Multipliers in $H^p(R^n)$, $0<p<\infty$

J. C. Peral and A. Torchinsky*

0. Introduction

In this paper we develop some recent results of Calderón and Torchinsky [2] concerning H^p multipliers in order to present sharp conditions in terms of directional derivatives of the multiplier function $m(\xi)$ which will assure that the associated translation invariant operator T defined by means of its Fourier transform by

$$(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$$

for $\hat{f}(\xi)$ in $C_0^\infty(\mathbf{R}^n)$ and $0 \notin \operatorname{supp}\hat{f}$, preserve the Hardy spaces $H^p(\mathbf{R}^n)$, $0<p<\infty$. In our context a tempered distribution f is in $H^p(\mathbf{R}^n)$ if

$$M(u,x) = \sup_{\varrho(x-y)\leqq t} |u(y,t)|$$

is in $L^p(\mathbf{R}^n)$, $\|f\|_{H^p} = \|M(u)\|_p$, $0<p<\infty$, where $\varrho(x)$ denotes the parabolic metric associated to the group $\{t^P\}_{t>0}$ with $(Px,x)\geqq(x,x)$, trace $P=\gamma$, and $u(y,t)=(f_*\varphi_t)(y)$ is an extension of f to $\mathbf{R}_+^{n+1}$ by means of convolution with a function $\varphi_t(y)=t^{-\gamma}\varphi(t^{-P}y)$ in the Schwartz class S with non-vanishing integral, see [1]. When $P=I, \gamma=n$ and $\varrho(x)=|x|$ these spaces coincide with the H^p spaces of several real variables considered in [5]. A bounded function $m(\xi)$ is an H^p multiplier with norm $\leqq K$ if $\|Tf\|_{H^p}\leqq K\|f\|_{H^p}$.

Since $H^p=L^p$ for $p>1$ and m is a multiplier in L^p if and only if it is an $L^{p'}$ multiplier with $1/p+1/p'=1$ we will assume throughout that $p\leqq 2$. Bounded functions $m(\xi)$ are the $L^2(\mathbf{R}^n)$ multipliers. We study here conditions on the smoothness of $m(\xi)$ and on its decay, together with its derivatives, at infinity that will imply that $m(\xi)$ is also a multiplier for some $p<2$.

* The first author was supported in part by a grant from Ministerio de Educación, Spain, and the second author by the National Science Foundation and a Faculty Research Grant from Indiana University.

1. Multiplier theorems

As is well known the function $m(\xi)=\theta(\xi)e^{i|\xi|^\varepsilon}$, $0<\varepsilon<1$, $\theta(\xi)$ vanishing near 0 and equal to 1 at infinity which satisfies

(1) $$|m(\xi)|+|\xi|^{1-\varepsilon}\left|\frac{\partial}{\partial\xi}m(\xi)\right| \leqq K$$

is only an $L^2(\mathbf{R}^n)$ multiplier. So no condition weaker than (1) above with $\varepsilon=0$ will insure positive results. It follows from Proposition 1.2 that if this is the case then indeed we have an $H^p(\mathbf{R}^n)$ multiplier for $1/p-1/2<1/n$. It is also well known, see Proposition 2.2, that there are such $m(\xi)$ which fail to be multipliers for $1/p-1/2>1/n$. Actually a stronger result follows in the spirit of [10] and [2] Theorem 4.7, for it suffices to assume a Hörmander type condition [7] for $|(\partial/\partial\xi)m(\xi)|$ in $L^{n+\varepsilon}(\mathbf{R}^n)$, see Proposition 1.1. We also point out that for radial functions $m(|\xi|)$ a local $L^2(\mathbf{R}^1)$ condition on $m'(t)$ will suffice but then the result holds for $1/p-1/2<1/2n$, see [8] and Theorem 1.4. We will also extend these results to functions $m(\xi)$ with k derivatives, $k\geqq 1$. Multiplier results for analytic H^p spaces were first discussed in [16]. A related result we consider in Proposition 2.4 is the following: given $1<p<2$ there exists a bounded, $C^\infty(\mathbf{R}^n)$ function $m(|\xi|)$ so that it is a multiplier in $L^q(\mathbf{R}^n)$ for $p<q<p'$ but it does not map $L^p(\mathbf{R}^n)$ into weak $L^p(\mathbf{R}^n)$ or $L^{p'}(\mathbf{R}^n)$ into weak $L^{p'}(\mathbf{R}^n)$. The construction of this example, which has been given in a more general situation in [4], requires some ideas of [13].

The results of [2] that we will use are the following. Let $d(t)$ be an infinitely differentiable non-decreasing function in $[0,\infty)$ such that $d(t)=2$ in $[0,1/2)$ and $d(t)=t$ for $t>3$. Then for a complex number z we define

$$(D_z f)^\wedge(\xi) = d(\varrho^*(\xi))^z \hat{f}(\xi)$$

where $\varrho^*(\xi)$ is the metric associated to the adjoint group $\{t^{P^*}\}_{t>0}$. Let $\hat{\eta}(\xi)=\hat{\eta}(\varrho^*(\xi))$ be a function in $C_0^\infty(\mathbf{R}^n)$ with support in $1/2<\varrho^*(\xi)<2$ and $\equiv 1$ in $1<\varrho^*(\xi)<3/2$. If $1/q=1/p-1/2$ and

(2) $$\|D_\lambda[m(t^{P^*}\xi)\hat{\eta}(\xi)]\|_q \leqq K<\infty, \quad t>0,$$

for some $\lambda>\gamma/q$ then m is a multiplier in $L^p(\mathbf{R}^n)$ with norm bounded by $c(K+\|m\|_\infty)$. If on the other hand

(3) $$\|D_\lambda[m(t^{P^*}\xi)\hat{\eta}(\xi)]\|_2 \leqq K<\infty, \quad t>0,$$

for some $\lambda>\gamma(1/p-1/2)$, and $p\leqq 1$, then m is an $H^p(\mathbf{R}^n)$ multiplier with norm bounded by $c(K+\|m\|_\infty)$.

We begin by obtaining a useful estimate in terms of directional derivatives of $m(\xi)$, similar in general character to those of [12], that will insure that (2) or (3) above hold.

Proposition 1.1. *Let $m(\xi)$ be a bounded function so that for vectors $v_1, \dots, v_j$, $1 \leqq j \leqq k$, $1<q<\infty$ and $t>0$ the directional derivatives of m satisfy*

$$\left[\int_{1\leqq\varrho^*(\xi)\leqq 2}\left|\left(\left(v_1, \frac{\partial}{\partial\xi}\right)\dots\left(v_j, \frac{\partial}{\partial\xi}\right)m\right)(t^{P^*}\xi)\right|^q d\xi\right]^{\frac{1}{q}} \leqq K\frac{(\Pi_{i=1}^{j}\varrho^*(v_i))}{t^j}.$$

Then $\|D_k[m(t^{P^*}\xi)\hat{\eta}(\xi)]\|_q \leqq c(\|m\|_\infty + K)$.

Proof. First some notation. Given a multi-index $\sigma=(\sigma_1, \dots, \sigma_n)$ denote by $(\partial/\partial x)^\sigma f(x)=(\partial^{|\sigma|}/\partial^{\sigma_1}x_1\dots\partial^{\sigma_n}x_n)f(x)$, and $x^\sigma=x_1^{\sigma_1}\dots x_n^{\sigma_n}$. $\partial/\partial\xi$ denotes the gradient. As is wellknown, for $\varrho^*(\xi)\geqq 1$ we have that $\varrho^*(\xi)\leqq|\xi|$ and $|(\partial/\partial\xi)^\sigma\varrho^*(\xi)|\leqq c\varrho^*(\xi)^{1-|\sigma|}$, see [1]. Thus we may apply for instance (2) above to

$$m_k(\xi) = \frac{d(\varrho^*(\xi))^k}{d(|\xi|)^k} \tag{4}$$

to obtain that $m_k(\xi)$ is a multiplier in $L^q(\mathbf{R}^n)$ for $1<q<\infty$. Also it may be seen that (cf. [17] Section 32)

$$d(|\xi|)^k = \hat{\varphi}(\xi)+|\xi|^k\hat{\mu}(\xi) \tag{5}$$

where $\varphi\in L^1(\mathbf{R}^n)$ and μ is a finite measure. Indeed we just choose a smooth function $\psi(\xi)=1$ for $|\xi|\leqq 3$ and vanishing for $|\xi|>4$ and then set $\hat{\varphi}(\xi)=d(|\xi|)^k\psi(\xi)$ and $\hat{\mu}(\xi)=1-\hat{\varphi}(\xi)$. Moreover

$$|\xi|^k = \sum_{|\sigma|=k}\left(\frac{\xi}{|\xi|}\right)^\sigma\xi^\sigma = \sum_{|\sigma|=k}R_\sigma(\xi)\xi^\sigma \tag{6}$$

where as is well-known each $R_\sigma(\xi)$ is a multiplier in $L^q(\mathbf{R}^n)$, $1<q<\infty$. Thus combining (4), (5) and (6) we obtain that

$$\|D_k[m(t^{P^*}\xi)\hat{\eta}(\xi)]\|_q \leqq c\|m\|_\infty + c\sum_{|\sigma|\leqq k}\left\|\left(\frac{\partial}{\partial\xi}\right)^\sigma(m(t^{P^*}\xi)\hat{\eta}(\xi))\right\|_q = c\|m\|_\infty + J. \tag{7}$$

Let $\chi(\xi)$ denote the characteristic function of $\{1\leqq\varrho^*(\xi)\leqq 2\}$. It is then readily seen that if $|\sigma|=j$

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\sigma[m(t^{P^*}\xi)\hat{\eta}(\xi)]\right| \leqq c\chi(\xi)\sum_{|\beta|\leqq j}\left|\left(\left(t^{P^*}v_1, \frac{\partial}{\partial\xi}\right)\dots\left(t^{P^*}v_{|\beta|}, \frac{\partial}{\partial\xi}\right)m(t^{P^*}\xi)\right)\right| \tag{8}$$

with $|v_i|=1$, $1\leqq i\leqq j$.

Substituting (8) in the corresponding term of J in (7) above we have

$$J \leqq c\sum_{0\leqq|\beta|\leqq|\sigma|\leqq k}\left[\int_{1\leqq\varrho^*(\xi)\leqq 2}\left|\left(\left(t^{P^*}v_1, \frac{\partial}{\partial\xi}\right)\dots\left(t^{P^*}v_{|\beta|}, \frac{\partial}{\partial\xi}\right)m\right)(t^{P^*}\xi)\right|^q d\xi\right]^{\frac{1}{q}}$$

$$\leqq cK\sum_{0<|\beta|\leqq|\sigma|\leqq k}\frac{(\Pi_{i=1}^{|\beta|}\varrho^*(t^{P^*}v_i))}{t^{|\beta|}} + c\|m\|_\infty \leqq c(K+\|m\|_\infty).$$

This completes our proof.

Our next result deals with functions m which have the smoothness, and decay discussed in the introduction.

Proposition 1.2. *Let* $m\in C^k(\mathbf{R}^n\setminus 0)$, *and suppose that for* $0\leqq j\leqq k$

$$\left|\left(\left(v_1, \frac{\partial}{\partial\xi}\right)\cdots\left(v_j, \frac{\partial}{\partial\xi}\right)m\right)(\xi)\right| \leqq \frac{K(\prod_{i=1}^{j}\varrho^*(v_i))}{\varrho^*(\xi)^j}$$

then m *is an* $H^p(\mathbf{R}^n)$ *multiplier for* $1/p-1/2<k/\gamma$, *with norm not exceeding* cK.

The proof follows at once from Proposition 1.1 and (2) and (3) above.

Our next Theorem extends this result to fractional decay as well.

Theorem 1.3. *Let* $m\in C^{k-1}(\mathbf{R}^n\setminus 0)$ *and suppose* $0<\alpha\leqq 1$ *is such that*

$$\left|m(x+h)-\sum_{|\sigma|\leqq k-1}\frac{1}{\sigma!}\left(\left(\frac{\partial}{\partial x}\right)^{\sigma}m\right)(x)h^{\sigma}\right| \leqq K\left(\frac{\varrho^*(h)}{\varrho^*(x)}\right)^{k-1+\alpha} \tag{9}$$

for $\varrho^*(x)\geqq 2\varrho^*(h)$. *Then* $m(\xi)$ *is an* $H^p(\mathbf{R}^n)$ *multiplier for* $1/p-1/2<(k-1+\alpha)/\gamma$ *with norm* $\leqq c(\|m\|_\infty+K)$.

Proof. As condition (9) is invariant under dilations $x\to s^{P^*}x,\ h\to s^{P^*}h$ we have that

$$\left|m(s^{P^*}(x+h))+\sum_{|\sigma|\leqq k-1}\frac{1}{\sigma!}\left(\left(\frac{\partial}{\partial x}\right)^{\sigma}m\right)(s^{P^*}x)(s^{P^*}h)^{\sigma}\right| \leqq K\left(\frac{\varrho^*(h)}{\varrho^*(x)}\right)^{k-1+\alpha} \tag{10}$$

with the same K as above independent of s. Let now $\varphi\in S(R^n)$ be supported in $\varrho(x)\leqq 1$, $\hat\varphi(\varrho^*(x))\dot{=}\hat\varphi(x)$ be such that $\hat\varphi(t^{P^*}x)\not\equiv 0$ as a function of $t>0$ for $x\neq 0$ and have all moments up to order $j+k-1$, where j is the smallest integer $\geqq k-1+\alpha$, equal to zero. If b is such that $\varrho(x)^b\leqq|x|$ for $|x|\leqq 1$ (see [1]) let $0<t<4^{-b}$. It then follows that if we set $M(x, s, t)=(m(s^{P^*}(y)\hat\eta(y))*\varphi_t)(x)$, then

$$|M(x, s, t)| \leqq c(K+\|m\|_\infty)t^{k-1+\alpha}\chi(x) \tag{11}$$

where χ is the characteristic function of $\{\varrho^*(x)<5\}$. Indeed, since the convolution is seen to vanish whenever $\chi(x)=0$, it only remains to show that the appropriate bound holds. Write

$$\hat\eta(x-y) = \sum_{|\sigma|<j}\frac{(-y)^{\sigma}}{\sigma!}\left(\frac{\partial}{\partial x}\right)^{\sigma}\hat\eta(x)+R(x, y),$$

where $|R(x, y)|\leqq c|y|^j\chi(x)$. We then have

$$\begin{aligned}M(x, s, t) &= \sum_{|\sigma|<j}\frac{1}{\sigma!}\left(\left(\frac{\partial}{\partial x}\right)^{\sigma}\hat\eta\right)(x)\int m(s^{P^*}(x-y))(-y)^{\sigma}\varphi_t(y)\,dy\\ &\quad+\int m(s^{P^*}(x-y))R(x, y)\varphi_t(y)\,dy\\ &= \sum_{|\sigma|<j}\frac{1}{\sigma!}\left(\frac{\partial}{\partial x}\right)^{\sigma}\hat\eta(x)I_{\sigma}(x, s, t)+J(x, s, t).\end{aligned}$$

Since m is bounded and $t<1$ we have

$$(12)\qquad |J(x,s,t)| \leqq c\|m\|_\infty \chi(x)\int |y|^j|\varphi_t(y)|\,dy$$
$$= c\|m\|_\infty \chi(x)\int |t^P y|^j|\varphi(y)|\,dy \leqq ct^j\|m\|_\infty\chi(x) \leqq ct^{k-1+\alpha}\|m\|_\infty\chi(x).$$

As for each $I_\sigma(x,s,t)$ we have

$$I_\sigma(x,s,t) = \int m(s^{P^*}(x-y)) - \sum_{|\beta|\leqq k-1}\frac{(s^{P^*}(-y))^\beta}{\beta!}\left(\left(\frac{\partial}{\partial x}\right)^\beta m\right)(s^{P^*}x)(-y)^\sigma\varphi_t(y)\,dy.$$

Now since we are only interested in those x in supp $\hat\eta$ and $\varphi_t(y)$ vanishes unless $\varrho(y)<t$ we have that $\varrho^*(y)\leqq|y|^{1/b}\leqq\varrho(y)^{1/b}\leqq t^{1/b}\leqq 1/4\leqq\varrho^*(x)/2$ for those x. So from (10) we obtain

$$|I_\sigma(x,s,t)| \leqq K\int\left(\frac{\varrho^*(y)}{\varrho^*(x)}\right)^{k-1+\alpha}|y^\sigma|\,|\varphi_t(y)|\,dy \leqq \frac{cKt^{k-1+\alpha}t^{|\sigma|}}{\varrho^*(x)^{k-1+\alpha}}.$$

Thus we have

$$(13)\qquad \left|\sum_{|\sigma|<j}\frac{1}{\sigma!}\left(\frac{\partial}{\partial x}\right)^\sigma \hat\eta(x)\,I_\sigma(x,s,t)\right| \leqq cKt^{k-1+\alpha}\chi(x).$$

Combining (12) and (13) we get (11) and we are ready to complete our proof. First notice that for $t\geqq 4^{-b}$ we have for any $q>1$

$$\|M(x,s,t)\|_q \leqq \|m\|_\infty\|\hat\eta\|_q\|\varphi_t\|_1 = c\|m\|_\infty.$$

Also for $t\leqq 4^{-b}$ and from (11) it follows that

$$\|M(x,s,t)\|_q \leqq c[K+\|m\|_\infty]t^{k-1+\alpha}.$$

Let $0<\delta<k-1+\alpha$. Then it is readily seen [1] Lemma 4.1, that there is a smooth function ψ so that for $\xi\neq 0$

$$\varrho^*(\xi)^\delta = \int_0^\infty t^{-\delta}\hat\varphi(t^{P^*}\xi)\hat\psi(t^{P^*}\xi)\frac{dt}{t}.$$

Therefore if $(\Lambda_\delta f)\hat{}\,(\xi)=\varrho^*(\xi)^\delta\hat f(\xi)$, then

$$(\Lambda_\delta[m(s^{P^*}y)\hat\eta(y)])\hat{}\,(\xi) = \int_0^\infty t^{-\delta}[m(s^{P^*}y)\hat\eta(y)]\hat{}\,(\xi)\hat\varphi(t^{P^*}\xi)\hat\psi(t^{P^*}\xi)\frac{dt}{t}$$

and for $q\geqq 1$ we have

$$\|\Lambda_\delta[m(s^{P^*}y)\hat\eta(y)\|_q \leqq \int_0^\infty \|M(x,s,t)*\psi_t\|_q t^{-\delta}\frac{dt}{t}$$

$$\leqq \|\psi\|_1\int_0^\infty\|M(x,s,t)\|_q t^{-\delta}\frac{dt}{t} \leqq c\|\psi\|_1\|m\|_\infty\int_0^{4^{-b}} t^{k-1+\alpha}t^{-\delta}\frac{dt}{t}$$

$$+c\|\psi\|_1(K+\|m\|_\infty)\int_{4^{-b}}^\infty t^{-\delta}\frac{dt}{t} = c(K+\|m\|_\infty).$$

But as in (5) it may be shown that there exist a function $\varphi \in L^1(\mathbf{R}^n)$ and a finite measure μ so that

$$d(\varrho^*(\xi))^\delta = \hat{\varphi}(\xi) + \varrho^*(\xi)^\delta \hat{\mu}(\xi)$$

and so independently of s

$$\|D_\delta[m(s^{P^*}y)\hat{\eta}(y)]\|_q \leqq c\|m\|_\infty + c\|\Lambda_\delta[m(s^{P^*}y)\hat{\eta}(y)]\|_q \tag{14}$$
$$\leqq c[K + \|m\|_\infty].$$

Suppose $p>1$. Let $\varepsilon = k-1+\alpha-\delta > 0$ and pick $2 \leqq q = \delta/(k-1+\alpha-2\varepsilon)$ so that $\gamma/q < \delta$. Then from (2) and (14) it follows that m is multiplier in $L^p(\mathbf{R}^n)$ for $1/p - 1/2 = (k-1+\alpha-2\varepsilon)/\gamma$. But $\varepsilon > 0$ is arbitrary, so that our conclusion follows in this case. If $p \leqq 1$ choose $q=2$ instead and combine (3) and (14) to obtain the desired conclusion also.

Our next theorem applies to radial functions m and allows the relaxation of the assumption $q \geqq 2$ in Proposition 1.1 to any $q>1$ in the Hörmander type conditions that appear.

Theorem 1.4. *Let $m(t)$ be a bounded function defined for $t \geqq 0$ with absolutely continuous derivatives up to order k and such that for some r, $1 < r \leqq \infty$, and all $s > 0$ we have*

$$\sum_{j=1}^{k} \left(s^{-1} \int_s^{2s} \left| u^j \left(\frac{d}{du}\right)^j m(u) \right|^r du \right)^{\frac{1}{r}} \leqq K.$$

Then the function $m(\varrho^*(\xi))$ is a multiplier for $1/p - 1/2 < (k - 1/r)/\gamma$ with norm not exceeding $c(K + \|m\|_\infty)$.

Proof. We begin by observing that for h and ξ the directional derivatives $((h, \partial/\partial\xi)\dots(h, \partial/\partial\xi))m(\varrho^*(\xi))$ of $m(\varrho^*(\xi))$ or order $j \leqq k$ are given by linear combinations of the form

$$\sum_{i=1}^{j} \left(\left(\frac{d}{dt}\right)^i m \right) (\varrho^*(\xi)) \quad I_i(h, \xi)$$

where the $I_i(h, \xi)$ are all possible linear combinations of products of the directional derivatives of $\varrho^*(\xi)$ of the form $[((h, \partial/\partial\xi)\dots(h, \partial/\partial\xi))\varrho^*(\xi)]$ where the order of each monomial does not exceed $k+1-i$ and for $\varrho^*(x) \geqq 2\varrho^*(h)$ and $0 \leqq s \leqq 1$

$$|I_i(h, x+sh)| \leqq c \left(\frac{\varrho^*(h)}{\varrho^*(x)} \right)^k \varrho^*(x+sh)^i. \tag{15}$$

Let now $M(x, h)$ denote the remainder of order k of the Taylor expansion of $m(x+h)$ about x, where $\varrho^*(x) \geqq 2\varrho^*(h)$. Then

$$M(x, h) = \int_0^1 \left(\left(h, \frac{\partial}{\partial x}\right) \dots \left(h, \frac{\partial}{\partial x}\right) m \right) (x+sh)\, k(1-s)^k\, ds. \tag{16}$$

Combining (15) and (16) it readily follows that

$$|M(x, h)| \leqq c\left(\frac{\varrho^*(h)}{\varrho^*(x)}\right)^k \sum_{i=1}^k \int_0^1 \left|\left(\left(\frac{d}{dt}\right)^i m\right)(\varrho^*(x+sh))\right| \varrho^*(x+sh)^i\, ds$$

$$= c\left(\frac{\varrho^*(h)}{\varrho^*(x)}\right)^k \sum_{i=1}^k J_i(x, h).$$

In each of the above integrals $J_i(x, h)$ we set $u=\varrho^*(x+sh)$, and then $du=|(h, \partial/\partial x)\varrho^*(x+sh)|\,ds$, and get

$$J_i(x, h) \leqq c\varrho^*(h)^{-1}\int_{\varrho^*(x)}^{\varrho^*(x)+\varrho^*(h)} \left|u^i\left(\frac{d}{du}\right)^i m(u)\right| du.$$

If $r=\infty$ then the conclusion follows at once from Proposition 1.2. If $r<\infty$ we apply Hölder's inequality to obtain

$$J_i(x, h) \leqq cK\varrho^*(h)^{-1+1/r'}\varrho^*(x)^{1/r} = cK\left(\frac{\varrho^*(x)}{\varrho^*(h)}\right)^{1/r}$$

and the conclusion follows now from Theorem 1.3.

2. Applications

Parabolic Riesz transforms and smooth functions homogenous of degree zero with respect to the metric $\varrho^*(\xi)$ are some of the multipliers covered by our results, the L^p results are better known and they are discussed, for example, in [14].

Another important class of examples are those multipliers which arise form some partial differential equations. For instance, as in [11] p. 205 let $\mathbf{R}^{n+5}=\{(x, y)\,|\,x\in\mathbf{R}^n, y\in\mathbf{R}^5\}$, denote by (ξ, η) the dual variables, and consider the differential operator $D=\partial^5/\partial y_1\ldots\partial y_5-\Delta_x$. Let $P=P^*$ be the diagonal matrix with entries $p_{ii}=5$, $1\leqq i\leqq n$ and $=2$ for $n+1\leqq i\leqq n+5$ so that $\gamma=5n+10$. Given g in $H^p(\mathbf{R}^{n+5})$ we wish to solve $Du=g$ and obtain estimates on u and its derivatives in appropriate $H^q(\mathbf{R}^{n+5})$ classes. For derivatives of lower order, the question may be settled by means of an argument similar to [2] Theorem 4.1. As for the estimates

$$\|Lu\|_{H^p(\mathbf{R}^{n+5})} \leqq c\|g\|_{H^p(\mathbf{R}^{n+5})}, \quad 0<p<\infty,$$

where L is a differential operator of the form $\partial^2/\partial x_j\partial x_k$ or $\partial^5/\partial^2 y_1\partial^3 y_2$ for instance, they are readily seen to follow from Proposition 1.2 by direct inspection of the multiplier $m(\xi, \tau)=$

$$\frac{\xi_j\xi_k}{i\tau_1\tau_2\tau_3\tau_4\tau_5-\sum_{h=1}^n \xi_h^2} \quad \text{and} \quad \frac{\tau_1^2\tau_2^3}{i\tau_1\tau_2\tau_3\tau_4\tau_5-\sum_{h=1}^n \xi_h^2}$$

respectively. Indeed these are smooth homogeneous functions of degree zero with respect to $\varrho^*(\xi,\tau)=\varrho(\xi,\tau)$, i.e. if $t^P(\xi,\tau)=(t^2\xi_1,\dots,t^2\xi_n,t^5\tau_1,\dots,t^5\tau_5)$, then $m(t^P(\xi,\tau))=m(\xi,\tau)$. Obviously the number 5 can be replaced by any odd number and the Laplacian Δ_x by a more general elliptic operator. The operator $\partial/\partial y_1-\Delta_x$ is also discussed in [3] pp. 601—605.

Still another class of examples corresponding to some strongly-weakly singular integrals [15] is as follows.

Proposition 2.1. *Let $F(s)$ be a possibly complex-valued function defined for $s>0$, vanishing near the origin and of class $C^k(R)$ with derivatives satisfying*

$$\left|s^j\left(\frac{d}{ds}\right)^j F(s)\right|\leqq K_1,\quad 0\leqq j\leqq k.$$

Further assume that $\varphi(\xi)$ is a positive, real valued function defined in $\mathbf{R}^n$ such that $\lim_{\varrho^*(\xi)\to\infty}\varphi(\xi)=\infty$ and for $0\leqq j\leqq k$

$$\left|\left(\left(v_1,\frac{\partial}{\partial\xi}\right)\dots\left(v_j,\frac{\partial}{\partial\xi}\right)\right)\varphi(\xi)\right|\leqq\frac{K_2\left(\prod_{i=1}^j\varrho^*(v_i)\right)\varphi(\xi)}{\varrho^*(\xi)^j}.$$

Then the function $m(\xi)=F(\varphi(\xi))$ is a multiplier for $1/p-1/2<k/\gamma$ with norm not exceeding $cK_1K_2^k$.

Proof. The proof of this proposition is similar to that of Theorem 1.4. As is readily seen we have

$$\left|\left(\left(v_1,\frac{\partial}{\partial\xi}\right)\dots\left(v_j,\frac{\partial}{\partial\xi}\right)\right)m(\xi)\right|\leqq cK_1K_2^j,\quad 0\leqq j\leqq k,$$

and we can therefore apply Proposition 1.2 to obtain the desired conclusion. Notice that a possible choice for $\varphi(\xi)$ is $\varrho^*(\xi)$.

Proposition 2.2. *Let $F(s)=\theta(s)e^{is^a}/s^b$, $a,b,s>0$, $\theta(s)$ a smooth positive function vanishing near zero and equal to 1 at infinity. Let k be the smallest integer $\geqq b/a$. Let $\varphi(\xi)$ be as in Proposition 2.1 and let $\psi(\xi)$ be a function in $\mathbf{R}^n$ so that for $0\leqq j\leqq k$ and any $\varepsilon>0$*

$$\lim_{\varrho^*(\xi)\to\infty}\frac{\left|\left(\left(v_1,\frac{\partial}{\partial\xi}\right)\dots\left(v_j,\frac{\partial}{\partial\xi}\right)\psi\right)(\xi)\right|\varrho^*(\xi)^j}{\varphi^\varepsilon(\xi)}=0 \tag{17}$$

Then $m(\xi)=F(\varphi(\xi))\psi(\xi)$ is a multiplier for $1/p-1/2<b/a\gamma$. This result cannot be improved.

Proof. First assume that $\psi(\xi)\equiv1$. If $b=ka$ for some integer k, then $F(s)$ verifies the assumptions of the preceeding Proposition and m is a multiplier for

$1/p-1/2<k/\gamma$ as we wished to show. If not, let k be the integer such that $(k-1)a<b<ka$ and consider the multiplier

$$m(\xi, z) = F(\varphi(\xi))\varphi(\xi)^{-z+b}.$$

When $\operatorname{Re} z=ja, m(\xi, ja+iv)$ is a multiplier for $1/p-1/2<j/\gamma$ for $j=k-1$ and k; when $k=1$ and $j=0$ we just mean $L^2(\mathbf{R}^n)$. Therefore by the theorem on analytic families of operators, see [2] Theorem 3.4, it follows that for $z=b, m(\xi, b)=m(\xi)$ is, a multiplier for $1/p-1/2<b/a\gamma$.

Let now $\psi(\xi)$ be arbitrary, $b=z$ and $2>p>2\gamma/(2+\gamma)$ be given. Consider the multiplier

$$m(\xi, z) = F(\varphi(\xi))\varphi(\xi)^{-z+a}\psi(\xi).$$

When $\operatorname{Re} z=\varepsilon>0$, then $m(\xi, \varepsilon+iv)$ is a bounded function and consequently an $L^2(\mathbf{R}^n)$ multiplier and when $\operatorname{Re} z=a+\delta, \delta>0$, from the assumptions on φ and ψ it readily follows from (17) that $m(\xi, \delta+iv)$ is an H^r multiplier for $2\geqq p>r>2\gamma/(2+\gamma)$. Let $(1/r-1/2)(1/p-1/2)=1+\eta$ and let $\varepsilon=a/2, \delta=a\eta/2$. Then by interpolation we have that for $z=a$ $m(\xi, a)=m(\xi)$ is a multiplier for H^q with $(a+\delta-\varepsilon)/(a-\varepsilon)=(1/q-1/2)/(1/p-1/2)=1+\delta/(a-\varepsilon)=1+\eta$ and the desired conclusion holds for $q=p$ as we wished to show. The proof for other values of b follows as in the preceeding Proposition.

We remark that a possible choice of $\psi(\xi)$ is $\ln \varphi(\xi)$. For $m(\xi)=\theta(\xi)e^{i|\xi|^a} \ln |\xi|/|\xi|^b$, with $a>1, b>0$, it is not hard to check that our result cannot be improved. Indeed it suffices to set $f(x)=|x|^{-n/p} (\ln |x|)^{-1}$ near zero and smooth at infinity, where $1/p-1/2=b/an$ and to use results from [9] and [19] to show that $m(\xi)\hat{f}(\xi)$ is not the Fourier transform of an $L^p(\mathbf{R}^n)$ function. An improvement on the $H^p(\mathbf{R}^n)$ result would imply a corresponding improvement of the $L^p(\mathbf{R}^n)$ result and this we have seen is not possible.

Possibly a more interesting example is the following.

Proposition 2.3 *Let $1<p<2$ be given and suppose that $n\geqq 3$. Let $J_\beta(t)$ denote the Bessel function of order β, see* [18], *and let a and s be parameters such that $0<a=(n-1)(1/p-1/2)+1/2$ and $s>1$. Set*

$$m(\xi) = J_{\frac{n-2}{2}}(|\xi|)\frac{[\ln(2+|\xi|)]^s}{|\xi|^a}.$$

Then the multiplier transformation associated to $m(\xi)$ is bounded in $L^r(\mathbf{R}^n)$ with $p<r<p'$ but fails to map $L^p(\mathbf{R}^n)$ into weak-$L^p(\mathbf{R}^n)$ *or $L^{p'}(\mathbf{R}^n)$ into* weak-$L^{p'}(\mathbf{R}^n)$.

Proof. It is clear that $a<(n-2)/2$ so $m(\xi)$ is a bounded function. It is shown in [13] that $J_{(n-2)/2}(|\xi|)/|\xi|^a$ is a bounded $L^r(\mathbf{R}^n)$ multiplier for $|1/r-1/2|\leqq(a+1/2)/(n-1)$ and by an argument similar to the one in the preceeding Proposition we can check that $m(\xi)$ is bounded in $L^r(\mathbf{R}^n)$ for $|1/r-1/2|<(a+1/2)/(n-1)$.

Let now $f(x)=|x|^{-n/p}(\ln|x|)^{-1}$ near zero and smooth at infinity, so that $f\in L^p(\mathbf{R}^n)$. Then

$$(Tf)\hat{}\,(\xi)=m(\xi)\hat{f}(\xi)\approx J_{\frac{n-2}{2}}(|\xi|)\frac{[\ln(2+|\xi|)]^{s-1}}{|\xi|^{a+n-n/p}}$$

at infinity. Thus if $\delta=a+1+n(1/2-1/p)=1-1/p>0$, for large values of x we have that $Tf(x)$ can be written as

$$\left(\frac{J_{\frac{n-2}{2}}(|\xi|)}{|\xi|^{\frac{n-2}{2}}}\right)^{\vee}*\left(\frac{(\ln[2+|\xi|])^{s-1}}{|\xi|^{\delta}}\right)^{\vee}(x)+\text{error}$$

where the error is negligible with respect to the first term. But then $Tf(x)$ is basically the radial function which is the convolution of the function $(\ln|x|)^{s-1}/|x|^{n-\delta}$ with the measure μ corresponding to the uniformly distributed mass over the unit sphere $|x|=1$. Let now $1/2<|x|<1$. A simple geometric argument readily shows that

$$|Tf(x)|\geqq\frac{c|\ln(1-|x|)|^{s-1}(1-|x|)^{n-1}}{(1-|x|)^{n-\delta}}\geqq\frac{c|\ln(1-|x|)|^{s-1}}{(1-|x|)^{1/p}}.$$

Therefore $|Tf(x)|^p\geqq c\,|\ln(1-|x|)|^{(s-1)p}/(1-|x|)$ for those values of x and our conclusion follows. Similarly for p'. This completes our proof.

References

1. Calderón, A. P., and Torchinsky, A., Parabolic maximal functions associated with a distribution, *Advances in Math.* **16** (1975), 1—64.
2. Calderón, A. P., and Torchinsky, A., Parabolic maximal functions associated with a distribution, II, *Advances in Math.* **24** (1977), 101—171.
3. Coifman, R. R., and Weiss, G., Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569—645.
4. Cowling, M. G., and Fournier, J. J. F., Inclusions and noninclusion of spaces of convolution operators, *Trans. Amer. Math. Soc.* **221** (1976), 59—95.
5. Fefferman, C., and Stein, E. M., H^p spaces of several real variables, *Acta Math.* **129** (1972), 137—193.
6. Hirschmann Jr., I. I., On multiplier transformations, *Duke Math. J.* **26** (1959), 221—242.
7. Hörmander, L., Estimates for translation-invariant operators in L^p spaces, *Acta Math.* **104** (1960), 93—140.
8. Igari, S., and Kuratsubo, S., A sufficient condition for L^p multipliers, *Pacific J. of Math.* **38** (1971), 85—88.
9. Ishii, H., On some Fourier multipliers and partial differencial equations, *Mathematica Japonicae* **19** (1974), 139—163.

10. LITTMAN, W., Multipliers in L^p and interpolation, *Bull. Amer. Math. Soc.* **71** (1965), 764—766.
11. LITTMAN, W., MCCARTHY, C., and RIVIÈRE, N. M., L^p multiplier theorems, *Studia Math.* **30** (1968), 193—217.
12. NAGEL, A., and STEIN, E. M., A new class of pseudo-differential operators, *Proc. Nat. Acad. Sci. U.S.A.* **75** (1978), 582—585.
13. PERAL, J. C., Doctoral Dissertation, Princeton University, 1978.
14. RIVIÈRE, N. M., Singular integrals and multiplier operators, *Arkiv Mat.* **9** (1971), 243—278.
15. SJÖLIN, P., L^p estimates for strongly singular convolution operators in R^n, *Arkiv Mat.* **14** (1976), 59—64.
16. STEIN, E. M., Classes H^p, multiplicateurs et fonctions de Littlewood-Paley, *C. R. Acad. Sci., Paris* **263** (1966), 716—719 and 780—781.
17. STEIN, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N. J., 1970.
18. STEIN, E. M., and WEISS, G., *Introduction to Fourier Analysis on Euclidean Space*, Princeton Univ. Press, Princeton, N. J., 1972.
19. WAINGER, S., Special trigonometric series in k-dimensions, *Memoirs of the Amer. Math. Soc.* **59**, (1965).

Received December 6, 1978

J. C. Peral and A. Torchinsky
Department of Mathematics
Princeton University
Princeton, New Jersey 08 540
U.S.A.

School of Mathematics
Institute for Advanced Study
Princeton, New Jersey 08 540
U.S.A.

Department of Mathematics
Indiana University
Bloomington, Indiana 47 405
U.S.A.