The Bergman projection over plane regions

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1. Introduction

Let $D \notin O_G$ (i.e., D has a nontrivial Green's function) be a plane region and let $L_p(D)$, $1<p<\infty$, be the usual Lebesgue space of functions on D, relative to the area Lebesgue measure $d\sigma(z)=dx\,dy$. For $1\leqq p<\infty$ and $q=p/p-1$, the pairing between $f \in L_p(D)$ and $g \in L_q(D)$ is given by $(f,g)=\int_D f(z)\overline{g(z)}\,d\sigma(z)$. The class of holomorphic functions on D will be denoted by $H(D)$ and we write $B_p(D)= L_p(D)\cap H(D)$. The Banach space $B_p(D)$ is called the *Bergman p-space* of D and its norm is given by $\|f\|_p=\{\int_D|f(z)|^p\,d\sigma(z)\}^{1/p}$. Let $K_D(z,\bar{\zeta})$ be the Bergman kernel of D and consider the "*Bergman projection*" (whenever is defined)

$$(P_D f)(\zeta)=(f, K_D(\cdot,\bar{\zeta}))=\int_D f(z)\overline{K_D(z,\bar{\zeta})}\,d\sigma(z).$$

When D is subjected to some mild smoothness requirements, it was shown in [5] and [6] that P_D is a bounded projection of $L_p(D)$ onto $B_p(D)$, for $1<p<\infty$. With this we have the decomposition $L_p(D)=B_p(D)\oplus B_q(D)^{\perp}$. This was done by exploiting an integral operator involving the "adjoint" [3] of the Bergman kernel. The latter has the required singularity of the theory of singular integrals.

Quite recently Solov'ev [12], using different methods, has announced a number of results on the boundedness of the Bergman projection provided D is a bounded finitely connected region with some smoothness requirements on its boundary. These results of [12] are similar to those of our previous work [5] and our present method of proof can be also applied to extend them to the more general case of $D \notin O_G$.

In this paper we extend our previous results [5, 6] to the more general case when $D \notin O_G$ and we investigate more extensively operators related to the "projection" P_D. We also characterize all regions $D \notin O_G$ for which P_D is a bounded projection on $L_p(D)$ by introducing a condition of the Muckenhoupt's type. The treatment of these problems will also yield new identities between certain relevant operators which may be of some interest and will enable us to identify the annihilator $B_q(D)^{\perp}$ of $B_p(D)$.

In § 2 we review some known results from the theory of singular integrals while in § 3 we introduce the Bergman—Schiffer transforms. In § 4 we establish the various relationships amongst the Bergman—Schiffer transforms, the Bergman projection and the Hilbert transform when D is an arbitrary region, $D \notin O_G$ (Theorem 1 and its corollaries). In § 5 we introduce the crucial class of regions W_p $(1<p<\infty)$ in terms of a universal cover mapping of a region. The fact that this definition is intrinsic is proved in Theorem 2. The intimate connection between this class and the boundedness of the Bergman projection is demonstrated in Theorem 3 and Corollary 2. A detailed study of the Bergman projection is conducted in § 6 (Theorems 4, 5 and their corollaries) while § 7 is devoted to the identification of the annhilator $B_q(D)^{\perp}$ as a Sobolev space and as an image of a Bergman—Schiffer transform (Proposition 4 and its corollaries).

2. Singular integrals

In this section we collect some known facts from the theory of singular integrals which will be needed in our work. Throughout this paper we shall restrict ourselves to the case $1<p<\infty$ with $q=p/p-1$. We shall consider certain transforms (integral operators) on $L_p(\mathbf{C})$. For a kernel $h(z, \zeta)$ we define the transform

$$(Hf)(\zeta) = \int_{\mathbf{C}} \overline{h(z, \zeta)} f(z)\, d\sigma(z)$$

and its (formal adjoint)

$$(H^* f)(\zeta) = \int_{\mathbf{C}} h(\zeta, z) f(z)\, d\sigma(z).$$

When the kernel $h(z, \zeta)$ is singular, the integrals shall always be taken in the principal value sense. Let χ_D be the characteristic function of the measurable subset D of $\mathbf{C}$ and write $H_D = H\chi_D$. Therefore, if H maps $L_p(\mathbf{C})$ into $L_p(\mathbf{C})$ we can view H_D as a mapping of $L_p(D)$ into $L_p(\mathbf{C})$ or $L_p(D)$ in a natural way. In particular, if H is bounded on $L_p(\mathbf{C})$ then H_D is bounded on $L_p(D)$. Also, $H_D^* = H^* \chi_D$ on $L_q(D)$.

We consider the following familiar transforms; the Cauchy transform

$$(Sf)(\zeta) = \frac{1}{\pi} \int_{\mathbf{C}} \frac{1}{\overline{z-\zeta}} f(z)\, d\sigma(z)$$

and the Hilbert transform

$$(Tf)(\zeta) = \frac{1}{\pi} \int_{\mathbf{C}} \frac{1}{\overline{(z-\zeta)}^2} f(z)\, d\sigma(z).$$

It is well known that these operators are bounded on $L_p(\mathbf{C})$ for any $1<p<\infty$, and, moreover

$$T^*T = TT^* = I \tag{2.1}$$

on $L_p(\mathbf{C})$ with I being the identity operator on $L_p(\mathbf{C})$. From this follows that

$$T^*T_D = T^*T\chi_D = (T^*T)_D = I_D,$$

$$TT_D^* = TT^*\chi_D = (TT^*)_D = I_D$$

on $L_p(D)$ with I_D being the identity operator on $L_p(D)$.

Let $z=x+iy$ and let f be a differentiable function near z. We write

$$f_z = \partial_z f = \tfrac{1}{2}(\partial_x f - i\partial_y f),$$

$$f_{\bar z} = \partial_{\bar z} f = \tfrac{1}{2}(\partial_x f + i\partial_y f).$$

By a use of Green's formula one easily obtains:

Proposition 1. *Let f be of class C^1 in* **C** *and assume it has a compact support. Then*

$$f = S^* f_{\bar z}, \quad f = -Sf_z$$

and

$$f_z = -T^* f_{\bar z}, \quad f_{\bar z} = -Tf_z.$$

We also note the following known fact: namely, if $f\in L_p(\mathbf{C})$ then Sf is absolutely continuous on almost every line parallel to either axis and, therefore, $(Sf)_z$ and $(Sf)_{\bar z}$ exist almost everywhere in **C**. In fact, $(Sf)_z=-f(z)$ and $(Sf)_{\bar z}=(Tf)(z)$ (for almost all z in **C**).

Let $\Delta=\{z\colon |z|<1\}$ denote the unit disk. We recall, that if L is a circle orthogonal to the boundary of $\Delta, \partial\Delta$, then the part of L in Δ is called a noneuclidean line in the Poincare's model of hyperbolic geometry. Naturally, a diameter of Δ is also a noneuclidean line and every noneuclidean line separates the noneuclidean plane Δ into two noneuclidean half planes. Any part of a noneuclidean line is called a noneuclidean segment.

By a noneuclidean polygon in Δ we shall mean a subset S of Δ bounded by a simple curve consisting of finitely many noneuclidean segments. The family of all such polygons is denoted by $N(\Delta)$. Evidently, this family remains invariant under the action of any $A\in \text{Möb}(\Delta)$ and it covers Δ.

Let λ be a non-negative locally integrable function on Δ. The space $L_p(\Delta\colon \lambda)$ stands for the class of functions on Δ for which

$$\|f\|_{L_p(\Delta:\lambda)} = \Big\{\int_\Delta |f(z)|^p \lambda(z)\, d\sigma(z)\Big\}^{1/p}$$

is finite. For future reference we shall record the following proposition which is due to Coifman and Fefferman [7] (see also [2]):

Proposition 2. *The Hilbert transform T_Δ is a bounded operator of $L_p(\Delta\colon \lambda)$ into $L_p(\Delta\colon \lambda)$ if and only if*

$$\sup_S \left[\frac{1}{|S|}\int_S \lambda(z)\,d\sigma(z)\right]\cdot\left[\frac{1}{|S|}\int_S \lambda(z)^{-\frac{1}{p-1}}\,d\sigma(z)\right]^{p-1} < \infty, \tag{2.2}$$

where the supremum is taken over all $S\in N(\Delta)$. Here $|S|=\sigma(S)$.

The condition (2.2) on λ is called the Muckenhoupt M_p-condition. The family $N(\Delta)$ can be, of course, replaced by a variety of other families (see, for example, [2]). The advantage of using noneuclidean geometry, however, is clearly demonstrated in defining the class W_p in § 5.

3. The Bergman—Schiffer transforms

Let $G=G_D(z,\zeta)$ be the Green's function of the region $D\notin O_G$. Thus

$$G_D(z,\zeta) = H(z,\zeta)-\log|z-\zeta|,$$

where $H=H(z,\zeta)$ is symmetric and harmonic in $(z,\zeta)\in D\times D$. The Bergman kernel $K(z,\bar\zeta)=K_D(z,\bar\zeta)$ is given by

$$K(z,\bar\zeta) = -\frac{2}{\pi}\partial_z\partial_{\bar\zeta}G$$

and its "adjoint" $L(z,\zeta)=L_D(z,\zeta)$ is

$$L(z,\zeta) = -\frac{2}{\pi}\partial_z\partial_\zeta G.$$

Therefore,

$$L(z,\zeta) = \frac{1}{\pi}\frac{1}{(z-\zeta)^2}-l(z,\zeta)$$

where

$$l(z,\zeta) = \frac{2}{\pi}\partial_z\partial_\zeta H$$

is symmetric and holomorphic in $(z,\zeta)\in D\times D$. The function $l(z,\zeta)$ is holomorphic in $(z,\zeta)\in \bar D\times\bar D$ whenever the boundary ∂D is analytic. For this and other related results, one is referred to Bergman and Schiffer [3]. Also, it can be shown that $l(z,\zeta)$ is identically zero if and only if D is a disk less (possibly) a set of zero inner capacity.

In analogy to the Bergman projection

$$(P_D f)(\zeta) = \int_D \overline{K(z,\bar\zeta)}f(z)\,d\sigma(z),$$

we also introduce the "*Bergman—Schiffer*" transforms

$$(Q_D f)(\zeta) = \int_D \overline{L(z,\zeta)} f(z)\, d\sigma(z)$$

and

$$(A_D f)(\zeta) = \int_D \overline{l(z,\zeta)} f(z)\, d\sigma(z).$$

Therefore,

$$T_D = Q_D + A_D. \tag{3.1}$$

Throughtout the remainder of this section we shall assume that the boundary ∂D is analytic. The class of all such regions will be denoted by **A**. In this case $l(z,\zeta)$ is holomorphic on $\bar{D}\times\bar{D}$ and $K(z,\bar{\zeta})$ is holomorphic in $(z,\bar{\zeta})$ for $(z,\zeta)\in\bar{D}\times D$. It is also a matter of a straightforward application of Green's formula to verify the following statement (see [3] for additional details):

Lemma 1. *Let $D\in\mathbf{A}$. Then, for $z,\zeta\in D$ we have*

$$\int_D \overline{l(\zeta,t)}\, l(z,t)\, d\sigma(t) = \frac{1}{\pi}\int_D \frac{1}{\overline{(t-\zeta)^2}}\, l(z,t)\, d\sigma(t) = K(z,\bar{\zeta}) - \Gamma(z,\bar{\zeta}) \tag{3.2}$$

where

$$\Gamma(z,\bar{\zeta}) = \frac{1}{\pi^2}\int_{E_D} \frac{d\sigma(t)}{(t-z)^2\,\overline{(t-\zeta)^2}}, \quad E_D \equiv \hat{\mathbf{C}} - \bar{D}. \tag{3.3}$$

Also

$$l(z,\zeta) = \int_D l(z,t) K(t,\bar{\zeta})\, d\sigma(t) = \frac{1}{\pi}\int_D \frac{1}{(z-t)^2} K(t,\bar{\zeta})\, d\sigma(t). \tag{3.4}$$

Clearly, $K(z,\bar{\zeta})$, $\Gamma(z,\bar{\zeta})$ and $K(z,\bar{\zeta})-\Gamma(z,\bar{\zeta})$ are Hermitian positive definite kernels on D. Also $\Gamma(z,\bar{\zeta})$ is holomorphic in $(z,\bar{\zeta})$ for $(z,\zeta)\in\bar{D}\times D$.

4. Identities for general regions

We now return to the general case namely, that D is a plane region, $D\notin O_G$. Let $\{D_n\}$ be a canonical exhaustion of D where each D_n belongs to **A**. The corresponding kernels of D_n, $K_n(z,\bar{\zeta})$ and $l_n(z,\zeta)$ converge strongly in $L_2(D)$ and hence uniformly on compacta of D to $K(z,\bar{\zeta})$ and $l(z,\zeta)$ respectively, when $n\to\infty$. For these, see [10] and [13]. From these also follows that Lemma 1 remains valid for the general case too, and, especially,

$$f(\zeta) = (f, K(\cdot,\bar{\zeta})), \quad \zeta\in D,$$

for any $f \in B_2(D)$. Here, of course, (3.3) holds and therefore, for a fixed $\zeta \in D$ we have

$$\Gamma(z, \bar{\zeta}) = \frac{1}{\pi} T^*_{E_D}\left[\frac{1}{\overline{(t-\zeta)^2}}\right](z), \quad E_D = \hat{\mathbf{C}} - \bar{D}.$$

Therefore,

$$\Gamma(z, \bar{\zeta}) = \frac{1}{\pi} T^*\left[\chi_{E_D}(t) \frac{1}{\overline{(t-\zeta)^2}}\right](z).$$

This function is in $B_2(D)$ for any fixed $\zeta \in D$. In fact $\Gamma(z, \bar{\zeta})$ is in $L_p(\mathbf{C})$ for any fixed $\zeta \in D$. Indeed,

$$\|\Gamma(\cdot, \bar{\zeta})\|_{L_p(\mathbf{C})} \leqq A_p \frac{1}{\pi} \left\|\chi_{E_D}(t) \frac{1}{\overline{(t-\zeta)^2}}\right\|_{L_p(\mathbf{C})} = A_p \frac{1}{\pi}\left\{\int_{E_D} \frac{d\sigma(t)}{|t-\zeta|^{2p}}\right\}^{1/p}.$$

Here A_p is a constant which can be taken as $A_p = k_1 p + k_2 (p-1)^{-1}$ with k_1, k_2 being positive constants independent of p $(1<p<\infty)$. When $p=2$, since T is an isometry on $L_2(\mathbf{C})$, we even have

$$\|\Gamma(\cdot, \bar{\zeta})\|_{L_2(\mathbf{C})} = \frac{1}{\pi} \left\|\chi_{E_D}(t) \frac{1}{\overline{(t-\zeta)^2}}\right\|_{L_2(\mathbf{C})} = \frac{1}{\pi}\left\{\int_{E_D} \frac{d\sigma(t)}{|t-\zeta|^{4}}\right\}^{1/2}.$$

We now introduce another transform

$$(\Lambda_D f)(\zeta) = (f, \Gamma(\cdot, \bar{\zeta})) = \int_D f(z) \overline{\Gamma(z, \bar{\zeta})}\, d\sigma(z).$$

Using (2.1), (3.3) and Fubini's theorem we have

$$\begin{aligned}(\Lambda_D f)(\zeta) &= \frac{1}{\pi}\int_{E_D} \frac{1}{(t-\zeta)^2}\left[\frac{1}{\pi}\int_D \frac{1}{\overline{(t-z)^2}} f(z)\, d\sigma(z)\right] d\sigma(t) \\ &= T^*[\chi_{E_D} T_D f](\zeta) = T^*[(1-\chi_D) T_D f](\zeta) \\ &= [(I_D - T_D^* T_D) f](\zeta)\end{aligned}$$

on $L_p(D)$. Hence

$$\Lambda_D = I_D - T_D^* T_D \tag{4.1}$$

on $L_p(D)$, and, Λ_D is a bounded self-adjoint operator on $L_p(D)$.

The operators Λ_D, P_D and Q_D are always bounded on $L_2(D)$ by virtue of the fact that $L_2(D)$ and $B_2(D)$ are both Hilbert spaces while the "geometric" operators T_D and Λ_D are bounded on $L_p(D)$. The following relationships amongst these operators will be needed in our work.

Lemma 2. *Let $D \notin O_G$, then on $L_2(D)$ the following hold:*

$$\Lambda_D^* \Lambda_D = \Lambda_D^* T_D, \tag{4.2}$$

$$\Lambda_D = T_D P_D = \Lambda_D P_D, \tag{4.3}$$

$$Q_D P_D = \Lambda_D^* Q_D = 0 \tag{4.4}$$

and

$$A_D = P_D - A_D^* A_D. \tag{4.5}$$

Proof. (4.2) follows from (3.2) and (4.3) follows from (3.4). Next, by (3.1) and (4.2), $A_D^* Q_D = A_D^*(T_D - A_D) = 0$. Also, by (3.1) and (4.3), $Q_D P_D = (T_D - A_D) P_D = 0$ and (4.4) follows. Finally, (4.5) follows from (3.2).

This lemma provides us with rather interesting identities involving the operators Q_D and P_D which are described in the following theorem:

Theorem 1. *Let $D \notin O_G$, then, on $L_2(D)$,*

$$Q_D = T_D(I_D - P_D) \tag{4.6}$$

and

$$I_D - P_D = T_D^* Q_D = Q_D^* Q_D. \tag{4.7}$$

Proof. According to (3.1) and (4.3), $Q_D = T_D - A_D = T_D - T_D P_D$ and (4.6) follows. Next, by (4.1), (4.2) and (4.5) we have

$$I_D - P_D = T_D^* T_D - A_D^* A_D = T_D^* T_D - A_D^* T_D$$

and thus, using (3.1), $I_D - P_D = Q_D^* T_D$. Conjugating the last identity we obtain the first half of (4.7). Further, by (3.1) and (4.4), we also have $T_D^* Q_D = (Q_D^* + A_D^*) Q_D = Q_D^* Q_D$ and the theorem follows.

The result $I_D - P_D = Q_D^* Q_D$ when D is assumed to be of class **A** was first proved by Block [4] by using different methods. Clearly, in view of (4.4) and (4.7), $P_D^2 = P_D$. Also $P_D f$ is in $B_2(D)$ for any $f \in L_2(D)$ and $P_D f = f$ for all $f \in B_2(D)$. The above results are valid on $L_2(D)$ and on $L_p(D)$, $p \neq 2$, P_D and Q_D are not necessarily bounded. However, we have the following simple, yet crucial corollary:

Corollary 1. *On $L_p(D)$ the boundedness of P_D is equivalent to the boundedness of Q_D.*

Proof. Assume P_D is bounded on $L_p(D)$. By (4.6) of Theorem 1, $Q_D = T_D - T_D P_D$ and so Q_D is bounded on $L_p(D)$ for, T_D is so. Conversely, assume Q_D is bounded on $L_p(D)$. By (4.7) of Theorem 1, $P_D = I_D - T_D^* Q_D$ and the result follows.

5. The class W_p

In view of Proposition 2, P_D is bounded on $L_p(D)$ provided the boundary ∂D of D is sufficiently smooth. Here, we shall characterize the class of all regions D for which Q_D and, therefore, P_D is bounded on $L_p(D)$.

Since $D \notin O_G$, D has the unit disk $\Delta = \{\omega : |\omega| < 1\}$ as its universal cover. Let $\pi : \Delta \to D$ be a universal covering map for D. Let Γ be the covering group of π, that is, Γ consists of all those $\gamma \in$ Möb (Δ) for which $\pi \circ \gamma = \pi$. Then, as is well known,

$$G_D(z, \zeta) = \sum_{\gamma \in \Gamma} \log \left| \frac{1-\gamma(\omega)\bar{\tau}}{\gamma(\omega)-\tau} \right|$$

with $z = \pi(\omega), \zeta = \pi(\tau)$; $\omega, \tau \in \Delta$. On the unit disk Δ,

$$G_\Delta(\omega, \tau) = \log \left| \frac{1-\omega\bar{\tau}}{\omega-\tau} \right|,$$

$$K_\Delta(\omega, \bar{\tau}) = \frac{1}{\pi} \frac{1}{(1-\omega\bar{\tau})^2}$$

and

$$L_\Delta(\omega, \tau) = \frac{1}{\pi} \frac{1}{(\omega-\tau)^2}.$$

Consequently,

$$G_D(z, \zeta) = \sum_{\gamma \in \Gamma} G_\Delta(\gamma(\omega), \tau),$$

$$K_D(z, \bar{\zeta}) \pi'(\omega) \overline{\pi'(\tau)} = \sum_{\gamma \in \Gamma} K_\Delta(\gamma(\omega), \bar{\tau}) \gamma'(\omega)$$

and

$$L_D(z, \zeta) \pi'(\omega) \pi'(\tau) = \sum_{\gamma \in \Gamma} L_\Delta(\gamma(\omega), \tau) \gamma'(\omega).$$

The above representations are evidently well defined and they are independent of the choice of the projection map π.

Let $\Omega = \Delta / \Gamma$ be a fundamental region of Γ. In this case $\pi|_\Omega$ is (modulo a set of measure zero) a homeomorphism of Ω onto D. Also, $\Delta = \bigcup_{\gamma \in \Gamma} \gamma(\Omega)$. Let $\Phi = (\pi|_\Omega)^{-1}$. For a measurable subset U of D, we write

$$\|\Phi'\|_{p:U} = \left\{ \int_U |\Phi'(z)|^p \, d\sigma(z) \right\}^{1/p}.$$

Let $N(\Omega)$ be the family of all noneuclidean polygons in Ω. Evidently, $N(\Omega)$ is a subfamily of $N(\Delta)$ and it covers Ω. By a (noneuclidean) polygon U of D we shall mean $U = \pi(V)$, where $V \in N(\Omega)$. The family of all such polygons is designated by $N(D)$. This definition of $N(D) = \pi[N(\Omega)]$ is independent of the projection π as the first part of the next theorem shows.

The region $D \notin O_G$ is said to belong to class W_p if

$$\sup_U \left[\frac{\|\varphi'\|_{p:U}}{\|\varphi'\|_{2:U}} \right] \cdot \left[\frac{\|\varphi'\|_{q:U}}{\|\varphi'\|_{2:U}} \right] < \infty,$$

where the supremum is taken over all $U \in N(D)$. Again, this definition is independent of the projection π as the following theorem shows:

Theorem 2. *Let* $\pi_1\colon \Delta\to D$ *be another cover map and let* Γ_1 *be its cover group. Let also* $\Omega_1=\Delta/\Gamma_1$ *be a fundamental region of* Γ_1 *with* $\psi=(\pi_1|_{\Omega_1})^{-1}$. *Then,* $N(D)=\pi[N(\Omega)]=\pi_1[N(\Omega_1)]$ *and*

$$\frac{1-r}{1+r}\|\Phi'\|_{p:U}\leqq\|\psi'\|_{p:U}\leqq\frac{1+r}{1-r}\|\varphi'\|_{p:U}$$

for all measurable subsets U *of* D. *Here,* r *is a constant in* $[0, 1)$ *which is independent of* U.

Proof. There exists an $A\in\text{Möb}(\Delta)$ so that $\pi_1\circ A=\pi$ and hence $\Gamma=A^{-1}\Gamma_1A$. Therefore, $\Omega_1=A(\Omega)$ and $\psi=A\circ\varphi$. Clearly, $A[N(\Omega)]=N(\Omega_1)$, and, therefore $\pi_1[N(\Omega_1)]=\pi[N(\Omega)]$. Next, $\psi'(z)=A'(\varphi(z))\varphi'(z)$ and

$$\|\psi'\|_{p:U}^p=\int_U|A'(\varphi(z))|^p|\varphi'(z)|^p\,d\sigma(z)$$

for $U\subset D$. Since $A\in\text{Möb}(\Delta)$, A is given by

$$A(\omega)=\alpha\frac{\omega-\tau_0}{1-\bar\tau_0\omega};\quad |\alpha|=1,\ |\tau_0|<1.$$

Now

$$A'(\omega)=(1-|\tau_0|^2)(1-\bar\tau_0\omega)^{-2}$$

and thus

$$\frac{1-|\tau_0|}{1+|\tau_0|}\leqq|A'(\omega)|\leqq\frac{1+|\tau_0|}{1-|\tau_0|}$$

for all $\omega\in\Delta$. Using the fact that $\omega=\varphi(z)\in\Omega\subset\Delta$ this, therefore, yields

$$\frac{1-|\tau_0|}{1+|\tau_0|}\|\varphi'\|_{p:U}\leqq\|\psi'\|_{p:U}\leqq\frac{1+|\tau_0|}{1-|\tau_0|}\|\varphi'\|_{p:U}.$$

The theorem now follows by setting $r=|\tau_0|$.

Theorem 3. Q_D *is bounded on* $L_p(D)$ *if and only if* $D\in W_p$.

Proof. By definition

$$\begin{aligned}(Q_Df)(\zeta)&=\int_D\overline{L_D(z,\zeta)}f(z)\,d\sigma(z)\\&=\int_\Omega\overline{L_D(z,\zeta)}f(z)|\pi'(\omega)|^2\,d\sigma(\omega)\\&=\frac{1}{\pi}\overline{\pi'(\tau)}^{-1}\int_\Omega\sum_{\gamma\in\Gamma}\overline{\frac{\gamma'(\omega)}{(\gamma\omega-\tau)^2}}f(\pi\omega)\pi'(\omega)\,d\sigma(\omega)\\&=\frac{1}{\pi}\overline{\pi'(\tau)}^{-1}\int_\Omega\sum_{\gamma\in\Gamma}\overline{\frac{\gamma'(\omega)}{(\gamma\omega-\tau)^2}}\,g(\omega)\,d\sigma(\omega)\end{aligned}$$

with $g=(f\circ\pi)\pi'$. Here, g is an automorphic form satisfying

$$g(\omega)=g(\gamma\omega)\gamma'(\omega)$$

for each $\omega\in\Delta$ and all $\gamma\in\Gamma$. Also

$$\|g\|_{L_p(\Delta;\mu)}=\|f\|_p;\ \mu(\omega)=|\pi'(\omega)|^{2-p}\chi_\Omega(\omega).$$

In what follows the interchange of sum and integral is justified by virtue of Fubini's theorem, and, we find that

$$\begin{aligned}(Q_Df)(\zeta)&=\frac{1}{\pi}\overline{\pi'(\tau)}^{-1}\sum_{\gamma\in\Gamma}\int_\Omega\frac{\overline{\gamma'(\omega)}}{\overline{(\gamma\omega-\tau)}^2}g(\omega)\,d\sigma(\omega)\\&=\frac{1}{\pi}\overline{\pi'(\tau)}^{-1}\sum_{\gamma\in\Gamma}\int_{\gamma\Omega}\frac{1}{\overline{(v-\tau)}^2}g(v)\,d\sigma(v)\\&=\frac{1}{\pi}\overline{\pi'(\tau)}^{-1}\int_\Delta\frac{1}{\overline{(v-\tau)}^2}g(v)\,d\sigma(v).\end{aligned}$$

Consequently,

$$(Q_Df)(z)=\overline{\pi'(\omega)}^{-1}(T_\Delta g)(\omega);\quad z=\pi(\omega)\in D. \tag{5.1}$$

Therefore,

$$\begin{aligned}\|Q_Df\|_p^p&=\int_D|\pi'(\omega)|^p|(T_\Delta g)(\omega)|^p\,d\sigma(z)\\&=\int_\Omega|(T_\Delta g)(\omega)|^p|\pi'(\omega)|^{2-p}\,d\sigma(\omega).\end{aligned}$$

The norm inequality $\|Q_Df\|_p^p\leqq C\|f\|_p^p$ is therefore equivalent to

$$\int_\Delta|(T_\Delta g)(\omega)|^p\mu(\omega)\,d\sigma(\omega)\leqq C\int_\Delta|g(\omega)|^p\mu(\omega)\,d\sigma(\omega)$$

with the weight

$$\mu(\omega)=\chi_\Omega(\omega)|\pi'(\omega)|^{2-p};\quad \omega\in\Delta.$$

This, in view of Proposition 2, is equivalent to

$$\sup_S\left[\frac{1}{|S|}\int_S\mu(\omega)\,d\sigma(\omega)\right]\cdot\left[\frac{1}{|S|}\int_S\mu(\omega)^{-\frac{1}{p-1}}d\sigma(\omega)\right]^{p-1}<\infty$$

or

$$\sup_S\frac{1}{|S|^p}\left[\int_{S\cap\Omega}|\pi'(\omega)|^{2-p}\,d\sigma(\omega)\right]\cdot\left[\int_{S\cap\Omega}|\pi'(\omega)|^{\frac{p-2}{p-1}}d\sigma(\omega)\right]^{p-1}<\infty,$$

where the supremum is taken over all $S\in N(\Delta)$. This evidently is equivalent to

$$\sup_V\frac{1}{|V|^p}\left[\int_V|\pi'(\omega)|^{2-p}\,d\sigma(\omega)\right]\cdot\left[\int_V|\pi'(\omega)|^{\frac{p-2}{p-1}}d\sigma(\omega)\right]^{p-1}<\infty, \tag{5.2}$$

where the supremum is taken over all $V \in N(\Omega)$. We write $U = \pi(V)$, $V \in N(\Omega)$. Then $U \in N(D)$ and

$$|V| = \int_V d\sigma(\omega) = \int_U |\varphi'(z)|^2 d\sigma(z) = \|\varphi'\|_{2:U}^2.$$

Also

$$\int_V |\pi'(\omega)|^{2-p} d\sigma(\omega) = \|\varphi'\|_{p:U}^p$$

and

$$\Big[\int_V |\pi'(\omega)|^{\frac{p-2}{p-1}} d\sigma(\omega)\Big]^{p-1} = \|\varphi'\|_{p:U}^p.$$

Therefore, condition (5.2) is equivalent to the condition that $D \in W_p$. This concludes the proof.

Corollary 2. *P_D is bounded on $L_p(D)$ if and only if $D \in W_p$.*

We also note that for any $D \notin O_G$, $D \in W_2$ and that $D \in W_p$ if and only if $D \in W_q$. The above results extend those obtained in our previous work [5, 6]. It is also clear that if ∂D is Dini smooth then $D \in W_p$ for any $p \in (1, \infty)$. In fact, the following stronger statement is also true. Assume ∂D is Dini smooth except at one point $c \in \partial D$. At the point c the boundary makes an angle with aperture π/α, $1/2 \leqq \alpha < \infty$. We denote the class of such domains by M_α. Then (see [6] for additional details).

Proposition 3. *Let $D \in M_\alpha$. If $\alpha \geqq 1$ then $D \in W_p$ for all $p \in (1, \infty)$. If $1/2 \leqq \alpha < 1$ then $D \in W_p$ if and only if $p \in (2/1+\alpha, 2/1-\alpha)$.*

We conclude this section by remarking that similarly to (5.1) we also have

$$(5.3) \qquad (P_D f)(z) = \pi'(\omega)^{-1}(P_\Delta g)(\omega); \quad z = \pi(\omega) \in D$$

with $g = (f \circ \pi)\pi'$. Recall that for any $\gamma \in \Gamma$ we have

$$(5.4) \qquad g(\omega) = g(\gamma\omega)\gamma'(\omega); \quad \omega \in \Delta.$$

Also $\|g\|_{L_p(\Delta:\mu)} = \|f\|_p$ with $\mu(\omega) = |\pi'(\omega)|^{2-p}\chi_\Omega(\omega)$, $\omega \in \Delta$.

6. The Bergman projection

Here we assume that the region $D \notin O_G$ is of class W_p so that Q_D and P_D are bounded on $L_p(D)$. Since $A_D = T_D - Q_D$ it also follows that A_D is bounded on $L_p(D)$. Moreover, in this case, it is also clear that the operator identities of Lemma 2 and Theorem 1 remain valid on $L_p(D)$ too. Especially, $P_D^2 = P_D$ on $L_p(D)$, in view of (4.4) and (4.7). Also, P_D is self-adjoint on $L_p(D)$ and thus $\|P_D\|_p = \|P_D\|_q$ with $\|P_D\|_2 = 1$.

Theorem 4. *Let $D\in W_p$. Then $P_D f\in B_p(D)$ for any $f\in L_p(D)$.*

Proof. Let $C_c^\infty(D)$ be the class of $C^\infty(D)$ functions with compact support inside D. According to Weyl's lemma it is sufficient to show that

$$(P_D f, \partial_z \bar{\psi}) = \int_D (P_D f)(z)\partial_z \psi(z)\, d\sigma(z) = 0$$

for every $\psi\in C_c^\infty(D)$. Let $f\in L_p(D)$, because of the density of $C_c^\infty(D)$ in $L_p(D)$ there exists a sequence $\{f_n\}\subset C_c^\infty(D)$ with $\lim_{n\to\infty}\|f_n-f\|_p=0$. Since also $f_n\in L_2(D)$ it follows that $P_D f_n\in B_2(D)$ and especially $P_D f_n\in H(D)$. Therefore,

$$(P_D f_n, \partial_z \bar{\psi}) = 0; \quad n = 1, 2, \dots,$$

for every $\psi\in C_c^\infty(D)$. Now, by the boundedness of P_D on $L_p(D)$,

$$|(P_D f, \partial_z \bar{\psi})| = |(P_D f - P_D f_n, \partial_z \bar{\psi})| \leqq A_p \|f-f_n\|_p \|\partial_z \bar{\psi}\|_q \xrightarrow[n\to\infty]{} 0$$

and the theorem follows.

Corollary 3. *Let $D\in W_p$ and let $\zeta\in D$ be fixed. Then $K(\cdot, \bar{\zeta})$ belongs to $B_p(D)\cap B_q(D)$.*

Proof. Since $D\in W_p$ we have $\|P_D f\|\leqq A_p\|f\|_p$ for each $f\in L_p(D)$. It follows by Fubini's theorem that $(P_D f)(z)$ converges absolutely for almost all $z\in D$. However, by Theorem 4, $P_D f$ is holomorphic in D and thus $(P_D f)(z)$ converges absolutely for all $z\in D$. Let $\zeta\in D$ be fixed and consider the linear functional $R(f)=(P_D f)(\zeta)$, $f\in L_p(D)$. This linear functional is bounded. Indeed, let $\{f_n\}\subset L_p(D)$ with $\lim_{n\to\infty}\|f_n-f\|_p=0$. Then $\lim_{n\to\infty}\|P_D f_n-P_D f\|_p=0$ and, since $P_D f_n$, $P_D f\in B_p(D)$ we obtain that $\lim_{n\to\infty} P_D f_n=P_D f$ uniformly on compacta of D. In particular, this shows that R is a bounded linear functional on $L_p(D)$. Consequently, in view of the Riesz representation theorem, $K(\cdot, \bar{\zeta})\in L_q(D)$. Since also $D\in W_q$ we have by duality $K(\cdot, \bar{\zeta})\in L_p(D)$. The corollary now follows by noting that $K(z, \bar{\zeta})$ is holomorphic in $z\in D$.

Theorem 5. *Let $D\in W_p$. Then $P_D f=f$ for every $f\in B_p(D)$.*

Proof. Let $f\in B_p(D)$. Since $(P_D f)(z)$ converges absolutely for all $z\in D$ it follows from (5.3) and (5.4) that $(P_\Delta g)(\omega)$ converges absolutely for all $\omega\in\Delta=\bigcup_{\gamma\in\Gamma}\gamma\Omega$ and is holomorphic in Δ. Also, since $f\in H(D)$ it follows from (5.4) that $g\in H(\Delta)$. Now, by the absolute convergence of $(P_\Delta g)(\omega)$ we have

$$\frac{1}{\pi}\int_\Delta \frac{|g(\tau)|}{|1-\bar{\tau}\omega|^2}\, d\sigma(\tau) < \infty$$

for every $\omega\in\Delta$. Especially, $g\in B_1(\Delta)$ and therefore

$$g(\omega)=\frac{1}{\pi}\int_{\Delta}\frac{g(\tau)}{(1-\bar{\tau}\omega)^2}\,d\sigma(\tau)$$

or $g(\omega)=(P_{\Delta}g)(\omega)$, $\omega\in\Delta$. This together with $g=(f\circ\pi)\pi'$ and (5.3) shows that

$$(P_D f)(z)=\pi'(\omega)^{-1}(P_{\Delta}g)(\omega)=\pi'(\omega)^{-1}g(\omega)=f(z);\quad z=\pi(\omega)\in D.$$

This concludes the proof.

We are now in a position to prove the following corollary:

Corollary 4. *Let $D\in W_p$, then we have the direct sum decomposition*

$$L_p(D)=B_p(D)\oplus B_q(D)^{\perp}.$$

Proof. For $f\in L_p(D)$, let $h=P_D f$ and $h^{\perp}=(I_D-P_D)f$. Hence $f=h+h^{\perp}$ and by Theorem 4, $h\in B_p(D)$. Let $g\in B_q(D)$, then $P_D g=g$ and by the self-adjointness of P_D

$$(h^{\perp},g)=\big((I_D-P_D)f,g\big)=(f,g)-(P_D f,g)=(f,g)-(f,P_D g)=0.$$

If $f\in B_p(D)\cap B_q(D)^{\perp}$, then $(f,g)=0$ for all $g\in B_q(D)$. However, by Corollary 3, $K(\cdot,\bar{\zeta})$ is in $B_q(D)$ and so by Theorem 5, $f(\zeta)=0$ for all $\zeta\in D$. This concludes the proof.

The rest of the results in the remainder of this section follow by standard arguments based mainly on the Hahn—Banach theorem and we include them here only for sake of completeness (see also [5]).

Corollary 5. *Let $D\in W_p$. The mapping R: $B_q(D)\to B_p(D)^*$ given by $R(g)=L_g$, where $L_g f=(f,g)$ for all $f\in B_p(D)$, defines an anti-linear isomorphism of $B_q(D)$ onto the dual of $B_p(D)$, $B_p(D)^*$.*

Corollary 6. *Let $D\in W_p$ and let $\{t_n\}$ be a dense sequence of points of D. Let $\Phi_n(z)=K(z,\bar{t}_n)$, $n=1,2,\dots$. Then the span of the Φ_n's $[\Phi_n]$ is dense in $B_p(D)$.*

Corollary 7. *Let $D\in W_p$ and let f_n, $f\in B_p(D)$, $n=1,2,\dots$. Suppose that $\{\|f_n\|_p\}$ is bounded, and, that $f_n(z)\to f(z)$ for each $z\in D$. Then $f_n\to f$ weakly in $B_p(D)$.*

Corollary 8. *Let $D\in W_p$. Suppose f_n, $f\in B_p(D)$ with $f_n(z)\to f(z)$ for each $z\in D$ and $\|f_n\|_p\to\|f\|_p$. Then $\|f_n-f\|_p\to 0$.*

7. The annihilator $B_q(D)^\perp$

The annihilator $B_q(D)^\perp$ of the Bergman space $B_p(D)$ may be identified as a Sobolev space. The region D is arbitrary here and not necessarily of class W_p nor necessarily $D \notin O_G$. This fact (Proposition 4 below) is rather known and it seems that the earliest reference for it appears in Havin [8]. For the special case that $D \in \mathrm{A}$ and $p=2$, this result was even mentioned earlier in Schriffer [11]. This identification has been also used by various authors (as, for example, [1, 8, 9]) in proving approximation theorems. For sake of completeness, however, we shall include a proof which uses Proposition 1.

The Sobolev space $W_p^1(D)$ stands for the class of all $L_p(D)$ functions $f(z)$ whose first partial derivatives (taken in the sense of distribution theory) $f_z(z)$ and $f_{\bar{z}}(z)$ also belong to $L_p(D)$. It is a Banach space normed by

$$\|f\|_{p,1} = \left\{\int_D (|f(z)|^2 + |f_z(z)|^2 + |f_{\bar{z}}(z)|^2)^{p/2} d\sigma(z)\right\}^{1/p}.$$

Trivially,

$$\max(\|f\|_p, \|f_z\|_p, \|f_{\bar{z}}\|_p) \leqq \|f\|_{p,1} < 2(\|f\|_p + \|f_z\|_p + \|f_{\bar{z}}\|_p). \tag{7.1}$$

We denote by $\mathring{W}_p^1(D)$ the closure of $C_c^\infty(D)$ in $W_p^1(D)$. As usual, a function $f \in W_p^1(D)$ is said to *vanish on* ∂D if $f \in \mathring{W}_p^1(D)$.

Let $R_p(D)$ be the $L_p(D)$-closure of the set $\{h_z \colon h \in C_c^\infty(D)\}$ and let $S_p(D) = \{h_z \in L_p(D) \colon h \in \mathring{W}_p^1(D)\}$. If $h_z \in S_p(D)$, then, clearly, $\lim_{n\to\infty} \|h_n - h\|_{p,1} = 0$ with $\{h_n\} \subset C_1^\infty(D)$. Consequently, using (7.1), $\lim_{n\to\infty} \|h_{n,z} - h_z\|_p = 0$ and therefore, $S_p(D) \subset R_p(D)$. Conversely, let $h_z \in R_p(D)$. Then $\lim_{n\to\infty} \|h_{n,z} - h_z\|_p = 0$ with $\{h_n\} \subset C_c^\infty(D)$. Especially, $\lim_{n,m\to\infty} \|h_{n,z} - h_{m,z}\|_p = 0$. In view of Proposition 1, $\lim_{n,m\to\infty} \|h_{n,\bar{z}} - h_{m,\bar{z}}\|_p = \lim_{n,m\to\infty} \|T_D(h_{n,z} - h_{m,z})\|_p = 0$ and $\lim_{n,m\to\infty} \|h_n - h_m\|_p = \lim_{n,m\to\infty} \|S_D(h_{n,z} - h_{m,z})\|_p = 0$. Therefore, using (7.1), $\lim_{n,m\to\infty} \|h_n - h_m\|_{p,1} = 0$. Hence, $h_z \in S_p(D)$, showing that $S_p(D) = R_p(D)$.

We now prove:

Proposition 4. $B_q(D)^\perp = S_p(D) = R_p(D)$.

Proof. Since $S_p(D) = R_p(D)$ it is sufficient to show that $B_q(D)^\perp = R_p(D)$. Moreover, since $B_q(D)$ and $R_p(D)$ are closed subspaces of the reflexive Banach spaces $L_q(D)$ and $L_p(D)$, respectively, it will suffice in showing $B_q(D) = R_p(D)^\perp$. Now, by Weyl's lemma, $R_p(D)^\perp \subset B_q(D)$. Conversely, let $f \in B_q(D)$ and let $h_z \in R_p(D)$. Let $\{h_n\} \subset C_c^\infty(D)$ with $\lim_{n\to\infty} \|h_{n,z} - h_z\|_p = 0$. Then

$$(h_{n,z}, f) = \int_D \partial_z h_n(z) \overline{f(z)}\, d\sigma(z) = -\int_D h_n(z) \partial_z \overline{f(z)}\, d\sigma(z) = 0$$

for each n and, consequently, $(h_z, f) = 0$. This concludes the proof.

The following corollary states that, for $D \in W_p$, each function of $L_p(D)$ is the direct sum of a holomorphic function and a complex derivative of a function in $W_p^1(D)$ which vanishes on the boundary.

Corollary 9. *Let* $D \in W_p$. *Then* $L_p(D) = B_p(D) \oplus S_p(D)$.

Proof. This follows from Corollary 4 and Proposition 4.

Corollary 10. *Let* $D \in W_p$. *Then* $S_p(D) = Q_D^* Q_D(L_p(D))$.

Proof. This follows from Theorem 1, Corollaries 1, 2, 4 and Proposition 4.

Bibliography

1. Bagby, T., Quasi topologies and rational approximation, *J. Functional Analysis* **10** (1972), 259—268.
2. Bekollé, D., and Bonami, A., In inégalités à poids pour le noyau de Bergman, *C. R. Acad. Sc. Paris* **286** (1978), 775—778.
3. Bergman, S., and Schiffer, M., Kernel functions and conformal mapping, *Compositio Math.* **8** (1951), 205, 249.
4. Block, I. E., Kernel function and class L^2, *Proc. Amer. Math. Soc.* **4** (1953), 110—117.
5. Burbea, J., Projections on Bergman spaces over plane domains, *Can. J. Math., to appear.*
6. Burbea, J., The Bergman projection on weighted norm spaces, *Can. J. Math., to appear.*
7. Coifman, R. R. and Fefferman, C., Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 241—250.
8. Havin, V. P., Approximation in the mean by analytic functions, *Doklady Akad. Nauk SSSR* **178** (1968), 1025—1028 (= Soviet Math. Doklady **9** (1968), 245—248).
9. Hedberg, L. I., Non-linear potentials and approximation in the mean by analytic functions, *Math. Z.* **129** (1972), 299—319.
10. Sario, L., An extremal method on arbitrary Riemann surfaces, *Trans. Amer. Math. Soc.* **73** (1952), 459—470.
11. Schiffer, M., Fredholm eigenvalues and conformal mapping, *Rend. Mat. Appl.* **22** (1963), 447—468.
12. Solov'ev, A. A., L^p-estimates of integral operators associated with spaces of analytic and harmonic functions, *Doklady, Akad. Nauk. SSSR* **240** (1978), 1301—1305 (= Soviet Math. Doklady **19** (1978), 764—768).
13. Vitranen, K. I., Über eine integraldarstellung von quadratisch integrierbaren analytischen Differentialen, *Ann. Acad. Sci. Fenn. A. I.* **69** (1950), 1—21.

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