# On the zeros of a class of generalised Dirichlet series — VI

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**1. Introduction.** This note is in the nature of an addendum to [3]. In [1] we stated that if we follow the method of [3] by working with the auxiliary coefficients $\Delta\left(\frac{X}{\lambda_n}\right)$ (where $X>0$, $\lambda_n>0$) in place of the auxiliary coefficients $\mathrm{Exp}\left(-\frac{\lambda_n}{X}\right)$, we get Theorem 1 below. The function $\Delta$ is defined for all $\chi>0$ by

$$\Delta(\chi) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \chi^w \,\mathrm{Exp}\,(w^{4k+2})\frac{dw}{w},$$

where $k$ is a positive integer which shall be a fixed constant. By moving the line of integration to $\mathrm{Re}\, w=A$ and $\mathrm{Re}\, w=-A$ we see that $\Delta(\chi)=O(\chi^A)$ and also $\Delta(\chi)=1+O(\chi^{-A})$ where $A$ is any positive constant and the $O$-constant depends only on $k$ and $A$.

**Theorem 1.** *Let* $0<\theta<\frac{1}{2}$ *and let* $\{a_n\}$ *be a sequence of complex numbers satisfying the inequalities*

$$|a_N| \leqq \left(\tfrac{1}{2}-\theta\right)^{-1} \quad \textit{and} \quad \left|\textstyle\sum_{m=1}^{N} a_m\right| \leqq \left(\tfrac{1}{2}-\theta\right)^{-1} N^{\theta}$$

*for* $N=1, 2, 3, \ldots$. *Then the number of zeros of the analytic function* $\zeta(s)+\sum_{n=1}^{\infty} a_n n^{-s}$ *in the region*

$$\sigma \geqq \tfrac{1}{4}+\tfrac{\theta}{2}, \quad T \leqq t \leqq 2T$$

*exceeds* $T(\log T)^{1-\varepsilon}$ *for all* $T\geqq T_0$, *where* $\varepsilon>0$ *is arbitrary and* $T_0$ *depends only on* $\theta$ *and* $\varepsilon$. *The same lower bound also holds for the derivatives (say the* $l^{\text{th}}$ *derivative of the analytic function in question) provided* $T_0$ *is allowed to depend on* $l$ *as well.*

The proof of this theorem, with some generalisations, will be given in § 3. However in § 2 we prove by the method of [3] yet another theorem of a sufficiently genreal nature, namely.

**Theorem 2.** *Let $\{a_n\}$ be a sequence of complex numbers such that the first non-zero $a_n$ is 1 and $|a_n| \leqq (n+1)^A$, where $A \geqq 1$ is a constant. The numbers $a_n$ can depend on the parameter $T$ to follow but the first $n$ say $n_0$ for which $a_{n_0}=1$ should not depend on $T$. Suppose $\sum_{n=1}^{\infty}(|a_n|^2 n^{-s})$ has a finite abscissa of convergence say $2\alpha$, and that $\sum_{n=1}^{\infty} a_n n^{-s}$ can be continued as an analytic function $F(s)$ in the region $\sigma \geqq \alpha - \frac{1}{A}$, $T \leqq t \leqq 2T$, and there $\max |F(s)| < T^A$ ($T$ being a parameter $\geqq 10$). Then a lower bound for the number of zeros of $F(s)$ in the region referred to is $T(\log T)^{1-\varepsilon} L^2$, where $\varepsilon > 0$ is arbitrary, $T \geqq T_0 = T_0(\varepsilon, A)$, and*

$$L = \frac{\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n) n^{2\alpha_2}} \left(\Delta\left(\frac{X}{n}\right)\right)^2\right)^2 \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\alpha_2}} \left(\Delta\left(\frac{X}{n}\right)\right)^2\right)^{-1}}{\left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n) n^{2\alpha_1}} \left(\Delta\left(\frac{X}{n}\right)\right)^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{|a_n|^2}{d(n) n^{2\alpha_3}} \left(\Delta\left(\frac{X}{n}\right)\right)^2\right)^{1/2}},$$

*where $\alpha_1, \alpha_2, \alpha_3$ are constants satisfying $2\alpha_2 = \alpha_1 + \alpha_3$, and $\alpha - \frac{1}{A} < \alpha_1 < \alpha_2 < \alpha_3 < \alpha$. It is further assumed the parameter $X$ satisfying the following two conditions exists and is defined, if it exists, by these conditions.*

(i) $$T^{\frac{1}{100A}} \leqq X \leqq T^{\frac{1}{10A}}$$

*and*

(ii) $$\sum_{X \leqq n \leqq 2X} |a_n|^2 \geqq X^{\xi - \eta},$$

*where*

$$\xi = \limsup_{\chi \to \infty} \left\{\log\left(\sum_{\chi \leqq n \leqq 2\chi} |a_n|^2\right)(\log \chi)^{-1}\right\},$$

*and $\eta$ is a sufficiently small positive constant depending on $\alpha_1, \alpha_2, \alpha_3$. Moreover $d(n)$ is defined as usual by $\zeta^2(s) = \sum_{n=1}^{\infty}(d(n) n^{-s})$.*

*Remark 1.* It is convenient to call $L^2$ as the loss factor. In the last remark in part $A$ of [3] we have stated the result with the loss factor $L_0^4$ where

$$L_0 = \frac{\sum_{n=1}^{\infty}\left(|a_n|^2 n^{-2\alpha_2}\left(\Delta\left(\frac{X}{n}\right)\right)^2\right)}{\sum_{n=1}^{\infty}\left(|a_n|^2 d(n) n^{-2\alpha_2}\left(\Delta\left(\frac{X}{n}\right)\right)^2\right)},$$

without proof. However the methods for obtaining this are sketched in sufficient detail there. The method of [3] actually leads to the loss factor $L_0^2$ and also to Theorem 2 above. It should be mentioned that in the last remark in part $A$ of [3] the condition $\frac{1}{X}\sum_{n \leqq X} |a_n|^2 \gg X^{\varepsilon}$ should read $\frac{1}{X}\sum_{n \leqq X} |a_n|^2 \gg X^{-\varepsilon}$.

*Remark 2.* As a nice application we can point out that $e^{2/T}\sum_p p^{-s}e^{-p/T}$ where $p$ runs through all primes has $>T(\log T)^{1-\varepsilon}$ zeros in $\sigma\geqq\frac{1}{2}-10^{-8}$, $T\leqq t\leqq 2T$ for all large $T$.

*Remark 3.* However if we apply Theorem 2 to $\zeta(s)$, $\sum_{n=1}^{\infty}\big(\mu(n)e^{-\frac{n}{T}}n^{-s}\big)$ or $\sum_{1\leqq n\leqq T}\big(\mu(n)n^{-s}\big)$ we cannot get such a nice lower bound. We get the lower bound $T(\log T)^{-\varepsilon}$. Here as usual $(\zeta(s))^{-1}=\sum_{n=1}^{\infty}\big(\mu(n)n^{-s}\big)$.

*Remark 4.* Define $d_j(n)$ by $(\zeta(s))^j=\sum_{n=1}^{\infty}\big(d_j(n)n^{-s}\big)$. Then it is possible to bring in the divisor function $d_j(n)$ ($j\geqq 1$ being an integer) or some such other functions in the lower bound for the number of zeros. We mention a result in this direction. A lower bound for the number of zeros is $T(\log T)^{1-\varepsilon}(L_1)^{\frac{j}{j-1}}$, where the loss factor $(L_1)^{\frac{j}{j-1}}$ is defined by

$$L_1=\frac{\left(\sum'\frac{|a_n|^2}{d_j(n)\,n^{2\alpha_2}}\left(\Delta\left(\frac{X}{n}\right)\right)^2\right)\left(\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\alpha_2}}\left(\Delta\left(\frac{X}{n}\right)\right)^2\right)^{-1}}{\left(\sum'\frac{|a_n|^2}{d_j(n)\,n^{2\alpha_1}}\left(\Delta\left(\frac{X}{n}\right)\right)^2\right)^{1/2}\left(\sum'\frac{|a_n|^2}{d_j(n)\,n^{2\alpha_3}}\left(\Delta\left(\frac{X}{n}\right)\right)^2\right)^{1/2}},$$

where the accent denotes the sum over any subsequence of $\{a_n\}$, provided the condition $\sum'_{X\leqq n\leqq 2X}|a_n|^2\geqq X^{\xi-\eta}$ is satisfied ($\xi$ being defined as before). In particular if $|a_n|=0$ or 1 and $\sum_{X\leqq n\leqq 2X}|a_n|^2\gg X$, we have, by taking $j=2$, and the accent to mean the restriction of the sum to those $n$ for which $d(n)$ lies between $(\log n)^{\log 2-\varepsilon}$ and $(\log n)^{\log 2+\varepsilon}$, (and using Hardy—Ramanujan theorem on round numbers which says that almost all $n$ have this property in an asymptotic sense), we get the lower bound $T(\log T)^{-\mu-\varepsilon}$ where $\mu=\log\left(\frac{4}{e}\right)$. Also another particular case $a_n=d_j(n)$ gives the lower bound $T(\log T)^{-j^2+1-\varepsilon}$.

**2. Proof of Theorem 2.** We use finite or infinite series of the type

$$\sum_{n=1}^{\infty}\left(a_n\Delta\left(\frac{X}{\lambda_n}\right)\lambda_n^{-s}\right)$$

(where the first $n$ (say $n_0$) for which $a_n\neq 0$ satisfies $a_{n_0}=1$ and further $n_0$ is independent of the range of $s$ in question, and further $a_n$ can depend on $T$ subject to $T\leqq t\leqq 2T$ and $|a_n|\leqq(n+1)^A$, $X>0$, $0<\frac{1}{A}<\lambda_1<\lambda_2<\lambda_3<\dots$, $0<\frac{1}{A}<\lambda_{n+1}-\lambda_n<A$ for $n=1,2,3,\dots$). We also use series of the type

$$\sum_{n=1}^{\infty}\left(a_n\left(\Delta\left(\frac{X}{\lambda_n}\right)-\Delta\left(\frac{Y}{\lambda_n}\right)\right)\lambda_n^{-s}\right),$$

where $0<Y\leqq X$. We prefer to call these Hardy polynomials of the first type and Hardy polynomials of the second type. Both these functions are not actually polynomials but entire functions. We first prove that these functions assume "large values on a big well spaced set of points". Next from this result we pass on to a similar result on the function represented by $\sum_{n=1}^{\infty}(a_n\lambda_n^{-s})$. From this using Theorem 3 of our earlier paper III of [2], we conclude that the function represented by $\sum_{n=1}^{\infty}(a_n\lambda_n^{-s})$ has "enough zeros". As stated already we follow the method of [3] closely. We begin with

**Lemma 1.** *We have,*

$$\frac{1}{T}\int_T^{2T}\left(\sum_{n=1}^{\infty}A_n\lambda_n^{-it}\right)\left(\sum_{n=1}^{\infty}\bar{B}_n\lambda_n^{it}\right)dt$$

$$=\sum_{n=1}^{\infty}A_n\bar{B}_n+O\left(\frac{1}{T}\left(\sum_{n=1}^{\infty}n|A_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}n\,|B_n|^2\right)^{1/2}\right),$$

*where* $0<\frac{1}{A}<\lambda_1<\lambda_2<\lambda_2<\dots$, $\frac{1}{A}<\lambda_{n+1}-\lambda_n<A$, *(where $A$ is a positive constant),* $(n=1,2,3,\dots)$, $\{A_n\}$ *and* $\{B_n\}$ *are two sequences of complex numbers* ($A_n$, $B_n$, $\lambda_n$ *independent of* $t$) *such that both side make sense. Moreover the O-constant depends only on* $A$.

*Remark.* This lemma is the special case of an important theorem of Montgomery and Vaughan. The special case is also important and for a simple proof of this see [4].

**Lemma 2.** *Let* $F_1(s)=\sum_{n=1}^{\infty}\left(a_n\lambda_n^{-s}\Delta\left(\frac{X}{\lambda_n}\right)\right)$. *Then, we have,*

$$\frac{1}{T}\int_T^{2T}|F_1(\sigma+it)|^2\,dt=\sum_{n=1}^{\infty}\left(1+O\left(\frac{\lambda_n}{T}\right)\right)\frac{|a_n|^2}{\lambda_n^2}\left(\Delta\left(\frac{X}{\lambda_n}\right)\right)^2$$

*Proof.* Follows from Lemma 1.

**Lemma 3.** *Let* $\lambda_n=n$. *Then, we have,*

$$\frac{1}{T}\int_T^{2T}|F_1(\sigma+it)|^4\,dt$$

$$\leqq\left(\sum_{n=1}^{\infty}\frac{d(n)|a_n|^2}{n^{2\sigma}}\left(\Delta\left(\frac{X}{n}\right)\right)^2\right)^2+O\left(\frac{1}{T}\left(\sum_{n=1}^{\infty}\frac{d(n)|a_n|^2}{n^{2\sigma-1}}\left(\Delta\left(\frac{X}{n}\right)\right)^2\right)^2\right).$$

*Proof.* Put $(F_1(s))^2=\sum_{n=1}^{\infty} B_n n^{-s}$. We have,

$$|B_n|^2 = \left|\sum_{n_1n_2=1}\left(a_{n_1}a_{n_2}\Delta\left(\frac{X}{n_1}\right)\Delta\left(\frac{X}{n_2}\right)\right)\right|^2$$

$$\leqq d(n)\sum_{n_1n_2=n}\left(\left|a_{n_1}\Delta\left(\frac{X}{n_1}\right)\right|^2\left|a_{n_2}\Delta\left(\frac{X}{n_2}\right)\right|^2\right)$$

$$\leqq \sum_{n_1n_2=n}\left\{\left(d(n_1)\left|a_{n_1}\Delta\left(\frac{X}{n_1}\right)\right|^2\right)\left(d(n_2)\left|a_{n_2}\Delta\left(\frac{X}{n_2}\right)\right|^2\right)\right\}.$$

From this and Lemma 1, Lemma 3 follows.

*Remark 1.* For $1\leqq Y\leqq X\leqq T$ put

$$F_2(s) = \sum_{n=1}^{\infty}\left\{\frac{a_n}{\lambda_n^s}\left(\Delta\left(\frac{X}{\lambda_n}\right)-\Delta\left(\frac{Y}{\lambda_n}\right)\right)\right\}.$$

Then

$$\frac{1}{T}\int_T^{2T}|F_2(\sigma+it)|^2 \geqq \frac{1}{10T}\int_T^{2T}|F_1(\sigma+it)|^2\,dt,$$

provided

$$\frac{1}{T}\int_T^{2T}\left|\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n^s}\Delta\left(\frac{Y}{\lambda_n}\right)\right|^2 dt$$

is small compared with the corresponding integral with $Y$ replaced by $X$. The same remark applies to the first power mean. As regards upper bounds, for $\frac{1}{T}\int_T^{2T}|F_2(s)|^4\,dt$ for instance we have trivially the upper bound

$$\frac{16}{T}\int_T^{2T}|F_1(s)|^4\,dt+\frac{16}{T}\int_T^{2T}\left|\sum_{n=1}^{\infty}\frac{a_n}{\lambda_n^s}\Delta\left(\frac{Y}{\lambda_n}\right)\right|^4 dt.$$

*Remark 2.* A useful version of Lemma 3 (for example that which helps to generalise Theorem 2) for general $\lambda_n$ is not known.

From now on we assume $T$ to be large enough.

**Lemma 4.** *Let* $S_2=\sum_{n=1}^{\infty}\left(\left|a_n\Delta\left(\frac{X}{n}\right)\right|^2 n^{-2\sigma}\right)$ *and* $S_3=\sum_{n=1}^{\infty}\left(d(n)\left|a_n\Delta\left(\frac{X}{n}\right)\right|^2 n^{-2\sigma}\right)$ *where $X$ is as in Theorem 2. Then the number of integers $M$ with $T\leqq M\leqq 2T-1$ for which*

$$\int_M^{M+1}|F_1(s)|^2\,dt > \frac{1}{10}S_2,\quad (s=\sigma+it),$$

*exceeds* $10^{-3}T(S_2S_3^{-1})^2$.

*Proof.* From Lemmas 2 and 3 we have

$$\int_T^{2T} |F_1(s)|^2 dt > \frac{T}{2} S_2 \quad \text{and} \quad \int_T^{2T} |F_1(s)|^4 dt < 2S_3^2.$$

From the first of these results, we have,

$$\sum{}' \int_M^{M+1} |F(s)|^2 dt > \frac{T}{10} S_2,$$

where the accent denotes the omission of those integrals over unit intervals of the form $(M, M+1)$ (where $M$ is an integer) which do not exceed $\frac{1}{10} S_2$. The lemma now follows by Holder's inequality.

**Lemma 5.** *Let* $\alpha - \frac{1}{A} < \sigma_0 < \sigma < \alpha$. *Then the number of integers* $M$ *with* $T \leqq M \leqq 2T-1$ *for which either*

$$\int_M^{M+1} |F(\sigma_0 + it)|^2 dt > c_0 S_2 X^{-2\sigma_0 + 2\sigma},$$

*or* $\int_M^{M+1} |F(\sigma + it)|^2 dt > c_0 S_2$, *exceeds* $T(S_2 S_3^{-1})^2 (\log T)^{-\varepsilon}$ *for every* $\varepsilon > 0$, *and a suitable constant* $c_0 > 0$, *for all* $T \geqq T_0(\varepsilon)$.

**Corollary.** *Suppose* $S_2$ *exceeds a fixed positive power of* $T$. *(This does happen under the hypothesis of Theorem 2). Then there exists* $T(S_2 S_3^{-1})^2 (\log T)^{-4\varepsilon}$ *integers* $M$ *in* $T \leqq M \leqq 2T-1$ *for which*

$$\int_M^{M+1} |F(\sigma_0 + it)|^2 dt > c_0' S_2,$$

*where* $c_0' > 0$ *is a certain constant.*

*Proof.* We have

$$F_1(s) = \frac{1}{2\pi i} \int F(s+w) X^W \operatorname{Exp}(W^{4k+2}) \frac{dW}{W}$$

where the integration is over a vertical line where $\operatorname{Re}(W)$ is fixed to be large enough and $k$ is a large positive integer constant depending on $\varepsilon$ and $A$. Let $M$ be any positive integer given by Lemma 4. We now cut off the portion $|\operatorname{Im} W| \geqq (\log T)$ with a small error and move the line of integration to such $W$ for which $\operatorname{Re}(s+W) = \sigma_0$. The residue at $W=0$ is $F(s)$. We integrate the mean square of the absolute value and get the lemma.

To deduce the corollary we start with

$$(F(s))^2 = \frac{1}{2\pi i} \int_R (F(W))^2 \operatorname{Exp}((W-s)^{4k+2}) \frac{dW}{W-s}$$

where the integration is over the rectangle $R$ with sides $\mathrm{Re}\, W=\sigma_0$, $\mathrm{Re}\, W=a$ large positive constant and $|\mathrm{Im}\, W|=\pm(\log T)^{\varepsilon}$. We now take the absolute values and integrate from $M$ to $M+1$. Every integer $M$ satisfying the second alternative in Lemma 5 gives rise to at least one integer $M'$ such that

$$\int_{M'}^{M'+1} |F(\sigma_0+it)|^2\, dt > c_0' S_2$$

where $c_0'$ is a positive constant and $|M-M'|\leqq(\log T)^{\varepsilon}$. This proves the corollary.

**Lemma 6.** *The number of zeros of $F(s)$ in the region* $\left\{T\leqq t\leqq 2T,\ \sigma\geqq\alpha-\frac{1}{A}\right\}$ *exceeds* $T(S_2 S_3^{-1})^2(\log T)^{1-\varepsilon}$.

*Proof.* By the Corollary to Lemma 5, there exist at least $\geqq\frac{1}{2}T(S_2 S_3^{-1})^2(\log T)^{-\varepsilon}$ points $\{\sigma_0+it_r\}=\{s_r\}$ which are well spaced i.e. $t_{r+1}-t_r\geqq 1$, at each of which $|F(\sigma_0+it)|$ exceeds a fixed positive constant power of $T$. But by Theorem 3 of paper III in [2] each such point gives rise to $\gg\log T$ zeros and $\varepsilon$ being arbitrary this proves the lemma.

Lemma 6 proves the result mentioned in Remark 1 below Theorem 2. We prove Theorem 2 by imitating the same idea, but with the first power mean lower bound and the mean square upper bound for $F_1(s)$. The rest of this section is devoted to the mean first power lower bound. This once again follows the method of [3].

**Lemma 7.** *Let* $F_3(s)=\sum_{n=1}^{\infty}\left(a_n(d(n))^{-1}n^{-s}\Delta\left(\frac{X}{n}\right)\right)$. *Then*

$$\int_T^{2T} |F_1(s)|\, dt \geqq \sum_{M=[T]+1}^{M=[2T]-2}\int_M^{M+1} |F_1(s)|\, dt$$

$$\geqq \frac{1}{D}\sum_I'\int_I |F_1(s) F_3(s)|\, dt,$$

*where $D>0$ is a free parameter (to be chosen later) and the sum is over those unit intervals $I$, for which* $\max_{t \text{ in } I}|F_3(s)|>0$.

*Proof.* Trivial.

**Lemma 8.** *We have,*

$$\sum_I \max_{t \text{ in } I} |F_3(s)|^4 = O(TS_4 S_5)$$

*where* $S_4=\sum_{n=1}^{\infty}\left(\left|a_n\Delta\left(\frac{X}{n}\right)\right|^2 (d(n))^{-1}n^{-2\sigma_1}\right)$, $S_5=\sum_{n=1}^{\infty}\left(\left|a_n\Delta\left(\frac{X}{n}\right)\right|^2 (d(n))^{-1}n^{-2\sigma_2}\right)$,

*and $\sigma_1$ and $\sigma_2$ are arbitrary constants such that $\sigma_1<\sigma<\sigma_2$ and $2\sigma=\sigma_1+\sigma_2$. Also $X$ is as in Theorem 2.*

*Proof.* Let $s_i$ be the points at which the maximum are attained. We use

$$(F_3(s_i))^4 = \frac{1}{2\pi i}\int_R (F_3(w))^4 Z^{w-s_i}\operatorname{Exp}((w-s_i)^2)\frac{dw}{w-s_i},$$

where $R$ is the rectangle bounded by the lines $\operatorname{Re} w=\sigma_1$, $\operatorname{Re} w=\sigma_2$, $\operatorname{Im} w=T-(\log T)$, $\operatorname{Im} w=2T+\log T$. Here $Z$ is a free parameter to be choosen later. From this taking absolute values and summing up with respect to $i$, we get,

$$\sum_I \max_{t \text{ in } I} |F_3(s)|^4$$

$$= O\big(T(Z^{\sigma_1-\sigma}J_1+Z^{\sigma_2-\sigma}J_2)+K\big)$$

where $J_1=\frac{1}{T}\int_{T-\log T}^{2T+\log T}|F_3(\sigma_1+it)|^4\,dt$ and $J_2=\frac{1}{T}\int_{T-\log T}^{2T+\log T}|F_3(\sigma_2+it)|^4\,dt.$ Further $K$ is small enough to be ignored for the choice of $Z$ which gives $Z^{\sigma_1-\sigma}J_1=Z^{\sigma_2-\sigma_1}J_2$. The lemma now follows from Lemma 3.

**Lemma 9.** *Put* $S_1=\sum_{n=1}^{\infty}\left(\left|a_n\Delta\left(\frac{X}{n}\right)\right|^2(d(n))^{-1}n^{-2\sigma}\right)$. *Then with $X$ satisfying the hypothesis of Theorem 2, we have,*

$$\frac{1}{T}\int_T^{2T}|F_1(s)|\,dt \geqq c_1 S_1^2(S_2 S_4 S_5)^{-1/2},$$

*where $c_1$ is a positive constant independent of $T$.*

*Proof.* By Hölder's inequality the last lower bound in Lemma 7 is (on using Lemma 8)

$$\frac{1}{D}\int_T^{2T}|F_1(s)F_3(s)|\,dt+O\left(\frac{T}{D^2}(S_2S_4S_5)^{1/2}\right)$$

$$\geqq \frac{1}{D}\left|\int_T^{2T}F_1(s)\overline{F_3(s)}\,dt\right|+O\left(\frac{T}{D^2}(S_2S_4S_5)^{1/2}\right)$$

$$\geqq \frac{T}{2D}S_1+O\left(\frac{T}{D^2}(S_2S_4S_5)^{1/2}\right),$$

on using Lemma 1. The lemma in question follows on choosing $\frac{1}{D}$ to be a suitable constant times $S_1(S_2S_4S_5)^{-1/2}$.

Theorem 2 can now be deduced from Lemmas 9 and 2, just as we deduced the result in Remark 1 below Theorem 2, from Lemmas 2 and 3. The result mentioned in Remark 4 below Theorem 2 can be deduced in the same way, but we have now to work with the function

$$F_4(s) = \sum{}' \left(a_n \Delta\left(\frac{X}{n}\right)(d(n))^{-1} n^{-s}\right)$$

in place of $F_3(s)$ and obtain an appropriate lower bound in Lemma 9.

**3. Proof of Theorem 1 and generalisations.** The notation of this section will be independent of the previous sections. We now begin by explaining a special type of Dirichlet series $\sum_{n=1}^{\infty} (a_n b_n \lambda_n^{-s})$ satisfying conditions (i) to (vii) below.

*Let $f(x)$ and $g(x)$ be positive real valued functions defined in $x \geqq 0$ satisfying*

(i) *$f(x)x^{\delta}$ is monotonic increasing and $f(x)x^{-\delta}$ is monotonic decreasing for every $\delta > 0$ and all $x \geqq x_0(\delta)$.*

(ii) $\lim_{x\to\infty} \dfrac{g(x)}{x} = 1.$

(iii) *For all $x \geqq 0$, $0 < a \leqq g'(x) \leqq b$ and $0 < a \leqq (g'(x))^2 - g(x)g'(x) \leqq b$ where $a$ and $b$ are constants.*

*Let $\{a_n\}$, $\{b_n\}$, $\{u_n\}$, $\{v_n\}$ be four infinite sequences satisfying the following conditions. $\{a_n\}$, $\{u_n\}$, $\{v_n\}$ are bounded sequences of complex numbers of which $\{u_n\}$ and $\{v_n\}$ are real and monotonic. We will set $\lambda_n = g(n) + u_n + v_n$ and assume that $\lambda_n > 0$ for all $n$.*

(iv) *$|b_n|$ lies between $af(n)$ and $bf(n)$ for all $n$.*

(v) *For all $X \geqq 1$, $\sum_{X \leqq n \leqq 2X} |b_{n+1} - b_n| \leqq bf(X)$.*

*We next assume that $\{a_n\}$ and $\{b_n\}$ satisfy one at least of the following two conditions* (vi) *and* (vii).

(vi) *Monotonicity condition.* $\text{Lim}_{x\to\infty} x^{-1} \sum_{n \leqq x} a_n = h$, *where $h$ is a non-zero constant (which may be complex) and further $|b_n|\lambda_n^{-\delta}$ is monotonic decreasing for every $\delta > 0$ and all $n \geqq n_0(\delta)$.*

(vii) *Real part condition. There exists an infinite arithmetic progression of positive integers such that if the accent denotes the restriction of the sum to these integers then,*

$$\liminf_{x\to\infty} \left(\frac{1}{x} \sum{}'_{x \leqq \lambda_n \leqq 2x,\, \mathrm{Re}\, a_n > 0} \mathrm{Re}\, a_n\right) > 0,$$

*and*

$$\lim_{x\to\infty} \left(\frac{1}{x} \sum{}'_{x \leqq \lambda_n \leqq 2x,\, \mathrm{Re}\, a_n < 0} \mathrm{Re}\, a_n\right) = 0.$$

Then we have the following Theorem 3.

**Theorem 3.** *Let* $F_1(s)=\sum_{n=1}^{\infty}\left(a_n b_n \Delta\left(\frac{T}{\lambda_n}\right)\lambda_n^{-s}\right)$. *Then for* $\sigma<\frac{1}{2}$ *and* $T\geqq 10$, *we have,*

$$\frac{1}{T}\int_T^{2T}|F_1(\sigma+it)|\,dt>c_2 T^{1/2-\sigma}f(T),$$

*where* $c_2>0$ *is a constant independent of* $T$.

*Also for* $1\leqq X\leqq T$, *we have,*

$$\frac{1}{T}\int_T^{2T}\left|\sum_{n=1}^{\infty}\left(a_n b_n \Delta\left(\frac{X}{\lambda_n}\right)\lambda_n^{-\sigma-it}\right)\right|^2 dt$$

$$<c_3\left(\sum_{\lambda_n\leqq X}|a_n b_n \lambda_n^{-\sigma}|^2+\frac{X^2}{T}\sum_{\lambda_n\geqq X}|a_n b_n \lambda_n^{-\sigma-1/2}|\right),$$

*where* $c_3>0$ *is a constant independent of* $T$ *and* $X$.

*Remark 1.* The first part of the theorem is nearly explained in [3]. The role of $F_3(s)$ of § 2 of the present paper is played by $F_5(s)=\sum^*_{\lambda_n\leqq D_0 T}(b_n\lambda_n^{-s})$ (where $D_0$ is a certain positive constant and $*$ denotes the sum restricted to the arithmetic progression of condition (vii) if it is satisfied, or all positive integers $n$ if condition (vi) is satisfied), which possesses a *gth* power mean with $g=g(\sigma)>2$ if $\sigma<\frac{1}{2}$ in the sense $\frac{1}{T}\int_T^{2T}|F_5(\sigma+it)|^g dt=O\left((T^{1/2-\sigma}f(T))^g\right)$. This *gth* power result is easily deducible from Lemma 6 of paper IV in [2], which is quoted in [3] as Theorem 4. The rest of the proof follows [3] except that $\operatorname{Exp}\left(-\frac{\lambda_n}{T}\right)$ is replaced by $\Delta\left(\frac{T}{\lambda_n}\right)$.

*Remark 2.* Let $\sigma>0$. Then *RHS* in the second inequality of theorem 3, is $\leqq c_4\left(\sum_{\lambda_n\leqq X}\frac{(f(n))^2}{n^{2\sigma}}+\frac{X^2}{T}\sum_{\lambda_n\geqq X}\frac{(f(n))^2}{n^{2\sigma+1}}\right)$. Using the fact that $f(n)n^{\delta}$ is monotonic increasing and $f(n)n^{-\delta}$ is monotonic decreasing for $n\geqq n_0(\delta)$, we see that this is $\leqq c_5 X^{1-2\sigma}(f(X))^2$. Further if $\mu$ is a constant satisfying $0<\mu<\frac{1}{2}-\sigma$, we see that $X^{1-2\sigma}(f(X))^2=X^{1-2\sigma-2\mu}(f(X)X^{\mu})^2\leqq X^{1-2\sigma-2\mu}(f(T)\,T^{\mu})^2=\left(\frac{X}{T}\right)^{1-2\sigma-2\mu}T^{1-2\sigma}(f(T))^2$ for $T\geqq T_0(\mu)$ and $X\geqq X_0(\mu)$. Thus if $T\ll X\ll T$ and $T\geqq T_0(\mu)$ the right hand side in the second inequality of theorem 3 is $O\left(\left(\frac{X}{T}\right)^{1-2\sigma-2\mu}T^{1-2\sigma}(f(T))^2\right)$ if $0<\sigma<\frac{1}{2}$ and $\mu=\frac{1}{4}-\frac{\sigma}{2}$. However the same result is true for all $\sigma<\frac{1}{2}$ and $\sigma>0$ is not used essentially.

We next state (as a corollary to Theorem 3 and the remarks below it),

**Lemma 10.** *Let* $F_2(s)=\sum_{n=1}^{\infty}\left(a_n b_n\left(\Delta\left(\frac{T}{\lambda_n}\right)-\Delta\left(\frac{DT}{\lambda_n}\right)\right)\lambda_n^{-s}\right)$, *where* $D>0$ *is a sufficiently small constant.*

*Then if* $\sigma<\frac{1}{2}$, *we have,*

$$\frac{1}{T}\int_{T}^{2T}|F_2(\sigma+it)|\,dt>c_6T^{1/2-\sigma}f(T)$$

*and*

$$\frac{1}{T}\int_{T}^{2T}|F_2(\sigma+it)|^2\,dt<c_7T^{1-2\sigma}(f(T))^2,$$

*where* $c_6$ *and* $c_7$ *are positive constants independent of* $T$.

As in [3] we deduce from this lemma

**Theorem 4.** *In the notation of Lemma* 10, *the number of integers* $M$ *in the range* $T\leqq M\leqq 2T-1$, *for which*

$$\int_{M}^{M+1}|F_2(\sigma+it)|\,dt>c_8T^{1/2-\sigma}f(T)$$

*exceeds* $c_9T$. *Here* $T\geqq 10$, *and* $c_8$ *and* $c_9$ *are positive constants independent of* $T$.

We now state the main theorem of this section.

**Theorem 5.** *Let* $T\geqq 10$ *and suppose that there exist positive constants* $\Phi$ $(\Phi<\frac{1}{2})$ *and* $A$ *such that the series* $F(s)=\sum_{n=1}^{\infty}(a_nb_n\lambda_n^{-s})$ *can be continued analytically in* $\sigma\geqq\Phi$, $T\leqq t\leqq 2T$ *and that* $\max|F(s)|$ *taken over this region does not exceed* $T^A$. *Let* $\sigma=\frac{1}{2}(\Phi+\frac{1}{2})$ *and* $\varepsilon>0$ *an arbitrary constant. Then the number of integers* $M$ *in the range* $T\leqq M\leqq 2T-1$ *for which*

$$\int_{M}^{M+1}|F(\sigma+it)|\,dt>c_{10}T^{1/2-\sigma}f(T),$$

*exceeds* $T(\log T)^{-\varepsilon}$ *provided further* $T\geqq T_0(\varepsilon)$. *Here* $\{a_n\}$, $\{b_n\}$, $\{\lambda_n\}$ *satisfy the conditions* (i) *to* (vii) *and* $c_{10}$ *is a positive constant independent of* $T$. *Further, (on using an earlier theorem of ours viz. theorem* 3 *of paper* III *in* [2]*) the number of zeros of* $F(s)$ *in* $\sigma\geqq\Phi$, $T\leqq t\leqq 2T$ *exceeds* $c_{11}T(\log T)^{1-\varepsilon}$ *where* $c_{11}$ *is a positive constant independent of* $T$.

*Remark 1.* Theorem 1 is the special case $a_n=1+\alpha_n$ (where $\alpha_n$ is the $a_n$ of theorem 1), $b_n=1$, $\lambda_n=n$. One can verify the conditions (i) to (vii) by taking $f(x)=g(x)=x$, and $\Phi$ to be any constant between $\theta$ and $\frac{1}{2}$ and $A$ to be 1. For the derivatives we have to take $b_n=(\log n)^l$.

*Remark 2.* It is easy to see that if we have good upper bounds for $\frac{1}{T}\int_T^{2T}|F(\sigma+it)|^2dt$ for $\sigma<\frac{1}{2}$, then we can improve the lower boundes $T(\log T)^{-\varepsilon}$ and $c_{11}T(\log T)^{1-\varepsilon}$ given by the theorem above.

*Proof of Theorem 5.* We have, by putting $s=\frac{1}{2}(\Phi+\frac{1}{2})+it$,

$$F_2(s)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s+W)T^W \operatorname{Exp}(W^{4k+2})\left(\frac{D^W-1}{W}\right)dW.$$

We cut off the portion $|\text{Im } W|\geqq(\log T)^{\varepsilon}$ with a small error, and move the rest of the line of integration to $\text{Re } W=0$. Let $M$ be given by Theorem 4. We now choose $k$ large, take absolute values both sides and integrate from $M$ to $M+1$ (confining to those $M$ in $T+\log T\leqq M\leqq 2T-\log T$). This proves Theorem 5 completely.

*Added in poof.* It is possible to replace the quality $(\log T)^{-\varepsilon}$ by a constant multiple of $(\log\log T)$ in every one of our theorems. Because we can replace the function $\operatorname{Exp}(w^{4R+2})$, (throughout) by $\operatorname{Exp}\left(\left(\operatorname{Sin}\left(\frac{W}{100\,A}\right)\right)^2\right)$. For example in place of $\Delta(x)$ we use the function

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^W \operatorname{Exp}\left(\left(\operatorname{Sin}\frac{W}{100\,A}\right)^2\right)\frac{dW}{W}.$$

Thus in Theorem 5, the number of genos is $\gg \frac{T\log T}{\log\log T}$.

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