On the density theorems of Borel and Furstenberg

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0. Introduction

Recently in [3] Furstenberg gave a generalization of the Borel density theorem [1] with a new proof. A careful examination of the method together with a number of additional observations enables one to prove the density theorem and related results for other large classes of groups or group actions, to unify in this way all known density results for real or complex Lie groups and to derive a number of new results. The method is closely analogous to that of [4] and [6] but involves the use of invariant measures under an induced linear action this time on projective space rather than linear or affine space. It has the feature of isolating exactly which properties of the action of G are relevant and of essentially removing H from consideration. We prove theorems about the linear action of G on V in three stages. First that a G-invariant measure on the Grassman manifold of V has its support contained in the G-fixed point set; then that if G/H has finite volume then every H-invariant subspace of V is automatically G-invariant; and finally that H is Zariski dense in G. The types of groups we deal with include minimally almost periodic groups, complex analytic groups, solvable analytic linear groups all of whose eigenvalues are real and, more generally, linear groups whose radical has this property and whose Levi-factor has no compact part.

1. Invariant measures and quasilinear actions

In § 1 we prove under very general circumstances (1.11) that given a continuous linear action $G \times V \to V$ on a real or complex vector space and a finite measure μ on $\mathcal{G}(V)$, the Grassman manifold, which is G-invariant under the induced action, then $\operatorname{supp}\mu$ is contained in the G-fixed points of $\mathcal{G}(V)$. In particular, this result

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holds for any linear action of a minimally almost periodic group, for any holomorphic action of a complex analytic group and for any real analytic subgroup of $Gl(V)$ with radical R such that G/R has no compact factors and R has only real eigenvalues (1.12).

Definition. As in [4] and [6], if $G\times X\rightarrow X$ is an action of a locally compact group G on a space X we shall write $X_c=\{x\in X$ with orbit G_x having compact closure$\}$ and X_{fix} (or X_{fix}^G)$=\{x\in X$ which are G-fixed$\}$.

Definition. Let V be a finite dimensional real or complex linear space of dimension n, $P(V)$ the corresponding projective space and $v\mapsto\bar{v}$ the canonical map $\pi\colon V-(0)\rightarrow P(V)$. $P(V)$ is a compact manifold. If W is a subspace of V of $\dim>0$, then $\overline{W}$ denotes the corresponding subvariety of $P(V)$. A finite union $\bigcup_{i=1}^r \overline{W}_i$ is called a quasi-linear variety (q. l. v.). Since $\overline{W}_i=P(W_i)$ is compact each q. l. v. is euclidean closed in $P(V)$.

Lemma 1.1. *If $A\subseteqq P(V)$ then there exists a unique minimal q. l. v. Q containing A.*

Proof. By considering $\pi^{-1}(A)$ it is enough to show that any subset $B\subseteqq V$ is contained in a unique minimal set of the form $\bigcup_{i=1}^r W_i$. Now B is contained in such a set, namely V. If we show there exists a smallest such set this will also imply uniqueness. Since each W_i is algebraic, a finite union $\bigcup_{i=1}^r W_i$ of such sets is again algebraic. An infinitely descending chain in V would correspond to an ascending sequence of ideals in $K[x_1, \dots, x_n]$ which is impossible by the Hilbert Basis Theorem.

Now if $g\in Gl(V)$ define $\bar{g}\colon P(V)\rightarrow P(V)$ by $\bar{g}(\bar{v})=\overline{g(v)}$. Routine calculations prove

Lemma 1.2. *$\bar{g}$ is well defined.*

$$\bar{g}\pi = \pi g.$$

$$\overline{g_1 g_2} = \bar{g}_1\bar{g}_2.$$

$$\text{if} \quad \gamma\neq 0 \quad \overline{\gamma g}=\bar{g}.$$

Now suppose one has a linear representation $G\times V\rightarrow V$ of G on V. By (1, 2) this induces a compatible action of G on $P(V)$ making the diagram below commutative.

$$\begin{array}{ccc} G\times V-(0) & \rightarrow & V-(0) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ G\times P(V) & \rightarrow & P(V) \end{array}$$

Lemma 1.3. *Let A be a G-invariant subset of $P(V)$ and $\bigcup_{i=1}^r \overline{W}_i$ the minimal q. l. v. containing A. Then G permutes $\{W_i\colon i=1, \dots, r\}$.*

Proof. Since $\bar{g}\pi = \pi g$ for $g \in Gl(V)$ we know $\pi^{-1}(A)$ is also G-invariant, $\pi^{-1}(A) \subseteq \bigcup_{i=1}^{r} W_i$ and this is the minimal linear variety containing it. But then

$$g(\pi^{-1}A) = \pi^{-1}(A) \subseteq g\Big(\bigcup_{i=1}^{r} W_i\Big) = \bigcup_{i=1}^{r} g \cdot W_i .$$

The latter is a linear variety for each $g \in G$. By minimality

$$\bigcup_{i=1}^{r} W_i \subseteq \bigcup_{i=1}^{r} g \cdot W_i \quad \text{for each} \quad g \in G.$$

This means

$$\bigcup_{i=1}^{r} W_i = \bigcup_{i=1}^{r} g \cdot W_i \quad \text{for all} \quad g.$$

The spaces involved in the unique linear variety containing a set are clearly also unique and $g \cdot W_i$ is one of them. Therefore $g \cdot W_i = W_j$ for some j.

Lemma 1.4. *Let* $g_k \in Gl(V)$ *and suppose* $\det g_k / \|g_k\|^n \to 0$ *as* $k \to 0$ *where* $\| \ \|$ *is any convenient norm on* End(V). *Then there exists a map* $\varphi\colon P(V) \to P(V)$ *such that* $\varphi(P(V))$ *is a proper q. l. v. of* $P(V)$ *and a subsequence* $\bar{g}_{k_i}$ *of* $\bar{g}_k$ *such that* $\bar{g}_{k_i} \to \varphi$ *pointwise.*

Proof. Let W be a nonzero subspace of V and consider $g_k | W\colon W \to V$. Denote by $\gamma_{k,W} = 1/\|g_k|W\|$. Then $\|\gamma_{k,W} \cdot g_k|W\| = 1$ for all k. Since $\{A | A\colon W \to V, \|A\| = 1\}$ is a compact set, there is a subsequence which we again call $\gamma_{k,W} g_k|W$ such that $\gamma_{k,W} g_k|W \to \sigma_W$ in norm and therefore pointwise on W. Here σ_W is a linear map $W \to V$. Since $\|\sigma_W\| = 1$, $\sigma_W \neq 0$. Now since π is continuous and for $w \in W$, $\gamma_{k,W} g_k|_W(w) \to \sigma_W(w)$ we have $\overline{\gamma_{k,W} g_k|_W(w)} \to \overline{\sigma_W(w)}$. But $\overline{\gamma_{k,W} g_k|_W(w)} = \bar{g}_k(\bar{w})$ so $\bar{g}_k(\bar{w}) \to \overline{\sigma_W(w)}$ pointwise on $\bar{W}$ and in particular for $\bar{w} \notin \overline{\operatorname{Ker} \sigma_W}$.

If $W = V$ we have $\gamma_k g_k \to \sigma_V$ and so $\det \gamma_k g_k = \gamma_k^n \det g_k = \det g_k / \|g_k\|^n \to \det \sigma_V$. Since this sequence tends to 0, σ_V is singular. Now inductively define subspaces $W_0, W_1, \ldots,$ of V by $W_0 = V$, $W_{i+1} = \operatorname{Ker} \sigma_{W_i}$, $i \geqq 0$. Then $\operatorname{Ker} \sigma_V < V$ since $\sigma_V \neq 0$. Similarly $W_{i+1} < W_i$ since $\sigma_{W_i} \neq 0$. Thus the sequence $V = W_0 > W_1 > \ldots$ $\ldots > W_i > W_{i+1} > \ldots$ must terminate after a certain number of steps at 0; $W_{i_0} = (0)$. For each i and finer and finer subsequences, which are again called g_k, $\bar{g}_k(\bar{w}_i) \to \overline{\sigma_{W_i}(w_i)}$ pointwise for $\bar{w}_i \in \bar{W}_i$. Define $\varphi\colon P(V) \to P(V)$ by $\varphi(\bar{v}) = \overline{\sigma_{W_i}(v)}$ if $v \in W_i \sim W_{i+1}$, $i = 0, \ldots, i_0 - 1$. If $\bar{v} = \bar{v}_1$ then $v = \gamma v_1$, $\gamma \neq 0$. If $v \in W_i \sim W_{i+1}$ the same is true of v_1 so $\overline{\sigma_{W_i}(v)} = \overline{\sigma_{W_i}(\gamma v_1)} = \overline{\gamma \sigma_{W_i}(v_1)} = \overline{\sigma_{W_i}(v_1)}$ since σ_{W_i} is linear. Thus $\varphi(\bar{v}) = \varphi(\bar{v}_1)$, φ is well defined and $\bar{g}_k \to \varphi$ pointwise on $P(V)$. Moreover $\varphi(P(V)) = \bigcup_{i=0}^{i_0-1} \overline{\sigma_{W_i}(W_i)}$ so the range of φ is a q. l. v. Since σ_V is singular $\sigma_V(V) < V$. For $i > 0$ $\sigma_{W_i}\colon W_i \to V$ so $\dim \sigma_{W_i}(W_i) \leqq \dim W_i < \dim V$ and $\sigma_{W_i}(W_i) < V$ for $i \geqq 0$. Now the union of a finite (or even countable) number of subspaces each of strictly lower dimension cannot equal V. For this it is clearly sufficient to take $K=\mathbf{R}$. If $V=\bigcup_i V_i$, then take a finite $\neq 0$ measure ν on V which is absolutely continuous with respect to Lebesgue measure, e.g., $d\nu=\exp(-\|x\|^2)\,dx$. Then by subadditivity $\nu(V)\leqq\sum\nu(V_i)=0$ since each $\nu(V_i)=0$, a contradiction. This means that $P(V)\neq\bigcup_{i=0}^{i_0-1}\overline{\sigma_{W_i}(W_i)}$ and the range of φ is proper.

Lemma 1.5. *Let $G\times X\to X$ be an action of G on a metric space X. Suppose there exists a sequence $g_k\in G$ and a closed subspace Y of X such that for each $x\in X$, $g_k x\to y(x)\in Y$ pointwise. Then each finite G-invariant measure μ on X has* $\operatorname{supp}\mu\subseteqq Y$.

Proof. Since $|\mu|$ is also G-invariant we may assume $\mu\geqq 0$. Let $D(x)=\operatorname{dist}(x, Y)$ where dist is an equivalent bounded metric on X. Then D is a bounded continuous nonnegative function on X and $D(x)=0$ if and only if $x\in Y$. Now $\int_X D(g_k x)\,d\mu(x)=\int_X D(x)\,d\mu(x)$ for all k. Since $g_k x\to y(x)$ we have $D(g_k x)\to D(y(x))=0$ pointwise on X. Now D is bounded so $|D(g_k x)|\leqq c$ for all k, x. The finiteness of μ together with the dominated convergence theorem shows $\int_X D(g_k x)\,d\mu(x)\to 0$, therefore $\int_X D(x)\,d\mu(x)=0$ so $D\equiv 0$ on $\operatorname{supp}\mu$. Since $D=0$ exactly on Y, $\operatorname{supp}\mu\subseteqq Y$.

Theorem 1.6. *Let $G\times V\to V$ be a linear action and $G\times P(V)\to P(V)$ be the associated action on projective space. Suppose*

(i) $V_c=V_{fix}$

(ii) *G has no closed subgroups of finite index,*

(iii) *For each G-invariant subspace W of V either the function*

$$g|W\mapsto\frac{\det_W(g|W)}{\|g|W\|^{\dim W}}$$

vanishes at ∞ or else G acts on W as scalars.

Then each finite G-invariant measure μ on $P(V)$ has $\operatorname{supp}\mu\subseteqq P(V)_{fix}$.

Remarks. In [9] we have given a number of conditions implying (i). If G is connected, (ii) holds. If $\det_W(g|W)=1$ for all G-invariant W then (iii) holds. This is true in particular if $G=[G, G]^-$ since $g\mapsto\det_W(g|W)$ is a continuous map $G\to\mathbf{R}^\times$ and is therefore trivial. If G is minimally almost periodic (m. a. p.) i.e., has no compact quotients then clearly, (i), (ii), and (iii) hold, (iii) because of the remarks above since any locally compact abelian group is maximally almost periodic.

Thus we have Furstenberg's formulation.

Corollary 1.7. *If G is minimally almost periodic then the conclusion of* (1.6) *holds. In particular if G is a semi-simple Lie group without compact factors this is so.*

In spite of the fact that nilpotent and indeed solvable groups can never be m. a. p. we also have

Corollary 1.8. *If* $G \times V \to V$ *is a unipotent representation of a connected group then the conclusion of* (1.6) *holds.*

Definition. A linear action $G \times V \to V$ is called of type E if all eigenvalues λ of every $g \in G$ lie off the unit circle except for $\lambda = 1$. If G is a solvable analytic group and its adjoint representation has this property one says G is of type E (see [2] or [12]).

As was remarked above, by [9], (i) holds for any type E linear action and in particular for a unipotent representation. By the above, (ii) holds. As for (iii), since all eigenvalues are equal to 1 this would have to be true on any G-invariant subspace W. In particular $\det_W(g|W) = 1$. This proves (1.8).

Proof of Theorem 1.6. If G acts on V by scalars then $P(V) = P(V)_{fix}$ and we are done. Otherwise, by (iii) $\det g_k / \|g_k\|^n \to 0$, so by (1.4) there exists $\varphi: P(V) \to P(V)$ such that $Q = \varphi(P(V))$ is a proper q. l. v. of $P(V)$ and $\bar{g}_k \to \varphi$ pointwise in $P(V)$. Since Q is closed supp $\mu \subseteqq Q$ by (1.5) and by (1.1) there exists a smallest q. l. v. which we shall call $S = \bigcup_{i=1}^r \overline{W}_i$ containing supp μ.

$$\operatorname{supp} \mu \subset S \subset Q < P(V).$$

Since μ is G-invariant so is supp μ. By (1.3) G permutes $\{W_i\}$. But there are only a finite number of W_i so each has a stability group of finite index. Moreover, since $G \times V \to V$ is continuous and the W_i are closed, the stability groups are also closed. By (ii) G leaves each W_i stable. Let W be any one of the W_i and consider the action of G on W. Condition (i) descends to this action; $W_c = W_{fix}$. (ii) is satisfied since it is a condition on G and not on the action. Condition (iii) clearly also descends. If we let $\nu = \mu | \overline{W}$ then we get a G-invariant measure on $P(W)$ and argue as before. Unless G acts on W by scalars we know there exists a proper q. l. v. T of $\overline{W} = P(W)$ such that supp $\nu \subseteqq T$. This contradicts the minimality of S. Otherwise G acts on W_i by scalars for each i. But then each $\overline{W}_i$ is G-fixed. This means $\operatorname{supp} \mu \subseteqq S \subseteqq \bigcup_{i=1}^r \overline{W}_i \subseteqq P(V)_{fix}$.

In (1.9) V is a complex vector space and G a complex analytic group. We shall call the jointly holomorphic action $G \times V \to V$ a complex linear action.

Lemma 1.9. *Let* $G \times V \to V$ *be a complex analytic linear action, then the conclusion of* (1.6) *holds.*

Proof. We first verify that conditions (i) and (ii) of (1.6) are always satisfied. By considering the representation restricted to the **C**-subspace V_c it is sufficient to prove that if a complex analytic representation ϱ is bounded, then ϱ is trivial. We may clearly consider G to be a subgroup of $Gl(V)$. For $X \in \mathfrak{g}$ the Lie algebra of G we have

$$\varrho_{\exp zX} = \operatorname{Exp} \varrho^\cdot(zX) = \operatorname{Exp} z\varrho^\cdot(X).$$

Since $\varrho_{\exp zX}$ is bounded for $z\in\mathbf{C}$ and the function $z\mapsto \operatorname{Exp} z\varrho^{\boldsymbol{\cdot}}(X)$ is holomorphic, it is constant by Liouville's theorem. Hence it's derivative $(d/dz)|_{z=0}=0=\varrho^{\boldsymbol{\cdot}}(X)$. This means $\varrho^{\boldsymbol{\cdot}}(\mathfrak{g})=0$. The equation above tells us that $\varrho_{\exp zX}=I$ for $X\in\mathfrak{g}$, $z\in\mathbf{C}$. In particular, $\varrho_g=I$ for all g in a canonical neighborhood U of 1. Since ϱ is a homomorphism and U generates G, $\varrho_g\equiv I$. (ii) follows from connectedness of G.

Since U generates G each g is of the form $\prod_{i=1}^{k} \operatorname{Exp} z_i X_i$ for some z_i, X_i and k. If μ is a G-invariant measure on $P(V)$ then it is invariant under $G_X=\{\exp zX\colon z\in\mathbf{C}\}$ for each $X\in\mathfrak{g}$. If we knew supp $\mu\subseteqq P(V)_{fix}^{G_X}$ for all X then by the above

$$\operatorname{supp}\mu \subseteqq \bigcap_{X\in\mathfrak{g}} P(V)_{fix}^{G_X} = P(V)_{fix}^{G}.$$

Thus we may assume G is a 1-parameter linear group $\{\operatorname{Exp} zX\}$. We show that in this situation (iii) holds and therefore by (1.6) the proof would be complete. Let W be a G-invariant subspace of V. Then W is X-invariant and we may as well assume $W=V$. We show $(\det g)/\|g\|^n\to 0$ as $\|g\|\to\infty$ or alternatively that $\|g\|^n/|\det g|\to\infty$ as $\|g\|$ does. But $\|g^n/\det g\|\leqq\|g\|^n/|\det g|$. If $\|\operatorname{Exp} zX\|\to\infty$ then, by continuity, for some subsequence $|z|\to\infty$. We show $\|g^n/\det g\|$ goes to ∞ as $\|g\|$ does. That is

$$\lim_{|z|\to\infty} \frac{\operatorname{Exp} znX}{\det(\operatorname{Exp} zX)} = \infty.$$

But this is a holomorphic function of z. By the maximal principle it tends to ∞ as $|z|$ does or it is constant. Thus $g^n/\det g=A\in\operatorname{End}_C V$. Taking $g=1$ we see that $A=I$ and $g^n=\det g I$. Let $H=\{g^n\colon g\in G\}$. Since $H=\{\operatorname{Exp} nzX\colon z\in\mathbf{C}\}=G$ we see that each $g\in G$ is a scalar. By (1.6) $\operatorname{supp}\mu\subseteqq P(V)_{fix}^{G_X}$.

Remark. One gets two immediate corollaries from the proof of (i).

1) A compact complex analytic group G is a torus.

2) If the complex analytic group G has a bounded faithful complex analytic representation, then G is trivial.

To see that 1) is true, take ϱ=adjoint representation. Then Ad is bounded so $G/Z(G)=\operatorname{Ad}(G)=1$ and $G=A(G)$. As a compact connected abelian Lie group G is a torus. 2) is clear.

Example 1.10. The converse of (1.6) fails.

Let $G=\left\{\begin{pmatrix}\lambda & 0\\ \mu & \lambda\end{pmatrix}=g\colon \lambda>0,\ \mu \text{ real}\right\}$. Then G is the identity component of a solvable algebraic subgroup of $Gl(2,\mathbf{R})$, and gives a type E action on $V=\mathbf{R}^2$. It follows that the action of G on V satisfies (i) and (ii) of (1.6) [9]. Clearly if $\lambda\to\infty$ and μ is fixed, then $|\det g|/\|g\|^2\to 1$. Thus (iii) is not satisfied.

Suppose $g\begin{pmatrix}a\\b\end{pmatrix}=\varphi(g)\begin{pmatrix}a\\b\end{pmatrix}$ i.e., $\begin{pmatrix}a\\b\end{pmatrix}$ spans a G-invariant subspace. Then $\varphi(g)=\lambda$ and $a=0$. Thus the only G invariant line is the one through $\begin{pmatrix}0\\1\end{pmatrix}$. Let $p_0=\begin{pmatrix}0\\1\end{pmatrix}^-$ be its image in projective space. Then $P(V)_{fix}=\{p_0\}$. If $v\in V$ is not on this line then $G\cdot v=\left\{\begin{pmatrix}x\\y\end{pmatrix}: x>0\right\}$ or $\left\{\begin{pmatrix}x\\y\end{pmatrix}: x<0\right\}$. So if $p_1=\bar{v}$, then $G\cdot p_1$ is the union of the images of these two sets. Since these subsets of $P(V)$ are equal, $P(V)=\{p_0\}\cup G\cdot p_1$, the disjoint union of two orbits.

Now let μ be a finite G-invariant measure on $P(V)$. If $p\in\operatorname{supp}\mu$ then $G\cdot p\subseteq \operatorname{supp}\mu$. We show $\mu(G\cdot p)=0$ if $p\in G\cdot p_1$. If not, $\mu(G\cdot p)>0$ and μ restricted to the open set $G\cdot p$ gives a finite G-invariant measure on $G\cdot p$. Now $G\cdot p= G/\operatorname{Stab}_G(p)$. We may assume $p=\bar{v}$ where $v=\begin{pmatrix}1\\0\end{pmatrix}$. Then $\operatorname{Stab}_G(p)=\left\{\begin{pmatrix}\lambda & 0\\0 & \lambda\end{pmatrix}\right\}$. This is a central and therefore normal subgroup of G and $G/\operatorname{Stab}_G(p)$ is clearly isomorphic to $\begin{pmatrix}1 & 0\\ \mu & 1\end{pmatrix}$, i.e., $\mathbf{R}$, the real line. Such a noncompact group cannot have a finite invariant (Haar) measure; therefore $\mu(G\cdot p_1)=\mu(G\cdot p)=0$. Since this set is open $\operatorname{supp}\mu\subseteq P(V)\sim G\cdot p_1=\{p_0\}=P(V)_{fix}$.

Hereafter we use the following notation. If ϱ is a representation of G on V and r is an integer $1\leqq r\leqq n=\dim V$ we shall denote by $\wedge^r\varrho$ the representation of G on $\wedge^r V$, the rth exterior power of V, given by $g\mapsto\varrho_g\wedge\ldots\wedge\varrho_g$. We can now give an extension of (1.6) to the higher Grassman spaces. Let $\mathscr{G}(V)$ denote the Grassman space of V. Then $\mathscr{G}(V)=\bigcup_{r=1}^n\mathscr{G}^r(V)$, the disjoint union of open sets where $\mathscr{G}^r(V)=\{r$ dimensional subspaces of $V\}$. Here $\mathscr{G}^1(V)$ equals $P(V)$. If $G\times V\to V$ is a linear action then there is an induced action of G on each $\mathscr{G}^r(V)$. Thus one has an action $G\times\mathscr{G}(V)\to\mathscr{G}(V)$ under which each $\mathscr{G}^r(V)$ is invariant.

Theorem 1.11. *Suppose the action $G\times V\to V$ has the property that $\Lambda^r\varrho$ satisfies the conditions of* (1.6). *Then each finite G-invariant measure μ on $\mathscr{G}^r(V)$ has* $\operatorname{supp}\mu\subseteq\mathscr{G}^r(V)_{fix}$. *If the action satisfies these conditions for every $r=1,\ldots,n$ then each finite G-invariant measure μ on $\mathscr{G}(V)$ has* $\operatorname{supp}\mu\subseteq\mathscr{G}(V)_{fix}$.

In particular, if G is m.a.p., if G is connected and ϱ is a unipotent representation, or if $G\times V\to V$ is a complex analytic linear action, then the conclusions of (1.11) *hold.*

Proof. It clearly suffices to prove the first statement, so let $G\times\mathscr{G}^r(V)\to\mathscr{G}^r(V)$ be the induced action. There is a canonical map $\varphi\colon\mathscr{G}^r(V)\to P(\Lambda^r V)$ (For an r dimensional subspace W of V choose a basis $\{w_1,\ldots,w_r\}$. Then $w_1\wedge\ldots\wedge w_r$ is a nonzero element of $\Lambda^r V$ and therefore the line through it gives a point in $P(\Lambda^r V)$.) Now $Gl(V)$ acts transitively and continuously on $\mathscr{G}^r(V)$. The latter is a quotient space $Gl(V)/\operatorname{Stab}_{Gl(V)}(W)$ where W is some fixed r-dimensional subspace of V. Let $\gamma\colon Gl(V)\to\mathscr{G}^r(V)$ the be corresponding projection. If $\{w_1,\ldots,w_r\}$ is a basis of W, then $\psi\colon g\mapsto gw_1\wedge\ldots\wedge gw_r$ is a map $Gl(V)\to\Lambda^r V$. Clearly the map φ factors as $\varphi_1\circ\pi$ where $\varphi_1\colon\mathcal{G}^r(V)\to\Lambda^r V-(0)$, $\pi\colon\Lambda^r(V)-(0)\to P(\Lambda^r V)$ and the diagram below is commutative.

$$\begin{array}{ccc} Gl(V) & \xrightarrow{\ \psi\ } & \Lambda^r V-(0) \\ \Big\downarrow{\scriptstyle\gamma} & \nearrow{\scriptstyle\varphi_1} & \Big\downarrow{\scriptstyle\pi} \\ \mathcal{G}^r(V) & \xrightarrow[\ \varphi\ ]{} & P(\Lambda^r V) \end{array}$$

since $\varphi_1\gamma(g)=\varphi_1(gW)=gw_1\wedge\ldots\wedge gw_r=\psi(g)$. To see that φ is smooth it is enough to show φ_1 is and hence that ψ is because γ and π are smooth. But clearly ψ is smooth since the map $g\mapsto gw_1\otimes\ldots\otimes gw_r$ is. Moreover φ intertwines the actions

$$\begin{array}{ccc} \mathcal{G}^r(V) & \xrightarrow{\ \varphi\ } & P(\Lambda^r V) \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle (g\wedge\ldots\wedge g)^-} \\ \mathcal{G}^r(V) & \xrightarrow[\ \varphi\ ]{} & P(\Lambda^r V) \end{array}$$

For let $g_1(W)$ be any point of $\mathcal{G}^r(V)$ and $g\in G$. Then $\varphi(gg_1(W))=(gg_1(w_1)\wedge\ldots\wedge gg_1(w_r))^-$, while $(g\wedge\ldots\wedge g)^-(\varphi(g_1W))=(g\wedge\ldots\wedge g)^-(g_1w_1\wedge\ldots\wedge g_1w_r)=((g\wedge\ldots\wedge g)(g_1w_1\wedge\ldots\wedge g_1w_r))^-=(gg_1w_1\wedge\ldots\wedge gg_1(w_r))^-$.

Since φ is a G-equivariant measurable function the measure μ can be pushed forward and can be regarded as a finite G-invariant measure on $P(\Lambda^r V)$ (supported on the image of $\mathcal{G}^r(V)$). By (1.6) $\operatorname{supp}\mu\subseteq$fixed point set. By G-equivariance $\operatorname{supp}\mu\subseteq\mathcal{G}^r(V)_{fix}$.

Concerning particular cases: If G is m.a.p. then because of the remarks following (1.6) any representation satisfies (i), (ii) and (iii). Also if ϱ is a complex analytic representation then so is $\Lambda^r\varrho$ for each r. Hence, by the proof of (1.9) each $\Lambda^r\varrho$ satisfies (i), (ii) and (iii). If ϱ is a unipotent representation of a connected group then so is $\Lambda^r\varrho$ for each r and hence by the proof of (1.8) the conditions are also satisfied. For if $A_i\in\operatorname{End}V_i$, $i=1,2$, the spectrum of $A_1\otimes A_2=\{\lambda_i\mu_j\colon i,j\}$ where $\{\lambda_i\}$ and $\{\mu_j\}$ are the spectrum of A_1 and A_2 respectively. It follows that if $A_1,\ldots,A_r$ are unipotent the same is true of $A_1\otimes\ldots\otimes A_r$ and therefore, since it acts on a quotient space, also of $A_1\wedge\ldots\wedge A_r$. These remarks also show that the exterior product of representations with only real eigenvalues again has the same property.

We conclude § 1 with a considerable extension of the unipotent case.

Theorem 1.12. *Let G be an analytic subgroup of $Gl(n,\mathbf{R})$ with radical R. If G/R has no compact factors and R has only real eigenvalues then each finite G-invariant measure μ on $\mathcal{G}(V)$ has* $\operatorname{supp}\mu\subseteq\mathcal{G}(V)_{fix}$.

Proof. By the remarks above, the Levi decomposition $G=R\cdot S$ and the fact that S has no compact factors we may assume by (1.7) that $G=R$. Such an action satisfies (i) and (ii) of (1.6) (See [9]). Since it is generated by 1-parameter groups we may assume G is itself a 1-parameter group $\{\operatorname{Exp} tX\colon t\in\mathbf{R}\}$ where

$$X=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ * & & \lambda_n\end{pmatrix}$$

has only real eigenvalues. As in (1.9) we may also assume $W=V$ and $|t|\to\infty$. We show $\det(\operatorname{Exp} tX)/\|\operatorname{Exp} tX\|^n\to 0$. Since

$$\|\operatorname{Exp} tX\|\geqq\left\|\begin{pmatrix}e^{t\lambda_1} & & 0\\ & \ddots & \\ 0 & & e^{t\lambda_n}\end{pmatrix}\right\|$$

we may clearly also assume X is diagonal. Now $\det(\operatorname{Exp} tX)=e^{t\cdot\operatorname{tr}X}$ so it suffices to show $|e^{t(n\lambda_i-\operatorname{tr}X)}|\to\infty$ as $|t|\to\infty$ for some i. If $t\to\infty$ choose λ_i to be the largest eigenvalue. Then $n\lambda_i\geqq\operatorname{tr}X$ and the exponential function tends to ∞ unless all λ_i are equal. Similarly if $t\to-\infty$ choose λ_i to be the smallest eigenvalue. If all λ_i are equal, G acts as scalars, proving that the action satisfies (iii).

This deals with the action of G on $P(V)$. To look at the higher $\mathcal{G}^r(V)$ we must verify that the linear group

$$H=\{g\wedge\ldots\wedge g\colon g\in G\}\subseteqq Gl(\Lambda^r V)$$

satisfies the hypothesis above. Since the map $g\mapsto g\wedge\ldots\wedge g$ is an analytic homomorphism, H is an analytic group, and its radical $R(H)$ equals the image of R. Therefore $H/R(H)$ has no compact factors. The remarks immediately preceding (1.12) tell us that $R(H)$ acts with only real eigenvalues. But then the proof of the first par of (1.12) shows that $\Lambda^r\varrho$ satisfies (i), (ii), and (iii). Since r was arbitrary an application of (1.11) completes the proof of (1.12).

2. Invariant subspaces and homogeneous spaces of finite volume

In this section we consider a continuous linear action $G\times V\to V$ of a locally compact group G on a finite dimensional real or complex vector space and a (euclidean) closed subgroup H of G such that G/H has finite volume. We show under general circumstances (2.1) that H-invariant subspaces of V must be G-invariant. In particular (2.2), this is so in the cases corresponding to those of § 1. As a consequence we prove that under these circumstances if G is an algebraic linear group then the identity component of the hull of H is normal in G.

Theorem 2.1. *Suppose G/H has a finite G-invariant measure and ϱ is a finite dimensional continuous representation of G on V such that ϱ together with all exterior powers $\Lambda^r\varrho$ satisfy conditions* (i), (ii) *and* (iii) *then each H-invariant subspace W of V is also G-invariant.*

Remarks. If G/H is merely compact but does not have finite volume then the conclusion of (2.1) need not hold [11]. Also if (G, ϱ) does not satisfy condition (iii) for all W then the conclusions of (1.6) and (2.1) do not necessarily hold. For one can always arrange that $\det_V g=1$ and therefore that $g\mapsto\det g/\|g\|^n$ vanishes at ∞ (by imbedding $Gl(n)$ in $Sl(n+1)$ and preserving invariant subspaces). However, one may not be able to do this so that $\det_W(g|W)=1$ for every invariant subspace W. In (2.2) b below a particular case is that of a solvable group acting with only real eigenvalues. However, there are examples of more general type E solvable linear actions where condition (iii) fails to hold and the conclusions of (1.6), (1.11), and (2.1) are false. An example of this type was pointed out to me in conversation by H. Abels. The reader will notice how the use of (1.11) in the proof of (2.1) essentially removes H from the field and makes this a question only about G and its action on V.

Proof of Theorem 2.1. If $\dim W=r$ form the rth Grassman space $\mathscr{G}^r(V)$ and consider the action $G\times\mathscr{G}^r(V)\to\mathscr{G}^r(V)$. W corresponds to a point $p_0\in\mathscr{G}^r(V)$. Since H leaves W stable, p_0 is H-fixed. So $H\subseteq\mathrm{Stab}_G(p_0)$. Since G/H has a finite G-invariant measure, the same is true of $G/\mathrm{Stab}_G(p_0)$. This means that $p_0\in\operatorname{supp}\mu$ for an appropriate G-invariant measure μ on $\mathscr{G}^r(V)$ where $\operatorname{supp}\mu=G\cdot p_0$. By (1.11) p_0 is G-fixed and this means that W is G-stable.

Theorem 2.2. *Suppose G/H has finite volume and ϱ is a continuous representation of G on V. If*

(a) *G is minimally almost periodic or*

(b) *G is an analytic subgroup of $Gl(V)$ with radical R, the elements of R have only real eigenvalues, and G/R has no compact factors, or*

(c) *$G\times V\to V$ is a complex analytic linear action,*

then any H-invariant subspace W of V is G-invariant.

Proof. In case (a) we merely verify that the action satisfies the conditions of (2.1). Cases (b) and (c) follow from the proofs of (1.12) and (1.11) respectively.

In (2.4) through (2.7) we deduce analogues of the remaining results of Borel [1] concerning the size of H in G, this time for any representation satisfying any of the various hypotheses of (2.2). With respect to (2.7), however, the result has already been proven in [4] and [6] (in more general form) both in cases (b) and (c). In fact, since in (b) R has real eigenvalues it is simply connected by (3.2) and of type E by (3.1) below. This together with the fact that G/R has no compact factors tells us by [4] and [6] that G has no automorphisms of bounded displacement. The same is true in case (c) ([4]), so we just give the proof in (a). Here G, H, ϱ and V are as in (2.2). We denote $Z_G(L)$ and $N_G(L)$ the centralizer and normalizer, respectively, of the subgroup L of G.

Lemma 2.3. *Let $G\times V\to V$ be a linear action and consider the action σ of G on* End V *formed from this given by $(g, T)\mapsto\varrho_g\cdot T$. Then the eigenvalues of the operators are contained in those of the corresponding ϱ_g for $g\in G$. The action is clearly continuous and complex analytic or faithful, respectively, if (G, ϱ) is.*

Proof. Suppose $\lambda\in\mathbf{C}$ is an eigenvalue and $T\neq 0\in M_n(\mathbf{C})$ the corresponding eigenvector. Then $\varrho_g T=\lambda T$ for that $g\in G$. Since $(\varrho_g-\lambda I)\cdot T=0$ we must have $\det(\varrho_g-\lambda I)=0$ otherwise $\varrho_g-\lambda I$ would be invertible. Multiplying by the inverse would imply $T=0$, a contradiction.

Corollary 2.4. *If ϱ is irreducible so is $\varrho|H$.*

This follows immediately from (2.2).

Corollary 2.5. *The linear span of $\varrho(G)$ equals the linear span of $\varrho(H)$.*

Proof. It suffices to show each ϱ_g is in the linear span of $\varrho(H)$. Consider the action σ defined in (2.3) and let $\mathscr{W}=$lin. span $\varrho(H)$. Since

$$\varrho_h\cdot\sum_i c_i\varrho_{h_i}=\sum_i c_i\varrho_{hh_i},$$

$\mathscr{W}$ is an H-invariant subspace of End V. If ϱ is either complex analytic or satisfies (2.2)b the same is true by (2.3) of σ. By (2.2) $\mathscr{W}$ is G-invariant. Since $I\in\mathscr{W}$ so does $\varrho_g\cdot I=\varrho_g$.

Corollary 2.6. *The centralizer of $\varrho(H)$ in* End V *equals the centralizer of $\varrho(G)$ in* End V. *In particular, $Z_{\varrho(G)}(\varrho(H))=Z(\varrho(G))$.*

Proof. If $T\in\operatorname{End} V$ and $T\varrho_h=\varrho_h T$ for all $h\in H$, then T also commutes with any linear combination of these elements. Therefore T commutes with $\varrho(G)$ by (2.5).

Corollary 2.7. *If G is an analytic minimally almost periodic group, then $Z_G(H)=Z(G)$.*

Proof. Let $g_0\in Z_G(H)$. Then $\operatorname{Ad} g_0\in Z_{\operatorname{Ad}(G)}(\operatorname{Ad}(H))$, which equals $Z(\operatorname{Ad}(G))$ by (2.6). This means that $[g_0, G]\subseteq\operatorname{Ker}\operatorname{Ad}=Z(G)$. Now let φ be defined on G by $\varphi(g)=[g_0, g]$. φ is continuous and takes values in $Z(G)$. Moreover

$$\varphi(g)\varphi(g')=g_0gg_0^{-1}g^{-1}g_0g'g_0^{-1}g'^{-1}=g_0g'g_0^{-1}g_0gg_0^{-1}g^{-1}g'^{-1}$$

since $[g_0, g]\in Z(G)$. The latter is clearly $\varphi(g'g)$. But $Z(G)$ is abelian so $\varphi(g)\varphi(g')=\varphi(g')\varphi(g)$. This means φ is a continuous homomorphism with values in an abelian group. Since G is a map, φ must be trivial and therefore $g_0\in Z(G)$.

We now specialize (2.2) to the case of the adjoint representation. Hereafter if A is a closed subgroup of $Gl(V)$ we denote by A_0 its euclidean identity component and $A^{\#}$ its algebraic hull. Hereafter, unless otherwise stated, connectedness and closure will always refer to the euclidean topology.

Corollary 2.8. *Let G be an analytic group and suppose that G/H has finite volume. If G is either m.a.p., nilpotent or complex analytic, then*

1) *Each analytic subgroup L of G normalized by H is normal in G.*

2) *In particular, if A is any closed subgroup of G containing H then A_0 is normal in G.*

3) *In particular, if G is an algebraic subgroup of $Gl(V)$ then $(H^{\#})_0$ is normal in G. For then $G \supseteqq H^{\#} \supseteqq H$ and $H^{\#}$ is closed since it is algebraic.*

4) *If $G/N_G(L)$ has finite volume where L is an analytic subgroup, then L is normal.*

This result substantially strengthens the corresponding one of S. P. Wang [14] (where $G/N_G(L)$ has proven to be compact) in these cases.

Lemma 2.9. *Let G be a solvable analytic subgroup of $Gl(n, \mathbf{R})$ such that every eigenvalue of each element is real. Then G has only real (positive) roots.*

Proof. Since G is analytic it is a subgroup of the real triangular group

$$\{(g_{ij}) | g_{ij} \text{ real}, \ g_{ij} = 0 \text{ if } j > i, \ g_{ii} > 0\}$$

by Lie's theorem. Its Lie algebra $\mathfrak{g}$ also consists of real triangular matrices X. As is well known, $\operatorname{Ad} g(X) = gXg^{-1}$. Let $\chi: G \to \mathbf{C}^{x}$ be a root and $X \neq 0 \in \mathfrak{g}^c$, the complexification of $\mathfrak{g}$, be the corresponding root vector. Then $gX = \chi(g) Xg$ for $g \in G$. When one calculates the ijth coordinate of each side, this yields

$$\sum_{k=j}^{i} g_{ik} X_{kj} = \chi(g) \sum_{k=j}^{i} X_{ik} g_{kj}.$$

Now consider the largest index j such that $X_{ij} \neq 0$ for some i. The equation above then yields $\chi(g) = g_{ii}/g_{jj} > 0$. Since $X \neq 0$, there must be such a pair.

Corollary 2.10. *Let G be an analytic subgroup of $Gl(V)$ such that G/R has no compact factors and* $\mathbf{R}$ *has only real eigenvalues and let H be a closed subgroup of G with G/H of finite volume. Then all the conclusions of* (2.8) *hold.*

Proof. We must show that one can replace G, R and H by the corresponding adjoint groups. Now as far as questions of invariant subspaces are concerned, one can always, by continuity, replace $\operatorname{Ad}(H)$ by $\operatorname{Ad}(H)^{-}$. By an elementary argument $\operatorname{Ad}(G)/\operatorname{Ad}(H)^{-}$ has finite volume. (In the case of a lattice Γ in (3.4) below one has actually the fact that $\operatorname{Ad}(\Gamma)$ itself is a lattice in $\operatorname{Ad}(G)$. This is proven in [5]). Now let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. Since $\operatorname{ad}(\mathfrak{g})/\operatorname{ad}(\mathfrak{r})$ is a quotient algebra of $\mathfrak{g}/\mathfrak{r}$ it clearly is semisimple without compact factors. This implies that ad $(\mathfrak{r})$ is the radical of ad $(\mathfrak{g})$. Thus $\operatorname{rad} \operatorname{Ad}(G)=\operatorname{Ad}(R)$, $\operatorname{Ad}(G)/\operatorname{Ad}(R)$ is semisimple without compact factors and we are reduced to the following

Lemma 2.11. *If the elements of R have only real eigenvalues then the roots of* $\operatorname{Ad}(R)$ *(acting on $\mathfrak{g}$) are all real.*

Proof. Suppose $X\neq 0\in\mathfrak{g}^c$, the complexification of $\mathfrak{g}$, is a root vector for the solvable analytic group $\operatorname{Ad}(R)$; $\operatorname{Ad}_r(X)=\chi(r)X$ where $\chi\colon R\to\mathbf{C}^\times$ is the corresponding root. Let $Y\in\mathfrak{r}$ and consider the corresponding 1-parameter group. Now $\operatorname{Ad}_{\exp tY}(X)=\operatorname{Exp} t(\operatorname{ad} Y)(X)=X+t[Y,X]+O(t^2)$. Thus

$$[Y,X]=\frac{\chi(\exp tY)-1}{t}\cdot X+O(t).$$

Taking the limit as $t\to 0$ yields

$$[Y,X]=\frac{d}{dt}\left(\chi(\exp tY)\right)\Big|_{t=0}\cdot X.$$

Consider the complex subalgebra of $\mathfrak{g}^c$ generated by $\mathfrak{r}$ and X. The above equation tells us that this algebra is $\mathfrak{r}^c+[X]$ and therefore this is a complex solvable Lie algebra of matrices. By Lie's theorem it can be simultaneously triangularized over $\mathbf{C}$. In particular $\mathfrak{r}$ and X can be simultaneously triangularized. Exponentiating, the same can be said of R and X. Since $\operatorname{Ad} r(X)=rXr^{-1}=\chi(r)X$ we can argue as in (2.9) to conclude that since $X\neq 0$, we have $\chi(r)\in\mathbf{R}_0^\times$ for all $r\in R$.

3. Zariski Density

In § 3 we prove that under various circumstances H is Zariski dense in G.

Theorem 3.1. *Let G be*

(a) *an algebraic subgroup of $Gl(n,\mathbf{R})$ which is m.a.p. or*

(b) *a unipotent analytic subgroup of $Gl(n,\mathbf{R})$ or*

(c) *a euclidean connected algebraic subgroup of $Gl(n,\mathbf{C})$.*

If G/H has finite volume, then H is Zariski dense in G.

Proof. In all cases G is algebraic. In case (b) this follows from Engels' theorem and the fact that here the exponential map is polynomial. Thus $H\subseteq H^\#\subseteq G$. Now $H^\#$ is euclidean closed and by [11], $G/H^\#$ has finite volume. Also $(H^\#)_0$ has finite index in $H^\#$ hence $G/(H^\#)_0$ also has finite volume. In all cases the hypotheses of

(2.8) are satisfied. Hence by (2.8) we know that $(H^{\#})_0$ is normal in G. Therefore $G/(H^{\#})_0$ is a compact group.

In case (a) this is impossible since G is map so $G/(H^{\#})_0=(1)$ and $G=(H^{\#})_0=H^{\#}$. In case (b) G is simply connected. Since $(H^{\#})_0$ is connected $G/(H^{\#})_0$ is also simply connected [7]. But as a quotient group of a connected solvable group $G/(H^{\#})_0$ is also connected and solvable and therefore a torus, $\mathbf{T}^n$. Since it is simply connected $n=0$ and $G=(H^{\#})_0=H^{\#}$. In case (c) we have $G/(H^{\#})_0$ is a compact complex analytic group. By the remarks following (1.9) $G/(H^{\#})_0$ is a torus. Thus $(H^{\#})_0\supseteqq[G, G]$. But then $H^{\#}$ also contains $[G, G]$ and so is normal in G. By [8], $G/H^{\#}$ is an algebraic linear group and therefore a complex analytic linear group. On the other hand, the map $G/(H^{\#})_0\to G/H^{\#}$ is surjective so $G/H^{\#}$ is compact. Since this has a faithful representation the remarks following (1.9) tell us $G/H^{\#}=(1)$ and $G=H^{\#}$.

In (3.1) (a) is due to Furstenberg, (b) is the classical result of Malcev and (c) was very recently proven by S. P. Wang in [14] (using quite different methods).

We next characterize those solvable algebraic groups which will be studied in (3.3). The present somewhat more general formulation of (3.2) was suggested by Gerhard Hochschild.

Proposition 3.2. *Let G be a solvable algebraic subgroup of $Gl(n, \mathbf{R})$. Then G_0 is simply connected iff each element of G_0 has only real eigenvalues.*

To prove (3.2) it is sufficient to prove (3.2. a) and (3.2. b) below.

3.2.a. *If G is an analytic subgroup of $Gl(n, \mathbf{R})$ and every element of G has only real eigenvalues then G is simply connected.*

One then takes for G the group G_0 of (3.2).

3.2.b. *If G is an algebraic subgroup of $Gl(n, \mathbf{R})$, $\chi\colon G\to\mathbf{C}^x$ a rational character and G_0 is simply connected then $\chi(G_0)\subseteqq\mathbf{R}_0^x$.*

By the Lie—Kolchin theorem one knows that G^0, the Zariski connected component of the G of (3.2), is in simultaneous triangular form over $\mathbf{C}$,

$$G^0=\begin{pmatrix}\lambda_1(g) & 0\\ * & \lambda_n(g)\end{pmatrix}.$$

But $G_0\subseteqq G^0$ and is its euclidean compact of 1. Taking $\chi=\lambda_i$ we find $\lambda_i(g)>0$ for all i and $g\in G_0$.

Proof of 3.2.a. Let K be a maximal compact subgroup of G. Since K is connected $K\subseteqq SO(n, \mathbf{R})$ and each element of K lies on a 1-parameter group of K. It follows that each element of K can be put in block diagonal form with blocks of the type $\begin{pmatrix} \operatorname{Cos}\theta & \operatorname{Sin}\theta \\ -\operatorname{Sin}\theta & \operatorname{Cos}\theta \end{pmatrix}$ or ones. Since the eigenvalues must be real, $K=(1)$ and G is simply connected by Iwasawa's Theorem.

Proof of 3.2.b. Let $\mathbf{C}^x=\mathbf{R}_0^x\times\mathbf{T}$ be the polar decomposition and τ denote the projection $\mathbf{C}^x\to\mathbf{T}$. Suppose $\chi(G_0)\subsetneqq\mathbf{R}_0^x$. Then $\tau\circ\chi\colon G_0\to\mathbf{T}$ is a non-trivial homomorphism of analytic groups. Hence its differential is not the zero map. Since $\mathbf{T}$ is 1-dimensional, it follows that $\tau\circ\chi(G_0)=\mathbf{T}$ and therefore $G_0/K\cong\mathbf{T}$ as a topological group where K denotes the kernel of $\tau\circ\chi$ on G_0. We regard $\mathbf{C}^x$ as the real algebraic group $\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right\}\subseteqq Gl(2,\mathbf{R})$ and $\mathbf{R}^x$ as the real algebraic subgroup $\left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\colon a\neq 0\right\}$. Since χ is rational $\chi^{-1}(\mathbf{R}^x)$ is algebraic. Since $\operatorname{Ker}\tau=\mathbf{R}_0^x$, $K=G_0\cap\chi^{-1}(\mathbf{R}_0^x)$. Now G/G_0 is finite hence so is $\chi^{-1}(\mathbf{R}_0^x)G_0/G_0\simeq\chi^{-1}(\mathbf{R}_0^x)/K$. But $\chi^{-1}(\mathbf{R}_0^x)$ has index 2 in $\chi^{-1}(\mathbf{R}^x)$. It follows that $\chi^{-1}(\mathbf{R}^x)/K$ and therefore since $K^{\#}\subseteqq\chi^{-1}(\mathbf{R}^x)$ that $K^{\#}/K$ is finite. Hence $K_0^{\#}=K_0$ so that $K/K_0=K/K_0^{\#}\subseteqq K^{\#}/K_0^{\#}$. Since the latter is finite so is K/K_0. But G_0 is simply connected, therefore $\Pi_1(G_0/K)=K/K_0$ is finite. On the other hand it is $\Pi_1(\mathbf{T})=\mathbf{Z}$, a contradiction.

Theorem 3.3. *Let G be a connected solvable algebraic subgroup of $Gl(n,\mathbf{R})$ such that each element has only real eigenvalues. If G/H has finite volume then H is Zariski dense.*

Proof. By the proof of (3.1) b $H_0^{\#}$ is normal in G and $G/(H^{\#})_0$ is a torus. Since G is simply connected by (3.2) and $H_0^{\#}$ is connected, $G/(H^{\#})_0$ is also simply connected and hence trivial.

Remark. For any closed subgroup H of G (as in (3.3)) let S denote the smallest analytic subgroup of G containing H. Since G is of type E by (3.2) such analytic subgroups exist by [13]. Following Mostow [10] one says H is analytically dense if $S=G$. Since G has only real roots, Theorem 2 of [13] tells us G/H is always homeomorphic to $\mathbf{R}^k\times S/H$ and S/H is compact. It is easy to see that this implies that the conditions G/H is compact, G/H has finite volume, and H is analytically dense in G are all equivalent. In contrast to the unipotent case however H being Zariski dense is definitely weaker.

Example. We now give a simple example of an abelian, in fact, diagonal subgroup of $Gl(2,\mathbf{R})$ and an analytic subgroup which is Zariski dense but which is not cocompact. Let $G=\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\colon \lambda,\mu>0\right\}$ and S be the 1-parameter subgroup

$$\left\{\begin{pmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{pmatrix}\colon t\in\mathbf{R}\right\}$$

where α is an irrational number. Then G is the identity component of an algebraic group. Let $p(\lambda, \mu)=\sum_{ij} a_{ij}\lambda^i\mu^j$ be one of the polynomials defining $S^\#$. Since $\lambda^\alpha=\mu$ on S we have $\sum_{ij} a_{ij}\lambda^{i+\alpha j}\equiv 0$. Now the exponents $i+\alpha j$ must be distinct because α is irrational. If $\sum_{k=1}^m \beta_k\lambda^{\alpha_k}\equiv 0$ for all $\lambda>0$ where α_k and β_k are real and $\alpha_1<\ldots<\alpha_m$, then all β_k must be 0. For the latter equals $\lambda^{\alpha_1}(\beta_1+\beta_2\lambda^{\alpha_2-\alpha_1}+\ldots+\beta_m\lambda^{\alpha_m-\alpha_1})$. Since the first factor is positive the second must be identically 0. Letting $\lambda\to 0$ we see that $\beta_1=0$ and then reason by induction on m. Thus $a_{ij}=0$ for all i, j and $p\equiv 0$. Since p was arbitrary $S^\#=G^\#$. On the other hand, G is simply connected and solvable so the analytic subgroup S is closed and clearly proper This means G/S is noncompact and has no finite invariant measure.

In the case of lattices one can go somewhat further using a result of H. C. Wang.

Theorem 3.4. *Let G be a connected algebraic subgroup of $Gl(n, \mathbf{R})$ such that each element in the radical R has only real eigenvalues and G/R has no compact factors. If Γ is a lattice in G then Γ is Zariski dense.*

Proof. By (8.28) of [11] $\Gamma\cap R$ is a lattice in R. By (3.3) $(\Gamma\cap R)^\#=R^\#=R$ since R is algebraic. Thus $\Gamma^\#\supseteq R$. Let $\pi\colon G\to G/R$ be the canonical map. By (8.27) of [11] $\pi(\Gamma)$ is a lattice in G/R. Since G/R is an algebraic linear group, the Borel density theorem (or (3.1) a) tells us that $\pi(\Gamma)^\#=G/R$. Now $\pi(\Gamma^\#)=\Gamma^\# R/R=\Gamma^\#/R$ is an algebraic subgroup of G/R which contains $\pi(\Gamma)$. This means $\pi(\Gamma^\#)=G/R$. This together with the fact that $\Gamma^\#\supseteq R$ means that $\Gamma^\#=G$.

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