Multipliers of H^p spaces

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Introduction

The purpose of this paper is to give non-periodic analogues of some results of Duren and Shields [2]. In the process, it is hoped that some of the arguments become clearer, and the key role played by the homogeneous Besov spaces will be highlighted. We describe the convoluteurs of H^p and $\dot{B}^{\alpha}_{ap}$ spaces into spaces of the same type as well as into FL^p spaces. The results were announced in [6].

Notation. We will define the Fourier transform by

$$\hat{f}(\xi) = \int \exp(-i\langle x, \xi\rangle) f(x)\, dx,$$

where

$$\langle x, \xi\rangle = x_1\xi_1 + \ldots + x_n\xi_n,$$

and when we wish to indicate its action on a space, we denote

$$FX = \{\hat{f} | f \in X\}.$$

The space $\mathbf{R}^{n+1}$ is considered as the Cartesian product $\mathbf{R}^n \times \mathbf{R}$, so that each $z \in \mathbf{R}^{n+1}$ can be written $z = (x, t)$, $x \in \mathbf{R}^n$, $t \in \mathbf{R}$. For a function u defined on $\mathbf{R}^{n+1}$ we write

$$\frac{\partial u}{\partial t} = D_{n+1} u.$$

For functions f defined on $\mathbf{R}^n$, we denote

$$\|f\|_p = \left(\int_{\mathbf{R}^n} |f(x)|^p\, dx\right)^{1/p}, \quad 0 < p < \infty,$$

$$= \operatorname{ess\,sup} |f(x)| \quad p = \infty,$$

and in general, when $\left(\int |\Phi(z)|^p\, dz\right)^{1/p}$ is written, it is to be interpreted as $\operatorname{ess\,sup} |\Phi(x)|$ for $p = \infty$. For a function u defined on $\mathbf{R}^{n+1}_+ = \{(x, t) | x \in \mathbf{R}^n, t > 0\}$, we set

$$M_p(u; t) = \|u(\cdot, t)\|_p,$$

and then we can define $\dot{B}^{\alpha}_{pq}$, the homogeneous Besov space, considered by Herz [5]. If k is the smallest non-negative integer greater than $\frac{1}{2}\alpha$ and u is the temperature with initial value f, we have $\dot{B}^{\alpha}_{pq}=\{f\in S' | (\int_0^\infty [t^{k-(\alpha/2)} M_p(D^k_{n+1}u;\, t)]^q t^{-1}\, dt)^{1/q}<\infty\}$, with the obvious seminorm, which is denoted $\dot{B}^{\alpha}_{pq}(f)$.

The Gauss—Weierstrass kernel is denoted by W, where

$$W(x, t) = (4\pi t)^{-(n/2)} \exp(-|x|^2/4t), \quad x\in \mathbf{R}^n, \quad t>0,$$

and the temperature u above is defined for any $f\in S'$ by

$$u(x, t) = \langle f, W(x-\cdot, t)\rangle.$$

The spaces K^{α}_{pq} are defined in [5], but we shall mainly use the alternate characterization given in [4].

The H^p spaces are defined for all $p>0$ in [3], and we use mainly the characterization on page 183.

In connection with the K^{α}_{pq} spaces, we denote $v(\xi)=b_n|\xi|^n$, where $\xi\in\mathbf{R}^n$ and b_n is the volume of the ball of radius 1 in R^n. Throughout A, B, C will denote constants whose value may change from line to line.

Finally, we write

$$Cv(X, Y) = \{k| \text{ if } f\in X,\ k*f\in Y \text{ and } \|k*f\|_Y \leqq B\|f\|_X\},$$

and write also

$$M(X, Y) = \{\hat{k} | k\in Cv(X, Y)\}.$$

1. Convoluteurs between H^p spaces and FL^p spaces

The following lemma provides us with a generous supply of test functions in the H^p spaces.

Lemma 1. *$f(x)=D^k_{n+1}W(x, s)\in H^p(R^n)$ iff $p>(n/n+2k)$ and*

$$\|D^k_{n+1}W(\circ, s)\|_{H^p} = Bs^{-k-(n/2p')} = Bs^{-k-n/2+(n/2p)}.$$

Proof. We note that $D^k_{n+1}W(x, t)=t^{-k}W(x, t)P(|x|^2/4t)$, where P is a polynomial of degree k. To apply Theorem 11 of [3], we must solve the heat equation in R^{n+1}_+ with initial value $f(x)$. By uniqueness the solution is

$$u(x, t) = D^k_{n+1}W(x, t+s),$$

which gives,

$$u^+(x) = \sup_{t>0} |D^k_{n+1}W(x, t+s)| = \sup_{t>0} \left|(t+s)^{-k}W(x, t+s)P\big(|x|^2/4(t+s)\big)\right|.$$

One checks easily

$$u^+(x) = B|x|^{-2k-n}, \quad |x|^2 \geqq As,$$
$$= Bs^{-k-n/2}|P(|x|^2/4s)| \exp(-|x|^2/4s), \quad |x|^2 \leqq As.$$

A trivial computation of the L^p norm then gives the result. The integral of u^+ converges at infinity iff $p>(n/n+2k)$.

The next lemma now allows us to characterize convoluteurs from H^p into FL^q.

Lemma 2. *If* $0<p<1$, $H^p \subseteqq \dot{B}_{11}^{n(1-1/p)}$.

This is proved on page 176 of [3].

For the convenience of the reader we recall the convolution theorem for Besov spaces.

Lemma 3. $\dot{B}^\alpha_{pq} * \dot{B}^\beta_{rs} \subseteqq \dot{B}^{\alpha+B}_{tu}$, *where*

$$1/t = 1/p+1/r-1, \quad 1/u = 1/s+1/q.$$

Our first theorem is an easy consequence of these results.

Theorem 1. *If* $0<p<1 \leqq q \leqq \infty$,

$$Cv(H^p, FL^q) = \{k | \hat{k} \in K_{q\infty}^{n(1/p-1)}\}.$$

Proof. If $k \in Cv(H^p, FL^q)$, then for any $f \in H^p$,

$$\|k*f\|_{FL^q} = \|\hat{k}\hat{f}\|_q \geqq B\|f\|_{H^p}.$$

Choose m so that $p>n/n+2m$, and take $f(x)=D^m_{n+1}W(x,s)$. The inequality becomes

$$\|\hat{k}(\xi)\exp(-s|\xi|^2)|\xi|^{2m}\|_q \leqq Bs^{-m-n/2+n/2p},$$

which says precisely that

$$\operatorname{ess\,sup} s^{1/2(2m+n-n/p)}\|\hat{k}(\xi)|\xi|^{2m}\exp(-s|\xi|^2)\|_q \leqq B,$$

and, except for the choice of constants, this is $\varkappa_{q\infty}^{2m,n/p-n}(\hat{k})$ in the notation of Flett [4]. For $2m>n(1/p-1)$, which we have assumed, it gives an equivalent norm on $K_{p\infty}^{n(1/p-1)}$.

Conversely suppose $\hat{k} \in K_{q\infty}^{n(1/p-1)}$. It follows from *Lemma 2* that if $f \in H^p$ $\hat{f} \in K_{\infty 1}^{n(1-1/p)}$ and by the multiplication theorem for K^α spaces due to Flett, $\hat{k}\hat{f} \in K^0_{q1} \subseteqq K^0_{qq} = L^q$.

Corollary 1.1. *Any of the following equivalent conditions is necessary and sufficient in order that a function m be a multiplier from* $H^p \to FL^q$, *where* $0<p<1 \leqq q \leqq \infty$.

(a) $$\Big(\int_{R \leqq |x| \leqq 2R} |m(x)|^q\, dx\Big)^{1/q} \leqq CR^{n(1-1/p)}, \quad \text{for all} \quad R>0.$$

(b) *For any* $\beta>n(1/p-1)$ *and any* $R>0$,

$$\left(\int_{|x|\leq R}(|x|^{\beta}|m(x)|)^{q}\,dx\right)^{1/q}\leqq C_{\beta}R^{\beta-n(1/p-1)}.$$

(c) *For any* $\beta>n(1/p-1)$ *and any* $R>0$,

$$\left(\int_{R_n}[|x|^{\beta}\exp(-4\pi^2|x|^2R)|m(x)|]^{q}\,dx\right)^{1/q}\leqq C_{\beta}R^{-1/2[\beta-n(1/p-1)]}$$

(d) $$\left(\int_{|x|>R}[|m(x)||x|^{n(1/p-1/q-1)}]^{q}\,dx\right)^{1/q}\leqq CR^{-n},$$

for all $R>0$.

Remarks 1) The same argument proves that

$$Cv(\dot{B}_{1s}^{n(1-1/p)},FL^{q})=\{k|\hat{k}\in K_{q\infty}^{n(1/p-1)}\},$$

$1\leqq s\leqq q$.

2) The theorems are more naturally stated for multipliers, and if we note that FL^1 is usually denoted $A(R^n)$, and FL^{∞} is the space of pseudomeasures denoted PM, we can restate the most important cases above in the form

$$M(H^p,A)=K_{1\infty}^{n(1/p-1)}$$
$$M(H^p,L^2)=K_{2\infty}^{n(1/p-1)}$$
$$M(H^p,PM)=K_{\infty\infty}^{n(1/p-1)},$$

if $0<p<1$.

When $p=1$, *Lemma 2* is not valid which causes a weakening of the theorem.

Theorem 2. *If* $2\leqq q\leqq\infty$, *then*

$$Cv(H^1,FL^q)=\{k|\hat{k}\in K_{q\infty}^{0}\}.$$

Proof. Repeat the proof of *Theorem 1* using now the fact that $H^1\subseteqq\dot{B}_{12}^{0}$. For $\hat{k}\in K_{q\infty}^{0}$, $f\in H^1$, the multiplication theorem gives only $\hat{k}\hat{f}\in K_{q2}^{0}$ which causes the restriction $2\leqq q\leqq\infty$.

Corollary 2.1. *Any of the following equivalent conditions is necessary and sufficient in order that a function m be a multiplier from* $H^1\to FL^q$, *where* $2\leqq q\leqq\infty$.

(a) $$\left(\int_{R\leqq|x|\leqq 2R}|m(x)|^{q}\,dx\right)^{1/q}\leqq C,\quad \textit{for all}\quad R>0.$$

(b) *For any* $\beta>0$ *and any* $R>0$,

$$\left(\int_{|x|\leqq R}[|x|^{\beta}|m(x)|]^{q}\,dx\right)^{1/q}\leqq C_{\beta}R^{\beta}.$$

(c) *For any* $\beta>0$ *and any* $R>0$,

$$\left(\int_{R_n}[|x|^{\beta}\exp(-4\pi^2|x|^2R)|m(x)|]^{q}\,dx\right)^{1/q}\leqq C_{\beta}R^{-(\beta/2)}.$$

(d) *For any* $R>0$

$$\left(\int_{|x|\geqq R} |m(x)|^q |x|^{-n}\, dx\right)^{1/q} \leqq CR^{-n},$$

Remarks 1) Again we can prove that

$$Cv(\dot{B}^0_{1s}, FL^q) = \{k \mid \hat{k} \in K^0_{q\infty}\}, \quad 1 \leqq s \leqq q, \quad 2 \leqq q \leqq \infty.$$

2) The case $q=2$ is interesting. It says that

$$M(H^1, L^2) = K^0_{2\infty} = \left\{m \,\middle|\, \sup_k \int_{(1/2)k\leqq|\xi|\leqq k} |m(\xi)|^2 d\xi \leqq B\right\}.$$

Theorem 3. *For* $1\leqq q\leqq 2$

$$K_{qr} \subsetneqq M(H^1, FL^q) \subsetneqq K^0_{q\infty},$$

where $1/r=1/q-1/2$.

The proof is immediate. Note that as $q\to 2$, $r\to+\infty$. Once again H^1 can be replaced by $\dot{B}^0_{1s}$, $1\leqq s\leqq q$. It follows from this result that the conditions given in Corollary 2.1 are necessary for multipliers; the sufficient conditions are given in the next corollary.

Corollary 3.1. *Any of the following equivalent conditions are sufficient for a function m to be a multiplier from* H^1 *into* FL^q *for* $1\leqq q\leqq 2$, $1/r=1/q-1/2$.
(a) *For any* $\beta>0$,

$$\int_0^\infty \left(t^{-q(\beta-\alpha)} \int_{|x|\leqq t} (|x|^\beta |m(x)|)^q\, dx\right)^{r/q} t^{-1}\, dt < +\infty.$$

(b) *For any* $\beta>0$,

$$\int_0^\infty \left(t^{-q\beta/2} \int_{R_n} (|x|^\beta \exp(-4\pi^2|x|^2 t)\,|m(x)|)^q\, dx\right)^{r/q} t^{-1}\, dt < +\infty.$$

(c) $\int_0^\infty \left(t^n \int_{|x|\geqq t} |m(x)|^q |x|^{-nq/r-n}\, dx\right)^{r/q} t^{-1}\, dt < +\infty.$

(d) If $b_j = \left(\int_{2^j\leqq|x|\leqq 2^{j+1}} |m(x)|^q\, dx\right)^{1/q}$, then $\{b_j\}\in l^r$.

The inclusion relation $H^p \subsetneqq \dot{B}^{n(1-1/p)}_{11}$, $0<p<1$, allows us to say something about the growth of the Fourier transform of an H^p function. We state this, and then note that this result is best possible in a certain sense. This follows as in Duren and Shields [2].

Theorem 4. (1) *If* $f\in H^p$, $0<p<1$, *then the least decreasing radial majorant of* $|\xi|^{-n/p}\hat{f}$ *is integrable.*
(2) *If* $f\in H^1$, *the least decreasing radial majorant of* $|\xi|^{-n/2}\hat{f}$ *is square integrable.*

Proof. For $f\in H^p$, $f\in \dot{B}^{n(1-1/p)}_{11}$ and by the Fourier transform theorem of [2], $\hat{f}\in K^{n(1-1/p)}_{\infty 1}$. By definition of the K^α_{pq} spaces this means that $v(\xi)^{1-1/p-1}\hat{f}\in {}_\infty L_1 \subsetneqq$

${}_1L_1=L^1$. Since ${}_\infty L_1$ is precisely the set of functions whose least decreasing radial majorant is in L^1, the result follows.

For $f\in H^1$, we know that $f\in\dot{B}^0_{12}$ so that the Fourier transform theorem implies that $\hat{f}\in K^0_{\infty 2}$, which gives the second result.

Remarks 1) A result of Fefferman (generalized by Björk, see [10] shows that if $f\in H^1$, then $|\xi|^{-n}\hat{f}\in L^1$. He also shows that the least decreasing radial majorant of $\hat{f}|\xi|^{-n}$ need not be integrable, and that Theorem 3 (ii) gives the best result about the least decreasing radial majorant. On the other hand, we should note that for $f\in\dot{B}^0_{11}$, then the least decreasing radial majorant of $|\xi|^{-n}\hat{f}$ is integrable.

2) Since the first draft of this paper appeared, this theorem has been partially generalized by Peetre [10]. He has shown that for $f\in H^p$, $v(\xi)^{1-2/p}\hat{f}\in L^p$ for $0<p\leqq 2$. However, our result is best possible for integrability and the result of Petree says nothing about the radial majorant. Incidentally in our terms the result of Peetre is proved by first showing that if $f\in H^p$, then $\hat{f}\in K^{n(1-1/p)}_{\infty 1}$. This is a simple consequence of our result that in fact $\hat{f}\in K^{n(1-1/p)}_{\infty 1}$ (an alternate proof may be given by noting that $e^{ix\xi}\in\dot{B}^\alpha_{\infty\infty}$ for any α real with $\dot{B}^\alpha_{\infty\infty}(e^{i\langle x,\xi\rangle})=B|\xi|^\alpha$, and that $(H^p)^*=\dot{B}^{n(1/p-1)}_{\infty\infty}$. The general result then follows by interpolation.

Corollary 4.1. *Suppose* $f\in H^p$, $0<p<1$. *Then for any* $|\alpha|\leqq k$, *with* k *an integer,* $k\leqq n(1/p-1)$,

$$\langle f, x^\alpha\rangle = 0.$$

Proof. This follows because $x^\alpha=0$ in $(H^p)^*$.

Next we indicate the sense in which *Theorem 4* is best possible.

Theorem 5. *If* g *is a function such that for every* $f\in H^p$, $\hat{f}(\xi)g(\xi)$ *is integrable, then there is a constant* B *(depending only on* g*) such that*

$$\int_{R\leqq|\xi|\leqq 2R}|\xi|^{n/p}|g(\xi)|\,d\xi\leqq B.$$

Proof. Our assumption implies that $g\in M(H^p, FL^1)$ by the closed graph theorem, but by *Theorem 2*, this space is $K^{n(1/p-1)}_{1\infty}$, and hence, $g(\xi)|\xi|^{n/p}\in{}_1L_\infty$, which gives the result.

It is perhaps worth while to note that our *Theorem 1* is not as strong as the corresponding result in Duren and Shields because of our restriction $1\leqq q$. These results can be extended to cover these spaces using the spaces $\dot{B}^\alpha_{pq}$ for $0<q<1$ (also $0<p<1$) developed by Peetre [11]; see also Triebel [13].

Another application of our method is a new version of Paley's theorem which makes use of sets lacunary in the sense of Herz. Recall that Herz says that a set $E\subseteqq R_n$ is lacunary if χ_E, the characteristic function of E, is in $K^0_{1\infty}$.

Corollary 5.1. *If E is a lacunary subset of R_n and $f\in H^p(R_n)$, $0<p<1$, then for any $1\leqq q\leqq\infty$,*

$$\left(\int_E \left||\xi|^{n(1-1/p)}\hat{f}(\xi)\right|^q d\xi\right)^{1/q}\leqq A\|f\|_p.$$

For $p=1$, this is true with $2\leqq q\leqq\infty$.

Proof. Since $\chi_E\in K^0_{q\infty}$ for any q ([4], p. 549), it follows that $v(\xi)^{-(1/p-1)}\chi_E\in K^{n(1/p-1)}_{q\infty}=M(H^p, FL^q)$ by *Theorem 1* and the result follows. For $p=1$, we must apply *Theorem 2.*

Paley's theorem is the special case $p=1$, $q=2$. We can also give results for nonlacunary sets with appropriate weights. The next result, if true for $p=1$, would have applications to the study of radial multipliers from $H^1\to FL^1$, as was pointed out to the author by D. Oberlin. Our methods do not give the result for $p=1$

Corollary 5.2. *If $f\in H^p$, $0<p<1$, then*

$$\sum_0^\infty\left(\int_{2^k\leqq|\xi|\leqq 2^k+1}|\hat{f}(\xi)|\,d\xi\right)2^{k(1-n/p)}\leqq A\|f\|_{H^p}.$$

Proof. This follows because if $E_k=\{\xi\,|\,2^k\leqq|\xi|\leqq 2^k+1\}$,

$$\sum 2^{-k(n-1)}\chi_{E_k}\in K^0_{1\infty}$$

and *Lemma 2* implies that if $f\in H^p$, $\hat{f}\in K^{n(1-1/p)}_{\infty 1}$, and the multiplication theorem for K^α_{pq} spaces gives

$$\sum 2^{-k(n-1)}\chi_{E_k}\hat{f}\in K^{n(1-1/p)}_{11},$$

which is precisely the theorem.

2. Convoluteurs between H^p spaces and Besov spaces

We have noted that the containing Banach space considered by Duren and Shields is the homogeneous Besov space $\dot{B}^{n(1-1/p)}_{11}$. The techniques previously used will now be applied to the characterization of convoluteurs between H^p spaces and several Besov spaces, including the containing Banach space.

Theorem 6. *If $p\leqq q$, then $C_v(\dot{B}^\alpha_{1p}, \dot{B}^\beta_{aq})=\dot{B}^{\beta-\alpha}_{a\infty}$.*

Proof. The convolution theorem for $\dot{B}^\alpha_{ap}$ spaces implies that

$$\dot{B}^{\beta-\alpha}_{a\infty}\subseteqq Cv(\dot{B}^\alpha_{1p}, \dot{B}^\beta_{ap})\subseteqq Cv(\dot{B}^\alpha_{1p}, \dot{B}^\beta_{aq}),$$

since $p\leqq q$.

Let p, q be arbitrary (we do not need $p\leqq q$ here) and suppose $k\in C_v(\dot{B}^\alpha_{1p}, \dot{B}^\beta_{aq})$. For any $f\in\dot{B}^\alpha_{1p}$, we have the inequality

$$\dot{B}^\beta_{aq}(k*f)\leqq C\dot{B}^\alpha_{1p}(f).$$

Let r be a nonnegative integer such that $r > +\frac{1}{2}|\alpha|$. For this r, $f(x) = D^r_{n+1} W(x, s)$ is in $\dot{B}^\alpha_{1p}$ and

$$\dot{B}^\alpha_{1p}(f) = Bs^{-(\alpha/2)-r}.$$

For this f, $k*f = D^r_{n+1}u(x, s)$, where u is the temperature with initial value k. To compute its $\dot{B}^\beta_{aq}$ norm, let L be a nonnegative integer such that $L > \frac{1}{2}\beta$ and then the B norm will be given by an integral involving the Lth time derivative of the solution of the heat equation with initial value $k*f$. By uniqueness, this solution is $D^r_{n+1}u(x, t+s)$, and it follows that

$$\dot{B}^\beta_{aq}(k*f) = \Big(\int_0^\infty [t^{L-\beta/2} M_a(D^{r+L}_{n+1}u;\ t+s)]^q t^{-1}\,dt\Big)^{1/q}.$$

This can be estimated since $\varrho \to M_a(D^{r+L}_{n+1}u;\ \varrho)$ is a decreasing function of ϱ, and thus

$$(s/2)^{L-\beta/2} M_a(D^{r+L}_{n+1}u;\ 2s)(\ln 2)^{1/q}$$
$$\leqq \Big(\int_{s/2}^s [t^{L-\frac{1}{2}\beta} M_a(D^{r+L}_{n+1}u;\ t+s)]^q t^{-1}\,dt\Big)^{1/q}$$
$$\leqq \dot{B}^\beta_{aq}(k*f) \leqq CBs^{-\alpha/2-r},$$

and collecting terms, we see that

$$s^{r+L-1/2(\beta-\alpha)} M_a(D^{r+L}_{n+1}u;\ s) \leqq B'C.$$

Since $r+L > 1/2(\beta-\alpha)$, $k \in \dot{B}^{\beta-\alpha}_{a\infty}$ and

$$\dot{B}^{\beta-\alpha}_{a\infty}(k) \leqq B'\|k\|,$$

where $\|k\|$ is the norm of k as a convoluteur from $\dot{B}^\alpha_{1p} \to B^\beta_{aq}$.

Theorem 7. *If* $0 < p < 1 \leqq q \leqq \infty$ *or if* $p = 1$, $2 \leqq q \leqq \infty$, *then*

$$Cv(H^p, \dot{B}^\beta_{aq}) = \dot{B}^{\beta-n(1-1/p)}_{a\infty}.$$

Proof. If $0 < p < 1$, $H^p \subseteqq \dot{B}^{n(1-1/p)}_{11}$, and thus

$$\dot{B}^{\beta-n(1-1/p)}_{a\infty} = Cv(\dot{B}^{n(1-1/p)}_{11}, \dot{B}^\beta_{aq}) \subseteqq C_v(H^p, \dot{B}^\beta_{aq}).$$

The converse follows by considering the test functions $f(x) = D^r_{n+1}W(x, s)$, where r is chosen so that $p > n/n+2r$, and noting that the norm of f in H^p is of the same order of magnitude as its norm in $\dot{B}^{n(1-1/p)}_{11}$.

Corollary 7.1. *If* $1 \leqq a < \infty$, $1 < q \leqq \infty$, $1 \leqq p < \infty$, *then*

$$Cv(\dot{B}^\alpha_{ap}, \dot{B}^\beta_{\infty q}) = \dot{B}^{\beta-\alpha}_{a',\infty}.$$

Proof. The convolution theorem for the Besov spaces gives one direction. The other direction follows by duality. If $k \in Cv(\dot{B}^\alpha_{ap}, \dot{B}^\beta_{\infty q})$, then

$$\dot{B}^\beta_{\infty,q}(k*f) \leqq C\dot{B}^\alpha_{ap}(f).$$

Now $B^{\beta}_{\infty q}=(B^{-\beta}_{1q'})*$, so it follows from functional analysis that

$$|\langle k*f, g\rangle| \leqq \dot{B}^{\beta}_{\infty q}(k*f)\dot{B}^{-\beta}_{1q'}(g)$$
$$\leqq C\dot{B}^{\alpha}_{ap}(f)\dot{B}^{-\beta}_{1q'}(g).$$

The left hand side also equals $\langle f, k*g\rangle$ and this gives

$$|\langle f, k*g\rangle| \leqq C\dot{B}^{\alpha}_{1p}(f)\dot{B}^{-\beta}_{1q'}(g),$$

i.e., $k*g$ defines a continuous linear functional on $\dot{B}^{\alpha}_{ap}$. This gives

$$k*g\in \dot{B}^{-\alpha}_{a',p'},$$

or alternately, $k\in Cv(\dot{B}^{-\beta}_{1q'}, \dot{B}^{-\alpha}_{a'p'})$ and we use the proof of *Theorem 5* (recall that this direction had no requirement on the second index) to conclude that

$$k\in \dot{B}^{\beta-\alpha}_{a'\infty}.$$

The fundamental inclusion between H^p spaces for $0<p<1$ and the homogeneous Besov spaces given by *Lemma 2*, combined with the fact that our test functions behave in the same manner in both spaces, leads to the next result. The case $p=1$ was proved already in [8].

Theorem 8. *If* $0<p<1\leqq q<\infty$, *or if* $p=1, 2\leqq q\leqq\infty$

$$Cv(H^p, L^q) = \dot{B}^{n(1/p-1)}_{q\infty}.$$

Proof. If $k\in Cv(H^p, L^q)$, then

$$\|k*f\|_q \leqq C\|f\|_p,$$

for each $f\in H^p(R^n)$; apply this with $f(x)=D^m_{n+1}W(x, s)$, where $p>\dfrac{n}{n+2m}$. Since $k*f=D^m_{n+1}u(\cdot, s)$, it follows that

$$\|D^m_{n+1}u(\cdot, s)\|_q \leqq B\|f\|_p = Bs^{-m-n/2+n/2p},$$

which is precisely the requirement that $k\in\dot{B}^{n(1/p-1)}_{q\infty}$.

Conversely, if $k\in\dot{B}^{n(1/p-1)}_{q\infty}$, $f\in H^p$, then *Lemma 2* and the convolution theorem for Besov spaces shows that $k*f\in\dot{B}^0_{q1}\subseteqq L^q$. For $p=1$, $f\in\dot{B}^0_{12}$ so that the convolution theorem gives $k*f\in\dot{B}^0_{q2}\subseteqq L^q$ for $2\leqq q\leqq\infty$.

This is an example of a theorem which is certainly capable of extension for $p\leqq q<1$. The above result shows that

$$s^{m+n/2-n/2p}\|D^m_{n+1}u(\cdot, s)\|_q \leqq B$$

and the announcement of Peetre indicates that this is an equivalent norm on $\dot{B}^{n(1/p-1)}_{q\infty}$.

3. Application of the above results

We give two applications of the above results. The first is a straightforward application of a general result relating topological tensor products and the space of translation invariant maps. The second answers a question raised by Jan-Erik Björk at the Nordic Summer School at Grebbestad.

Our first result shows that the Besov spaces can be built up from the H^p spaces by the operations of duality, topological tensor products and interpolation. Given two quasinormed spaces X and Y, we define

$$X*Y = \left\{\sum_{i=1}^{\infty} f_i * g_i \middle| f_i \in X,\ g_i \in Y,\ \sum \|f_i\|_X \|g_i\|_Y < \infty\right\}$$

and equip this linear space with the norm

$$|||h||| = \inf\left\{\sum \|f_i\| \|g_i\| \,\middle|\, h = \sum f_i * g_i\right\}.$$

Theorem 9. *The identity map from*

$$H^p * L^{q'} \to \dot{B}^{n(1-1/p)}_{q'1}$$

is bicontinuous for $0<p<1\leqq q<\infty$, *or* $p=1$, $2\leqq q<\infty$.

Proof. The map is well-defined by *Lemma 2*, the fact that $L^q \subseteqq B^0_{q\infty}$ for any q and the convolution theorem for the Besov spaces. We also get the estimate

$$\dot{B}^{n(1-1/p)}_{q'1}(f_i * g_i) \leqq C\dot{B}^{n(1-1/p)}_{11}(f_i)\dot{B}^0_{q'\infty}(g_i)$$

$$\leqq C\|f_i\|_{H^p}\|g_i\|_{L^{q'}},$$

and thus if $h=\sum_i f_i * g_i$,

$$\dot{B}^{n(1-1/p)}_{q'1}(h) \leqq C|||h|||.$$

If $p=1$, then $H^1 \subseteqq \dot{B}^0_{12}$ and since $1\leqq q'\leqq 2$, $L^{q'} \subseteqq \dot{B}^0_{q2}$, and we get the corresponding result.

Conversely, if $h\in H^p * L^{q'}$, there is an $F\in(H^p * L^{q'})*$ such that $F(h)=|||h|||$, $\|F\|=1$. This is because $H^p * L^{q'}$ is a normed space even though H^p is only quasinormed. Now we apply the next proposition. The next result is well-known in greater generality but we include its proof for completeness.

Proposition. $Cv(H^p, L^q)=(H^p * L^{q'})^*$, $1<q<\infty$.

Proof. Suppose $k\in Cv(H^p, L^q)$. We want to define a continuous linear functional on $H^p * L^{q'}$, and of course it suffices to do this for $f\in H^p$, $g\in L^{q'}$. Then $k*f\in L^q$, and it makes sense to define

$$F(f*g) = \int k*f(-x)g(x)\,dx, = k*f*g(0),$$

and extend linearly. The definition of the norm assures us that there is no problem with convergence, and the fact that this is $k*f*g(0)$ assures that it is well-defined. If $h\in H^p*L^{q'}$,

$$|F(h)| = \left|\sum_i \int k*f_i(-x)g_i(x)\,dx\right|$$

$$\leqq \sum_i \|k*f_i\|_q \|g_i\|_{q'} \leqq \|k\| \sum \|f_i\|_p \|g_i\|_{q'} \leqq \|k\|\,\|h\|.$$

Conversely, if F is a bounded linear functional on $H^p*L^{q'}$, fix $f\in H^p$ and consider the map $T\colon g\to F(f*g)$. It is a linear map such that

$$|T(g)| \leqq \|F\|\,\|f*g\| \leqq \|F\|\,\|f\|_p \|g\|_{q'}.$$

By the characterization of the dual of L^q, $\exists!\ Tf(x)$ in $L^q(R^n)$ such that

$$F(f*g) = \int Tf(-x)g(x)\,dx.$$

Uniqueness implies that T is linear and

$$\int T(\tau_h f)(-x)g(x)\,dx = F(\tau_h f*g) = F(f*\tau_h g)$$
$$= \int Tf(-x)g(x-h)\,dx = \int T_h Tf(-x)g(x)\,dx,$$

and since this holds for all $g\in L^{q'}$,

$$T(\tau_h f) = \tau_h(Tf),$$

so T defines a translation invariant operator from $H^p\to L^{q'}$. To conclude the proof of *Theorem 9*, we note that

$$|||h||| = F(h) = k*h(0) = \int k(-x)h(x)\,dx$$

and $k\in Cv(H^p, L^q)=\dot B_{q\infty}^{n(1/p-1)}$, by *Theorem 7*, while $h\in \dot B_{q'1}^{n(1-1/p)}$, which gives

$$|||h||| \leqq B\dot B_{q\infty}^{n(1/p-1)}(k)\|h\|_{\dot B_{q'1}^{n(1-1/p)}}.$$

At the Nordic Summer School in Grebbestad, Jan-Erik Björk asked whether *Theorem 1.1* of his paper [1] could be extended to allow

$$\int_{2^j\leqq|\xi|\leqq 2^{j+1}} |\hat v(\xi)|^2\,d\xi \leqq A^2$$

in place of $|\hat v(\xi)|\leqq C(1+|\xi|^2)^{-n/4}$. Our *Theorem 2* gives this immediately.

Theorem 10. *Suppose* $v\in E'(R^n)$ *satisfies*

$$|\hat v(\xi)| \leqq A$$

$$\int_{2^j\leqq|\xi|\leqq 2^{j+1}} |\hat v(\xi)|^2\,d\xi \leqq A^2, \quad -\infty<j<\infty.$$

Then there is a constant A_v *such that*

$$\|v*f\|_{BMO} \leqq A_v\|f\|_\infty, \quad \text{for every } f\in C_0^\infty.$$

Note. As remarked in *Lemma 1.1* of [1] this implies that on $H^1 \cap C_0^\infty(R^n)$, $\|v*f\|_{H^1} \leqq A_v \|f\|_{H^1}$, and by further remarks in that paper (see also [8]),

$$\|v*f\|_p \leqq A_v \|f\|_p, \quad 1 < p < \infty.$$

Proof. We normalize and assume $A=1$, $\operatorname{supp} v \subset \{|x| \leqq 1\}$. *Lemma 1.3* of [1] implies that for large cubes

$$\frac{1}{|Q|}\int_Q |v*f(x)|\,dx \leqq 3^{n/2}\|f\|_\infty.$$

It remains to consider small cubes Q, centered at the origin, of volume $|Q| \leqq \delta_v^n$. We set, with Björk, $f_1 = f$ for $|y_\nu| < 2$, $\nu = 1, \ldots, n$, and $f_i = 0$ otherwise, and note that

$$v*f(x) = v*f_1(x), \quad \text{for} \quad x \in Q.$$

Our assumption on v implies, by *Theorem 2*, that

$$v : H^1 \to L^2,$$

and hence $v : L^2 \to BM0$. We estimate

$$\|v*f_1\|_{BM0} \leqq C\|f_1\|_2$$

and then, in particular for our Q,

$$\frac{1}{|Q|}\int_Q |v*f_1(x) - \lambda|\,dx \leqq C\|f_1\|_2 \leqq C4^{n/2}\|f\|_\infty$$

$$= \frac{1}{|Q|}\int_Q |v*f(x) - \lambda|\,dx \leqq C4^{n/2}\|f\|_\infty.$$

4. The case $p \leqq 2 \leqq q$

The appearance of the Lipschitz spaces as sufficient ([7], [12]) or necessary [8] conditions for convoluteurs will now be shown to be closely related to Sobolev type theorems. We need the following lemmas.

Lemma 3. *$Cv(\dot{B}^\alpha_{ps}, \dot{B}^\beta_{qr}) = Cv(\dot{B}^{\alpha+r}_{ps}, \dot{B}^{\beta+r}_{qr})$, for any α, β, r real, $1 \leqq p, q, r, s \leqq \infty$.*

Proof. This is an immediate consequence of the fact that $R^r : \dot{B}^\alpha_{ps} \to \dot{B}^{\alpha+r}_{ps}$ for any α.

Lemma 4. *If $X_1 \subseteqq Y_1$, $X_2 \supseteqq Y_2$, where both inclusions are topological, then*

$$Cv(Y_1, Y_2) \subseteqq Cv(X_1, X_2).$$

This is trivial.

Lemma 5. *If* $1 \leqq s \leqq \infty$, $Cv(L^p, L^q) \subseteq Cv(\dot{B}^{\alpha}_{ps}, \dot{B}^{\alpha}_{qs})$, *for* α *real*, $1 \leqq p \leqq q \leqq \infty$.

Proof. By *Lemma 3*, we may assume that $\alpha < 0$. Let $k \in Cv(L^p, L^q)$ and consider an arbitrary element of $\dot{B}^{\alpha}_{ps}$. Its norm is computed by forming the temperature u with initial value f, and computing

$$\dot{B}^{\alpha}_{ps}(f) = \Big(\int_0^\infty [t^{-\alpha/2} M_p(u; t)]^s t^{-1}\, dt\Big)^{1/s},$$

in the case $1 \leqq s < \infty$. The $\dot{B}^{\alpha}_{ps}$ norm of $k*f$ is computed by forming the temperature w with initial value $k*f$, and note that $w = k*u$. Hence,

$$\dot{B}^{\alpha}_{qs}(k*f) = \Big(\int_0^\infty [t^{-\alpha/2} M_q(u, t)]^q t^{-1}\, dt\Big)^{1/q},$$

and since $w(\cdot, t) = k*u(\cdot, t)$ with $k \in Cv(L^p, L^q)$,

$$M_q(w; t) \leqq \|k\|\, M_p(u; t),$$

and thus

$$\dot{B}^{\alpha}_{qs}(k*f) \leqq B\|k\|\, \dot{B}^{\alpha}_{ps}(f).$$

The converse of this theorem is not true. Indeed we've seen in *Theorem 6* that in some cases the right hand side is a homogeneous Besov space, and as we have investigated in [9] homogeneous Besov spaces are not invariant under multiplication by $e^{i\langle x, h\rangle}$ while $Cv(L^p, L^q)$ is invariant. Using the results of [9] we can prove that

$$Cv(L^p, L^q) = \{k \mid e^{i\langle x,h\rangle} k \in Cv(\dot{B}^0_{p\infty}, \dot{B}^0_{q\infty}) \quad \text{for all} \quad h \in R^n,$$

with

$$\|e^{i\langle \cdot, h\rangle} k\| \leqq C\|k\|\},$$

for $1 < p \leqq q \leqq \infty$.

These results allow us to give a quick and enlightening proof of some cases of *Theorem 3* of [5].

Theorem 11. *Suppose* $\beta \leqq \alpha$, $1/a - \alpha/n = 1/b - \beta/n$ *and* $p \leqq q$. *Then* $\dot{B}^{\alpha}_{ap} \subseteq \dot{B}^{\beta}_{bq}$

Proof. If $\beta = \alpha$, there is nothing to prove. Note that since $0 \leqq \frac{1}{a} \leqq 1$ with a corresponding inequality for $1/b$, we see that

$$\alpha - \beta = n(1/a - 1/b) \leqq n.$$

First, we suppose that $\alpha - \beta < n$. The method of our proof requires us to assume that $1 < a,\ b < \infty$. Rewriting the equality above, we see that $\dfrac{1}{a} - \dfrac{\alpha-\beta}{n} = \dfrac{1}{b}$, and hence by Sobolev's inequality

$$R^{\alpha-\beta}\colon L^a \to L^b$$

so $R^{\alpha-\beta}\in Cv(\dot{B}^{\lambda}_{as}, \dot{B}^{\lambda}_{bs})$ for any λ real. In particular

$$\dot{B}^{\alpha}_{ap} = R^{\alpha-\beta}(\dot{B}^{\beta}_{ap}) \subseteqq \dot{B}^{\beta}_{bp} \subseteqq \dot{B}^{\beta}_{qp}.$$

If $\alpha-\beta=n$, we have $a=1$ and this method cannot apply.

We can also recover many of the results of [7, 8, 12] by combining *Lemma 5* with the convolution theorem for homogeneous Besov spaces.

Theorem 12. *For* $1<p\leqq 2\leqq q<\infty$,

$$Cv(L^p, L^q) = Cv(\dot{B}^0_{p2}, \dot{B}^0_{q2}).$$

Proof. Lemma 5 gives the inclusion of the left hand side in the right hand side; the reverse inclusions follow from $L^p\subseteqq B^0_{p2}$ and $B^p_{q2}\subseteqq L^q$ and an application of *Lemma 4.*

Theorem 12 of [7] follows immediately from the above result and the convolution theorem for Besov spaces, while *Theorem 11* requires the inclusion relations between Besov spaces and L^p spaces, which are versions of Sobolev's theorems.

The above results combined with results from [3], and [5] show that Calderon—Zygmund operators preserve the Besov spaces. Indeed as remarked on page 150 of [3], both operators satisfying the Mikhlin—Hörmander condition and Calderon—Zygmund operators map $H^1\to H^1$. Thus by the remark after *Theorem 8*, they belong to $\dot{B}^0_{1\infty}$. By the convolution theorem for Besov spaces they map $\dot{B}^{\alpha}_{pq}\to\dot{B}^{\alpha}_{pq}$ for any α real. For $\alpha>0$, since they also map $L^p\to L^p$, $1<p<\infty$, we see that they map $B^{\alpha}_{pq}\to B^{\alpha}_{pq}$, $1<p<\infty$, but by duality the same result follows for $\alpha<0$. Interpolation gives the result at $\alpha=0$. Note that they preserve the homogeneous Besov spaces even for $p=1$ but for the inhomogeneous Besov spaces we must have $1<p<\infty$.

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