The oblique derivative problem II

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0. Introduction

Let Ω be a bounded domain in R^n, $n \geqq 3$, of class $C^{2+\lambda}$ and consider a unit vector field l on the boundary $\partial\Omega$. We will consider the following boundary value problem for the second order elliptic operator $\mathscr{L}$ in Ω: Find a solution of $\mathscr{L}u=g$ in Ω such that $\partial u/\partial l=f$ on $\partial\Omega$. When l is a conormal to $\partial\Omega$ with respect to $\mathscr{L}$, this is the Neumann problem and it is well known that if the oblique vector field l never becomes tangential, then the problem is still elliptic.

In this article we continue our investigation of the *degenerating* problem, i.e. when l can be tangential to $\partial\Omega$. Writing $l=\alpha\hat{n}+X$ where $\hat{n}$ is the outer conormal and X is tangential, we see that degeneracy occurs precisely when the scalar function α is zero. In [5] we investigated the case when $\alpha \geqq 0$ on $\partial\Omega$ and now we intend to include a case when α changes its sign. Then, however, the problem, as stated above is not well posed. In fact, it is known that one has to allow singularities of the solutions or prescribe their values on certain subsets of $\partial\Omega$. For a historical review, and for references to results by other authors, we refer to [5].

The results given here are improvements of those in the author's thesis [4]. What makes them specific is that we work in function classes of Hölder type and that we do not make the usual assumption of lower dimensionality of the set where α vanishes.

1. Basic assumptions and notations

In order to describe the situation we introduce some further notation. We assume that $l \in C^{1+\lambda}(\partial\Omega)$ and hence there are unique integral curves to X through every $p \in \partial\Omega$. These will be called *X-curves* and we denote the maximal X-curve through p by γ_p. It will also be convenient to use the symbol γ_p^A for the component

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of $\gamma_p \cap A$ which contains p. Here A is any subset of $\partial\Omega$. We associate with γ_p the standard parametrization

$$s \to \tilde{x}_p(s) \quad \text{where} \quad \tilde{x}_p(0) = p \quad \text{and} \quad \frac{d}{ds}\tilde{x}_p(s) = X \circ \tilde{x}_p(s).$$

Similarly, if L denotes a C^1-extension of l to $\bar{\Omega}$, then the concept of *L-curves*, the notation Γ_p for the maximal L-curve through $p \in \bar{\Omega}$ and Γ_p^A for the component of $\Gamma_p \cap A$ which contains p, and a standard parametrization $s \to x_p(s)$ for Γ_p are introduced in an analogous way.

The *set of tangency* for l is

$$H = \{p \in \partial\Omega \colon \alpha(p) = 0\}.$$

Let k_0 be the *suprеmum of all lengths* of connected X-curves within H. We require that k_0 be finite.

Let $\partial\Omega$ be the union of two closed sets $\mathscr{A}_+$ and $\mathscr{A}_-$ such that $\alpha \geqq 0$ in $\mathscr{A}_+$ and $\alpha \leqq 0$ in $\mathscr{A}_-$ and such that $\mathscr{M} = \mathscr{A}_+ \cap \mathscr{A}_- \subset H$ is an orientable manifold of dimension $n-2$, where n is the dimension of the space. We assume that X satisfies a *transversality condition* of the following kind:

There is a neighbourhood U of H in $\partial\Omega$ such that for any $p \in \mathscr{M}$ either

(i) γ_p^U intersects $\mathscr{M}$ in one point, the *angle* between $X(p)$ and $\mathscr{M}$ is bounded away from zero by $\theta_0 > 0$ and α changes its sign *from minus to plus* along γ_p^U,

or

(ii) γ_p^U intersects $\mathscr{M}$ only once and ∂H twice, and α changes its sign *from plus to minus* along γ_p^U.

The conditions (i) and (ii) separate $\mathscr{M}$ into two disjoint manifolds $\mathscr{M}_+$ and $\mathscr{M}_-$ respectively. Note that Maz'ja in [3] was able to investigate a case where our transversality conditions were replaced by conditions on the behaviour of X in a neighbourhood of the subset of H where X is tangential to H. In [3], however, H is of the dimension $n-2$ and $k_0 = 0$.

The *operator* $\mathscr{L}$ has the form

$$a_{ij} D_i D_j + b_i D_i + c$$

where D_i represents differentiation with respect to x_i and the summation convention is used. We assume that $\mathscr{L}$ is uniformly elliptic, that the coefficients are in $C^{\lambda}(\bar{\Omega})$ but that the a_{ij}: s have $C^{1+\lambda}$ – regularity in a neighbourhood of H in $\bar{\Omega}$ and are $C^{2+\lambda}$ in a neighbourhood of $\mathscr{M}_- \setminus \operatorname{int} H$. We also require that $c \leqq 0$ in a neighbourhood of $\mathscr{M}_-$.

To conclude our list of requirements we assume that l is in $C^{2+\lambda}$ in a neighbourhood of H, that the manifold $\mathscr{M}_+$ is of class $C^{2+\lambda}$, that $\mathscr{M}_-$ is $C^{3+\lambda}$ in a neighbourhood of $\mathscr{M}_- \setminus \operatorname{int} H$, and that $\partial\Omega$ is $C^{3+\lambda}$ near H. Finally, we impose a *mono-*

tonicity condition on the X-curves through $\mathcal{M}_-$. In fact, if $p\in\mathcal{M}_-\setminus\operatorname{int} H$, then we require that $\operatorname{dist}(\tilde{x}_p(s),\mathcal{M}_-)$ is monotonic for $s\geqq 0$ and $s\leqq 0$ such that $\tilde{x}_p(s)$ belongs to a fixed neighbourhood of $\mathcal{M}_-$.

In various estimates we will use *seminorms* $[\ ]_{k+\lambda}$ and $[\ ]_k$, and *norms* $\|\ \|_{k+\lambda}$ and $\|\ \|_k$. These are defined as in Agmon—Douglis—Nirenberg [1] (where, however, $\|\ \|$ is written $|\ |$). For every neighbourhood W of H in $\bar{\Omega}$ and every extension L of l to $\bar{W}$ we denote by $S(W, L)$ the *Banach space of functions* (f, g, h) in $C^{1+\lambda}(\partial\Omega)\times C^{\lambda}(\bar{\Omega})\times C^{2+\lambda}(\mathcal{M}_+)$ such that $f\in C^{2+\lambda}(\partial\Omega\cap\bar{W})$ and $\partial g/\partial L\in C^{\lambda}(\bar{W})$. The norm in $S(W, L)$ is given by

$$\|(f, g, h)\| = \|f\|_{1+\lambda}^{\partial\Omega}+\|f\|_{2+\lambda}^{\partial\Omega\cap W}+\|g\|_{\lambda}^{\Omega}+\left\|\frac{\partial g}{\partial L}\right\|_{\lambda}^{W}+\|h\|_{2+\lambda}^{\mathcal{M}_+}.$$

2. Results

We are now able to formulate the main results:

Theorem 1. *There is a positive constant m, depending on* $\Omega, a_{ij}, \|l\|_1^{\partial\Omega}, \mathcal{M}_+$ *and* θ_0 *such that the following holds:*

Let W be a neighbourhood of H in $\bar{\Omega}$. Then if $k_0<m$ there is an extension L of l to $\bar{W}$ and a closed subspace $\tilde{S}$ of finite co-dimension in $S(W, L)$ such that for every $(f, g, h)\in\tilde{S}$ we may find a bounded solution $u\in C_{\mathrm{loc}}^{2+\lambda}(\bar{\Omega}\setminus\mathcal{M}_-)$ of

$$\text{(P)}\quad \begin{cases} \mathcal{L}u = g & in \quad \Omega \\ \dfrac{\partial u}{\partial l} = f & on \quad \partial\Omega\setminus\mathcal{M}_- \\ u = h & on \quad \mathcal{M}_+ \end{cases}$$

Moreover, $\operatorname{codim}\tilde{S}=\dim\mathcal{N}$ *where* $\mathcal{N}$ *is the kernel*

$$\Big\{u\in C_{\mathrm{loc}}^{1}(\bar{\Omega}\setminus\mathcal{M}_-)\cap C^2(\Omega)\cap L^\infty(\Omega)\colon\ \mathcal{L}u = 0 \ in\ \Omega,$$

$$\frac{\partial u}{\partial l} = 0 \ on\ \partial\Omega\setminus\mathcal{M}_- \quad and \quad u = 0 \ on\ \mathcal{M}_+\Big\}.$$

If the coefficient c in $\mathcal{L}$ is non-positive and if c takes on negative values or $\mathcal{M}_+$ is non-empty, then (P) *has a unique solution for all $f\in C^{1+\lambda}(\partial\Omega)$, $g\in C^{\lambda}(\bar{\Omega})$, $h\in C^{2+\lambda}(\mathcal{M}_+)$ such that f is in $C^{2+\lambda}$ in some neighbourhood of H on $\partial\Omega$ and $\partial g/\partial L$ is in C^{λ} in some neighbourhood of H in $\bar{\Omega}$.*

Our method of proof also gives a second result which should be compared with works by Janusjauskas (see [2]) for "L-convex" domains and operators $\mathcal{L}$ which "commute with $\partial/\partial L$ as for lower order terms". Note that our regularity requirements are much weaker.

Theorem 2. *Assume that $\mathcal{M}_-$ is empty and that there is an extension L of l to all of $\bar{\Omega}$ such that*

$$\mathcal{L}\frac{\partial}{\partial L}-\frac{\partial}{\partial L}\mathcal{L} = \mathcal{P}_1+\mathcal{P}_2\frac{\partial}{\partial L}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are first order operators. Furthermore, assume that $\mathcal{M}_+$ is the boundary of a manifold Λ which is such that Ω is a union of connected L-curves, all intersecting Λ once and under uniformly positive angle. Then Theorem 1 *is true for any value of k_0 and the solutions are of class $C^{2+\lambda}(\bar{\Omega})$.*

For the proof of this result, see Remark 3.2.

3. Two lemmas

Let L be a unit vector field of class $C^{2+\lambda}$, and assume that Λ is an $(n-1)$-dimensional surface in $\mathbf{R}^n$ of class $C^{2+\lambda}$ with $C^{2+\lambda}$ boundary $\partial\Lambda$. Suppose that L is defined in a neighbourhood of $\bar{\Lambda}$ such that at any point $p\in\Lambda$ the angle between $L(p)$ and Λ is $\theta(p)\geqq\theta_1>0$.

Consider a family $\{W^\delta\}_{0\leqq\delta\leqq\delta_0}$ of domains with the properties:

(i) W^δ is the union of connected L-curves, originating in Λ and with length at most δ.
(ii) $W^\delta\supset\Lambda$
(iii) For any $p\in\delta^{-1}\cdot\partial W^\delta$ the intersection $(\delta^{-1}\cdot\partial W^\delta)\cap B(p)$ with a ball around p with radius equal to one can be mapped onto a hemisphere of radius one via a $C^{2+\lambda}$ transformation $\mathcal{D}$ such that $\mathcal{D}$ and $\mathcal{D}^{-1}$ both have $C^{2+\lambda}$-norms bounded by a constant d_0 which does not depend on δ.

Then we can prove

Lemma 3.1. *Assume that the coefficients of $\mathcal{L}$ and their derivatives with respect to L are in $C^\lambda\big(\overline{\bigcup_{\delta\leqq\delta_0}W^\delta}\big)$ and that $c<0$. Then there is a positive number $\delta_1\leqq\delta_0$ such that for all $0<\delta<\delta_1$ the problem*

$$(\mathrm{P}_\delta)\quad\begin{cases}\mathcal{L}u = g & in \quad W^\delta\\ \dfrac{\partial u}{\partial L}=f & on \quad \partial W^\delta\\ u=h & on \quad \partial\Lambda\end{cases}$$

has a unique solution $u\in C^{2+\lambda}(\overline{W^\delta})$ if g and $\partial g/\partial L$ belong to $C^\lambda(\overline{W^\delta})$, if $f\in C^{2+\lambda}(\partial W^\delta)$ and if $h\in C^{2+\lambda}(\partial\Lambda)$.

The solution satisfies an inequality

$$\|u\|_{2+\lambda}^{W^\delta} \leqq C\cdot\left\{\|g\|_\lambda^{W^\delta}+\left\|\frac{\partial g}{\partial L}\right\|_\lambda^{W^\delta}+\|f\|_{2+\lambda}^{\partial W^\delta}+\|h\|_{2+\lambda}^{\partial\Lambda}\right\} \tag{3.1}$$

where C depends on δ, $\mathscr{L}$, L, Λ, θ_1 and d_0.

Proof. We will use functions of the type

$$u(p) = v\big(x_p(-s)\big)+\int_0^s w\circ x_p(\tau-s)\,d\tau \tag{3.2}$$

where $-s=-s(p)$ is the unique parameter value on Γ_p such that $x_p(-s)\in\Lambda$.

If we chose v and w with boundary values h and f respectively, then u, defined by (3.2), satisfies the two boundary conditions of (P_δ). Moreover, the choice can be made in such a way that the C^λ-norms of $\mathscr{L}u$ and $\partial\mathscr{L}u/\partial L$ are bounded by a constant times the expression on the right hand side of (3.1). We may thus confine ourselves to the case when *f and g are identically zero.*

The action of $\mathscr{L}$ on functions which are constant along L-curves defines an elliptic operator $\mathscr{L}'$ in Λ for which the Dirichlet problem is uniquely solvable. Hence, in (3.2) we can take v as the solution of $\mathscr{L}'v=g$ in Λ for which $v=0$ on $\partial\Lambda$. For w we choose the solution of $\mathscr{L}w=\partial g/\partial L$ in W^δ which is zero on the boundary. The lemma will be proved if we can show that the map $\mathscr{A}\colon g\to\mathscr{L}u$ from X_λ to X_λ has range $\mathscr{R}(\mathscr{A})=X_\lambda$. Here X_λ is the Banach space $\{g\in C^\lambda(\overline{W}^\delta)\colon \partial g/\partial L\in C^\lambda(\overline{W}^\delta)\}$ with norm

$$\|g\|_\lambda' = \delta^{1+\lambda}\left[\frac{\partial g}{\partial L}\right]_\lambda+\delta\left\|\frac{\partial g}{\partial L}\right\|_0+\delta^\lambda[g]_\lambda+\|g\|_0.$$

The proof will proceed in two steps:

Step 1: We prove that $\mathscr{A}=I+\mathscr{T}+\mathscr{K}$ where I is the identity, $\mathscr{K}$ is compact and $\mathscr{T}$ has norm less than one if δ is small enough.

Step 2: By the Riesz—Schauder theory $\mathscr{A}$ has full range iff the kernel is trivial. We prove that $\mathscr{L}u=0$ implies that $g=0$.

The commutator $\mathscr{L}\,\partial/\partial L-\partial\mathscr{L}/\partial L$ can be decomposed into $\tilde{\mathscr{L}}+\mathscr{P}_1+\mathscr{P}_2\partial/\partial L$ where $\tilde{\mathscr{L}}$ is of the second order and $\mathscr{P}_1$ and $\mathscr{P}_2$ are first order operators, all with C^λ-coefficients. Integrating $\partial\mathscr{L}u/\partial L$ along Γ_p we get

$$\mathscr{L}u(p) = \mathscr{L}u(p')+\int_0^s\left(\frac{\partial g}{\partial L}-\tilde{\mathscr{L}}u-\mathscr{P}_1u-\mathscr{P}_2w\right)\circ x_{p'}(\tau)\,d\tau$$

where $p'=x_p(-s)$ and $\mathscr{L}u(p')=\mathscr{L}'v(p')+\mathscr{Q}w(p')$ with a first order operator $\mathscr{Q}$. Hence

$$(\mathscr{A}g-g)(p) = \mathscr{Q}w(p')-\int_0^s(\mathscr{P}_1u+\mathscr{P}_2w+\tilde{\mathscr{L}}u)\circ x_{p'}(\tau)\,d\tau$$

and we define $\mathscr{T}g$ and $\mathscr{K}g$ as the functions

$$T(p) = -\int_0^s \tilde{\mathscr{L}} u \circ x_{p'}(\tau)\, d\tau$$

and

$$K(p) = \mathscr{Q}w(p') - \int_0^s (\mathscr{P}_1 u + \mathscr{P}_2 w) \circ x_{p'}(\tau)\, d\tau.$$

We first prove that $\mathscr{K}$ is compact. In fact let $\lambda' < \lambda$, note that

$$\|K\|'_\lambda \leqq C \cdot \{\|w\|^{W^\delta}_{1+\lambda} + \|v\|^{A}_{1+\lambda}\} \leqq C \cdot \{\|w\|^{W^\delta}_{2+\lambda'} + \|v\|^{A}_{2+\lambda'}\} \leqq C \cdot \|g\|'_{\lambda'}$$

and use the fact that X_λ is compactly embedded in $X_{\lambda'}$.

Now, $\|T\|_0 \leqq \delta \cdot \|\tilde{\mathscr{L}} u\|_0$ and $[T]_\lambda \leqq C \cdot \delta \cdot [\tilde{\mathscr{L}} u]_\lambda + C \cdot \delta^{1-\lambda} \cdot \|\tilde{\mathscr{L}} u\|_0$ because of the assumption that $|s| \leqq \delta$ in W^δ. Here the constant C possibly depends on δ_0 but not on δ. Hence

$$\|T\|'_\lambda \leqq C \cdot \{\delta^{1+\lambda} [\tilde{\mathscr{L}} u]_\lambda + \delta \cdot \|\tilde{\mathscr{L}} u\|_0\}.$$

Using (3.2) we get

$$\|T\|'_\lambda \leqq C \cdot \delta \cdot \{\delta^\lambda [v]^{A}_{2+\lambda} + \|v\|^{A}_2 + \delta^{1+\lambda} [w]^{W^\delta}_{2+\lambda} + \delta [w]^{W^\delta}_2 + \|w\|^{W^\delta}_1\}. \tag{3.3}$$

To estimate the right hand side of this we consider the function w' in $\delta^{-1} W^\delta$ defined by the relation $w'(x) = w(\delta x)$. First we note that w' satisfies

$$a_{ij}(\delta x) w'_{ij}(x) + \delta b_i(\delta x) w'_i(x) + \delta^2 c(\delta x) w'(x) = \delta^2 \frac{\partial g}{\partial L}(\delta x)$$

and hence by the assumption (iii) for $\{W^\delta\}$ and Theorem 7.3 in Agmon—Douglis—Nirenberg [1] it follows that

$$\|w'\|_{2+\lambda} \leqq C \cdot \delta^2 \cdot \left\{ \left\| \frac{\partial g}{\partial L}(\delta x) \right\|_\lambda + \|w'\|_0 \right\}$$

with all norms taken over $\delta^{-1} W^\delta$ and C independent of δ. For w in the domain W^δ this means

$$\delta^{2+\lambda} \cdot [w]_{2+\lambda} + \delta^2 \cdot [w]_2 + \delta \cdot \|w\|_1 \leqq C \cdot \left\{ \delta^{2+\lambda} \cdot \left[\frac{\partial g}{\partial L} \right]_\lambda + \delta^2 \cdot \left\| \frac{\partial g}{\partial L} \right\|_0 + \|w\|_0 \right\}$$

with C not depending on δ.

Now the maximum principle applied to $w - c_0 \cdot (\delta^2 - (\mathrm{dist}\,(\cdot, A))^2) \cdot \|\partial g/\partial L\|_0$ proves that $|w(p)| \leqq c_0 \cdot \delta^2 \cdot \|\partial g/\partial L\|_0$ and hence the terms on the right hand side of (3.3) which contain w can be estimated by $C \cdot \delta \cdot \|g\|'_\lambda$. Another use of the a priori estimates, Theorem 7.3 in [1], proves that there is a bound for the terms containing v which is of the same type. Hence the norm of $\mathscr{T}$ is $C\cdot\delta$ which clearly can be made less than one.

It remains to prove that $\mathscr{A}g=0$ implies $g=0$. In fact, the function u corresponding to g and definied by (3.2) is then a solution of (P_δ) with zero data. By Lemma 2.2 of [5] such a solution has its extreme values on $\partial\Lambda$. But there u is zero and hence u vanishes identically in W^δ. From this it follows that $w=\partial u/\partial L$ is zero in W^δ and that v vanishes in all of Λ. But then $g=\mathscr{L}'v=0$ in Λ and $\partial g/\partial L=\mathscr{L}w=0$ in W^δ. Consequently, $g=0$ in W^δ. The estimate (3.1) follows from the fact that $\mathscr{A}$ is invertible. QED

Remark 3.2. Theorem 2 is a corollary to the proof above since in case $\tilde{\mathscr{L}}$ vanishes, the operator $\mathscr{T}$ is zero and hence no bound on δ_1 is necessary.

We conclude this section by citing a result from [5] which will be of great importance in Section 5.

Lemma 3.3. (Theorem 3.2 of [5]) *Assume that $\alpha\geqq 0$ on $\partial\Omega$, that $c\leqq 0$ and that $c\not\equiv 0$. Then there is a constant $m_1>0$ depending only on Ω, $\|l\|_1^{\partial\Omega}$ and $\mathscr{L}$ such that the following is true as soon as $k_0<m_1$:*

Let Σ be a fixed neighbourhood of H in $\bar{\Omega}$ and assume that $f\in C^{1+\lambda}(\partial\Omega)\cap C^{2+\lambda}(\bar{\Sigma}\cap\partial\Omega)$, $g\in C^\lambda(\bar{\Omega})$ and $\partial g/\partial L\in C^\lambda(\bar{\Sigma})$ where L is a C^1-extension of l to $\bar{\Omega}$ which is $C^{2+\lambda}$ in a neighbourhood of H. Then there is a solution $u\in C^{2+\lambda}(\bar{\Omega})$ of $\mathscr{L}u=g$ in Ω with $\partial u/\partial l=f$ on $\partial\Omega$ satisfying the inequality

$$\|u\|_{2+\lambda}^{\Omega}\leqq C\cdot\left\{\|g\|_\lambda^\Omega+\left\|\frac{\partial g}{\partial L}\right\|_\lambda^\Sigma+\|f\|_{1+\lambda}^{\partial\Omega}+\|f\|_{2+\lambda}^{\partial\Omega\cap\Sigma}\right\}. \tag{3.4}$$

Remark 3.4. If H consists of several disjoint closed sets H_i with associated maximal lengths k_i of X-curves, then the Lemma 3.3 is true if *the k_i: s are* less than a constant *depending on the local behaviour of Ω, $\|l\|_1$ and $\mathscr{L}$* in a neighbourhood of the H_i: s. This follows by means of a suitable partition of unity.

4. Proof of the uniqueness assertion

The main result of this section is

Lemma 4.1. *Assume that the coefficient c of $\mathscr{L}$ is non-positive and that c does not vanish identically or $\mathscr{M}_+$ is non-empty. Let u be a bounded solution of $\mathscr{L}u=0$ of class $C^2(\Omega)\cap C^1(\bar{\Omega}\setminus\mathscr{M}_-)$ which satisfies the boundary conditions $\partial u/\partial l=0$ in $\partial\Omega\setminus\mathscr{M}_-$ and $u=0$ on $\mathscr{M}_+$. Then u vanishes identically in Ω.*

For the proof we need a special barrier type function, given by

Proposition 4.2. *There is a neighbourhood of* $\mathcal{M}_-\setminus \operatorname{int} H$ *in which a function* w *is defined and has the properties*

(A) *w is of class C^2 outside $\mathcal{M}_-$*

(B) $\mathcal{L}w \leqq 0$

(C) $\dfrac{\partial w}{\partial l} \geqq 0$ *in* $\mathcal{A}_+\setminus\mathcal{M}_-$ *and*

$\dfrac{\partial w}{\partial l} \leqq 0$ *in* $\mathcal{A}_-\setminus\mathcal{M}_-$.

(D) *There are positive constants c_1 and c_2 such that*

$$c_1 w(p) \leqq -\log \operatorname{dist}(p, \mathcal{M}_-) \leqq c_2 w(p).$$

Proof. Let s, t, p' be coordinates in a neighbourhood of $\mathcal{M}_-\setminus \operatorname{int} H$ such that p' can be identified with coordinates in $\mathcal{M}_-$ and $\partial\Omega$ is given by $t=0$. Since we have imposed a monotonicity condition on the X-curves (see Section 1) we can choose s in such a way that $\partial s/\partial X \geqq 0$ on $\partial\Omega$ near $\mathcal{M}_-$.

In the expression for $\mathcal{L}$ in the new coordinates we single out the part which contains all second order derivatives with respect to s and t:

$$a_{ss}\frac{\partial^2}{\partial s^2}+2a_{st}\frac{\partial^2}{\partial s\,\partial t}+a_{tt}\frac{\partial^2}{\partial t^2}.$$

Now we define the function φ as the non-negative solution of

$$\varphi^2 = s^2-2\frac{a_{st}}{a_{tt}}st+\frac{a_{ss}}{a_{tt}}t^2$$

and let

$$w = -\log\varphi\cdot\exp(-\varphi).$$

It is readily verified that

$$\mathcal{L}w = \frac{a_{ss}a_{tt}-a_{st}^2}{a_{tt}}\cdot\frac{\log\varphi}{\varphi}+O(1/\varphi)$$

as $\varphi\to 0$, and hence that w satisfies (B).

Furthermore, since $\hat{n}$ is the outer conormal to $\partial\Omega$ with respect to $\mathcal{L}$ it follows that

$$\frac{\partial\varphi}{\partial l} = \operatorname{sgn} s\cdot\frac{\partial s}{\partial X}$$

near $\mathcal{M}_-\setminus \operatorname{int} H$ on $\partial\Omega\setminus\mathcal{M}_-$. Hence

$$\frac{\partial w}{\partial l} = (-1/s+\operatorname{sgn} s\cdot\log|s|)\cdot\exp(-|s|)\cdot\frac{\partial s}{\partial X}$$

which means that (C) is true. By the very construction of w it follows that (A) holds, and (D) follows from the ellipticity of $\mathscr{L}$. QED

Proof of Lemma 4.1. Since u is bounded it follows that

$$\sup_{\Omega} u = \sup_{\partial\Omega\setminus\mathscr{M}_-} u.$$

Assume that this supremum, m, is positive. The value m cannot be attained at any point in $\partial\Omega\setminus\mathscr{M}_-$ since this would imply that u takes on the value m in $\mathscr{M}_+$. (See Lemma 2.2 of [5].) Hence, there is a point $p\in\mathscr{M}_-$ such that

$$m = \limsup_{\partial\Omega\setminus\mathscr{M}_-\ni p'\to p} u(p').$$

We claim that p cannot belong to the interior of H. In fact, u is constant along $\gamma_p^{H\setminus\mathscr{M}_-}$ and if it has a maximum, then $\partial u/\partial n$ is positive along that arc and just as in [5], Lemma 2.2, we get a contradiction via $\partial u/\partial X = -\alpha\partial u/\partial n$.

Fix a neighbourhood W of $\mathscr{M}_-\setminus\operatorname{int} H$ in $\bar{\Omega}$ so small that Proposition 4.2 can be applied in $\overline{W}$. Put $m'=\sup_{\Omega\setminus W} u$. Then $m'<m$ and thus for all sufficiently small $\varepsilon>0$, the function $v=u-\varepsilon w$ is less than $(m'+m)/2$ on $\partial W\cap\Omega$. Since $w(p)$ tends to $+\infty$ as p approaches $\mathscr{M}_-$, it follows that $v<(m'+m)/2$ on $\partial W_\varrho\cap\Omega$ too, where $W_\varrho=\{p\in W\colon \operatorname{dist}(p,\mathscr{M}_-)<\varrho\}$ and $\varrho>0$ is sufficiently small.

On the other hand, on $\partial(W\setminus W_\varrho)\cap\partial\Omega$ we have $\partial v/\partial l=-\varepsilon\partial w/\partial l$ which is non-positive in $\mathscr{A}_+$ and non-negative in $\mathscr{A}_-$. Hence $v<(m'+m)/2$ in all of $W\setminus W_\varrho$ for all $\varrho>0$. First let ϱ tend to zero and then ε. Hence $u\leqq(m'+m)/2$ in W contradicting $\sup_W u=m$. Since the same discussion can be applied to $-u$ too, it follows that m has to be zero. QED

From the proof of Proposition 4.2 it follows that w does not depend on α. Moreover, for small distances to $\mathscr{M}_-$ we may give a more precise estimate for $\mathscr{L}w$ than that in (B). In fact

Proposition 4.3. *The barrier function w of* Proposition 4.2 *depends only on X, $\mathscr{M}_-$, Ω, and the principle part of $\mathscr{L}$. Moreover, for sufficiently small φ*

$$\mathscr{L}w \leqq C_0\cdot\frac{\log\varphi}{\varphi}$$

where C_0 is a positive constant and φ is equivalent to the distance to $\mathscr{M}_-$.

5. Proof of Theorem 1 in a special case

In this section we assume that H is a neighbourhood of $\mathcal{M}_-$ in $\partial\Omega$. We also require that *the coefficient c in $\mathcal{L}$ is non-positive* and not identically zero.

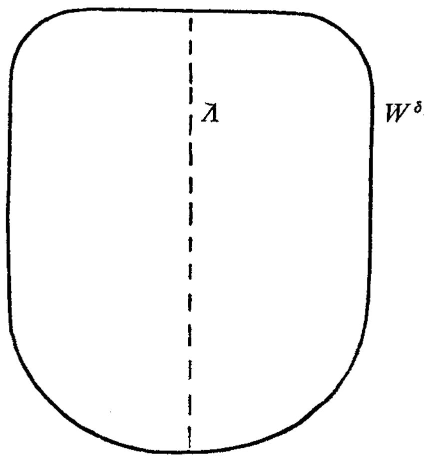
We will construct local solutions u_0 outside a neighbourhood of $\mathcal{M}$, u_1 in a neighbourhood of $\mathcal{M}_+$ and u_2 near $\mathcal{M}_-$. A special kind of a partition of unity in Ω will then enable us to combine the local terms to a global, approximate solution of (P). From now on we assume that f, g and h are fixed functions such that $f\in C^{1+\lambda}(\partial\Omega)\cap C^{2+\lambda}(\bar{\Sigma}\cap\partial\Omega)$, $g\in C^{\lambda}(\bar{\Omega})\cap C^{1+\lambda}(\bar{\Sigma})$ and $h\in C^{2+\lambda}(\mathcal{M}_+)$ where Σ is a fixed neighbourhood of H in $\bar{\Omega}$.

Construction of u_0: Choose an open neighbourhood $\mathcal{V}_1$ of $\mathcal{M}_+$ in $\partial\Omega$ and an open neighbourhood $\mathcal{V}_2$ of $\mathcal{M}_-$ in $\partial\Omega$ such that $H\setminus\mathcal{V}_1$ has a positive distance to $\mathcal{V}_1$ and $\bar{\mathcal{V}}_2\subset\operatorname{int} H$. Obviously, there is a unit vector field l' which is plus or minus l outside $\mathcal{V}_1\cup\mathcal{V}_2$ such that $\alpha'=l'\cdot n\geqq 0$ on $\partial\Omega$, the tangential component of l' is parallel to X and $l'\in C^{1+\lambda}$ is of class $C^{2+\lambda}$ in $\bar{\Sigma}\cap\partial\Omega$. In order to be able to apply Lemma 3.3 with l' as the directional field on the boundary, we must know that the lengths of X-curves in $H'=\{p\in\partial\Omega\colon \alpha'(p)=0\}$ are sufficiently small. However, since the X-curves are supposed to have zero length in $\partial H\cap U$ according to (ii) in the definition of $\mathcal{M}_-$ in Section 1 and since $\alpha'>0$ in $\mathcal{V}_1$ this is always possible to achieve. In fact, by Remark 3.4 we can apply Lemma 3.3 locally and thus the existence of u_0 is guaranteed by the smallness of k_0 independently of the choice of l' in the problem which we now are going to formulate.
Put $\tilde{f}=f$ in $\mathcal{A}_+$ and $=-f$ in $\mathcal{A}_-$. We choose a smooth function η in $\partial\Omega$ which is zero in a neighbourhood of $\mathcal{M}$ but equals one in a neighbourhood of $\partial\Omega\setminus(\mathcal{V}_1\cup\mathcal{V}_2)$. Denote the function $\eta\cdot\tilde{f}$ by f' and let u_0 be the unique solution of $\mathcal{L}u=g$ in Ω with $\partial u/\partial l'=f'$ on $\partial\Omega$.

Construction of u_1: To obtain u_1 we will use Lemma 3.1. Hence we have to define Λ and $\{W^\delta\}$. Let $\underline{n}$ be a smooth extension of the normal vector field $\underline{\hat{n}}$ to a neighbourhood of $\mathcal{M}_+$. Fix a sufficiently small $\delta_0>0$ and let Λ be the union of integral curves to $\underline{n}$ in $\bar{\Omega}$, originating in $\mathcal{M}_+$ and with length δ_0. We also introduce the function $t(p)$ in Λ as arclength on the integral curve to $\underline{n}$ through p, normalized in such a way that $t=0$ on $\mathcal{M}_+$. Let $\tilde{v}$ be a $C^{2+\lambda}$ unit vector field in a neighbourhood of Λ such that $\tilde{v}$ is normal to Λ and tangential to $\partial\Omega$. Introduce the coordinate function $s(p)$ as arclength (with sign) on integral curves to $\tilde{v}$, normalized in such a way that $s=0$ on Λ. This is well defined for all p on $\tilde{v}$-curves originating in Λ. Thus to every such p there is the coordinate $s(p)$, the coordinate $t(p)=t(p')$ where p' is the intersection of the $\tilde{v}$-curve through p with Λ, and the point $p''(p)\in\mathcal{M}_+$ which is the intersection of the $\underline{n}$-curve through p' with $\mathcal{M}_+$.

The construction of $\{W^\delta\}$ is thus reduced to the definition of a corresponding family of two-dimensional domains. Let $\mathscr{C}_1$ be a C^3-curve in the (s, t)-plane which consists of the line $\{t=0, 0\leqq s\leqq 1\}$, a convex arc from $(1,0)$ to $(2, 1)$ and then the line $\{s=2, 1\leqq t\leqq 2\}$.

Let $\mathscr{C}_2$ be a C^3-curve in the (s, t)-plane which consists of the two straight lines $\{s=2$ or $-2, 2\leqq t\leqq 3\}$ and a convex arc, symmetric with respect to the t-axis, which has strictly positive curvature at $s=t=0$. Hence we get W^δ as a combination of two curves congruent to $\delta\cdot\mathscr{C}_1$, one curve congruent to $\delta\cdot\mathscr{C}_2$ and two straight lines.



Furthermore, if k_0 and $\mathscr{V}_1$ are small enough, then $\partial W^\delta\supset\supset\mathscr{V}_1$ for some $\delta>k_0$ for which the problem (P_δ) in Lemma 3.1 is solvable when L is an extension of l to W^{δ_0} such that L is tangential to W^{δ_0} at $\partial\Lambda$ and has uniformly positive angle with Λ.

Construction of u_2*:* Define u_2 on $\partial\Omega\setminus\mathscr{M}_-$ as u_0 outside $\mathscr{V}_2$ and for $p\in\mathscr{V}_2$ as

$$u_2(p) = u_0(p')+\int_0^s f\circ\tilde{x}_{p'}(\tau)\,d\tau \tag{5.1}$$

where $p'\in H\setminus\mathscr{V}_2$, $p=\tilde{x}_{p'}(s)$ and the X-curve from p' to p is completely contained in $H\setminus\mathscr{M}_-$. This in general produces a singularity on $\mathscr{M}_-$. Extend u_2 to all of $\bar{\Omega}$ by solving $\mathscr{L}u=g$ in Ω with boundary values as above.

Construction of a partition of unity: Let ψ_1 be equal to one on a neighbourhood of $H\cap\mathscr{V}_1$ but equal to zero in $\partial\Omega\setminus\mathscr{V}_1$. Since l is not tangential to $\partial\Omega$ in $\mathscr{V}_1\setminus H$ we may extend ψ_1 to $\bar{\Omega}$ such that $\psi_1\in C^{2+\lambda}(\bar{\Omega})$, supp $\psi_1\subset W^\delta\cdot\partial\psi_1/\partial L\in C^{2+\lambda}(\overline{W^\delta})$, $0\leqq\psi_1\leqq 1$, and $\partial\psi_1/\partial l=0$ on $\partial\Omega$.

Let ψ_2 have support in a neighbourhood of $\mathscr{M}_-$ such that $\operatorname{supp}\psi_2\cap\partial\Omega\subset H$ and $\psi_2=1$ in a neighbourhood of $\bar{\mathscr{V}}_2$. We may take ψ_2 in $C^{2+\lambda}(\bar{\Omega})$ such that $\partial\psi_2/\partial L'\in C^{2+\lambda}(\bar{\Omega})$ for a fixed extension L' of l' and such that $0\leqq\psi_2\leqq 1$.

Now we define the *approximate global solution:*

$$u = (1-\psi_1-\psi_2)u_0+\psi_1 u_1+\psi_2 u_2. \tag{5.2}$$

We will prove that $\partial u/\partial l=f$ on $\partial\Omega\setminus\mathscr{M}_-$, that $u=h$ on $\mathscr{M}_+$ and that $\mathscr{L}u=g+\mathscr{K}$ where $\mathscr{K}$ corresponds to a compact linear perturbation in some space.

It is obvious that $u=h$ on $\mathscr{M}_+$ since on that manifold $\psi_1=1$ but $\psi_2=0$, and u_1 equals h there by definition.

Since $\partial u_0/\partial l'=f'$ it follows that $\partial u_0/\partial l=f$ except in a subset of $\mathscr{V}_1$, the closure of which is also contained in $\mathscr{V}_1$, or in $\mathscr{V}_2$. Hence, if the subset of $\mathscr{V}_1$ in which $\psi_1=1$ covers the first of these exceptional sets it follows that $\partial u/\partial l=f$ except possibly in $\operatorname{supp}\psi_2\cap\partial\Omega$. However, in this set $\psi_1=0$ and $l=X$. Outside $\mathscr{V}_2$, $u_0=u_2$ and hence $\partial u/\partial l=f$ there. In $\mathscr{V}_2$, $\psi_2=1$ and hence $u=u_2$. Thus by (5.1), $\partial u_2/\partial X=f$ and $\partial u/\partial l=f$.

Now $\mathscr{L}u=g+\mathscr{P}_0u_0+\mathscr{P}_1u_1+\mathscr{P}_2u_2$ where the $\mathscr{P}_i$: *s* are first order operators with supports in $\operatorname{supp}(\operatorname{grad}\psi_1)\cup\operatorname{supp}(\operatorname{grad}\psi_2)$, $\operatorname{supp}(\operatorname{grad}\psi_1)$ and $\operatorname{supp}(\operatorname{grad}\psi_2)$ respectively. For the construction of u_0 we only had to require that $\partial g/\partial L'\in C^\lambda(\bar{\Sigma}')$ where Σ' is a neighbourhood of $H\setminus\mathscr{V}_1$ in $\bar{\Omega}$. Similarly, the construction of u_1 was possible, provided only that $\partial g/\partial L\in C^\lambda(\bar{W}^\delta)$. Note that $\mathscr{L}u$ has the same regularity since the $\mathscr{P}_i$: *s* are of the first order. This means that we can assume that $f=0$ and $h=0$ in the sequel since the construction above enables us to reduce the problem (P) to that situation by subtracting a function u according to (5.2).

Now introduce the Banach space

$$S=\left\{g\in C^\lambda(\bar{\Omega})\colon \frac{\partial g}{\partial L}\in C^\lambda(\bar{W}^\delta),\ \frac{\partial g}{\partial L'}\in C^\lambda(\bar{\Sigma}')\right\}$$

with norm

$$\|g\|_\lambda' = \|g\|_\lambda^\Omega+\left\|\frac{\partial g}{\partial L}\right\|_\lambda^{W^\delta}+\left\|\frac{\partial g}{\partial L'}\right\|_\lambda^{\Sigma'}$$

and *define the linear map* $\mathscr{A}$ *in S by* $\mathscr{A}\colon g\to\mathscr{L}u$ where u is obtained from (5.2) via the constructions of u_0, u_1 and u_2. Then $\mathscr{A}=I+\mathscr{K}$ where as before $\mathscr{K}g=\mathscr{P}_0u_0+\mathscr{P}_1u_1+\mathscr{P}_2u_2$. We claim that $\mathscr{K}$ *is compact in S.*

In fact, the map $g\to u_i$, $i=0,1,2$, is continuous from S into $C^{2+\lambda}$ and therefore the map $g\to\mathscr{K}g$ is continuous from S into $C^{1+\lambda}$. We will prove that the maps $g\to\partial\mathscr{K}g/\partial L'|_{\Sigma'}$ and $g\to\partial\mathscr{K}g/\partial L|_{W^\delta}$ are continuous from S into $C^{1+\lambda}$. From this the compactness of $\mathscr{K}$ will follow. However, in Σ', $\partial/\partial L'\,\mathscr{K}g=\mathscr{P}_0'\,\partial u_0/\partial L'+\mathscr{P}_0''u_0+\mathscr{P}_2'\,\partial u_2/\partial L+\mathscr{P}_2''u_2$ where the operators are of the first order. We thus have to consider

$\partial u_0/\partial L'$ and $\partial u_2/\partial L'$ in $\operatorname{supp}(\operatorname{grad}\psi_2)$. Since u_0 and u_2 satisfy the weak identity

$$\mathcal{L}\frac{\partial u_i}{\partial L'}=\frac{\partial g}{\partial L'}+\mathcal{Q}u_i$$

where $\mathcal{Q}$ is of the second order, it follows from Theorem 9.3 in Agmon—Douglis—Nirenberg [1] that $\partial u_i/\partial L'\in C^{2+\lambda}$ in any open domain in Σ' which meets $\partial\Omega$ in a set where $l'=\pm l$ and that there is an a priori estimate

$$\left\|\frac{\partial u_i}{\partial L'}\right\|_{2+\lambda}\leqq C\cdot\left\{\left\|\frac{\partial g}{\partial L'}\right\|_{\lambda}^{\Sigma'}+\|u_i\|_{2+\lambda}\right\}\leqq C\cdot\|g\|_{\lambda}'.$$

A similar argument applies to $\partial\mathcal{K}g/\partial L$ in W^δ.

By the Riesz—Schauder theory, it follows that *the range* $\mathcal{R}(\mathcal{A})$ *of* $\mathcal{A}=I+\mathcal{K}$ *is all of* S if and only if the kernel $\mathcal{N}(\mathcal{A})$ is trivial. Thus, assume that $\mathcal{A}g=0$, i.e. $\mathcal{L}u=0$. Since $\partial u/\partial l=0$ in $\partial\Omega\setminus\mathcal{M}_-$ and $u=0$ on $\mathcal{M}_+$ it follows from Lemma 4.1 that $u=0$ in Ω. We will prove that this implies that $g=0$. In fact, this will follow if we can prove that $u_0=u_1$ in W^δ and $u_0=u_2$ in Ω when $u=0$, since then $u_0=u=0$ and $g=\mathcal{L}u_0=0$.

First, consider $v=u_0-u_2$ in Ω. Since $\mathcal{L}v=0$ in Ω and since by construction, $u_0=u_2$ in $\partial\Omega\setminus\mathcal{V}_2$, we only have to show that $\partial v/\partial l'=0$ in $\mathcal{V}_2$ to conclude that $v=0$ in Ω by the maximum principle. But this follows from the fact that $u_2=0$ in $\{\psi_2=1\}$ which is a neighbourhood of $\mathcal{V}_2$ by construction. Hence, $\partial u_2/\partial l'=0$ and since by definition $\partial u_0/\partial l'=0$ it follows that $\partial v/\partial l'=0$.

Next, consider $w=u_0-u_1$ in W^δ. This difference satisfies $\mathcal{L}w=0$ and by construction $w=0$ on $\partial\Lambda$ and $\partial w/\partial L=0$ in $\partial W^\delta\setminus\partial\Omega$. On $\partial W^\delta\cap\partial\Omega$ we note that $\partial u_0/\partial l'=0$ but that $\partial u_1/\partial l=0$. However, if $l'=\pm l$ this means that $\partial w/\partial l=0$ and if $l'\neq\pm l$ then $\psi_1=1$. By $(1-\psi_1)\cdot u_0+\psi_1\cdot u_1=0$ it thus follows that $u_1=0$ in a neighbourhood of $\{l'\neq\pm l\}$ in $\bar{\Omega}$ from which $\partial u_1/\partial l'=0$, too. But these conditions on the boundary imply that $w=0$.

This completes the proof of Theorem 1 in the special case. In the next section we apply an approximation procedure to get the general result.

6. Proof of Theorem 1 in the general case

It is enough to prove the theorem under the hypothesis that $f=0$ and $h=0$. Let $\delta_k\searrow 0$ and take smooth functions η_k on $\partial\Omega$ such that $0\leqq\eta_k\leqq 1$, $\eta_k(p)=1$ if $\operatorname{dist}(p,\mathcal{M}_-)>\delta_k$ and $\eta_k=0$ in a neighbourhood of $\mathcal{M}_-$. Replacing α by $\eta_k\cdot\alpha$ we get new fields l_k.

Using the monotonicity of the X-curves in a neighbourhood of $\mathcal{M}_-\setminus\operatorname{int}H$ we can choose $\{\eta_k\}$ such that the X-curves within $\partial H_k\setminus H$, where $H_k=$

$\{p\in\partial\Omega\colon (\eta_k\alpha)(p)=0\}$, have zero length. Hence, there is a unique solution of $(\mathscr{L}-a)u_k=g$ in Ω such that $\partial u_k/\partial l_k=0$ on $\partial\Omega\setminus\mathscr{M}_-$ and $u_k=0$ on $\mathscr{M}_+$. Here a is a smooth function, chosen such that $c-a\leqq 0$ in Ω, $c-a<0$ at some point in Ω, and $a=0$ in a neighbourhood of $\mathscr{M}_-$.

Since we want to select a converging subsequence of $\{u_k\}$ we first prove the following result:

Lemma 6.1. *Let Σ be a neighbourhood of $\mathscr{M}_-$ in $\bar{\Omega}$. Then there is a constant C which does not depend on g or k such that for sufficiently large k*

$$\|u_k\|_{2+\lambda}^{\Omega\setminus\Sigma}+\|u_k\|_{L^\infty(\Omega)} \leqq C\left\{\|g\|_\lambda^\Omega+\left\|\frac{\partial g}{\partial L_k}\right\|_\lambda^W\right\} \tag{6.1}$$

where W is a fixed neighbourhood of H in $\bar{\Omega}$.

Proof. Application of Agmon—Douglis—Nirenberg [1] outside H and use of the technique in the previous article [5], Section 3.1 and in Section 3 of this article shows, that we only have to consider the estimates in a neighbourhood of the components H_- of H which contain $\mathscr{M}_-$.

From a weak identity $\mathscr{L}(\partial u/\partial L_k)=\partial g/\partial L_k+\mathscr{P}u$ with a second order operator $\mathscr{P}$, and from Lemma A.2 in [5] it follows that

$$\|u_k\|_{2+\lambda}^{\Sigma'} \leqq C\cdot\left\{\varepsilon\cdot\|u_k\|_{2+\lambda}^{\Sigma''}+\|g\|_\lambda^\Omega+\left\|\frac{\partial g}{\partial L_k}\right\|_\lambda^W\right\}$$

where $\Sigma'\subset\subset\Sigma''$ are neighbourhoods of $H_-\setminus\Sigma$.

However, from local estimates for the Dirichlet problem and the fact that $\partial u_k/\partial X=0$ in H, a bound for the $C^{2+\lambda}$-norm of u_k up to $\partial\Omega$ in the interior of H_- is easy to get. Hence,

$$\|u_k\|_{2+\lambda}^{\Omega\setminus\Sigma} \leqq C\cdot\left\{\|g\|_\lambda^\Omega+\left\|\frac{\partial g}{\partial L_k}\right\|_\lambda^W+\|u_k\|_{L^\infty(\Omega)}\right\}.$$

Next we claim that in a neighbourhood U of Σ,

$$|u_k| \leqq C\cdot\max|g|+\sup_{\Omega\setminus\Sigma}|u_k|. \tag{6.2}$$

In fact, let w be the barrier function according to Proposition 4.2. Let U_ϱ be the set $\{p\in\Omega\colon \operatorname{dist}(p,\mathscr{M}_-)<\varrho\}$ and choose $\varrho_\nu=\delta_k^\nu$, $\nu=1,2,\dots$. The function $v=u-\varepsilon w$ satisfies (see Propositions 4.2 and 4.3)

$$\mathscr{L}v = g-\varepsilon\mathscr{L}w \geqq g-\varepsilon C_0\frac{\log\varphi}{\varphi} \quad \text{in } U_{\varrho_\nu},$$

$$\frac{\partial v}{\partial l_k} = -\varepsilon\frac{\partial w}{\partial l_k} \quad \text{on } \partial\Omega\cap U_{\varrho_1} \quad \text{and}$$

$$v \leqq u \quad \text{on } \partial U_{\varrho_\nu}.$$

If we choose ε to be $-C_0^{-1}\max|g|\cdot\varrho_\nu/\log\varrho_\nu$ we find that $\mathscr{L}v\geqq 0$ in U_{ϱ_ν} and hence that $u-\varepsilon w\leqq\sup_{\partial U_{\varrho_\nu}}u$ in U_{ϱ_ν} (cf. the proof of Lemma 4.1). Thus $u\leqq\sup_{\partial U_{\varrho_\nu}}u+(C_0\cdot c_1)^{-1}\cdot\varrho_\nu\cdot\max|g|$ in $U_{\varrho_\nu}\setminus U_{\varrho_{\nu+1}}$ from which it follows that $u\leqq\sup_{\partial U_{\varrho_1}}u+C\cdot\delta_k\cdot\max|g|$ in U_{ϱ_1}. A similar argument applies in $U\setminus U_{\varrho_1}$ and this proves (6.2).

Hence we get

$$\|u_k\|_{2+\lambda}^{\Omega\setminus\Sigma}+\|u_k\|_{L^\infty(\Omega)}\leqq C\cdot\left\{\|g\|_\lambda^\Omega+\left\|\frac{\partial g}{\partial L_k}\right\|_\lambda^W+\|u_k\|_{L^\infty(\Omega\setminus\Sigma)}\right\}$$

from which the lemma follows by a compactness argument. QED

Choosing a sequence of domains $\Sigma_k\setminus\mathscr{M}_-$ and applying a diagonal procedure we get a subsequence $\{u_{k'}\}$ which converges in $C^{2+\lambda}(\bar\Omega\setminus\Sigma)$ for every $\Sigma\supset\mathscr{M}_-$ and which is bounded. Hence the limit function u satisfies $\mathscr{L}u=g$ in Ω, $\partial u/\partial l=0$ on $\partial\Omega\setminus\mathscr{M}_-$ and $u=0$ on $\mathscr{M}_+$. Furthermore, u is bounded.

Tending to the limit in (6.1) we get

$$\|u\|_{2+\lambda}^{\Omega\setminus\Sigma}+\|u\|_{L^\infty(\Omega)}\leqq C(\Sigma)\cdot\left\{\|g\|_\lambda^\Omega+\left\|\frac{\partial g}{\partial L}\right\|_\lambda^W\right\} \tag{6.3}$$

if the extensions L_k of l_k are chosen such that they converge to a fixed extension L of l. In the same way as Theorem 3.2 of [5] was proved from Theorem 1 in [5], it follows from (6.3) that there is a unique solution of $(\mathscr{L}-a)\,u=g$ with boundary conditions as before, if only $g\in C^\lambda(\bar\Omega)$ and $\partial g/\partial L\in C^\lambda(\bar W)$. Finally from (6.3) and the fact that a is zero in a neighbourhood of $\mathscr{M}_-$ it follows that the map $\mathscr{A}\colon g\to\mathscr{L}u=g+a\cdot u$ is the sum of the identity map in $S'(W,L)=\{g\in C^\lambda(\bar\Omega)\colon\partial g/\partial L\in C^\lambda(\bar W)\}$ and a compact operator. Hence Theorem 1 follows from the Riesz—Schauder theory. In particular, the result on the codimension of $\tilde S$ follows from the observation that if $u\in\mathscr{N}$ then $v=u$ is the unique solution of $\mathscr{L}v-av=-au$ with zero boundary data.

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References

1. AGMON, S., DOUGLIS, A., NIRENBERG, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I. *Comm. Pure Appl. Math.* **12** (1959), 623—727.
2. Янушаускас А. И., Глобальные методы исследования задачи о наклонной производной для эллиптических уравнений. (Global methods of studying the oblique derivative problem for elliptic equations.) *Diff. uravn.* **12** (1976), 159—171.
3. MAZ'JA, V. G., On a degenerating problem with directional derivative. *Math. USSR Sbornik* **16** (1972), 429—469.
4. WINZELL, B., Solutions of second order elliptic partial differential equations with prescribed directional derivative on the boundary. *Linköping Studies in Science and Technology, Dissertations* No **003** (1975).
5. WINZELL, B., The oblique derivative problem I. *Math. Ann.* **229** (1977), 267—278.

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