Primes in short intervals

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1. Introduction

We shall be concerned with the problem of estimating the number of primes in an interval of the form $[x-y, x]$ with $y=x^{\theta}$, $\frac{1}{2}<\theta<1$. In 1930 Hoheisel [3] proved that

$$(1) \qquad \pi(x)-\pi(x-y) \sim y/\log x$$

if $\theta>\theta_0=1-1/33000$, and Heilbronn [2] showed that one may take $\theta_0=1-1/250$. A substantial improvement became possible owing to I. M. Vinogradov's work on Weyl sums, which led to the following result on the zero-free region for Riemann's zeta-function (see [1], or [11], p. 114): there exist two constants $A>0$ and $B<1$ such that for all zeros $\varrho=\beta+i\gamma$ of $\zeta(s)$

$$(2) \qquad \beta < 1-A\big(\log(|\gamma|+2)\big)^{-B}.$$

Since then the problem has been essentially connected with the zero-density estimate

$$(3) \qquad N(\alpha, T) \ll T^{b(1-\alpha)}\log^c T,$$

where $T\geqq 3$ and $N(\alpha, T)$ denotes the number of zeros $\varrho=\beta+i\gamma$ of $\zeta(s)$ such that $\beta\geqq\alpha$ and $|\gamma|\leqq T$. Ingham [5] showed that if (3) holds uniformly for $\frac{1}{2}\leqq\theta\leqq 1$, then one may take $\theta_0=1-1/b$, and he gave also the relation $b\leqq 2+4c$, where c is any constant such that for $t\geqq 3$ and some constant d

$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \ll t^c \log^d t.$$

From $c=1/6$ Ingham obtained $b=8/3$ and $\theta_0=5/8$. The best known result $c=173/1067$ due to Kolesnik [8] leads to a very small improvement. Recently Montgomery [9] found a new approach to density theorems (the Halász—Montgomery method) and without using any bound for $\zeta(\frac{1}{2}+it)$ he proved (3) with $b=5/2$, thus getting $\theta_0=3/5$. Later Huxley [4] refined the method and obtained $b=12/5$, $\theta_0=7/12$. Any further improvement of the constant b seems to require essentially new ideas.

Our aim is to prove the following result.

Theorem. *For any* $\theta \geqq 13/23$ *and all sufficiently large* x *we have*

$$\pi(x)-\pi(x-x^{\theta}) > \frac{1}{177}\frac{x^{\theta}}{\log x}. \tag{4}$$

By elaborating the method the constant 13/23 could be reduced. We return to this question in the end of the paper.

The main novelty in our approach is a combination of a certain version of the linear sieve (see Lemma 1) with analytic methods. We lose the asymptotic formula (1) because we work with inequalities in the sieve part of the proof. Huxley's density theorem is used in proving a "weighted" density estimate (see Lemma 3) which plays the same role as (3) in the classical method.

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2. Sieve results

Let $\frac{1}{2}<\theta<1$, $y=x^{\theta}$, $2<v<3$, $z=x^{1/v}$, $P(z)=\prod_{p<z} p$ and

$$\mathcal{A} = \{a \mid x-y < a \leqq x\}.$$

We denote

$$S(\mathcal{A}, z) = \left|\{a\in\mathcal{A} \mid (a, P(z)) = 1\}\right|$$

with the aim of getting a lower bound by the sieve method. For this we shall require the following result (see [6]).

Lemma 1. *Let* $M\geqq 2$, $N\geqq 2$ *and* $z\geqq 2$. *Then, for any* $\varepsilon>0$ *we have*

$$S(\mathcal{A}, z) > V(z)y\big(f(s)-E\big)-2^{\varepsilon^{-10}}R(\mathcal{A}; M, N),$$

where

$$V(z) = \prod_{p<z}\left(1-\frac{1}{p}\right) \sim e^{-C}/\log z$$

by Mertens' prime number theorem (C is the Euler constant), $s=\log MN/\log z$,

$$f(s) = 2e^{C}s^{-1}\log(s-1) \quad \text{for} \quad 2 \leqq s \leqq 4,$$

$$E = E(\varepsilon, s, M, N) \ll \varepsilon s+\varepsilon^{-8}(\log MN)^{-1/3},$$

and the remainder term $R(\mathscr{A}; M, N)$ *is of the form*

$$R(\mathscr{A}; M, N) = \sum_{m<M} \sum_{n<N} a_m b_n r(\mathscr{A}, mn)$$

with some coefficients a_m, b_n *bounded by* 1 *in absolute value and depending at most on* M, N, z *and* ε, *and with*

$$r(\mathscr{A}, d) = \left[\frac{x}{d}\right] - \left[\frac{x-y}{d}\right] - \frac{y}{d}.$$

The remainder term $R(\mathscr{A}; M, N)$ is required to be $o(y/\log x)$. Its flexible form allows one to prove this with values of MN larger than those that would be possible if it had its traditional form.

Lemma 2. *Let* $5/9<\theta<1$ *and* $0<\alpha<\min(1/2, (5\theta-1)/4)$. *Then there exists a number* $\eta=\eta(\theta, \alpha)>0$ *such that*

$$R(\mathscr{A}; x^\alpha, x^\alpha) \ll yx^{-\eta},$$

the implied constant depending at most on θ *and* α.

Proof. We shall use the classical mean value theorem for Dirichlet polynomials (see [10], Theorem 6.1), which asserts that

$$\int_{T_0}^{T_0+T} \left|\sum_{n=1}^N a_n n^{-it}\right|^2 dt = \left(T+O(N)\right) \sum_{n=1}^N |a_n|^2,$$

and the Halász—Montgomery inequality (ibid., Theorem 8.2) which asserts that

$$\sum_{r=1}^R \left|\sum_{n=1}^N a_n n^{-s_r}\right|^2 \ll (N+RT^{1/2}\log T) \sum_{n=1}^N |a_n|^2,$$

where the points $s_r=\sigma_r+it_r$ have the properties that $\sigma_r \geqq 0, 0\leqq t_r \leqq T$, and $|t_r-t_{r'}|\geqq 1$ for $r\neq r'$.

We shall also use various estimates for $\zeta(s)$ in the critical strip, mainly due to Hardy and Littlewood. Let $a(\sigma)$ and $b(\sigma)$ be functions, defined in the interval $[\frac{1}{2}, 1]$, such that for any given $\varepsilon>0$

$$|\zeta(\sigma+it)| \ll t^{a(\sigma)+\varepsilon},$$

$$\int_{-T}^{T} |\zeta(\sigma+it)|^{1/(2b(\sigma)+\varepsilon)} dt \ll T^{1+\varepsilon},$$

uniformly for $t\geqq 2$, resp. $T\geqq 2$. By Theorems 5.8 and 7.10 of [11] we have

$$a(1/2) \leqq 1/6, \quad a(3/4) \leqq 1/16, \quad a(\sigma) = o(1-\sigma) \quad \text{as} \quad \sigma \to 1-,$$

$$b(1/2) \leqq 1/8, \quad b(5/8) \leqq 1/12, \quad b(3/4) \leqq 1/20, \quad b(1) = 0.$$

Further, we may suppose that $a(\sigma)$ and $b(\sigma)$ are convex functions; the convexity of $a(\sigma)$ is a standard result, and for $b(\sigma)$ the assertion follows from a two-variable convexity theorem (see [11], p. 203). Now it is easy to verify that in the whole interval $[\frac{1}{2}, 1]$

(5) $$\left(\sigma-\frac{1}{2}\right)a(\sigma)+(1-\sigma)b(\sigma) \leqq \frac{1}{8}(1-\sigma).$$

To prove the lemma it is sufficient to estimate the sums

$$R(\mathscr{A};\, M,\, N) = \sum_{M\leqq m<2M}\sum_{N\leqq n<2N} a_m b_n r(\mathscr{A},\, mn)$$

with $2\leqq M,\, N\leqq x^\alpha$. Let $A(s)$ and $B(s)$ denote the Dirichlet generating functions of the sequences $\{a_m\}_{M\leqq m<2M}$ and $\{b_n\}_{N\leqq n<2N}$, respectively. By Perron's integral formula we obtain

$$R(\mathscr{A};\, M,\, N) = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\zeta(s)A(s)B(s)\frac{x^s-(x-y)^s}{s}\,ds+O(yx^{-\eta})$$

$$\ll yx^{\sigma-1}\int_{-T}^{T}|\zeta(\sigma+it)A(\sigma+it)B(\sigma+it)|\,dt+yx^{-\eta},$$

with some $T\in\left[\frac{1}{2}x^{1+2\eta}/y,\, x^{1+2\eta}/y\right]$ and arbitrary $\sigma\in[\frac{1}{2}, 1)$. In the first line we used the fact that in any interval $[T_1, 2T_1]$ with $T_1\geqq 2$ there is a number T such that

$$\int_{1/2}^{2}|\zeta(\sigma\pm iT)|\,d\sigma \ll \log T.$$

To estimate the integral of $|\zeta AB|$ it is useful to consider the ranges of small and large values of $\zeta(s)$ separately. Put $a=a(\sigma)$, $b=b(\sigma)$. In the range

$$C_1 = \left\{-T \leqq t \leqq T \,\middle|\, |\zeta(\sigma+it)| \leqq T^{b+\eta}\right\}$$

we use the mean value theorem for Dirichlet polynomials, getting

$$\int_{C_1}|\zeta AB| \ll T^{b+\eta}\left(\int|A|^2\right)^{1/2}\left(\int|B|^2\right)^{1/2}$$

$$\ll T^{b+\eta}(M+T)^{1/2}(N+T)^{1/2}(MN)^{1/2-\sigma}.$$

To treat the complementary set C_2 we choose points $s_r=\sigma+it_r$, $t_r\in C_2$, $|t_r-t_{r'}|>1$ in such a way that

$$\int_{C_2}|\zeta AB| \ll \sum_{r=1}^{R}|\zeta AB(s_r)|.$$

We know (see [10], Theorem 10.3 for the case $\sigma=1/2$) that

$$\sum_{r=1}^{R}|\zeta(s_r)|^{1/(2b+\varepsilon)} \ll T^{1+\varepsilon}.$$

For $V\geqq T^{b+\eta}$, consider those points s_r for which $V\leqq|\zeta(s_r)|<2V$. From the above we see that there are $\ll T^{1+\varepsilon}V^{-1/(2b+\varepsilon)}$ such points, so that by the Halász—Montgomery inequality

$$\sum_{r,\, V\leqq|\zeta(s_r)|<2V}|\zeta AB| \ll V\left(\sum|A|^2\right)^{1/2}\left(\sum|B|^2\right)^{1/2}$$

$$\ll V(M+T^{3/2}V^{-1/(2b+\varepsilon)})^{1/2}(N+T^{3/2}V^{-1/(2b+\varepsilon)})^{1/2}(MN)^{1/2-\sigma}T^{\varepsilon}.$$

We sum this over those V of the form $V=2^k$, $T^{b+\eta}\leqq V\leqq T^{a+\varepsilon}$, and deduce that

$$\int_{C_2}|\zeta AB|\ll T^{a+\eta}(MN)^{1-\sigma}+T^{b+\eta}(M+T)^{1/2}(N+T)^{1/2}(MN)^{1/2-\sigma}.$$

Hence

$$R(\mathscr{A};\,M,N)\ll yx^{\sigma-1+3\eta}\{(x/y)^a(MN)^{1-\sigma}+(x/y)^{b+1/2}x^{\alpha/2}(MN)^{1/2-\sigma}+(x/y)^{b+1}(MN)^{1/2-\sigma}\}. \tag{6}$$

If

$$(x/y)^{a(1/2)}\leqq(x/MN)^{1/2}x^{-4\eta},$$

we take $\sigma=1/2$ and use the inequalities $b(1/2)\leqq 1/8$, $\theta>5/9$, $\alpha<(5\theta-1)/4$, getting

$$R(\mathscr{A};\,M,N)\ll yx^{-\eta}. \tag{7}$$

Otherwise let σ be a root of the equation

$$(x/y)^{a(\sigma)}=(x/MN)^{1-\sigma}x^{-4\eta}.$$

Since $a(\sigma)=o(1-\sigma)$ as $\sigma\to 1-$, the above equation has a solution σ for which $1/2<\sigma<1-\varepsilon$ with some $\varepsilon=\varepsilon(\theta,\alpha)>0$, provided that η is sufficiently small. Hence by (6) and (5) we obtain again (7). This completes the proof of the lemma.

From Lemmas 1 and 2 we get

$$S(\mathscr{A},x^{1/v})>\frac{y}{\log x}\{\lambda(\theta,v)-\varepsilon\} \tag{8}$$

with

$$\lambda(\theta,v)=\frac{4}{5\theta-1}\log\left(\frac{5\theta-1}{2}v-1\right),$$

provided that $5/9<\theta<3/5$, $\frac{4}{5\theta-1}<v<\frac{8}{5\theta-1}$ and $x>x_0(\varepsilon,\theta)$.

Denote by $\mathscr{A}_d$ the set of the elements of $\mathscr{A}$ which are divisible by d, and for $2<u<(1-\theta)^{-1}$ let

$$T(\mathscr{A};\,x^{1/v},x^{1/u})=\sum\nolimits_{x^{1/u}\leqq p<x^{1/2}}S(\mathscr{A}_p,x^{1/v}),$$

with the aim of estimating this from above by the sieve method as well. Take any P form the interval $[x^{1/u},x^{1/2}]$ and let $\mathscr{B}$ be the sequence of elements a of $\mathscr{A}$, each repeated once for each $p|a$ with $P\leqq p<2P$. By Theorem 1 of [6] when applied to $\mathscr{B}$ we obtain

$$\sum\nolimits_{P\leqq p<2P}S(\mathscr{A}_p,x^{1/v})<V(z)y\left(\sum\nolimits_{P\leqq p<2P}p^{-1}\right)\left(F(\log B/\log z)+\Phi(\varepsilon,B)\right)+2^{\varepsilon^{-10}}R(\mathscr{B};\,P,B^{2/3},B^{1/3}), \tag{9}$$

where $B\geqq 2$, $F(s)=2e^C/s$ for $0<s\leqq 3$, $\Phi(\varepsilon,B)\ll\varepsilon+\varepsilon^{-8}(\log B)^{-1/3}$ and

$$R(\mathscr{B};\,P,M,N)=\sum\nolimits_{P\leqq p<2P}\sum\nolimits_{m<M}\sum\nolimits_{n<N}\alpha_m\beta_n r(\mathscr{A},pmn)$$

with the coefficients α_m, β_n bounded by 1 in absolute value and depending at most on B, P, v and ε.

Letting $P(s)$, $G(s)$ and $H(s)$ be the generating functions for the sequence of primes p from the interval $P \leqq p < 2P$ and for the sequences $\{\alpha_m\}_{m<B^{2/3}}$ and $\{\beta_n\}_{n<B^{1/3}}$, respectively, we obtain by Perron's integral formula

$$R(\mathscr{B};\, P,\, B^{2/3}, B^{1/3}) \ll yx^{-1/2}\int_{1/2-iT}^{1/2+iT} |\zeta(s)P(s)G(s)H(s)|\,|ds| + yx^{-\eta}$$

with some $T \in \left[\frac{1}{2}x^{1+2\eta}/y,\, x^{1+2\eta}/y\right]$ and any $\eta > 0$ if we suppose that $BP \leqq x$.

By the fourth moment estimate for $\zeta(s)$ we see that the set

$$L_1 = \left\{s = 1/2+it \,\big|\, |t| \leqq T,\ |\zeta(s)| \leqq T^{1/8}\right\}$$

has measure $\ll T^{1/2}\log^4 T$; by the mean value theorem for Dirichlet polynomials we see that

$$\begin{aligned}\int_{L_1} |\zeta PGH| &\ll \Big(\int_{L_1} |\zeta|^8\Big)^{1/8}\Big(\int |P|^2\Big)^{1/2}\Big(\int |G|^4\Big)^{1/4}\Big(\int |H|^8\Big)^{1/8}\\ &\ll T^{1/8+1/16}(P+T)^{1/2}(B^{4/3}+T)^{1/4}(B^{4/3}+T)^{1/8}\log^4 x\\ &\ll \big(T^{3/16}(PB)^{1/2}+T^{9/16}P^{1/2}\big)\log^4 x.\end{aligned}$$

To treat the remaining range L_2 we choose well-spaced points $s_r = 1/2+it_r \in L_2$ so that

$$\int |\zeta PGH| \ll \sum\nolimits_r |PGH(s_r)|.$$

We consider those s_r for which $V \leqq |\zeta(s_r)| < 2V$. There are $\ll TV^{-4}\log^4 T$ such s_r; by Hölder's inequality the sum over these s_r is

$$\ll (V^4 T\log^4 T)^{1/8}\big(\sum |P|^2\big)^{1/2}\big(\sum |G|^4\big)^{1/4}\big(\sum |H|^8\big)^{1/8}.$$

We apply the mean value theorem for Dirichlet polynomials to P, and the Halász—Montgomery inequality to G^2 and to H^4, to see that the above is

$$\ll V^{1/2}T^{1/8}(T+P)^{1/2}(B^{4/3}+T^{3/2}V^{-4})^{1/4}(B^{4/3}+T^{3/2}V^{-4})^{1/8}\log^4 x.$$

Summing over $V = 2^k$, $T^{1/8} \leqq V \leqq T^{1/6}$ we see that

$$\int_{L_2} |\zeta PGH| \ll \big(T^{5/24}(PB)^{1/2}+T^{9/16}P^{1/2}\big)\log^4 x.$$

Hence we obtain for $PB \leqq x^{\beta}$ and $\beta < (5\theta+7)/12$

$$R(\mathscr{B};\, P,\, B^{2/3}, B^{1/3}) \ll yx^{-\eta} \tag{10}$$

with $\eta = \eta(\theta, \beta) > 0$. Combining (9) and (10) and summing over P, we get by partial summation

$$T(\mathscr{A};\, x^{1/v}, x^{1/u}) < \frac{y}{\log x}\big(\Lambda(\theta, u)+\varepsilon\big) \tag{11}$$

with

$$A(\theta, u) = \frac{24}{7+5\theta} \log\left(\frac{(7+5\theta)u-12}{2(1+5\theta)}\right)$$

provided $(5\theta+7)/12 < 1/u + 3/v$ and $x > x_0(\varepsilon, \theta)$.

The integers $a \in \mathscr{A}$ having a prime factor in the interval $[x^{1/v}, x^{1/u}]$ will be treated by function-theoretic methods. Using the arguments familiar from Hoheisel type theorems, we arrive at the problem of estimating the number of zeros of the zeta-function in the rectangle $\alpha \leqq \sigma < 1$, $|t| \leqq T$, weighted by the generating function of the primes $p \in [x^{1/v}, x^{1/u}]$.

3. The weighted density estimate

It turns out that if some Dirichlet polynomial weights are attached to the zeros $\varrho = \beta + i\gamma$ of the zeta-function, the resulting "weighted" density estimate is better than the usual one, sometimes as good as the density conjecture (see [7]).

Lemma 3. *Let a_q be a sequence of complex numbers bounded by 1 in absolute value, and put*

$$K(s) = \sum_{Q \leqq q < 2Q} a_q q^{-s}.$$

Then

$$\sum_{|\gamma| \leqq T,\, \beta \geqq \alpha} |K(\varrho)| \ll (T^{16/5} Q^{-1} + T^{6/5} Q)^{1-\alpha} \log^A T \tag{12}$$

for $0 \leqq \alpha \leqq 1$, $Q \geqq T^{4/5}$.

Proof. The above formulation of the lemma and its proof are due to H. L. Montgomery. By Cauchy's inequality we see that

$$\sum_{|\gamma| \leqq T,\, \beta \geqq \alpha} |K(\varrho)| \leqq N(\alpha, T)^{1/2} \Big(\sum_{|\gamma| \leqq T,\, \beta \geqq \alpha} |K(\varrho)|^2\Big)^{1/2}. \tag{13}$$

For $N(\alpha, T)$ we have the bounds of Ingham [5] and Huxley [4], namely

$$N(\alpha, T) \ll \begin{cases} T^{(3-3\alpha)/(2-\alpha)} \log^A T & 0 \leqq \alpha \leqq 3/4, \\ T^{(3-3\alpha)/(3\alpha-1)} \log^A T & 3/4 \leqq \alpha \leqq 1. \end{cases} \tag{14}$$

We can now estimate the second factor in (13) by appealing to the mean value theorem for Dirichlet series, which gives

$$\sum |K(\varrho)|^2 \ll (T+Q) Q^{1-2\alpha} \log^A T, \tag{15}$$

since the number of γ with $t \leqq \gamma \leqq t+1$ is $\ll \log T$. From (14) we see that

$$N(\alpha, T) \ll T^{12(1-\alpha)/5} \log^A T \tag{16}$$

for $0 \leqq \alpha \leqq 1$; this with (13) and (15) gives

$$\sum |K(\varrho)| \ll (T^{(17-12\alpha)/10} Q^{1/2-\alpha} + T^{6(1-\alpha)/5} Q^{1-\alpha}) \log^A T.$$

Hence we have (12) when either $Q \geqq T$ or $\alpha \leqq 3/4$. It remains to treat the case when $T^{4/5} \leqq Q \leqq T$, $3/4 \leqq \alpha \leqq 1$. If in this case we use the full strength of (14) in place of the weaker estimate (16) we obtain the sharper bound

$$\sum |K(\varrho)| \ll T^{1/(3\alpha-1)} Q^{1/2-\alpha} \log^A T.$$

In comparison with the desired bound $T^{(16-16\alpha)/5} Q^{\alpha-1}$ we see that the worst case is when $Q = T^{4/5}$. It is then easy to check that the above bound gives (12) when $3/4 \leqq \alpha \leqq 5/6$. Now suppose that $5/6 \leqq \alpha \leqq 1$, $T^{4/5} \leqq Q \leqq T$. Instead of using (15) we apply the Halász—Montgomery inequality, which gives

$$\sum |K(\varrho)|^2 \ll (Q + T^{1/2} N(\alpha, T)) Q^{1-2\alpha} \log^A T. \tag{17}$$

Combining this with (13), (16), we see that

$$\sum |K(\varrho)| \ll (T^{(6-6\alpha)/5} Q^{1-\alpha} + T^{(53-48\alpha)/20} Q^{1/2-\alpha}) \log^A T.$$

To compare the second term with (12) we again note that the worst case is when $Q = T^{4/5}$; we then find that the above gives (12) when $13/16 \leqq \alpha \leqq 1$. This includes the desired range. On comparing (15), (17) we see that we should use (17) when $N(\alpha, T) \leqq T^{1/2}$. Thus on the basis of the available knowledge provided by (14), we see that it is optimal to use (17) for $\alpha < 7/9$, and (15) for $\alpha \leqq 7/9$.

4. The interval $[x^{1/v}, x^{1/u}]$

Letting

$$V(\mathscr{A};\, x^{1/v}, x^{1/u}) = \sum_{x^{1/v} \leqq p < x^{1/u}} S(\mathscr{A}_p, x^{1/v}),$$

we obtain

$$\begin{aligned} V(\mathscr{A};\, x^{1/v}, x^{1/u}) &= \sum_{x^{1/v} \leqq p < x^{1/u}} \left(\pi\left(\frac{x}{p}\right) - \pi\left(\frac{x-y}{p}\right)\right) \\ &\sim \sum_{x^{1/v} \leqq p < x^{1/u}} \frac{y}{p \log (x/p)} \\ &\sim \frac{y}{\log x} \log\left(\frac{v-1}{u-1}\right) = \frac{y}{\log x} \Omega(u, v) \end{aligned} \tag{18}$$

say, the penultimate line being formed heuristically. In order to justify this we consider sums of the type

$$\Psi(x, y;\, P) = \sum_{P \leqq p < 2P} \left(\psi\left(\frac{x}{p}\right) - \psi\left(\frac{x-y}{p}\right)\right)$$

with $x^{1/v} \leqq P < x^{1/u}$. Let $\gamma = 1 + (\log x)^{-1}$, $2 \leqq T \leqq x$ and $K(s)$ the generating function for the primes p form the interval $[P, 2P)$. By Perron's integral formula we obtain

$$\Psi(x, y; P) = \frac{1}{2\pi i} \int_{\gamma - iT}^{\gamma + iT} -\frac{\zeta'}{\zeta}(s) K(s) \frac{x^s - (x-y)^s}{s} \, ds + O(x^{1+\varepsilon}/T)$$

$$= yK(1) - \sum_{|\mathrm{Im}\,\varrho| \leqq T} K(\varrho) \frac{x^\varrho - (x-y)^\varrho}{\varrho} + O(x^{1+\varepsilon}/T)$$

by moving the integration to the line $\sigma = -1/2$. We assumed that

$$\left| \frac{\zeta'}{\zeta} (\sigma \pm iT) \right| \ll \log^2 T$$

uniformly for all $-1/2 \leqq \sigma \leqq \gamma$; it is well known that this is true at least for one T from each interval of length 1. Choosing $T = x^{1+2\varepsilon}/y$, we see by Lemma 3 and (2) that the sum over the zeros is $\ll y(\log x)^{-4}$ provided

$$\frac{16}{5}(1-\theta) < 1 + 1/v, \tag{19}$$

$$\frac{6}{5}(1-\theta) < 1 - 1/u. \tag{20}$$

Therefore, by partial summation, we have proved the asymptotic formula (18) if u, v and θ satisfy the conditions (19), (20). Obviously the optimal values of u, v are those which make the interval $[x^{1/v}, x^{1/u}]$ as long as possible. Gathering together (8), (11) and (18), we conclude that if $5/9 < \theta \leqq 7/12$ and there are parameters u and v satisfying (19), (20) and

$$\lambda(\theta, v) - \Lambda(\theta, v) - \Omega(u, v) > 1/177, \tag{21}$$

then

$$\pi(x) - \pi(x - x^\theta) > \frac{1}{177} \frac{x^\theta}{\log x}$$

for all sufficiently large x. To verify (21) it is sufficient to take u and v which equalize (19) and (20). A computation then shows that for all $\theta \in [13/23, 7/12]$ the inequality (21) holds.

5. Remarks

There are several ways in which the constant 13/23 can be reduced. First of all, Lemma 1 can be sharpened as follows. The method of [6] actually gives a lower bound for

$$S(\mathscr{A}, z) - \sum_{(MN/p_1)^{1/3} \leqq p_2 < p_1 < z} S(\mathscr{A}_{p_1 p_2}, p_2) - \ldots$$

In Lemma 1 only the first term was taken into account. However, by the same arguments as in (18) one can get asymptotic formulae for the parts of the other terms corresponding to $p_1, \ldots, p_{2r}$ with $x^{1/v} \leqq p_1 p_2 \ldots p_{2r} < x^{1/u}$. This extra contribution weakens the condition (21) so that a smaller number θ satisfies it.

Another possibility is offered by a recent elegant identity due to R. C. Vaughan [12] for sums involving von Mangoldt's function. It can be utilized both in the weighted density estimate and in the sieve. By Vaughan's identity our $K(s)$ can be transformed into double sums of a more suitable form; in that way one may make use of the fact that the sum $K(s)$ is over primes. As a result the formula (18) can be proved for a wider interval $[x^{1/v}, x^{1/u}]$. In the sieve the same identity leads to a better estimate for $T(\mathscr{A}; x^{1/v}, x^{1/u})$ since $B = x^{7/12} y^{5/12}/P$ can be replaced by $B = x^{-1/2} y^{5/2}/P$.

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