On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on $\mathbf{C}^n$

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We shall prove that a locally polar set in $\mathbf{C}^n$ is globally polar which generalizes a well-known result from potential theory for subharmonic functions and answers a question posed by Lelong [2]. Our method differs from the ones frequently used in potential theory, since it seems that there is a lack in the representation of plurisubharmonic functions by kernels, and the main step in our proof is to find, to every given function which is analytic in a ball, polynomials which are sufficiently small on the set where the given function is small (Proposition). From this the theorem will follow (Lemma 3) because locally a plurisubharmonic function is a Hartogs function. A consequence of the theorem is that an analytic set is globally polar and the theorem also has applications in the theory for capacities and extremal functions in $\mathbf{C}^n$. See for example Siciak [3].

Definition. A set $D \subset \mathbf{C}^n$ is called *locally polar* if there exist, to every $z \in D$, an open set $V_z \subset \mathbf{C}^n$ and $u_z \in PSH(V_z)$, where $PSH(V_z)$ denotes the set of all plurisubharmonic functions in V_z, so that $z \in V_z$ and such that $u_z/V_z \cap D$, the restriction of u_z to $V_z \cap D$, is equal to $-\infty$. D is *globally polar* if we can take $V_z = \mathbf{C}^n$. For details see [2].

We shall give $\mathbf{C}^n$ the sup-norm and we shall let $\mathscr{H}(V)$, where $V \subset \mathbf{C}^n$ is open, denote the set of all analytic functions on V. We note that f has a Taylor series expansion $f(z) = \sum a_r z^r$ if $f \in \mathscr{H}(B(0, S))$, where $B(0, S)$ is the open ball in $\mathbf{C}^n$ with centre 0 and radius S, $a_r \in \mathbf{C}$, $r = (r_1, \dots, r_n)$ is a multi-index and $z^r = z_1^{r_1} \dots z^{r_n}$ where $z = (z_1, \dots, z_n) \in \mathbf{C}^n$.

Theorem. *A set $D \subset \mathbf{C}^n$ is globally polar if and only if D is locally polar.*

From the theorem we obviously have the following,

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Corollary. *An analytic subset of an open set in* $\mathbf{C}^n$ *is globally polar.*

We note that the "only if" part of the theorem is evident. For the rest of the proof we need a number of lemmas.

Let $D\subset\mathbf{C}^n$ be a locally polar set. From the definition it follows that, for every $z\in D$, there exist $r_z>0$ and $u_z\in PSH(B(z, r_z))$ such that $u_z/B(z, r_z)\cap D=-\infty$.

Let from now on z be fixed. We shall first show that $B(z, r_z/32)\cap D$ is a globally polar set. Without loss of generality we may assume that $z=0$ and $r_0=4$. To avoid too many subscripts we shall write u instead of u_0 and it is obvious that we can take u such that $u(z)<0$ when $\|z\|\leqq 2$.

From Bremermann [1] we easily get the following:

Lemma 1. *We can write* $u(z)=\overline{\lim}_{z'\to z}\overline{\lim}_{j\to\infty}(1/j)\log|f_j(z')|$ *where*

$$f_j(z)\in\mathscr{H}(B(0,4)) \quad \textit{and} \quad \|f_j\|_{B(0,2)}=\sup_{\|z\|\leqq 2}|f_j(z)|\leqq 1.$$

Proof. From [1] it follows that

$$H=\{(z,w)\in\mathbf{C}^{n+1};\ z\in B(0,4) \text{ and } |w|<e^{-u(z)}\}$$

is an open pseudoconvex set. Since $u<0$ when $\|z\|\leqq 2$ we have that $K=\{(z,w);\ \|z\|\leqq 2 \text{ and } |w|\leqq 1\}$ is a compact subset of H. The theorem of Bremermann—Norguet—Oka gives that there exists an $f\in\mathscr{H}(H)$ which cannot be continued over H and so that $\|f\|_K=\sup_{(z,w)\in K}|f(z,w)|<1$. We can write $f(z,w)=\sum w^j f_j(z)$ where $f_j\in\mathscr{H}(B(0,4))$ and

$$u(z)=\overline{\lim_{z'\to z}}\ \overline{\lim_{j\to\infty}}\,(1/j)\log|f_j(z')|$$

according to [1]. Since $\|f\|_K<1$ it follows that $\|f_j\|_{B(0,2)}<1$ which completes the proof. Q.E.D.

There exists an integer $q>0$ such that $\sup_{\|z\|\leqq 1/4}u(z)>-q+1$. Hence there exists an infinite set $S\subset\mathbf{N}$ so that $\|f_j\|_{B(0,1/4)}>e^{-qj}$ when $j\in S$. Since $\overline{\lim}_{z'\to z}\overline{\lim}_{j\in S}(1/j)\log|f_j(z')|\leqq u(z)$ we may assume that equality holds, i.e. u is defined by $(f_j)_{j\in S}$. We may also assume that $(nj)^{2n}<2^j$ when $j\in S$.

Next we will find, to every f_j, a polynomial g_j of degree i_j such that $|g_j(z)|^{1/i_j}$ is small when $|f_j(z)|^{1/j}$ is small. We cannot expect the Taylor series to give such a good approximation in the set where $(f_j)^{1/j}$ is small or such a good approximation for example on a ball and have to find other methods.

Put $N(s)=\{f\in\mathscr{H}(B(0,3));\ \|f\|_{B(0,1)}\leqq 1 \text{ and } |f(0)|>e^{-s}\}$. We note that there exists, for every $j\in S$, $x^j\in B(0,1/4)$ such that $f_j'(z)=f_j(z-x^j)\in N(qj)$ since $f_j\in\mathscr{H}(B(0,4))$, $\|f_j\|_{B(0,2)}\leqq 1$ and $\|f_j\|_{B(0,1/4)}>e^{-qj}$.

Proposition. *Let* $f\in N(j)$, *where* $j\in\mathbf{R}^+$ *is so big that* $(nj)^{2n}<2^j$, *and let* $\varphi>100$. *Then there exists a polynomial* g *such that* $1\leqq\|g\|_{B(0,1)}\leqq 2^i$, *where* i *is the degree* *of* g, *and so that* $|g(z)|<\exp(-Ci\varphi^{1/n})$ *when* $|f(z)|<\exp(-j\varphi)$ *and* $\|z\|\leqq 1/2$, *where* $C=1/2\cdot 10^3 n$.

Proof. First we note that it is no restriction to assume that $\varphi^{1/n}$ is an integer, because if the proposition is true for every such φ with $C=1/(10^3 n)$ (as we shall prove), then it holds for every $\varphi>100$ with C as in the proposition, since then $[\varphi^{1/n}]-1>1$, where [ ] denotes the integer part. It is also easy to see that we may suppose that j is an integer.

Furthermore, we may assume that $\varphi\leqq j$ since we can always raise f to the power φ and if g exists relative to $f^\varphi\in N(j\varphi)$ as in the proposition, g also has the desired properties relative to f.

Let $f(z)=\sum a_r z^r$ and let $M\subset \mathbf{N}^n$ be the set $M=\{r;\, r_s<j\varphi\}$. It is clear that M contains exactly $j^n\varphi^n$ different elements. Put $Q(z)=\sum_{r\in M} x_r z^r$ and $H(z)=f(z)Q(z)=\sum d_r z^r$, where $d_r=\sum_{t\in M} a_{r-t}x_t$ where we put $a_{r-t}=0$ if $\min_s(r_s-t_s)<0$.

Now $(d_r=0)_{r\in M}$ is a system of linear equations in the variables x_t and with coefficients a_r. There are $j^n\varphi^n$ variables and equations. Let $D(M)$ be the determinant of the system.

We note that $H(z)$ is small when $f(z)$ is small since H is a product of a polynomial and f. We shall show that x_t can be chosen so that $d_r=0$ when $\|r\|=\sum_1^n r_s\leqq j\cdot\varphi/2$ and $\max r_s>A=i/n=100j\varphi^{(n-1)/n}$ and so that at least one d_r, with $\|r\|\leqq i$, is big (at least bigger than $e^{-j\cdot\varphi/10}$). Then it will follow that $G(z)=\sum_{\max r_s\leqq A} d_r z^r$ is small when $f(z)$ is small, since G is almost H, and that G has the desired properties, i.e. G is not small on the unit ball. That the variables x_t can be taken in the way described above follows from the fact that if all d_r are small when $\|r\|\leqq i$ then the system of equations $\{d_r=0\}_{r\in M}$ can be slightly changed so that the new system has a non-trivial solution, thus the determinant of the new system is zero since the system has as many variables as equations. But then it follows that there exists a submatrix of $\{d_r=0\}_{r\in M}$ with a determinant which is much bigger than that of $\{d_r=0\}$ and since $D(M)$ is big, a repetition of this argument will lead to a contradiction because $|a_r|\leqq 1$.

We shall first show that $D(M)=(f(0))^{j^n\varphi^n}$. This follows from the fact that the coefficient for x_t in d_t is $a_0=f(0)$ and because the coefficient for x_t in d_r is 0 if $r_s<t_s$ for some $s\in(1,\dots,n)$. Hence the matrix belonging to the system $(d_r=0)_{r\in M}$ is zero on one side of the diagonal and with diagonal elements equal to $f(0)$ which gives that $D(M)=(f(0))^{j^n\varphi^n}$.

Let N and $N'\subset M$ be such that $\tau(N)=\tau(N')$, where τ denotes the number of elements. Let $\{d^{(N')}=0\}_{r\in N}$ be the system of linear equations $\sum_{t\in N'} a_{r-t}x_t=0$, $r\in N$ and let $D(N,N')$ denote its determinant which exists since $\tau(N)=\tau(N')$. We have that

(1) $|D(N, N')| < (j^n\varphi^n)^{j^n\varphi^n} < e^{j^{n+1}\varphi^n}$ if $j^{2n} \leqq e^j$ since $\varphi \leqq j$, the number of equations in the system is less or equal to $(j\varphi)^n$ and since $|a_r| \leqq 1$ because $f \in N(j)$.

Let M_k and $N_k \subset M$ be such that

a) $\tau(M_k) = \tau(N_k) = \tau(M) - k = j^n\varphi^n - k$,

b) $r \in M_k$ if $r \in M$ and $\max_s r_s > 100j\varphi^{(n-1)/n} = A$,

c) $|D(M_k, N_k)| > \exp(kj\varphi/10 - j^{n+1}\varphi^n)$.

$M_0 = N_0 = M$ fulfil the requirements, since $|D(M, M)| = |D(M)| = |f(0)|^{j^n\varphi^n} > e^{-j^{n+1}\varphi^n}$ (Since $f \in N(j)$).

According to (1) there exists a biggest integer m so that M_m and N_m exist and satisfy the conditions a)—c). We also have from (1) that

$$m < 20j^n\varphi^{n-1}. \tag{2}$$

There exists $r^0 \in M_m$ such that $\max_s r_s^0 \leqq A = 100j\varphi^{(n-1)/n}$. This follows because there are $(A+1)^n > 100j^n\varphi^{n-1} > m$ different $r \in M$ with $\max_s r_s \leqq A$, since $A < j\varphi$ if $\varphi > 100$.

Put $M_{m+1} = M_m \setminus \{r^0\}$. The system of linear equations

$$\sum_{t \in N_m} a_{r-t}x_t = 0, \quad r \in M_{m+1}$$

has a nontrivial solution, since the number of variables x_t is $\tau(N_m) = \tau(M) - m$ and the number of equations is $\tau(M_{m+1}) = \tau(M) - m - 1$. Let $\{u_t\}$ be a solution such that $\max_t |u_t| = 1$ and take $t^0 \in N_m$ so that $|u_{t^0}| = 1$.

We shall now prove that

$$|\bar{d}_{r^0}| = \left|\sum_{t \in N_m} a_{r^0-t}u_t\right| \geqq e^{-j\varphi/10}. \tag{3}$$

Put $b_{r^0,t^0} = a_{r^0-t^0} - (\sum_{t \in N_m} a_{r^0-t}u_t)/u_{t^0}$ and $b_{r,t} = a_{r-t}$ when $r \neq r^0$ or $t \neq t^0$. We have $\sum_{t \in N_m} b_{r^0,t}u_t = a_{r^0-t^0}u_{t^0} - \sum_{t \in N_m} a_{r^0-t}u_t + \sum_{t \in N_m, t \neq t^0} a_{r^0-t}u_t = 0$ and $\sum_{t \in N_m} b_{r,t}u_t = \sum_{t \in N_m} a_{r-t}u_t = 0$, when $r \in M_{m+1}$ according to the choice of $\{u_t\}$. Thus the system of linear equations $\sum_{t \in N_m} b_{r,t}x_t = 0$, $r \in M_m$ has the nontrivial solution $\{u_t\}$, hence the determinant D of the system, which exists since the number of variables is equal to the number of equations $(\tau(M_m) = \tau(N_m))$, is zero. Put $N_{m+1} = N_m \setminus \{t^0\}$. Then $D = D(M_m, N_m) + (b_{r^0,t^0} - a_{r^0-t^0}) \cdot D(M_{m+1}, N_{m+1}) = 0$ since $b_{r,t} = a_{r-t}$ when $r \neq r^0$ or $t \neq t^0$. Trivially it follows that

a) $\tau(M_{m+1}) = \tau(N_{m+1}) = \tau(M) - m - 1$

b) $r \in M_{m+1}$ if $r \in M$ and $\max_s r_s > A$, because $\max_s r_s^0 \leqq A$ and $M_m = M_{m+1} \cup \{r^0\}$.

Because of the choice of m (m is the biggest integer so that a)—c) are fulfilled for any sets M_m and $N_m \subset M$), we must have that $|D(M_{m+1}, N_{m+1})| =$

$|b_{r^0,t^0}-a_{r^0-t^0}|^{-1}\ |D(M_m, N_m)|\leqq\exp\big((m+1)j\varphi/10-j^{n+1}\varphi^n\big)$ hence that $|b_{r^0,t^0}-a_{r^0,t^0}|\geqq e^{-j\varphi/10}$ because of c). But $|b_{r^0,t^0}-a_{r^0-t^0}|=\big|\sum_{t\in N_m} a_{r^0-t}u_t\big|/|u_{t^0}|=|\bar{d}_{r^0}|$, since $|u_{t^0}|=1$, and thus (3) is established.

We shall now proceed to construct the polynomial g in the proposition.

Let $\bar{H}(z)$, $\bar{Q}(z)$ (resp. $\bar{d}_r$) be the functions (resp. complex numbers) which are obtained from $H(z)$, $Q(z)$ (resp. d_r) when we replace the complex variables $\{x_t\}$ by the complex numbers $\{u_t\}$. Then $|\bar{Q}(z)|\leqq(j\varphi)^n$ when $\|z\|\leqq 1$ since $|u_t|\leqq 1$. Hence $|\bar{H}(z)|<(j\varphi)^n e^{-j\varphi}$ if $|f(z)|<e^{-j\varphi}$ and $\|z\|\leqq 1$.

Put $G(z)=\sum_{\max_s r_s\leqq A}\bar{d}_r z_r$ and $\|r\|=\sum_1^n r_s$. Then $\bar{H}(z)-G(z)=\sum_{\|r\|>j\varphi/2}\bar{d}_r z^r$, because $\|r\|<j\varphi/2$ when $\max_s r_s\leqq A$ and $\varphi>100$, and because $d_r=0$ if $\|r\|\leqq j\varphi/2$ and $\max r_s>A$. The last assertion follows from the fact that $r\in M$ if $\|r\|\leqq j\varphi/2$, hence that $r\in M_m$ and also $r\in M_{m+1}$ according to b), if $\max r_s>A$, and from the fact that $\bar{d}_r=0$ when $r\in M_{m+1}$ (the construction of $\{u_t\}$). For every r we also have that $|\bar{d}_r|\leqq(j\varphi)^n$ since $|u_t|\leqq 1$ and $|a_r|\leqq 1$ $\big(f\in N(j)\big)$. Thus $|\bar{H}(z)-G(z)|\leqq\sum_{\|r\|>j\varphi/2}(j\varphi)^n 2^{-\|r\|}$ if $\|z\|\leqq 1/2$ since we have given $\mathbf{C}^n$ the sup-norm. But $\sum_{\|r\|>j\varphi/2}2^{-\|r\|}<\sum_{l=j\varphi/2}^{\infty} l^n 2^{-l}<2^{-j\varphi/2}e^{j\varphi/5}$ since $100\leqq\varphi\leqq j$ and $j^{2n}<e^j<e^{j\varphi/5}$. Hence $|\bar{H}(z)-G(z)|\leqq e^{-j\varphi/4}$ and so $|G(z)|<e^{-j\varphi/5}$ when $\|z\|\leqq 1/2$ and $|f(z)|<e^{-j\varphi}$ since then, according to the above, $|\bar{H}(z)|<(j\varphi)^n e^{-j\varphi}<e^{-j\varphi+j/5}$.

Put $d=\max_{r_s\leqq A}|\bar{d}_r|$. We have that $d>e^{-j\varphi/10}$ since $|\bar{d}_{r^0}|>e^{-j\varphi/10}$ according to (3) $(\max r_s^0\leqq A)$. Finally put $g(z)=d^{-1}G(z)$.

Then $g\in P_i(\mathbf{C}^n)$ where $i=An$, since $G\in P_i(\mathbf{C}^n)$. It is also true that $1\leqq\|g\|_{B(0,1)}\leqq 2^i$ because $\max_{r,r_s<A}d^{-1}|\bar{d}_r|=1$ and because $(nA)^n<2^i$ (Since $(jn)^{2n}<2^j$). We have further that

$$|g(z)|\leqq e^{j\varphi/10-j\varphi/5}=e^{-j\varphi/10}=\exp(i\varphi^{1/n}/n10^3)\quad\text{when}\quad |f(z)|\leqq e^{-j\varphi}$$

and $\|z\|\leqq 1/2$, since $d^{-1}\leqq e^{j\varphi/10}$ and $|G(z)|<e^{-j\varphi/5}$ in that case. Thus g has the properties in the proposition which completes the proof. Q.E.D.

Proof of the theorem continued. Take, for every f_j' (defined as before the Proposition) and every integer $r\geqq 10$, $i(j,r)\in\mathbf{N}$ and $g_{j,r}\in P_{i(j,r)}(\mathbf{C}^n)$ as in the Proposition such that

(1) $|g_{j,r}(z)|<\exp\big(-Ci(j,r)r^{2r}\big)$ when $|f_j'(z)|<\exp(-jqr^{2nr})$ and $\|z\|\leqq 1/2$.

Put $t_j=\prod_{r=10}^j i(j,r)$ and $e_{j,r}(z)=\big(g_{j,r}(z+x^j)\big)^{t_j/i(j,r)}$. We note that

(2) $$2^{-t_j}\leqq\|e_{j,r}\|_{B(0,1)}\leqq 4^{t_j}$$

since $$\sup_{\|z\|\leqq 3/4}|e_{j,r}(z-x^j)|=\sup_{\|z\|\leqq 3/4}|g_{j,r}(z)|^{t_j/i(j,r)}\geqq t_j^{-1}(3/4)^{t_j}>2^{-t_j}$$

and since

$$\sup_{\|z\|\leqq 5/4}|e_{j,r}(z-x^j)|\leqq t_j(5/4)^{t_j}2^{t_j}<4^{t_j}\quad\text{because}\quad 1\leqq\|g_{j,r}\|_{B(0,1)}\leqq 2^{i(j,r)}.$$

We also note that

(1)′ $|e_{j,r}(z)|<\exp(-Ct_jr^{2r})$ when $|f_j(z)|<\exp(-jqr^{2nr})$ and $\|z\|\leqq 1/4$ which follows from (1).

Put $h_j(z)=\prod_{10\leqq r\leqq j}(e_{j,r}(z))^{r^{-r}}$ and finally

$$v(z) = \overline{\lim_{z'\to z}}\ \overline{\lim_{j\in S,\, j\to\infty}}\ (1/t_j)\log|h_j(z')|$$

where S is as before the Proposition.

Lemma 2. *$v\in PSH(\mathbf{C}^n)$ and $v(z)=-\infty$ if $z\in D\cap B(0, 1/8)$.*

Proof. We have that $v(z)<8k$ when $\|z\|\leqq k\geqq 1$ because (2) gives that $\|e_{j,r}\|_{B(0,k)}\leqq t_jk^{t_j}4^{t_j}<(8k)^{t_j}$ and because $\sum_{r\geqq 10}r^{-r}<1$.

Put $D_{j,r}=\{z\in B(0,1);\ |e_{j,r}(z)|^{1/r^rt_j}\leqq 1-2^{-r}\}$ and let $L(D_{j,r})$ be the Lebesgue measure of $D_{j,r}$.

Because of (2) there exists $y^{j,r}\in B(0,1)$ such that $|e_{j,r}(y^{j,r})|^{1/r^rt_j}\geqq 2^{-r^{-r}}$ and since $\log|e_{j,r}(y^{j,r}+z)|$ is plurisubharmonic we then have that

$$\frac{1}{(4\pi)^n}\int_{\|z\|\leqq 2}(1/r^rt_j)\log|e_{j,r}(y^{j,r}+z)|\,dz \geqq -r^{-r}\log 2.$$

Furthermore, since $\|e_{j,r}\|_{B(0,3)}\leqq 24^{t_j}$ according to the above we have that

$$\frac{1}{(4\pi)^n}\int_{\|z\|\leqq 2}(1/r^rt_j)\log|e_{j,r}(y^{j,r}+z)|\,dz \leqq r^{-r}\log 24+L(D_{j,r})\log(1-2^{-r}).$$

Together with the inequality above this gives that

$$L(D_{j,r}) \leqq r^{-r}\log 48/\log(1-2^{-r}) < r^{-r+1}2^{r+1} < 2^{-r} \quad \text{if} \quad r\geqq 10.$$

Thus it follows from the construction of h_j that $\|h_j\|_{B(0,1)}>2^{-t_j}$ since $B(0,1)\setminus\bigcup_{r\geqq 10}D_{j,r}$ is not empty because $\prod_{r\geqq 10}(1-2^{-r})>1/2$. Hence [1] or Hartogs' Lemma gives that $v\not\equiv-\infty$, that is, $v\in PSH(\mathbf{C}^n)$.

We shall now show that $v(z)=-\infty$ when $z\in D\cap B(0, 1/8)$. Assume that this is not true. Then there exist $z\in D\cap B(0, 1/8)$ and a constant $-\infty<T<0$ such that $v(z)>T+1$. Hence there exist, for every $m\in\mathbf{N}$, a vector $z^m\in B(0, 1/4)$ and an infinite set $S_m\subset S$ such that $z^m\to z$ as $m\to\infty$ and so that $|h_j(z^m)|\geqq e^{Tt_j}$ when $j\in S_m$.

Take $l\in\mathbf{N}$ so big that $-l^lC<T-2$ where C is defined in the Proposition. According to (2), $|e_{j,r}(z^m)|<e^{2t_j}$ and hence $\prod_{10\leqq r\leqq j,\, r\neq l}|e_{j,r}(z^m)|^{r^{-r}}<e^{2t_j}$. But $|h_j(z^m)|\geqq e^{Tt_j}$ when $j\in S_m$ so it follows that $|e_{j,l}(z^m)|\geqq\exp((T-2)t_jl^l)>\exp(-Ct_jl^l)$. Thus (1) gives that $|f_j(z^m)|\geqq\exp(-jql^{2nl})$ when $j\in S_m$. That implies that $u(z)>-ql^{2nl}$ since $u(z)=\overline{\lim}_{z'\to z}\overline{\lim}_{j\to\infty}(1/j)\log|f_j(z')|$ which contradicts the fact that $z\in D\cap B(0, 1/8)$ and completes the proof. Q.E.D.

We have proved that $D \cap B(0, 1/8)$ is globally polar hence that $D \cap B(z, r_{z/32})$ is globally polar. Since z is arbitrarily taken in D it is enough to prove the following lemma to complete the proof of the theorem.

Lemma 3. *If there exists, to every $z \in D$, a ball $B(z, r_z)$ such that $D \cap B(z, r_z)$ is globally polar then D is globally polar.*

Proof. Obviously $\bigcup_{z \in D} B(z, r_z)$ is open and hence σ-compact. Thus there exist countably many $z^j \in D$ such that $D \subset B(z^j, r_{z^j})$. But it is well known and easily seen that a countable union of globally polar sets is a globally polar set, which proves the Lemma and thus completes the proof of the Theorem. Q.E.D.

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