Capacities and extremal plurisubharmonic functions on subset of $\mathbf{C}^n$

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1. Introduction

Let U be a bounded open subset of $\mathbf{C}^n$. Denote by $\mathrm{PSH}(U)$ the plurisubharmonic functions on U. We construct a capacity D (in Choquet's sense, cf. Choquet [6]) on U, related to the complex structure. A compact set K is of vanishing D-capacity if and only if K is $\mathbf{C}^n$-polar. (See Josefson [8] and Lelong [9] for definitions and results concerning $\mathbf{C}^n$-polar sets). We also show that, in many cases, the D-capacity of a compact set K is related to the extremal plurisubharmonic function

$$\varlimsup_{z'\to z} \sup \{\varphi(z');\ \varphi \leqq 0;\ \varphi|_K \leqq -1,\ \varphi\in \mathrm{PSH}(U)\}$$

studied by Bedford [2], Siciak [10], Zaharjuta [12] and others.

2. Capacitary functionals

Definition. Let S be a compact space. Denote by $T(S)$ and $C(S)$ the real-valued and the continuous real-valued functions on S respectivly. A *capacitary functional* L is a mapping

$$T(S) \xrightarrow{L} [0, +\infty]$$

such that $L(1) < +\infty$ and

i) $|\alpha| L(h) = L(\alpha h),\ \forall \alpha\in\mathbf{R},\ \forall h\in T(S)$.

ii) $L(h_1+h_2) \leqq L(h_1)+L(h_2)$.

iii) If $0 \leqq h_1 \leqq h_2$ then $L(h_1) \leqq L(h_2)$.

iv) If $h_n\in C(S),\ h_n \geqq h_{n+1} \geqq 0,\ n\in\mathbf{N}$ then $\lim L(h_n) = L(\lim h_n)$.

* Supported by the Swedish Natural Science Research Council contract no. 3435—100.

Remark. Every capacitary functional is continuous on $C(S)$.

Remark. Anger [1] studies capacitary functional without condition iv).

Definition. Let S be a compact space and L a capacitary functional. Define a set of measures M_L by

$$M_L = \left\{\mu \geqq 0;\ \int h\, d\mu \leqq L(h),\ \forall h \in C(S)\right\}$$

and $\hat{L}$ by

$$\hat{L}(h) = \sup_{\mu \in M_L} \int^* |h|\, d\mu.$$

Remark. Since $0 \leqq \int d\mu \leqq L(1) < +\infty \forall \mu \in M_L$, M_L is weakly compact.

Proposition 2.1. *For every non-negative upper semicontinuous function h we have $\hat{L}(h) = L(h)$.*

Proof. If h is continuous then it follows from the Hahn—Banach theorem that there is a measure μ such that

i) $\int h\, d\mu = L(h)$

ii) $\int f\, d\mu \leqq L(f) \quad \forall 0 \leqq f \in C(S)$

and we can choose μ to be positive (cf. Anger [1]). Hence, we have proved the statement for continuous functions.

Let now h be any upper semicontinuous function on S and choose a decreasing sequence $\{h_n\}_{n=1}^{\infty}$ of continuous functions on S with $\lim_{n \to +\infty} h_n = h$. Since $\hat{L}(h_n) = L(h_n)$, $\forall n \in \mathbf{N}$ it is by iv) enough to prove that

$$\lim_{n \to +\infty} \hat{L}(h_n) = \hat{L}(h).$$

Choose $\mu_n \in M_L$ so that $\hat{L}(h_n) = \int h_n\, d\mu_n$, $n \in \mathbf{N}$. We can assume that μ_n tends to μ weakly where $\mu \in M_L$. For each fixed m we then have

$$\int h_m\, d\mu = \lim_{n \to +\infty} \int h_m\, d\mu_n \geqq \lim_{n \to +\infty} \int h_n\, d\mu_n = \lim_{n \to +\infty} \hat{L}(h_n)$$

so

$$\int h\, d\mu \geqq \lim_{n \to +\infty} \hat{L}(h_n) \geqq \int h\, d\mu$$

which proves the proposition.

Corollary 2.2. *Let M be a weakly compact set of measures. Put*

$$A(f) = \sup_{\mu \in M} \int^* |f|\, d\mu, \quad f \in T(S).$$

Then $A(f)$ is a capacitary functional and

$$P(S) \ni E \mapsto A(\chi_E)$$

is a capacity. Here $P(S)$ denotes the subsets of S and χ_E the characteristic function of E.

Corollary 2.3. *For every non-negative universally measurable function h on S we have $\hat{L}(h) \leqq L(h)$.*

Proposition 2.4. *Let S be a compact space and F a convex cone of negative upper semicontinuous functions on S containing all the negative constants. Let R be a negative continuous linear form on F. Then $\bar{R}(h) = \inf \{R(\varphi);\ \varphi \in F, \varphi \leqq -|h|\}$ is a capacitary functional.*

Proof. We restrict ourselves to iv) in the definition of capacitary functional. The verification of i)—iii) is easy and will be omitted.

Assume that $\{h_n\}_{n=1}^{\infty}$ is a decreasing sequence of non-negative continuous functions on S. By iii) it is clear that $\bar{R}(h_n) \geqq \bar{R}(h)$ where $\lim_{n\to+\infty} h_n = h (\geqq 0)$. Given $\varepsilon > 0$ choose $\varphi \in F$ such that $\varphi < -h + \varepsilon$ and $\bar{R}(h) + \varepsilon > R(\varphi)$. Now $S = \bigcup_n \{\varphi + h_n < \varepsilon\}$ so there is an n_ε such that $\varphi - \varepsilon < -h_{n_\varepsilon}$ on S. Hence

$$\bar{R}(h_n) \leqq R(\varphi - \varepsilon) = R(\varphi) + \varepsilon R(-1), \quad n \geqq n_\varepsilon$$

so

$$\bar{R}(h) \leqq \bar{R}(h_n) \leqq \bar{R}(h) + \varepsilon\big(1 + R(-1)\big), \quad n \geqq n_\varepsilon$$

which proves the proposition.

Definition. Let z be a fixed point in the compact set S; we also write z for the linear form on $T(S)$ defined by

$$T(S) \ni h \mapsto -h(z).$$

Then $\bar{z}$ denotes the capacitary functional constructed in Proposition 2.4, $M_z = \{\mu \geqq 0;\ \int h\, d\mu \leqq \bar{z}(h), h \in C(S)\}$ and $\hat{z}(h) = \sup_{\mu \in M_z} \int^* |h|\, d\mu$ (cf. the definition just before Proposition 2.1.)

Proposition 2.5. *If $\mu \in M_z$ then*

$$\int \hat{\xi}(\chi_E)\, d\mu(\xi) \leqq \hat{z}(\chi_E)$$

for every universally capacitable set E.

Proof. Let K be any compact subset of S. Then $\hat{z}(\chi_K) = \bar{z}(\chi_K)$ by Proposition 2.1.

Now, if $\varphi \in F$ and $\mu \in M_z$ then $\varphi(z) \leqq \int \varphi\, d\mu$. To see this choose $\{h_n\}_{n=1}^{\infty}$ a decreasing sequence of continuous functions on S with $\lim_{n\to+\infty} h_n = \varphi$. Then

$$-\int \varphi\, d\mu = \lim \int -h_n\, d\mu \leqq \lim_{n\to+\infty} \bar{z}(-h_n) \leqq \bar{z}(-\varphi) = -\varphi(z).$$

Hence

$$\int \bar{z}(\chi_K)\, d\mu \leqq \inf\Big\{-\int \varphi\, d\mu;\ \varphi \in F;\ \varphi \leqq -\chi_K\Big\}$$
$$\leqq \inf\{-\varphi(z);\ \varphi \in F;\ \varphi \leqq -\chi_K\} = \bar{z}(\chi_K)$$

so

$$\int \hat{\xi}(\chi_K)\, d\mu(\xi) \leqq \hat{z}(\chi_K)$$

and both expressions are capacities which completes the proof of the proposition.

Corollary 2.6. $\hat{z}(\chi_E)=\hat{z}(\xi\mapsto\hat{\xi}(\chi_E))$.

Corollary 2.7. *If E is universally capacitable and if $\hat{z}(\chi_E)$ is upper semicontinuous, then* $\hat{z}(\chi_E)=\bar{z}(\chi_E)$.

Proof. $\bar{z}(\hat{z}(\chi_E))\geqq\bar{z}(\chi_E)\geqq\hat{z}(\chi_E)=\hat{z}(\hat{z}(\chi_E))=\bar{z}(\hat{z}(\chi_E))$ so $\hat{z}(\chi_E)=\bar{z}(\chi_E)$.

Corollary 2.8. $\mu\in M_z\Leftrightarrow\varphi(z)\leqq\int\varphi\, d\mu\ \ \forall\varphi\in F$.

Proof. It follows from the proof of Proposition 2.5 that if $\mu\in M_z$ then $\varphi(z)\leqq\int\varphi\, d\mu$ for every $\varphi\in F$. Conversely, assume that $\varphi(z)\leqq\int\varphi\, d\mu$ for every $\varphi\in F$. Given $0\leqq h\in C(S)$, choose $\varphi\in F$ with $\varphi\leqq -h$. Then $\int h\, d\mu\leqq\int -\varphi\, d\mu\leqq -\varphi(z)$ and it follows that $\int h\, d\mu\leqq\bar{z}(h)$ so $\mu\in M_z$ by definition.

Corollary 2.9. *For any bounded non-negative function h we have* $\hat{z}(h)\leqq\bar{z}(h)$.

Proof. Given $\mu\in M_z$. If $0\leqq h$ is a bounded function and if $\varphi\in F$; $\varphi\leqq -h$ then

$$\int^* h\, d\mu \leqq \int -\varphi\, d\mu \leqq -\varphi(z)$$

so $\hat{z}(h)\leqq\bar{z}(h)$.

3. Extremal plurisubharmonic functions in a bounded set in $\mathbf{C}^n$

Let U be any bounded set in $\mathbf{C}^n$. Denote by F the convex cone

$$F=\{\varphi\in \mathrm{PSH}(U);\ \varphi\leqq 0,\ \lim_{z'\to z}\varphi(z') \text{ exists } \forall z\in\partial U\}.$$

(The value $-\infty$ is allowed.) We want to apply the result of the preceeding section and so we put for $z\in\overline{U}$

$$\bar{z}(h)=\inf\{-\varphi(z);\ \varphi\in F,\ \varphi\leqq -|h|\}$$

$$M_z=\{\mu\geqq 0;\ \int h\, d\mu\leqq\bar{z}(h),\ \forall 0\leqq h\in C(\overline{U})\}$$

$$\hat{z}(h)=\sup_{\mu\in M_z}\int h\, d\mu.$$

We already know that $\hat{z}(h)\leqq\bar{z}(h)$ for any bounded function with equality if h is upper semicontinuous.

Consider now the family $(\hat{z}(\chi_E))_{E\in P(\overline{U})}$.

Since $\hat{z}(\chi_K)=\bar{z}(\chi_K)$ for every compact set K and since $\bar{z}(\chi_E)$ is Lebesgue-measurable for every E it is clear that $\hat{z}(\chi_K)$ is Lebesgue measurable. By a proof, similar to that of Theorem 3.5 in Cegrell [4] we have the following proposition.

Proposition 3.10. *The set function d defined by*

$$P(U)\ni E \overset{d}{\longmapsto} \int_U \hat{z}(\chi_E)\,dz$$

is a subadditive capacity on U.

Proposition 3.11. *If* $E\subset U$ *is* $\mathbf{C}^n$*-polar then* $d(E)=0$. *If* $E=\bigcup_{n=1}^{\infty} K_n$ *where* $d(E)=0$ *and* K_n *are compact subsets of U then E is* $\mathbf{C}^n$*-polar.*

Proof. If E is $\mathbf{C}^n$-polar then $\bar{z}(\chi_E)=0$ a.e. (cf. Siciak [10]) and since $\hat{z}(\chi_E)\leqq \bar{z}(\chi_E)$ (Corollary 2.9) we have $d(E)=0$.

On the other hand, if $E=\bigcup_{n=1}^{\infty} K_n$ where K_n are compacts and $d(E)=0$, we have to prove that E is $\mathbf{C}^n$-polar. It is enough to prove that K_n is $\mathbf{C}^n$-polar. But $\hat{z}(\chi_{K_n})=\bar{z}(\chi_{K_n})$ by Proposition 2.1 and we conclude that K_n is $\mathbf{C}^n$-polar (cf. Siciak [10]).

Remark. Proposition 3.11 is also proved by Gamelin and Sibony [7, p. 62].

Theorem 3.12. *Denote by L the capacitary functional*

$$h \mapsto L(h) = \int_U \bar{z}(h)\,dz,$$

by D the capacity

$$P(U)\ni E \mapsto D(E) = \sup_{\mu\in M_L} \mu^*(E)$$

and by d the capacity defined in proposition 3.10. *Then* $d(E)=D(E)$ *for every universally capacitable set.*

Proof. We know that d and D are capacities, so it is sufficient to prove that $d(K)=D(K)$ for every compact set K. But then, again by Proposition 2.1, we have

$$D(K) = \sup_{\mu\in M_L} \mu(K) = \int_U \bar{z}(\chi_K)\,dz = \int_U \hat{z}(\chi_K)\,dz = d(K).$$

Remark. We know that $L(\chi_E)=0$ if and only if E is $\mathbf{C}^n$-polar but we do not know if $D(E)=0$ implies that E is $\mathbf{C}^n$-polar. This problem is equivalent to the following problem:

If a Borel set has the property that every compact subset in $\mathbf{C}^n$-polar, is the set then necessarily $\mathbf{C}^n$-polar? Thus, this question has a positive answer if and only if there is a capacity vanishing exactly on the $\mathbf{C}^n$-polar sets. *In that case D is such a capacity.* In any case, the following proposition *is easily proved.*

Proposition 3.13. *Let E be a subset of U. Then E is* $\mathbf{C}^n$*-polar if and only if there is a decreasing sequence* $\{O_n\}_{n=1}^{\infty}$ *of open sets containing E such that* $\lim_{n\to+\infty} D(O_n)=0$.

4. Extremal plurisubharmonic functions in regular domains

Let U be an open and bounded subset of $\mathbf{C}^n$. If there is a $\psi \in \mathrm{PSH}(U)$ such that

$$A_\alpha = \{z \in U;\ \psi(z) < \alpha\}$$

is relatively compact in U for every $\alpha < 0$ and $U = \bigcup_{\alpha<0} A_\alpha$ then U is called *P-regular* (cf. Siciak [10] and Zaharjuta [12]) or *hyperconvex* (cf. Berg [3] and Stehlé [11]). If ψ can be extended to a plurisubharmonic function in a neighborhood of $\bar{U}$ we say that U is *essential* or *amply P-regular* (cf. Zaharjuta [13]).

Consider the following extremal plurisubharmonic function (cf. Bedford [2], Siciak [10] and Zaharjuta [12]):

$$h_E(z) = \sup\{\varphi(z);\ \varphi \in \mathrm{PSH}(U);\ \varphi \leqq 0,\ \varphi|_E \leqq -1\}.$$

If U is P-regular and K compact in U then it is clear that

$$h_K(z) = -\bar{z}(\chi_K) = -\hat{z}(\chi_K)$$

and since

$$P(U) \ni E \mapsto \hat{z}(\chi_E)$$

is a capacity we have proved the following theorem, which gives a partial answer to a conjecture posed by Siciack in [10, p. 149].

Theorem 4.14. *Assume that U is P-regular and that $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of compact subsets of U such that $\bigcup_{n=1}^{\infty} K_n = K_\infty$ is compact in U. Then* $\lim_{n\to+\infty} h_{K_n} = h_{K_\infty}$.

Remark. This is a partial answer to a conjecture posed by Siciak [10, p. 149]. This result is also used implicitly in Zaharjuta [12, Lemma 6].

Lemma 4.15. *Assume that U is P-regular and that K is compact in U. Then $\hat{z}(\chi_K)$ is upper semicontinuous.*

Proof. Given $\varepsilon > 0$. Let N be any compact subset of U and choose M such that $M\psi \leqq \varphi - \varepsilon$ on N where $\varphi \in \mathrm{PSH}(U)$, $\varphi \leqq 0$ and $\varphi|_K = -1$. Choose $(\varphi_n)_{n=1}^{\infty}$ a decreasing sequence of negative continuous plurisubharmonic functions, defined near $\{M\psi < -\varepsilon\}$, with limit φ. Take n so large that $\varphi_n - \varepsilon < -1$ on K and put

$$\theta = \begin{cases} \sup(\varphi_n - \varepsilon, M\psi), & z \in \{M\psi < -\varepsilon\} \\ M\psi, & z \in \{M\psi \geqq -\varepsilon\}. \end{cases}$$

It is clear that $\theta \in \mathrm{PSH}(U)$ and that $\theta \leqq -1$ on K.

Furthermore, θ is continuous on N since $\theta = \varphi_n - \varepsilon$ on N. Thus h_K can be written as a supremum of continuous functions on N. Since N was any compact subset of U and since $\hat{z}(\chi_K) = -h_K(z)$ the lemma is proved.

Remark. See also Zaharjuta [13, Theorem 3.1].

Corollary 4.16. *It follows from Lemma* 4.15 *that if U is P-regular, then* $(\hat{z}(\chi_E))_{E\in P(N)}$ *is a swarm for every compact subset N of U.*

In particular, by Theorem 3.6 in Cegrell [4] $\hat{z}(\chi_E)$ is a universally capacitable function for every universally capacitable set E.

Note. A class of non-negative functions $(L_E)_{E\in P(V)}$, is called a swarm if the following two conditions hold:

i) $E\mapsto L_E(x)$ is a capacity for every fixed x,

ii) L_K is an upper semi-continuous function with compact support for every compact set K.

For the rest of this section we assume that U is amply P-regular in $\mathbf{C}^2$. We wish to study the connections between the capacity C defined in Bedford [2] and the capacity D defined in Section 3. It is not known if C is a capacity in Choquet's sense.

Theorem 4.17. *To every compact subset N of U there is a constant* C_N *such that*

$$C(K) \leqq C_N D(K)$$

for every compact subset K of N.

Proof. Let N be any compact subset of U. By the same method as in Cegrell [5, Section 6], we prove that there is a constant C_N such that

$$\int_N dd^c\varphi\Lambda\, dd^c\eta \leqq C_N \|\varphi\|_U |\eta|_U \qquad (*)$$

for all $\varphi\in \mathrm{PSH}(U)\cap L^\infty(U)$ and all $\eta\in\mathrm{PSH}(U)\cap L^1(U)$. Here $\|\cdot\|_U$ is the L^∞-norm and $|\cdot|_U$ the L^1-norm. To prove $(*)$ we choose θ to be a testfunction on U which is equal to 1 near N and $0\leqq\theta\leqq 1$. We then have

$$0\leqq \int_N dd^c\varphi\Lambda\, dd^c\eta \leqq \int_U \varphi\, dd^c\eta\Lambda\, dd^c\theta.$$

The right hand side of this inequality defines, for θ fixed, a bilinear form on $[\delta\text{-PSH}(U)\cap L^\infty(U)]\times\delta\text{-PSH}(U)$. (See [5] for the notation of delta-plurisubharmonic functions). To prove continuity of this form, it is enough to prove separate continuity and this follows from Cegrell [5, Theorem 2.3.1] since

$$\int_U \varphi\, dd^c\eta\Lambda dd^c\theta \geqq 0 \quad \text{for} \quad \varphi,\eta\in\mathrm{PSH}(U),\ \varphi\in L^\infty(U).$$

This completes the proof of $(*)$.

Now, if K is compact in N, we have

$$C(K) \leqq \int_N dd^c(-\hat{z}(\chi_K)) \wedge dd^c(-\hat{z}(\chi_K)) \leqq C_N \int_U \hat{z}(\chi_K)\, dz$$

so $C(K) \leqq C_N D(K)$.

Remark. By Bedford [2, Proposition 4.1] we have for every pair of compact subsets K and K' of N:

$$\int \frac{(dd^c(-\hat{z}(\chi_{K'})))^2 \chi_K}{C_N} \leqq \int \frac{(dd^c(-\hat{z}(\chi_K)))^2 \chi_K}{C_N}$$

so, by Theorem 4.17, for every K compact in N the measures

$$\frac{(dd^c(-\hat{z}(\chi_K)))^2}{C_N} \chi_N \in M_D$$

where $M_D = \{\mu \geqq 0;\ \mu(k) \leqq D(K),\ \forall K \text{ compact in } U\}$. (Compare with Theorem 3.12.)

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Received August 22, 1979

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