_2'+v)-E(t_1'+v)\big)e^{-(v/2G_1)^2}dv \gg U \tag{7.3}$$

for

$$t_1-\frac{2}{3}XL^{-2} \leqq t_1' \leqq t_1-\frac{1}{3}XL^{-2}, \quad t_2+\frac{1}{3}XL^{-2} \leqq t_2' \leqq t_2+\frac{2}{3}XL^{-2}. \tag{7.4}$$

Having fixed G_1, we define

$$K_1 = TG_1^{-2} = YL^8$$

as in Lemma 4. By this lemma there exists a number $\eta_0 \in \left[\frac{1}{2}G_1, G_1\right]$ such that

$$G_1^{-2}\int_{T-K_1}^{T+K_1}\Big\{\int_{-H_1}^{H_1}\big(E(t+v+\eta_0)-E(t+v)\big)e^{-(v/2G_1)^2}dv\Big\}^2 dt \tag{7.5}$$

$$\ll T^{\varepsilon}\sup_{0\leqq\eta\leqq G_1}\ \sup_{K_1\leqq Z\leqq T/8} K_1^2Z^{-2}\int_{T/2\pi-Z}^{T/2\pi+Z}(\Delta^*(x+\eta)-\Delta^*(x))^2dx + T^{\varepsilon}(T^4G_1^{-17}+TG_1^{-2}).$$

Let t_1' be a number such that not only t_1' but also all numbers in the interval $[t_1'-G_1, t_1'+G_1]$ satisfy the condition (7.4) for t_1'. There exists an integer J such that the number $t_2'=t_1'+J\eta_0$ and all numbers in the interval $[t_2'-G_1, t_2'+G_1]$ satisfy the condition (7.4) for t_2'.

We now apply (7.3) substituting $t_1'+\xi$ (resp. $t_2'+\xi$) for t_1' (resp. t_2'); here ξ is an arbitrary number such that $|\xi|\leqq G_1$. It follows that

$$U \ll G_1^{-1}\sum_{j=1}^{J}\int_{-H_1}^{H_1}\big(E(t_1'+j\eta_0+\xi+v)-E(t_1'+(j-1)\eta_0+\xi+v)\big)e^{-(v/2G_1)^2}dv.$$

We square both sides, apply Cauchy's inequality to the j-sum, integrate with respect to ξ, and finally sum over all intervals A_i under consideration (say R_1 in number). Taking into account that $J \ll AG_1^{-1}$, together with the facts that the intervals A_i

are well-spaced and lie in the interval $\left[T-\frac{1}{2}K_1,\ T+\frac{1}{2}K_1\right]$, we obtain

$$R_1 U^2 G_1 \ll A G_1^{-3} \int_{T-K_1}^{T+K_1} \left\{ \int_{-H_1}^{H_1} \left(E(t+v+\eta_0)-E(t+v)\right) e^{-(v/2G_1)^2}\, dv \right\}^2 dt.$$

The assertion of the corollary now follows from this and (7.5).

8. Proof of Corollary 2

We apply Corollary 1 with

$$X = A = V^2 L^{-3}.$$

We cover the interval $[T-TV^{-4},\ T+TV^{-4}]$, contained in the interval $[T-TX^{-2},\ T+TX^{-2}]$, by subintervals of length A, deleting those intervals which do not contain any point t_i. These intervals are classified by the condition that the j'th class consists of those intervals which contain at least 2^j but less than 2^{j+1} points t_i. We estimate the cardinality of the j'th class.

By (1.18) the function $E(t)$ increases in each interval $[t_i-L^2,\ t_i+L^2]$ an amount $\gg V^2L^{-1}$. The positive variation of $E(t)$ in an interval of the j'th class is therefore $\gg(1+2^jL^{-2})V^2L^{-1}$. On the other hand, the negative variation in the same interval is $\ll AL \ll V^2L^{-2}$ by (7.1). Hence we may take in Corollary 1

$$U \gg (1+2^jL^{-2})V^2L^{-1}.$$

Corollary 1 now gives an upper bound for the cardinality of the j'th class. Multiplying this by 2^j and summing over j, we complete the proof of the corollary.

9. Proof of Corollary 3

We observed already in the introduction that (1.37) follows immediately from Corollary 2. Hence only (1.38) requires a proof.

We commence by showing that $|E(t)|$ cannot be too large throughout a given interval $[T-T_0,\ T+T_0]$. Let $T_0\in[T^{1/3},\ T^{2/5}]$, and define $E_1(x)$ as in (1.29) by choosing $G=T_0T^{-1/2}L^{-1}$. Then $E_1(x)$ is given by the formula (4.2), whence by a trivial estimation

(9.1) $$E_1(T^{1/2}) \ll T^{1/2+\varepsilon}T_0^{-1/2}.$$

By the definition of $E_1(x)$ this implies that there exists a number $t\in[T-T_0,\ T+T_0]$ such that

(9.2) $$|E(t)| \ll T^{1/2+\varepsilon}T_0^{-1/2}.$$

We are going to prove that for suitable T_0 the estimate (9.2) is valid in the whole interval $[T-T_0,\ T+T_0]$, when the constant implied by the symbol $\ll$ is somewhat enlargened, say doubled. If this is not true, then

$$\sup_{T-T_0\leqq t_1,t_2\leqq T+T_0} |E(t_1)-E(t_2)| \gg T^{1/2+\varepsilon}T_0^{-1/2}.$$

We now apply Corollary 1 with

$$A = 2T_0, \quad X = U = T^{1/2+\varepsilon}T_0^{-1/2}.$$

Using in (1.35) the hypothesis (1.36), we obtain

$$1 \ll T_0 T^{3/2+\varepsilon} X^{-6} \ll T_0^4 T^{-3/2}.$$

This is impossible for $T_0=cT^{3/8}$, with c a suitable constant. Thus (9.2) holds in the whole interval $[T-T_0,\ T+T_0]$. This proves (1.38).

The result could be somewhat sharpened by applying the theory of exponential sums in (9.1) instead of the trivial estimate.

10. On the estimation of $E(T)$

We show briefly how the estimate (1.16) with $\theta_2\leqq 1/3$ follows from Atkinson's formula.

We begin with the trivial observation that if $1\leqq t_1\leqq T\leqq t_2\leqq 2T$, then

$$I(t_1) \leqq I(T) \leqq I(t_2),$$

whence by (1.1)

$$E(t_1)+O\big((T-t_1)\log T\big) \leqq E(T) \leqq E(t_2)+O\big((t_2-T)\log T\big). \tag{10.1}$$

Let Y be a parameter with $T^{1/4}L^{-1}\leqq Y\leqq T^{1/3}L^{-1}$, and let $G=T^{-1/2}YL^{-2}$. With this value of G we define $E_1(x)$ by (1.29). Then it follows from (10.1) that

$$E_1\big((T-Y)^{1/2}\big)+O(YL) \leqq E(T) \leqq E_1\big((T+Y)^{1/2}\big)+O(YL).$$

The values of $E_1\big((T\pm Y)^{1/2}\big)$ are given by (4.2), where $M=TY^{-2}L^6$. Using also partial summation and putting $X=YL$, we arrive at the following result.

Lemma 5. *Let* $T^{1/4}\leqq X\leqq T^{1/3}$ *and* $M=TX^{-2}L^8$. *Then*

$$E(T) \ll X+T^{1/4} \sup_{|t-T|\leqq X}\ \sup_{\xi\leqq M} \Big|\sum\nolimits_{n\leqq\xi}(-1)^n d(n) n^{-3/4}\cos\big(f(t,n)\big)\Big|. \tag{10.2}$$

The partial summation allowed us to drop the factors $e(t,n)$ and $r(t^{1/2},n)$.

Choosing $X=T^{1/3}$ and estimating the sum in (10.2) trivially, we get $E(T)\ll T^{1/3+\varepsilon}$. Sharper estimates are obtained by applying van der Corput's method and choosing the parameter X optimally.

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