The trace of potentials on general sets

A. Jonsson

1. Introduction

Two classical imbedding theorems characterize the restriction of the Bessel potential space $L_\alpha^p(R^n)$ and the Besov space $\Lambda_\alpha^{p,q}(R^n)$ to linear subvarieties of R^n. The space $L_\alpha^p(R^n)$ is defined in § 2.3, and concerning Besov spaces we recall here only that for $0<\alpha<1$, $1\leqq p<\infty$, a function f belongs to $\Lambda_\alpha^{p,p}(R^n)$ if and only if the norm

$$\|f\|_{\Lambda_\alpha^{p,p}(R^n)} = \|f\|_p + \left(\iint_{|x-y|<1} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\,dx\,dy\right)^{1/p} \tag{1.1}$$

is finite. It turns out that $L_\alpha^p(R^n)$ and $\Lambda_\alpha^{p,p}(R^n)$ have the same restriction space to linear subvarieties R^d of R^n. The result is that

$$\Lambda_\alpha^{p,q}(R^n)|_{R^d} = \Lambda_\beta^{p,q}(R^d), \quad 1 \leqq p,\, q \leqq \infty,\ 1 \leqq d < n \tag{1.2}$$

and

$$L_\alpha^p(R^n)|_{R^d} = \Lambda_\beta^{p,p}(R^d), \quad 1 < p < \infty,\ 1 \leqq d < n \tag{1.3}$$

where $\beta=\alpha-(n-d)/p>0$.

Here, by e.g. (1.3) we mean that if $f\in L_\alpha^p(R^n)$, then the pointwise restriction of f to R^d (cf. § 2.4 below) belongs to $\Lambda_\beta^{p,p}(R^d)$, and conversely, that every $f\in \Lambda_\beta^{p,p}(R^d)$ can be extended to R^n so that it is a function in $L_\alpha^p(R^n)$. Also, the imbedding operators involved here are continuous. For a more precise statement we refer to [6], Chap VI, § 4. In its final form, (1.2) is due to Besov [2], and (1.3) to Stein [8], but these papers were preceded by papers considering various special cases.

In [4] and [5] the author in joint work with H. Wallin generalized (1.2) by considering the restriction of $\Lambda_\alpha^{p,q}(R^n)$ to more general closed sets. The aim of this paper is to prove a similar generalization of (1.3). For more of the background to these generalizations, and for a large list of references of interest in this connection we refer to [4] or [10].

We are able to characterize the restriction of $L^p_\alpha(R^n)$ to a large class of sets, the so called d-sets, whose definition is given in § 2.2. A linear subvariety R^d is a d-set, and it can be shown that e.g. Cantor-sets, and thus some sets of non-integer dimension, are d-sets, and that sets which are minimally smooth boundaries of open sets in the terminology of [6], p. 189 are d-sets (see [4], where also more examples are given).

To each d-set F, there is in a natural way associated a unique "d-dimensional" measure μ supported by F, which in case $F=R^d$ is the Lebesgue measure on R^d (see § 2.2). Now, let F be a d-set and μ this measure. Then, for $0<\alpha<1$, a function f belongs to the *generalized Besov space* $B^p_\alpha(F)$ if and only if the norm (cf. (1.1))

$$\|f\|_{p,\alpha,F} = \|f\|_{p,\mu}+\left(\iint_{|x-y|<1} \frac{|f(x)-f(y)|^p}{|x-y|^{d+\alpha p}}\, d\mu(x)\, d\mu(y)\right)^{1/p} \tag{1.4}$$

is finite. The general definition of $B^p_\alpha(F)$ for non-integer α is given in § 2.2.

The main result of this paper is that if F is a d-set, then

$$L^p_\alpha(R^n)|_F = B^p_\beta(F), \quad 0<d<n, \ 1<p<\infty, \tag{1.5}$$

where $\beta=\alpha-(n-d)/p>0$ and β non-integer (Theorem 3). Since R^d is a d-set, this result contains (1.3) in case β is a non-integer. Actually we prove more than (1.5); both the restriction part and the extension part of this theorem are given in a more general form (Theorem 1 and Theorem 2). In [4] we proved that $\Lambda^{p,p}_\alpha(R^n)|_F= B^p_\beta(F)$, $1\leqq p<\infty$, $0<d\leqq n$, so it turns out that $\Lambda^{p,p}_\alpha(R^n)$ and $L^p_\alpha(R^n)$ have the same restriction to a d-set F, as long as $0<d<n$ and $1<p<\infty$.

2. Definitions and statement of theorems

2.1 *The norm* $\|\cdot\|_{p,\alpha,\mu,d}$. This norm, introduced in [4], is a generalization of the classical Besov-norm, suited for the study of Besov-type spaces on general closed sets. It may also be seen as a generalization of the norm used in the Whitney extension theorem (see [4]).

Let μ be a positive measure on R^n, let $0<d\leqq n$, $\alpha>0$, α noninteger, $1\leqq p<\infty$ and let k be the integer such that $k<\alpha<k+1$. For any collection $\{f_j\}_{|j|\leqq k}$, where the functions f_j are measurable with respect to μ and defined μ-a.e. on the support of μ, define r_j by

$$f_j(x) = \sum\nolimits_{|j+l|\leqq k} \frac{f_{j+l}(y)}{l!}(x-y)^l + r_j(x, y), \quad x, y \in \operatorname{supp} \mu.$$

Here we use the usual multiindex notation, $j=(j_1, j_2, \ldots, j_n)$, $l=(l_1, l_2, \ldots, l_n)$, $l! = l_1!\, l_2! \ldots l_n!$, $|j|=j_1+j_2+\ldots+j_n$, and $x^l=x_1^{l_1}x_2^{l_2}\ldots x_n^{l_n}$.

We now define the norm $\|f\|_{p,\alpha,\mu,d}$ of $f=\{f_j\}_{|j|\leqq k}$ by

$$\|f\|_{p,\alpha,\mu,d} = \sum_{|j|\leqq k}\left(\|f_j\|_{p,\mu} + \left(\iint_{|x-y|<1} \frac{|r_j(x,y)|^p}{|x-y|^{d+(\alpha-|j|)p}}\, d\mu(x)\, d\mu(y)\right)^{1/p}\right) \quad (2.1)$$

where $\|f_j\|_{p,\mu}$ denotes the $L^p(\mu)$-norm.

2.2 *The space $B^p_\alpha(F)$, F d-set.* Following [4], we call a closed set F a *d-set*, if there exists a measure μ supported by F, such that for some $r_0>0$

$$\mu(B(x,r)) \leqq c_1 r^d, \quad x\in R^n, \quad r\leqq r_0, \quad (2.2)$$

and

$$c_3 \geqq \mu(B(x,r)) \geqq c_2 r^d, \quad x\in F, \quad r\leqq r_0. \quad (2.3)$$

Here c_1 and c_2 are constants, and $B(x,r)$ denotes the ball with center x and radius r. (Of course, the upper bound c_3 is superfluous in this definition, but we include it for future reference. Note also that the constant r_0 may be taken arbitrarily big; we then just get different constants c_1, c_2 and c_3.) We saw in [4] that measures μ satisfying (2.2) and (2.3) give equivalent norms $\|\cdot\|_{p,\alpha,\mu,d}$, and that the restriction $\Lambda_d|_F$ of the d-dimensional Hausdorff measure to F satisfies (2.2) and (2.3). Having this in mind, it is natural to define *the generalized Besov space $B^p_\alpha(F)$* in the following way:

Let F be a d-set, and let d, α, k and p be as in § 2.1. Then $B^p_\alpha(F)$ is the space of all functions $f=\{f_j\}_{|j|\leqq k}$ with finite norm $\|f\|_{p,\alpha,F}$ given by $\|f\|_{p,\alpha,F}=\|f\|_{p,\alpha,\Lambda_d|_F,d}$.

Since R^n is an n-set, and Λ_n is the n-dimensional Lebesgue measure it is obvious by comparing (1.1) and (2.1) that $B^p_\alpha(R^n)=\Lambda^{p,p}_\alpha(R^n)$ for $\alpha<1$, and we proved in [4] that in the general case, the functions f_j, $|j|\geqq 1$, are uniquely determined by f_0 if $\{f_j\}_{|j|\leqq k}\in B^p_\alpha(R^n)$, by means of $D^j f_0=f_j$, and that $B^p_\alpha(R^n)=\Lambda^{p,p}_\alpha(R^n)$ with equivalent norms.

2.3 *The space $L^p_\alpha(R^n)$ of Bessel potentials* is defined for $\alpha>0$ and $1\leqq p\leqq\infty$. It consists of all functions of the form $f(x)=(\text{a.e.})=\int G_\alpha(x-y)g(y)\,dy$, where $g\in L^p(R^n)$, and G_α is the Bessel kernel of order α. This kernel may be defined by its Fourier transform, $\hat{G}_\alpha(\xi)=(1+|\xi|^2)^{-\alpha/2}$; for more details about G_α, see § 2.7. The norm in L^p_α is defined by $\|f\|_{L^p_\alpha(R^n)}=\|g\|_p$.

2.4 Since we consider the behaviour of functions defined a.e. in R^n on subsets of n-dimensional Lebesgue measure zero, we need the concept of a strictly defined function f. If f is a locally integrable function on R^n, we define the corrected function $\bar{f}$ by

$$\bar{f}(x) = \lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(t)\, dt,$$

at every point where the limit exists. According to a fundamental theorem by Lebesgue, $f=\bar{f}$ a.e. We say that f is *strictly defined* if f is redefined, if necessary, so that $f=\bar{f}$ at every point where the limit exists.

2.5 We now state the results of this paper.

Theorem 1. *(Restriction theorem) Let* $1<p<\infty$, $0<d<n$, $\beta=\alpha-(n-d)/p$, $k<\beta<k+1$, *where k is a non-negative integer, and let* μ *be a measure satisfying* (2.2). *Then, for all* $f\in L^p_\alpha(R^n)$,

$$\|\{D^j f\}_{|j|\leqq k}\|_{p,\beta,\mu,d} \leqq c\|f\|_{L^p_\alpha(R^n)}, \tag{2.4}$$

where the partial derivatives $D^j f$ *are strictly defined for* $|j|\leqq k$, *and where the constant c depends only on* α, β, μ, d *and* p.

Theorem 2. *(Extension theorem) Let* $1\leqq p<\infty$, *let* α, β, d *and* k *be as in Theorem 1, let F be a closed set, and let* μ *be a measure supported by F satisfying* (2.3). *Then there exists a linear operator E, such that for every* $f=\{f_j\}_{|j|\leqq k}$ *with* $\|f\|_{p,\beta,\mu,d}<\infty$ *we have*

(a) $\|Ef\|_{L^p_\alpha(R^n)}\leqq c\|f\|_{p,\beta,\mu,d}$, *where c depends only on* α, β, μ, d *and* p,

(b) *Ef is an extension of f in the sense that the corrected functions* $D^j(Ef)$ *coincide* μ-a.e. *with* f_j *for* $|j|\leqq k$.

If F is a closed set supporting a measure μ satisfying both (2.2) and (2.3), then clearly Theorem 1 and Theorem 2 give a complete characterization of the restriction of $L^p_\alpha(R^n)$ to F, and as a corollary to Theorem 1 and Theorem 2 we have:

Theorem 3. *Let F be a d-set, let* $0<d<n$, $1<p<\infty$, $\beta=\alpha-(n-d)/p>0$, β *non-integer. Then*

$$L^p_\alpha(R^n)|_F = B^p_\beta(F),$$

where the meaning of the statement is that the conclusions in Theorem 1 and Theorem 2 hold with $\|\cdot\|_{p,\alpha,\mu,d}$ *replaced by* $\|\cdot\|_{p,\beta,F}$.

2.6 *Remarks on the proofs.* We carry out the proofs in detail only for $\beta<1$, since the formulas are for $\beta<1$ less heavy, and thus easier to read (the expression (2.1) for $\|\cdot\|_{p,\beta,\mu,d}$ is e.g. reduced to (1.4)). The case $\beta>1$, however, is not much harder, and we explain in separate paragraphs which essential changes must be done to carry out the proofs in the general case. Also, the theorems in [4] were proved in full generality, and we are able to refer to [4] for same ideas and formulas needed in the case $\beta>1$.

The general idea of the proof of the restriction theorem is the following. The essential part of the proof is, when $\beta<1$, to prove that the double integral of (1.4) but with α in (1.4) replaced by $\beta=\alpha-(n-d)/p$, is less than a constant times $\|f\|_{L^p_\alpha}=\|g\|_p$, where $f=G_\alpha*g$, $g\in L^p$.

Writing the double integral in the form

$$\iint |x-y|^{n-2d}\Big(|x-y|^{-\alpha}\Big|\int (G_\alpha(x-t)-G_\alpha(y-t))\,g(t)\,dt\Big|\Big)^p d\mu(x)\,d\mu(y),$$

we see that this follows if we prove that the operator

$$T\colon L^p(R^n) \to L^p(|x-y|^{n-2d}\,d\mu(x)\,d\mu(y)) \quad \text{given by}$$

$$Tg(x,y) = |x-y|^{-\alpha}\Big|\int (G_\alpha(x-t)-G_\alpha(y-t))g(t)\,dt\Big|$$

is bounded, and we want to show this using the Marcinkiewicz interpolation theorem. However, one runs into difficulties for some values of the parameters (cf. the remark after (3.13)), and to come over this, we have to make some preliminary calculations (§ 3.2) before defining an operator which is proved to be continuous using interpolation.

(If we just wanted to prove the classical restriction theorem, the case when F is an hyperplane, using this approach, these difficulties could have been overcome easily, using that we get an equivalent definition of Besov spaces using higher differences, (see [6], p. 153), and then the calculations in § 3.2 would have been essentially superfluous.)

Concerning notation, it should be mentioned that in the proofs c denotes different constants at most times times it appears.

2.7 *Preliminaries.* The following lemma, due to Calderon and Zygmund, is essential in the proof of the restriction theorem (see e.g. [3]).

Lemma 2.1. *Let* $u\in L^1$ *and let* $\delta>0$. *Then we can write* $u=g+\sum_1^\infty g_i$, *where* $\|g\|_1+\sum_1^\infty \|g_i\|_1\leqq 3\|u\|_1$, $|g(x)|\leqq 2^n\delta$ a.e., $\int g_i=0$, *and for certain disjoint cubes* Q_i, $g_i(x)=0$, $x\notin Q_i$ *and* $\sum_1^\infty m(Q_i)\leqq \delta^{-1}\|u\|_1$.

Here $m(Q_i)$ denotes the Lebesgue measure of Q_i. When reading the calculations below, it will also be convenient to have the following simple lemma in mind.

Lemma 2.2. *Let* $0<d\leqq n$, *and let* μ *be a positive measure satisfying* (2.2). *Then we have*

$$\int_{|x-t|\leqq a} |x-t|^{-\gamma}\,d\mu(t) = O(a^{d-\gamma}) \quad \textit{if} \quad d>\gamma,\ a\leqq r_0 \tag{2.5}$$

and

$$\int_{a<|x-t|\leqq b} |x-t|^{-\gamma}\,d\mu(t) = O(a^{d-\gamma}) \quad \textit{if} \quad d<\gamma,\ b\leqq r_0. \tag{2.6}$$

Here O stands for a constant depending on c_1, γ, and d.

Proof. If we write

$$\int_{|x-t|\leqq a} |x-t|^{-\gamma}\,d\mu(t) = \int_0^a r^{-\gamma}\,d\mu(B(x,r))$$

and make a partial integration, we get (2.5). In a similar way (2.6) is proved.

Finally we list some properties of the Bessel kernel G_α (see e.g. [1]). The kernel G_α is a positive, decreasing function of $|x|$, analytic on $R^n\setminus\{0\}$, satisfying, for a number c_1 not depending on x,

$$|D^j G_\alpha(x)| \leqq c_1 |x|^{\alpha-|j|-n} \quad \text{for} \quad a < n+|j| \tag{2.7}$$

$$|D^j G_\alpha(x)| \leqq c_1 \log\frac{1}{|x|}, \quad 0<|x|<1, \quad \text{for} \quad \alpha = n+|j| \tag{2.8}$$

$$D^j G_\alpha(x) \text{ is finite and continuous at } x=0 \text{ for } \alpha > n+|j| \tag{2.9}$$

and, for all derivatives

$$|D^j G_\alpha(x)| \leqq c_1 e^{-c_2|x|}, \quad 1 \leqq |x| < \infty \quad \text{for some} \quad c_2 > 0. \tag{2.10}$$

3. Proof of the restriction theorem

3.1. We assume that $0<\beta=\alpha-(n-d)/p<1$ (compare § 2.6 and § 3.4), and shall prove that

$$I=\iint_{|x-y|<1} |x-y|^{-d-\beta p}\Big(\int |G_\alpha(x-t)-G_\alpha(y-t)|\,|f(t)|\,dt\Big)^p d\mu(x)\,d\mu(y) \leqq c\|f\|_p^p \tag{3.1}$$

and that $\int \big|\int G_\alpha(x-t)f(t)\,dt\big|^p\,d\mu(x) \leqq c\,\|f\|_p^p$; these inequalities give (2.4) for $\beta<1$. The proof of the latter inequality is straightforward, and it can be found in [4], (Lemma 8.4), so we shall here concentrate on the main problem, the proof of (3.1).

3.2. We shall first carry out some preliminary estimates for the left member I of (3.1) (compare § 2.6). Let I_1, I_2 and I_3 be as I, but with the t-integration taken over $|x-t|<2|x-y|$, $2|x-y|\leqq|x-t|<2$, and $|x-t|\geqq 2$ respectively.

If $\alpha<n$ (for the case $\alpha\geqq n$, see § 3.4), we have by (2.7) that $|G_\alpha(x-t)-G_\alpha(y-t)|\leqq c(|x-t|^{\alpha-n}+|y-t|^{\alpha-n})$, and observing that $\{t\,|\,|x-t|<2|x-y|\}\subset\{t\,|\,|y-t|<3|x-y|\}$, we deduce that

$$I_1 \leqq c\iint_{|x-y|<1} |x-y|^{-d-\beta p}\Big(\int_{|x-t|<3|x-y|} |x-t|^{\alpha-n}\,|f(t)|\,dt\Big)^p d\mu(x)\,d\mu(y).$$

By writing the right member of this inequality on the form

$$\int d\mu(x)\sum_{i=0}^{\infty}\int_{2^{-(i+1)}\leqq|x-y|<2^{-i}} (\ )\,d\mu(y),$$

it is easy to realize that

$$I_1 \leqq c\int d\mu(x)\int_0^2 r^{-1-\beta p}\Big(\int_{|x-t|<3r} |x-t|^{\alpha-n}\,|f(t)|\,dt\Big)^p dr. \tag{3.2}$$

In order to estimate I_2, we first observe that using the mean value theorem we get, for some $0<\delta<1$, $|G_\alpha(x-t)-G_\alpha(y-t)|\leqq|x-y|\cdot\big|\text{grad}\, G_\alpha\big(x-t+\delta(y-x)\big)\big|\leqq$

(by (2.7)) $\leqq c|x-y|\cdot|x-t+\delta(y-x)|^{\alpha-n-1}\leqq c|x-y|\cdot|x-t|^{\alpha-n-1}$, where the last inequality is valid if $|x-y|<|x-t|/2$. Thus we have

$$I_2 \leqq c\iint_{|x-y|<1}|x-y|^{-d-\beta p}\Big(|x-y|\int_{|x-y|\leqq|x-t|<2}|x-t|^{\alpha-n-1}|f(t)|\,dt\Big)^p d\mu(x)\,d\mu(y),$$

and similarly as above one obtains that

$$I_2 \leqq c\int d\mu(x)\int_0^{1/2} r^{-1-\beta p}\Big(r\int_{r\leqq|x-t|<2}|x-t|^{\alpha-n-1}|f(t)|\,dt\Big)^p dr. \tag{3.3}$$

For reasons which will be evident soon, we sometimes want a factor $|x-t|^{-\gamma}$ in this expression.

So, let $\gamma>0$, and put the expression $|x-t|^{-\gamma}\left(r^\gamma+\gamma\int_r^{|x-t|}s^{\gamma-1}\,ds\right)=1$ under the integral sign of $\int_{r\leqq|x-t|<2}|x-t|^{\alpha-n-1}|f(t)|\,dt$, and reverse the order of integration. Then we get

$$\int_{r\leqq|x-t|<2}|x-t|^{\alpha-n-1}|f(t)|\,dt$$

$$=\int_r^2\gamma s^{\gamma-1}\,ds\int_{s\leqq|x-t|<2}|x-t|^{\alpha-n-1-\gamma}|f(t)|\,dt+r^\gamma\int_{r\leqq|x-t|<2}|x-t|^{\alpha-n-1-\gamma}|f(t)|\,dt \tag{3.4}$$

Put $A(s)=\gamma s^{\gamma-1}\int_{s\leqq|x-t|<2}|x-t|^{\alpha-n-1-\gamma}|f(t)|dt$ if $s\leqq 2$, and $A(s)=0$, $s\geqq 2$. Hardy's inequality then gives (see e.g. [6] p. 272)

$$\int_0^{1/2} r^{-1-\beta p+p}\Big(\int_r^2 A(s)\,ds\Big)^p dr \leqq \int_0^\infty r^{-1-\beta p+p}\Big(\int_r^\infty A(s)\,ds\Big)^p dr$$

$$\leqq c\int_0^\infty (rA(r))^p r^{-1-\beta p+p}\,dr = c\int_0^2 r^{\gamma p-1-\beta p+p}\Big(\int_{r\leqq|x-t|<2}|x-t|^{\alpha-n-1-\gamma}|f(t)|\,dt\Big)^p dr.$$

Combining this with (3.3) and (3.4) we see that

$$I_2 \leqq c\int d\mu(x)\int_0^2 r^{-1-\beta p}\Big(r^{\gamma+1}\int_{r\leqq|x-t|<2}|x-t|^{\alpha-n-1-\gamma}|f(t)|\,dt\Big)^p dr. \tag{3.5}$$

From the mean value theorem, we obtain in the same way as above, but now using (2.10) as an estimate for the derivatives of G_α, that $|G_\alpha(x-t)-G_\alpha(y-t)|\leqq c|x-y|e^{-c_2|x-t|/2}$ if $|x-y|\leqq|x-t|/2$ and $|x-t|\geqq 2$. Using Hölder's inequality we obtain e.g.

$$I_3=\iint|x-y|^{-d-\beta p}\Big(\int_{|x-t|\geqq 2}|G_\alpha(x-t)-G_\alpha(y-t)|\,|f(t)|\,dt\Big)^p d\mu(x)\,d\mu(y)$$

$$\leqq c\iint|x-y|^{d-\beta p+p}\int_{|x-t|\geqq 2}e^{-c_2|x-t|p/4}|f(t)|^p\,dt\,d\mu(x)\,d\mu(y).$$

Recalling from [4], (formula (2.1)) that $\mu(B(x,r))\leqq cr^n$, $r\geqq r_0$, we can easily show that the last integral above is less than $c\|f\|_p^p$, so we have $I_3\leqq c\|f\|_p^p$.

We collect the estimates above in the following way. Let $0<r<2$, $\gamma\geqq 0$ and $\alpha<n$, and define a function K, depending on α, n, γ, r, for $s>0$ by $K(s)=s^{\alpha-n}$,

$s \leqq r$, $K(s)=r^{\gamma+1}s^{\alpha-n-\gamma-1}$, $r \leqq s \leqq 2$, $K(s)=r^{\gamma+1}2^{\alpha-n-\gamma-1}(3-s)$, $2 \leqq s \leqq 3$, and $K(s)=0$, $s \geqq 3$. For future reference, we remark here that we have

$$K(s) \leqq cr^{\gamma+1}s^{\alpha-n-\gamma-1} \leqq crs^{\alpha-n-1} \leqq cs^{\alpha-n}, \quad r \leqq s, \tag{3.6}$$

and

$$\left|\frac{d}{ds}K(s)\right| \leqq cr^{\gamma+1}s^{\alpha-n-\gamma-2} \leqq cs^{\alpha-n-1}, \quad r < s, s \neq 2, s \neq 3. \tag{3.7}$$

Clearly, by (3.2) and (3.5), both I_1 and I_2 are less than a constant times $\int d\mu(x) \int_0^2 r^{-1-\beta p}(\int K(|x-t|)\,|f(t)|\,dt)^p\,dr$, and since the left member I of (3.1) satisfies $I \leqq c(I_1+I_2+I_3)$, and since $I_3 \leqq c\|f\|_p^p$, (3.1) follows if we prove that (recall that $\beta=\alpha-(n-d)/p$)

$$\int d\mu(x)\int_0^2 r^{-1+n-d}\left(r^{-\alpha}\int K(|x-t|)\,|f(t)|\,dt\right)^p dr \leqq c\|f\|_p^p.$$

3.3. Thus, by the argument in § 3.2, our task is to prove that the operator $T\colon L^p(R^n) \to L^p(\mu_1)$ given by

$$Tf(x, r) = r^{-\alpha}\int K(|x-t|)\,|f(t)|\,dt$$

is bounded, where we have put

$$d\mu_1 = r^{-1+n-d}\,d\mu(x)\,dr, \quad r < 2.$$

This follows from the Marcinkiewicz interpolation theorem (see e.g. [9], Chap. V), if we prove that T is of weak type (p, p) for all p with $0<\alpha-(n-d)/p<1$, i.e. we shall prove that

$$\mu_1\{(x, r)\,|\,Tf(x, r) > \sigma\} \leqq \left(\frac{c\|f\|_p}{\sigma}\right)^p \tag{3.8}$$

for all $\sigma>0$, where c is a constant, independent of $f \in L^p(R^n)$. Using Hölder's inequality we see that, with $0<a<1$, we have

$$|Tf(x, r)|^p \leqq r^{-\alpha p}\int K(|x-t|)^{ap}|f(t)|^p\,dt\left(\int K(|x-t|)^{(1-a)p'}\,dt\right)^{p/p'},$$

where $\frac{1}{p}+\frac{1}{p'}=1$. By integrating over $|x-t|<r$ and $|x-t| \geqq r$ separately, using in the latter case that $K(s) \leqq crs^{\alpha-n-1}$, $s \geqq r$, one immediately verifies that $\int K(|x-t|)^{(1-a)p'}\,dt$ is less than $r^{n-(n-\alpha)(1-a)p'}$ if

$$0 < \frac{n}{(1-a)p'}-(n-\alpha) < 1, \tag{3.9}$$

which gives

$$(Tf(x, r))^p \leqq cr^{-n+ap(n-\alpha)}\int K(|x-t|)^{ap}|f(t)|^p\,dt. \tag{3.10}$$

Denote the right member of (3.10) by $H(x, r)$.

Now we apply Lemma 2.1 with $u=|f|^p$ and $\delta=\sigma^p c_0 2^{-n}$, where c_0 is a constant to be chosen later, and thus write $|f|^p$ on the form $g+\sum g_i$, where $|g|\leqq c_0\sigma^p$. Let $H_1(x,r)$ and $H_2(x,r)$ be the right member of (3.10) with $|f|^p$ replaced by g and $\sum g_i$ respectively. We then have

$$\mu_1\{(x,r)|\,T(x,r)>\sigma\}=\mu_1\{(x,r)\,|\,T(x,r)^p>\sigma^p\}$$

$$\leqq \mu_1\{(x,r)\,|\,H(x,r)>\sigma^p\}\leqq \mu_1\{(x,r)|\,H_1(x,r)>\sigma^p/2\}+\mu_1\{(x,r)|H_2(x,r)>\sigma^p/2\}.$$

To prove (3.8), it is thus sufficient to prove that

$$\mu_1\{(x,r)|\,H_1(x,r)>\sigma^p/2\}=0 \tag{3.11}$$

and

$$\mu_1\{(x,r)|\,H_2(x,r)>\sigma^p/2\}\leqq (c\|f\|_p/\sigma)^p. \tag{3.12}$$

Estimates on H_1. Integrating over $|x-t|<r$ and $|x-t|\geqq r$ separately, using in the latter case the first estimate in (3.6), we clearly get $|H_1(x,r)|\leqq cr^{-n+ap(n-\alpha)}\,c_0\sigma^p\int K(|x-t|)^{ap}\,dt\leqq cc_0\sigma^p=\sigma^p/2$, where the equality is valid if the constant c_0 in advance is chosen to be equal to 1/2c. The calculation is valid if

$$0<n/ap-(n-\alpha)<\gamma+1. \tag{3.13}$$

This gives (3.11).

Remark (cf. § 2.6). In general, the constant a can not be chosen so that this is fulfilled with $\gamma=0$.

Estimates on H_2. Recall that H_2 is the right member of (3.10) with $|f|^p$ replaced by $\sum_{i=1}^\infty g_i$, where the functions g_i satisfy $\int g_i=0$, $g_i=0$ outside a cube Q_i, whose diameter and center we henceforth call r_i and m_i respectively, $\sum r_i^n\leqq (2\sqrt{n})^n\|f\|_p^p/c_0\sigma^p$, and $\sum\|g_i\|_1\leqq c\|f\|_p^p$.

Put $E=\cup E_i$, where $E_i=\{(x,r)|\,|x-m_i|<2r_i,\,r<r_i\}$. We shall below show that

$$\iint_{\complement E}|H_2(x,r)|\,d\mu_1(x,r)\leqq c\|f\|_p^p \tag{3.14}$$

which gives $\mu_1\{(x,r)\in\complement E|\,|H_2(x,r)|>\sigma^p/2\}\leqq c(\|f\|_p/\sigma)^p$. Observe next that, since $\mu(B(x,r))\leqq cr^d$,

$$\mu_1(E)\leqq \sum_{i=1}^\infty\int_{|x-m_i|<2r_i}d\mu(x)\int_{r<r_i}r^{n-d-1}\,dr\leqq c\sum_{i=1}^\infty r_i^{d+n-d}\leqq c(\|f\|_p/\sigma)^p.$$

Note that here we used that d is strictly less than n. Together with (3.14), this gives (3.12).

In order to prove (3.14), we take the sum sign of $\sum g_i$ outside the integrals, and then use that $\complement E\subset\complement E_i$, and get

$$\iint_{\complement E}|H_2(x,r)|\,d\mu_1\leqq c\sum_{i=1}^{\infty}\iint_{\complement E_i}r^{-n+ap(n-\alpha)}\left|\int K(|x-t|)^{ap}g_i(t)\,dt\right|d\mu_1$$
$$=c\sum_{i=1}^{\infty}(A_i+B_i),$$

where in A_i and B_i the integration with respect to μ_1 is taken over $\{(x,r)|r_i\leqq r<2\}$ and $\{(x,r)|\,|x-m_i|\geqq 2r_i,\,r<r_i\}$, respectively. Since $\int g_i(t)\,dt=0$, we may subtract $K(|x-m_i|)^{ap}$ from the integrand in the t-integration when estimating A_i, and after changing the order of integration we get

$$A_i\leqq\int_{Q_i}|g_i(t)|\,dt\int_{r_i}^{2}r^{-1-d+ap(n-\alpha)}\,dr\int\left|K(|x-t|)^{ap}-K(|x-m_i|)^{ap}\right|d\mu(x).$$

Here we first perform the integration with respect to $d\mu(x)$, and integrate over $|x-t|<2|t-m_i|$ and $|x-t|\geqq 2|t-m_i|$ separately. If $|x-t|\geqq 2|t-m_i|$ we have $\left|K(|x-t|)^{ap}-K(|x-m_i|)^{ap}\right|\leqq|t-m_i|\sup_{\xi\in L}|\operatorname{grad}h(\xi)|$, where $h(x)=K(|x|)^{ap}$ and L is the line segment between $x-t$ and $x-m_i$. Since, by (3.6), (3.7) and the definition of K, $K(s)\leqq cs^{\alpha-n}$ and $|dK(s)/ds|\leqq cs^{\alpha-n-1}$, $s\neq r$, $s\neq 2$, $s\neq 3$, we have $|dK(s)^{ap}/ds|\leqq cs^{(\alpha-n)(ap-1)+\alpha-n-1}=cs^{(\alpha-n)ap-1}$ which gives $|\operatorname{grad}h(\xi)|\leqq c|\xi|^{(\alpha-n)ap-1}$. This gives (see below), for $t\in Q_i$, $r<2$,

$$\int_{|x-t|\geqq 2|t-m_i|}\left|K(|x-t|)^{ap}-K(|x-m_i|)^{ap}\right|d\mu(x)$$
$$\leqq c|t-m_i|\int_{2|t-m_i|\leqq|x-t|\leqq 4}(|x-t|/2)^{(\alpha-n)ap-1}\,d\mu(x)\leqq c|t-m_i|^{d+(\alpha-n)ap}.$$

We also have

$$\int_{|x-t|<2|t-m_i|}\left|K(|x-t|)^{ap}-K(|x-m_i|)^{ap}\right|d\mu(x)$$
$$\leqq c\int_{|x-t|<2|t-m_i|}|x-t|^{(\alpha-n)ap}\,d\mu(x)+c\int_{|x-m_i|<3|t-m_i|}|x-m_i|^{(\alpha-n)ap}\,d\mu(x)$$
$$\leqq c|t-m_i|^{d+(\alpha-n)ap}.$$

The integrations here may be performed by means of Lemma 2.2, and the calculations hold if $0<d-(n-\alpha)ap<1$, which is satisfied if

$$0<d/ap-(n-\alpha)<1,\tag{3.15}$$

since $d-(n-\alpha)ap<1$ is trivial if $ap>1$, since then $d-(n-\alpha)ap<\alpha-(n-d)<\alpha-(n-d)/p=\beta<1$. Clearly, since $|t-m_i|\leqq r_i$, under (3.15) this gives $A_i\leqq c\int|g_i(t)|\,dt$.

B_i is estimated more straightforwardly, we have (since $|t-m_i|\leqq r_i$) $B_i\leqq c\int_{Q_i}|g_i(t)|\,dt\int_0^{r_i}r^{-1-d+ap(n-\alpha)}\,dr\int_{r_i\leqq|x-t|<3}r^{ap}|x-t|^{(\alpha-n-1)ap}\,d\mu(x)\leqq c\int|g_i(t)|\,dt$ if (3.15) holds.

Thus $\sum_{i=1}^{\infty}(A_i+B_i)\leqq c\sum_{i=1}^{\infty}\|g_i\|_1\leqq c\|f\|_p^p$, and (3.14) is proved, if we show that the constant a may be chosen so that (3.9), (3.13) and (3.15) are satisfied. Clearly, (3.13) holds if (3.15) holds and γ is chosen big enough, and the fact that a can be chosen to satisfy (3.9) and (3.15) was proved in [4], p. 86, so we omit it here. This completes the proof of the theorem under the assumption that $\alpha<n$, $\beta<1$.

3.4. *The case* $\alpha>n$, $\beta<1$. The only place in the proof above where we used the assumption $\alpha<n$ was in the estimation of I_1, and consequently it also affects the definition of K. Since $\beta=\alpha-(n-d)/p<1$ we have $\alpha<n+1$, and if $n<\alpha<n+1$ it is known that $|G_\alpha(x-t)-G_\alpha(y-t)|\leqq c|x-y|^{\alpha-n}$ for $|x-t|\leqq 2|x-y|$ (see e.g. [4], p. 82). Using this estimate, we obtain

$$I_1\leqq\int d\mu(x)\int_0^2 r^{-1-\beta p}\left(r^{\alpha-n}\int_{|x-t|<2r}|f(t)|\,dt\right)^p,$$

and if we now define K for $s\leqq r$ by $K(s)=r^{\alpha-n}$, the proof runs essentially as before.

The case $k<\beta<k+1$, α *non-integer.* The desired inequality (3.1) is now replaced by inequalities of type

$$\iint_{|x-y|<1}|x-y|^{-d-(\beta-|j|)p}\left(\int|A_j(x,y,t)|\,|f(t)|\,dt\right)^p d\mu(x)\,d\mu(y)\leqq c\|f\|_p^p,$$

where $A_j(x,y,t)=D^jG_\alpha(x-t)-\sum_{|j+l|\leqq k}D^{j+l}G_\alpha(y-t)(x-y)^l/l!$. The terms corresponding to I_2 and I_3 are estimated as before, the only essential difference being that the use of the mean value theorem in § 3.2 is replaced by an application of the Lagrange remainder in Taylor's formula. When $\alpha<n+|j|$ the term corresponding to I_1 in § 3.2 is estimated in a similar way as I_1: Each term in $A_j(x,y,t)$ is estimated separately. When $\alpha>n+|j|$ the term corresponding to I_1 must be given a more careful treatment, compare [4], pp. 181—182. This leads to defining K by e.g. $K(s)=r^{k-|j|}s^{\alpha-n-k}$, $s\leqq r$, and

$$K(s)=r^{k-|j|+\gamma+1}s^{\alpha-n-k-\gamma-1},\quad s\geqq r,\quad\text{if}\quad\alpha<n+k.$$

The case $\alpha\geqq n$, α *integer.* This case may now be obtained from the case α non-integer using complex interpolation. If α is an integer, $k<\beta=\alpha-(n-d)/p<k+1$, choose α_i, $i=1,2$, α_i non-integer, $k<\beta_i=\alpha_i-(n-d)/p<k+1$ so that $\alpha_1<\alpha<\alpha_2$. We consider the case $k=0$. Then the operator $T\colon L^p_{\alpha_i}(R^n)\to L^p|x-y|^{-d-\beta_i p}\,d\mu(x)\,d\mu(y)$ given by $Tf=f(x)-f(y)$ is bounded by what is already proved. Let t be given by $\alpha=t\alpha_1+(1-t)\alpha_2$. Then, from the general theory, see e.g. [9], Chap V, Section 5, in particular § 5.4 and § 5.7, T is bounded from L^p_α to $L^p|x-y|^{-\gamma}\,d\mu(x)\,d\mu(y)$, where $\gamma=(d+\beta_1 p)t+(d+\beta_2 p)(1-t)=d+\beta p$.

4. Proof of the extension theorem

4.1. *The extension operator.* We assume throughout the proof that $0<\beta=\alpha-(n-d)/p<1$; for a comment on the general case, see § 4.5. We use exactly the same extension operator of Whitney type as in [4], and to define it, we need the same partition of unity as in the Whitney extension theorem. This partition is obtained in the following way (see [6], Chap VI, Section 2).

Let F be a given closed set. Then there exists a collection of closed cubes Q_k, with centers x_k and diameters l_k, and sides parallel to the axes, with the following properties:

a) $\complement F=\cup Q_k$

b) The interior of the cubes are mutually disjoint.

c) For a cube Q_k, let $d(Q_k, F)$ denote its distance to F. Then

$$l_k \leqq d(Q_k, F) \leqq 4l_k. \tag{4.1}$$

This partition also has the following properties:

d) Suppose Q_k and Q_ν touch. Then

$$l_k/4 \leqq l_\nu \leqq 4l_k. \tag{4.2}$$

e) Let ε be a fix number satisfying $0<\varepsilon<1/4$, and let Q_k^* denote the cube which has the same center as Q_k, but expanded by the factor $1+\varepsilon$. Then each point in $\complement F$ is contained in at most N_0 cubes Q_k^*, where N_0 is a fixed number. Furthermore, Q_k^* intersects a cube Q_ν only if Q_k touches Q_ν.

Let ψ next be a C^∞-function satisfying $0\leqq\psi\leqq 1$, $\psi(x)=1$, $x\in Q$ and $\psi(x)=0$, $x\notin(1+\varepsilon)Q$ where Q denotes the cube centered at the origin with sides of length 1 parallel to the axes. Define ψ_k by $\psi_k(x)=\psi\big((x-x_k)/s_k\big)$, where $s_k=l_k/\sqrt{n}$ is the side of Q_k, and then φ_k by $\varphi_k(x)=\psi_k(x)/\sum\psi_k(x)$. Then $\varphi_k(x)=0$ if $x\notin Q_k^*$, $\sum\varphi_k(x)=1$, $x\in\complement F$, and it is easy to show that for any multi-index j we have

$$|D^j\varphi_k(x)| \leqq A_j l_k^{-|j|}, \tag{4.3}$$

where A_j is a constant.

Let now μ be a measure, supported by F, satisfying (2.3), and let f be a function defined on F and summable with respect to μ on bounded sets. Define an operator E' by

$$E'f(x) = \textstyle\sum_i \varphi_i(x)c_i\displaystyle\int_{|t-x_i|\leqq 6l_i} f(t)\,d\mu(t), \quad x\in\complement F, \tag{4.5}$$

where c_i is defined by $c_i^{-1}=\mu\big(B(x_i, 6l_i)\big)$.

For convenience, we will below often denote the function $E'f$ by f. It should be noted, that $E'f$ is defined a.e. on R^n by this formula, since a set F carrying a measure satisfying (2.3) must have Lebesgue measure zero, see the proof of [4], Prop. 1.1.

We note here, for future reference, that the lower bound in (2.3) for μ allows us to get an upper estimate for c_i in the following way. From (4.1) we see that there exists a point $p_i \in F$ with $|p_i - x_i| \leqq 5l_i$. This gives $\mu(B(x_i, 6l_i)) \geqq \mu(B(p_i, l_i)) \geqq c_2 l_i^d$ if $l_i \leqq r_0$ or

$$c_i = \left(\mu(B(x_i, 6l_i))\right)^{-1} \leqq c_2^{-1} l_i^{-d} \quad \text{if} \quad l_i \leqq r_0. \tag{4.6}$$

Next fix a function φ such that $\varphi \in C^\infty$, $\varphi(x)=1$ if $d(x, F) \leqq 3$, $\varphi(x)=0$ if $d(x, F) \geqq 4$, and such that $D^j \varphi$ is bounded for every j. The extension operator E used in the theorem is now defined by $Ef(x) = \varphi(x) E' f(x)$.

4.2. *Lemmas.* The first lemma below gives the fundamental estimates on the extended function in terms of the given function f on F. It is a variant of [4], Lemma 5.2, and thus essentially proved in [4], but we include a proof here also, since the mechanism of the lemma becomes more clear here, due to the fact that we have stated the lemma with less generality. Recall that $\complement F = \cup Q_i$, where Q_i are cubes with diameters l_i and centers x_i. Below the center of Q_i is also sometimes denoted by y_i.

Lemma 4.1. *Let F be a closed set, let μ be a measure supported by F satisfying* (2.3), *let f be defined* μ-a.e. *on F and summable with respect to μ on bounded sets, let $1 \leqq p < \infty$, and let $f = E' f$ be given by* (4.5). *Let also $x \in Q_i$ and $y \in Q_\nu$ be points with distance from F not greater than* 4, *and put*

$$J(x_i, y_\nu) = \iint_{|t-x_i| \leqq 30 l_i,\ |s-y_\nu| \leqq 30 l_\nu} |f(t)-f(s)|^p \, d\mu(t)\, d\mu(s).$$

a) *Then for any multiindex j with $|j|>0$ we have*

$$|D^j f(x)|^p \leqq c l_i^{-|j|p-d} l_\nu^{-d} J(x_i, y_\nu). \tag{4.7}$$

b) *We have*

$$|f(x)-f(y)|^p \leqq c l_i^{-d} l_\nu^{-d} J(x_i, y_\nu).$$

Proof. The assumption that the distance from x and y to F is bounded by a fix constant, will allow us in the proof not to bother about the constant r_0 appearing in (2.3), as r_0 may be assumed arbitrarily big (see after (2.3)). For convenience, we first make the following *change of notation: We assume that $x \in Q_I$* and $y \in Q_N$, and shall consequently prove that the lemma holds with i and ν replaced by I and N, respectively.

We first prove statement a). Recall that $c_i = \mu(B(x_i, 6l_i))^{-1}$. Since $\sum \varphi_i(x) = 1$, $x \in \complement F$, we have $\sum_i D^j \varphi_i(x) = 0$, $j \neq 0$, and we can write

$$D^j f(x) = \sum\nolimits_i D^j \varphi_i(x) c_i \int_{|t-x_i| \leqq 6l_i} \left(f(t)-f(s)\right) d\mu(t),$$

and using Hölder's inequality we obtain

$$|D^j f(x)| \leqq \sum_i |D^j \varphi_i(x)| c_i^{1/p} \Big(\int_{|t-x_i|\leqq 6l_i} |f(t)-f(s)|^p \, d\mu(t)\Big)^{1/p}.$$

Let Q_i be a cube touching Q_I. Then by (4.6) and (4.2) we have

$$c_i \leqq c_2^{-1} l_i^{-d} \leqq c l_I^{-d},$$

and by (4.2)

$$|t-x_I| \leqq |t-x_i|+|x_i-x_I| \leqq 6l_i+l_i+l_I \leqq 30 l_I \quad \text{if} \quad |t-x_i| \leqq 6l_i.$$

Since $\varphi_i(x)\neq 0$ only if $x\in Q_i^*$, and $x\in Q_i^*$ iff Q_i and Q_I touch, it follows that these estimaties hold for the at most N_0 numbers i such that $\varphi_i(x)\neq 0$. Together with (4.2) and (4.3) this gives

$$|D^j f(x)|^p \leqq c l_I^{-d-|j|p} \int_{|t-x_I|\leqq 30 l_I} |f(t)-f(s)|^p \, d\mu(t).$$

Integrating this inequality with respect to $d\mu(s)$ over $B(y_N, 30l_N)$ we obtain the desired estimate (4.7), since clearly, by (4.6), $\mu\big(B(y_N, 30l_N)\big) \geqq c l_N^d$.
Next we prove b). Since $\sum \varphi_i(x)=1$, we clearly have

$$f(x)-f(y) = \sum \varphi_i(x) c_i \int_{|t-x_i|\leqq 6l_i} (f(t)-f(y)) \, d\mu(t)$$

$$= \sum_i \sum_\nu \varphi_i(x)\varphi_\nu(y) c_i c_\nu \iint_{|t-x_i|\leqq 6l_i,\, |t-y_\nu|\leqq 6l_\nu} (f(t)-f(s)) \, d\mu(t)\, d\mu(s).$$

Using Hölder's inequality and estimating as in part a), we immediately arrive at the desired formula.

Remark. For $j=0$ we have, instead of (4.7), the easily provable estimate $|f(x)|^p \leqq c l_i^{-d} \int_{|t-x_i|\leqq 30 l_i} |f(t)|^p \, d\mu(t)$.

The next lemma is our main tool when we shall put the local estimates of Lemma 4.1 together, in order to show that our extension belongs to $L_\alpha^p(R^n)$. For a proof of the lemma, see [4], the proof of the formulas (5.16) and (5.17).

Below, we put $h_I=2^{-I}$ and $\Delta_I=\{x \mid h_{I+1}\leqq d(x, F)<h_I\}$, I integer. The following observation will be needed later. If i and I are such that Q_i intersects Δ_I, we obtain from (4.1) that $(h_{I+1}-l_i)/4 \leqq l_i \leqq h_I$, and hence

$$h_I/10 \leqq l_i \leqq h_I \quad \text{if} \quad Q_i \cap \Delta_I \neq \emptyset. \tag{4.8}$$

Lemma 4.2. *Let F, μ, f, p, and $J(x_i, y_\nu)$ be as in Lemma 4.1, and let $J^a(x_i, y_\nu)$ be as $J(x_i, y_\nu)$, but with the number 30 replaced by $a>0$.*

a) *If g is given by $g(x)=J^a(x_i, x_i)$, $x\in \operatorname{int} Q_i$, then*

$$\int_{x\in\Delta_I} g(x) \leqq c h_I^n \iint_{|t-s|<2ah_I} |f(t)-f(s)|^p \, d\mu(t)\, d\mu(s). \tag{4.9}$$

b) *If* g *is given by* $g(x, y)=J(x_i, y_v)$, $x\in\operatorname{int} Q_i$, $y\in\operatorname{int} Q_v$, *then for* h_I, $h_N\leqq c_0 h_K$ *we have*

$$\iint_{x\in\Delta_I, y\in\Delta_N, |x-y|<h_K} g(x, y)\,dx\,dy \leqq c h_I^n h_N^n \iint_{|t-s|<(1+62c_0)h_K} |f(t)-f(s)|^p\,d\mu(t)\,d\mu(s). \tag{4.10}$$

4.3. When we prove that our extension belongs to $L^p_\alpha(R^n)$, we shall use the following equivalent characterization of L^p_α (see [7]). Let $0<\alpha<2$ and $1<p<\infty$. Then $f\in L^p_\alpha(R^n)$ if and only if $f\in L^p(R^n)$ and

$$D^\varepsilon_\alpha f(x)=\int_{|t|\geqq\varepsilon} |t|^{-n-\alpha}(f(x+t)-f(x))\,dt$$

converges in L^p as $\varepsilon\to 0$ to a limit $D_\alpha f$. Also, then $c_1\|f\|_{L^p_\alpha}\leqq\|f\|_p+\|D_\alpha f\|_p\leqq c_2\|f\|_{L^p_\alpha}$. This characterization may be used for all $\alpha>0$, if one makes a reduction using the following fact ([6], p. 136). A function f belongs to L^p_α, $0<m<\alpha$, m integer, if and only if $D^j f\in L^p_{\alpha-m}$, $|j|\leqq m$, and we have

$$c_1\|f\|_{L^p_\alpha} \leqq \sum\nolimits_{|j|\leqq m}\|D^j f\|_{L^p_{\alpha-m}} \leqq c_2\|f\|_{L^p_\alpha}.$$

(Actually, the term in the middle of this inequality may be replaced by $\sum_{|j|<m}\|D^j f\|_p+\sum_{|j|=m}\|D^j f\|_{L^p_{\alpha-m}}$, this may e.g. be seen using various inclusion results for L^p_α and Sobolev spaces.) Here $D^j f$ denotes derivatives in the distribution sense.

In order to prove that $D^\varepsilon_\alpha f$ converges in L^p, it is clearly sufficient to prove that $S_\alpha f(x)=\int |t|^{-n-\alpha}\,|\Delta^2_t f(x)|\,dt$ belongs to $L^p(R^n)$, where $\Delta^2_t f(x)$ denotes $f(x+t)-2f(x)+f(x-t)$; then we also have $\|D_\alpha(f)\|_p\leqq\frac{1}{2}\|S_\alpha f\|_p$. Also, if $f\in L^p$, by Minkowski's inequality, we at once have $\left\|\int_{|t|\geqq 1}|t|^{-n-\alpha}\,|\Delta^2_t f(x)|\,dt\right\|_p\leqq \int_{|t|\geqq 1}|t|^{-n-\alpha}\,\|\Delta^2_t f(x)\|_p\,dt\leqq c\|f\|_p$.

4.4. We are now ready to prove the extension theorem. Let f be as in Theorem 2, $0<\beta=\alpha-(n-d)/p<1$, and let the integer m be given by $m+1\leqq\alpha<m+2$ if $\alpha\geqq 1$, and $m=0$ if $\alpha<1$. We shall show that the function Ef defined in § 4.1 satisfies the conditions of Theorem 2. We showed in [4] that Ef is an extension of f in the sense of part b) of Theorem 2, and that $\|Ef\|_{\Lambda^p_\alpha(R^n)}\leqq c\|f\|_{p,\beta,\mu,d}$, and thus in particular that $\|D^j(Ef)\|_p\leqq c\|f\|_{p,\beta,\mu,d}$, $|j|\leqq m$, and, consequently, also that the distribution derivatives $D^j(Ef)$ coinside a.e. with the pointwise derivatives. By the argument in § 4.3. it remains to show that

$$\int\Big(\int_{|t|<1}|t|^{-n-\alpha+m}|\Delta^2_t D^j(Ef)(x)|\,dt\Big)^p dx \leqq c\|f\|^p_{p,\beta,\mu,d},\quad |j|\leqq m, \tag{4.11}$$

where D^j means pointwise derivatives.

Recalling that $Ef=0$ if $d(x, F)>4$ and the definition of Δ_I given before Lemma 4.2, we see that the left number of (4.11) is less than or equal to

$\sum_{I=-3}^{\infty}\int_{x\in\Delta_I} 2^{p-1}(A_I(x)^p+B_I(x)^p)\,dx$, where

$$A_I(x)=\int_{|t|<h_I/4}|t|^{-n-\alpha+m}|\Delta_t^2 D^j(Ef)(x)|\,dt,\quad x\in\Delta_I$$

and B_I is the same integral over $h_I/4\leqq|t|<1$. Note that in $A_I(x)$ and $B_I(x)$, for $I\geqq -1$, we have $\Delta_t^2 D^j(Ef)(x)=\Delta_t^2 D^j(E'f)(x)=\Delta_t^2 D^j f(x)$ for all points in the domain of integration. Also $B_I(x)=0$, $I=-2,-3$. We consider first $A_I(x)$ for $I\geqq -1$ (for the case $I=-2$ and $I=-3$, see the comment after the formula (4.12)). Using the mean value theorem twice, we see that

$$|\Delta_t^2 D^j f(x)|\leqq c|t|^2\sum_{|l|=2}\sup_{\xi\in L}|D^{j+l}f(\xi)|,\ |t|\leqq h_I/4,\ x\in\Delta_I,$$

where L is the line segment between $x-t$ and $x+t$. If $x\in\Delta_I\cap Q_i$ and, say, $\xi\in Q_{\nu_0}$, one can realize that (see [4], § 6.3 for details) $h_I/20<l_{\nu_0}\leqq 5h_I/4$, and that $|t-x_{\nu_0}|\leqq 30l_{\nu_0}$ implies $|t-x_i|\leqq 400l_i$. In view of this, it follows, using part a) of Lemma 4.1 with $i=\nu=\nu_0$, that

$$|\Delta_t^2 D^j f(x)|\leqq c|t|^2 h_I^{-(|j|+2)-2d/p}(J'(x_i,x_i))^{1/p},\quad x\in\Delta_I\cap Q_i$$

where $J'(x_i,x_i)$ is as $J(x_i,x_i)$, but with $30l_i$ replaced by $400l_i$. Thus we have

$$(A_I(x))^p\leqq c\Big(\int_{|t|<h_I/4}|t|^{-n-\alpha+m+2}\,dt\Big)^p h_I^{-(|j|+2)p-2d}J'(x_i,x_i)$$
$$\leqq (\text{since } m>\alpha-2 \text{ and } |j|\leqq m)\leqq ch_I^{-\alpha p-2d}J'(x_i,x_i),\quad x\in\Delta_I\cap Q_i.$$

By (4.9) we obtain that

$$\int_{x\in\Delta_I}(A_I(x))^p\,dx\leqq ch_I^{n-\alpha p-2d}\iint_{|t-s|\leqq 800h_I}|f(t)-f(s)|^p\,d\mu(t)\,d\mu(s). \tag{4.12}$$

We remark here that one can see that a similar formula holds for $I=-2$, $I=-3$, and that this formula immediately gives $\int_{x\in\Delta_I}(A_I(x))^p\,dx\leqq\|f\|_{p,\beta,\mu,d}^p$, $I=-2$, -3. (However, the term $(D^{j+1}\varphi(\xi))(f(\xi))$ which appears when using the mean value theorem is estimated a little differently, and formula (5.13) in [4] is useful in this connection.) Performing the summation of (4.12) with respect to I, it is not hard to see (compare [5], Lemma 5.3 and Remark 3.1) that

$$\sum_{I=-1}^{\infty}(A_I(x))^p\leqq c\iint_{|t-s|\leqq 1600}|t-s|^{-\alpha p+n-2d}|f(t)-f(s)|^p\,d\mu(t)\,d\mu(s)\leqq c\|f\|_{p,\beta,\mu,d}^p.$$

To estimate B_I, we first make the following estimates, where $\varepsilon>0$, and where we use Hölder's inequality in the second inequality

$$(B_I(x))^p\leqq 2^p\Big(\int_{h_I/4\leqq|t|<1}|t|^{-n-\alpha+m}|D^j f(x+t)-D^j f(x)|\,dt\Big)^p$$
$$\leqq ch_I^{-\varepsilon p}\int_{h_I/4\leqq|t|<1}|t|^{-n-\alpha p+mp+\varepsilon p}|D^j f(x+t)-D^j f(x)|^p\,dt$$
$$\leqq ch_I^{-\varepsilon p}\sum_{K=0}^{I+1}h_K^{-n-\alpha p+mp+\varepsilon p}\int_{|t|<h_K}|D^j f(x+t)-D^j f(x)|^p\,dt$$
$$=ch_I^{-\varepsilon p}\sum_{K=0}^{I+1}h_K^{-n-\alpha p+mp+\varepsilon p}\sum_{N=K-2}^{\infty}\int_{|x-y|<h_K,\,y\in\Delta_N}|D^j f(x)-D^j f(y)|^p\,dy.$$

Part a) of Lemma 4.2, or part b) if $|j|=0$, shows that $|D^j f(x)-D^j f(y)|^p \leqq c l_i^{-d} l_\nu^{-d}(l_i^{-|j|p}+l_\nu^{-|j|p})J(x_i, y_\nu)$, $x\in Q_i$, $y\in Q_\nu$, and using (4.8) and (4.10) we get

$$\sum_{I=-1}^{\infty}\int_{x\in\Delta_I}(B_I(x))^p\,dx \leqq c\sum_{I=-1}^{\infty} h_I^{-\varepsilon p}\sum_{K=0}^{I+1}\sum_{N=K-2}^{\infty} h_K^{-n-\alpha p+mp+\varepsilon p} h_I^{n-d} h_N^{n-d}$$
$$\times(h_I^{-|j|p}+h_N^{-|j|p})\iint_{|t-s|\leqq 300h_K}|f(t)-f(s)|^p\,d\mu(t)\,d\mu(s).$$

After writing the sum as $\sum_{K=0}^{\infty}\sum_{I=K-1}^{\infty}\sum_{N=K-2}^{\infty}$, we perform the summations with respect to I and N, and obtain, since $n-d>0$ and $n-d-|j|p>0$ for $|j|\leqq m$, (this follows from $m\leqq\alpha-1$ and $\alpha-(n-d)/p<1$), if ε is chosen small enough, that the sum is less than

$$c\sum_{K=0}^{\infty} h_K^{n-2d-\alpha p}\iint_{|t-s|\leqq 300h_K}|f(t)-f(s)|^p\,d\mu(t)\,d\mu(s)$$
$$\leqq c\iint_{|t-s|\leqq 300}|t-s|^{n-2d-\alpha p}|f(t)-f(s)|^p\,d\mu(t)\,d\mu(s)\leqq c\|f\|_{p,\beta,\mu,d}^p.$$

This concludes the proof of the theorem for $0<\beta<1$.

4.5. *The case* $k<\beta<k+1$, *k integer*, $k>0$. The operator E' defined by (4.5) is in this case replaced by

$$E'(\{f_j\}_{|j|\leqq k})(x)=\sum_i \varphi_i(x)c_i\int_{|t-x_i|\leqq 6l_i} p(x,t)\,d\mu(t),$$

where $p(x,t)=\sum_{|j|\leqq k}\big((x-t)^j/j!\big)f_j(t)$. The essential change in the proof is that a) and b) of Lemma 4.1 are replaced by (we put $f=E'(\{f_j\}_{|j|\leqq k})$)

a) $|D^j f(x)|^p\leqq c\sum_{|u|\leqq k} l_i^{(|u|-|j|)p-d}\, l_\nu^{-d} J_u(x_i, y_\nu)$, $\quad |j|>k$

b) $|D^j f(x)-D^j f(y)|^p\leqq c\sum_{|u|\leqq k} l_i^{-d} l_\nu^{-d}(l_i^{(|u|-|j|)p}+l_\nu^{(|u|-|j|)p})J_u(x_i, y_\nu)$, $\quad |j|=k$

where $J_u(x_i, y_\nu)$ is as in Lemma 4.2, but with $f(t)-f(s)$ replaced by $r_u(t,s)$ (cf § 2.1). This is a variant of a special case of [4], Lemma 5.2. Also, in (4.9) and (4.10) the expression $f(t)-f(s)$ is replaced by $r_u(t,s)$, the integer m in § 4.4. shall be defined by $m+1\leqq\alpha<m+2$ if $\alpha\geqq k+1$, $m=k$, $k<\alpha<k+1$, and we prove (4.11) only for $|j|=m$, which is sufficient due to the discussion in § 4.3.

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Alf Jonsson
Umeå Universitet
Inst. f. matematik och statistik
S—901 87 Umeå
Sweden